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On the decomposition of lattices over orders

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0. Introduction.

We shall extend two basic theorems on decomposition of lattices over orders—'Roiter-Jacobinski Divisibility Theorem' and 'Jacobinski-Swan Cancellation Theorem'—to an arbitary R-order Λ over an arbitary Dedekind domain R. The point is that we do not assume the ambient algebra $A=K\Lambda$ to be separable over the quotient field K of R.

0.0. As for terminology, we mostly follow that of [1] and [2]. However, for a maximal ideal P of R, the suffix P like R_P always denotes the P-adic completion rather than the localization.

A left Λ-lattice L' will be called a local direct summand of another Λ-lattice L if L'_P is a direct summand of L_P for any maximal ideal P.

Write $KL \gg KL'$ if every A-indecomposable direct summand of KL' occurs strictly oftener in KL than in KL'.

Write $M \sim L$ if $L_P \cong M_P$ for any P.

THEOREM 1 (Roiter-Jacobinski type Divisibility). *Suppose that L' is a local direct summand of L. Then*

(i) *L has a direct summand M' such that $M' \sim L'$.*

(ii) *If $KL \gg KL'$, then L' itself is a direct summand of L.*

THEOREM 2 (Jacobinski-Swan type Cancellation). *Assume that the K-algebra $B=\mathrm{End}_A KL$ has the "strong approximation". Then the following cancellation law* (c) *holds.*

(c) *If L' is a local direct summand of $nL=L\oplus L\oplus\cdots\oplus L$ (n-times), then $L\oplus L' \cong M\oplus L'$ implies $L\cong M$.*

0.1. Remark on Theorem 1. (i) is known if A is separable over K (cf. [1] 31.12.) (ii) is known if A is separable over K and moreover K is a global field, i.e., K is a finite extension of the rational number field $\boldsymbol{Q}$ or of the rational function field $\boldsymbol{F}_q(T)$ (cf. [1] 31.32, [4], [6].)

The current proof of (i) heavily depends on the existence of maximal orders, while the proof of (ii) depends on Jordan-Zassenhaus Theorem.

To avoid the use of maximal orders, generalizing the elementary subgroup $E(n, C)$ of $GL(n, C)=M(n, C)^{\times}$, we consider the "elementary subgroup" $E_{\boldsymbol{e}}(B)$ of $B^{\times}$ associated to a given finite set $\boldsymbol{e}$ consisting of mutually orthogonal idempotents of B:

$$E_{\boldsymbol{e}}(B):=\langle 1+eBe' ;\ e,\ e'\in\boldsymbol{e},\ e\neq e'\rangle .$$

Using an almost obvious fact (1.2.1) that $E_{\boldsymbol{e}}(B)$ is always dense in the elementary subgroup $E_{\boldsymbol{e}}(B\otimes\boldsymbol{A})$ of the adelized ring $B\otimes\boldsymbol{A}$, we can reduce the proof of Theorem 1 to an almost local problem (2.0) depending only upon KL and KL' rather than L and L'. This problem is easily solved by applying the well known Lemma of Bass which states: if C is semi-local, then, by the usual embedding $C^{\times}\subset GL(n, C)$, $GL(n, C)=E(n, C)C^{\times}$. In our proof, claims (i) and (ii) are derived simultaneously.

0.2. Remark on Theorem 2. The theorem is known again under the assumption that K is a global field and A is separable over K (cf. [2] 51.28.) Beside that, there is a result of Drozd-Swan (cf. [7] 16.7, [3]), which is closely related to ours and will be recalled at the end of this paragraph. In the known case, the "strong approximation" is in the sense of Eichler-Kneser (cf. [5]), for the norm 1 subgroup $B^{(1)}$ of $B^{\times}$. We shall modify the sense of "strong approximation" by replacing $B^{(1)}$ with the group of Vaserstein $\tilde{E}(B)$ defined as

$$\tilde{E}(B):=\langle (1+xy)(1+yx)^{-1} ;\ x,\ y\in B,\ 1+xy\in B^{\times}\rangle .$$

The group $\tilde{E}(B)$ coincides with $\tilde{E}(1, B, B)$ of [8], and contains $[B^{\times}, B^{\times}]$. If A is separable and K is a global field, $\tilde{E}(B)=B^{(1)}=[B^{\times}, B^{\times}]$.

We say that B has the "strong approximation" if $\tilde{E}(B)$ is dense in $\tilde{E}(B\otimes\boldsymbol{A})$. Our Theorem 2 follows directly from a result of Vaserstein ([8] Th. 3.6) which states: if C is semi-local, then $E(n, C)\cap C^{\times}=\tilde{E}(C)$ for $n\geqq 2$. We do not discuss in this paper, the interesting problem of finding out when "strong approximation" holds. Thus our extension in Theorem 2 remains rather formal. However it still gives us some gain, say, if $B=M(n, C)$ by some K-algebra C with $n\geqq 2$, then our "strong approximation" trivially holds for B (1.2.2). In particular our Theorem 2 includes the above mentioned result of Drozd-Swan.

0.3. Restatements of Theorems. Let $\mathcal{G}(L)$ denote the genus of L, namely $\mathcal{G}(L)$ is the set of all Λ-isomorphism classes of Λ-lattices M such that $M\sim L$. Theorem 1 can be restated as:

THEOREM 1'. *Suppose* $M\in\mathcal{G}(L'\oplus L'')$. *Then*

(i) $M\cong M'\oplus M''$ *by some* $M'\in\mathcal{G}(L')$ *and* $M''\in\mathcal{G}(L'')$

(ii) *If* $KM\gg KL'$, *then* $M\cong L'\oplus M''$.

When Theorem 1 (ii) is granted, the cancellation law (c) of Theorem 2 can be restated as

(c′) The map $X \mapsto X \oplus (n-1)L$ induces an injection $\mathcal{G}(L) \to \mathcal{G}(nL)$ for any $n \geqq 1$.

1. Adeles and Ideles.

Let R be a Dedekind domain and K be its quotient field. Let $\boldsymbol{A}$ denote the (finite) adele ring of K, namely, the restricted direct product $\prod' K_P$ (w.r.t R_P) of the topological rings K_P with respect to the subrings R_P, $\boldsymbol{A} = \{a = (a_P) \in \prod K_P;\ a_P \in R_P$ for almost all $P\}$. As usual we consider $\boldsymbol{A}$ to contain (diagonally embedded) K and to be a K-algebra. Let B be a finite dimensional K-algebra. The adelization of B is, by definition, the K-algebra $B \otimes_K \boldsymbol{A}$, emdowed with the initial topology for the family of mappings $f \otimes \mathrm{id}_{\boldsymbol{A}}: B \otimes \boldsymbol{A} \to \boldsymbol{A}$, $f \in \mathrm{Hom}_K(B, K)$, or equivalently the topology from the identification $B \otimes \boldsymbol{A} \cong \boldsymbol{A} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}$ by any choice of K-basis of B. It is a topological ring and contains B through the embedding $b \mapsto b \otimes 1$. The K-algebra morphism $\theta: B \otimes \boldsymbol{A} \to \prod B_P$, $b \otimes a \mapsto (b \otimes a_P)$ induces an isomorphism of topological rings as well as of bi-B-modules:

$$\theta: B \otimes_K \boldsymbol{A} \xrightarrow{\sim} B_{\boldsymbol{A}} := \textstyle\prod' B_P \text{ (w.r.t } \Gamma_P),\ x \mapsto (x_P),$$

where Γ is any R-order of B. We shall identify $B \otimes \boldsymbol{A}$ with $B_{\boldsymbol{A}}$ and x with (x_P) by θ.

1.1. The idele group $(B \otimes \boldsymbol{A})^{\times} = B_{\boldsymbol{A}}^{\times}$ of B is, by definition, the topological group $\prod'(B_P)^{\times}$ (w.r.t $(\Gamma_P)^{\times}$). Explicitly, a fundamental system of neighbourhoods of 0 in $B_{\boldsymbol{A}}$ (resp. of 1 in $(B_{\boldsymbol{A}})^{\times}$) is given by

$$U^{+}(S, n) = \prod_{P \in S} P^n \Gamma_P \times \prod_{P \notin S} \Gamma_P \text{ (resp. } U^{\times}(S, n) = \prod_{P \in S} (1 + P^n \Gamma_P) \times \prod_{P \notin S} (\Gamma_P)^{\times}),$$

where S runs over all finite set of maximal ideals and n runs over all positive integers.

1.1.1. *Suppose H is a subgroup of $(B \otimes \boldsymbol{A})^{\times} = (B_{\boldsymbol{A}})^{\times}$ having the following property:*

(b) *If $x = (x_P) \in H$ and $x_P \in \Gamma_P$, then $x_P \in (\Gamma_P)^{\times}$.*
Then the induced topology on H from the adele topology of $B \otimes \boldsymbol{A}$ coincides with the induced topology on H from the idele topology of $(B \otimes \boldsymbol{A})^{\times}$.

PROOF. (b) implies $H \cap (1 + U^{+}(S, n)) = H \cap U^{\times}(S, n)$.

1.2. Let $\boldsymbol{e}$ be a finite set of orthogonal idempotents in B. Identifying $\boldsymbol{e} \otimes 1$ with $\boldsymbol{e}$, along with the elementary subgroup $E_{\boldsymbol{e}}(B)$ of 0.1, we can consider $E_{\boldsymbol{e}}(B_P) = E_{\boldsymbol{e}}(B \otimes K_P)$ or $E_{\boldsymbol{e}}(B \otimes \boldsymbol{A})$. Put

$$\mathcal{E}_e(B) := (B_A)^\times \cap \prod E_e(B_P).$$

$E_e(B\otimes A)$ is obviously a subgroup of $\mathcal{E}_e(B)$. In some cases it is known that these two groups coincide, but in general we do not know whether they coincide or not. However, since $E_e(B\otimes A)$ contains each quasi factor $E_e(B_P)$, for any open subgroup $\mathcal{U}$ of $(B_A)^\times$, we have

(1) $E_e(B\otimes A)\mathcal{U}=\mathcal{E}_e(B)\mathcal{U}$.

1.2.1. LEMMA. *$E_e(B)$ is dense in $E_e(B\otimes A)$ in the idele topology. It is also dense in $\mathcal{E}_e(B)$.*

PROOF. By Chinese Remainder Theorem, B is dense in $B\otimes A$, and eBe' is dense in $e(B\otimes A)e'$. Hence $1+eBe'$ is dense in $1+e(B\otimes A)e'$ in the adele topology. Since any element of $e(B\otimes A)e'$ is nilpotent, the group $H=1+e(B\otimes A)e'$ has the property (b) of 1.1.1. Thus $1+eBe'$ is dense in $1+e(B\otimes A)e'$ in the idele topology. This obviously implies that $E_e(B)$ is dense in $E_e(B\otimes A)$. It is also dense in $\mathcal{E}_e(B)$ by (1).

1.2.2. *Let $\tilde{E}(B)$ be the group of Vaserstein as in* 0.2. *Suppose that B is the total matrix algebra $M(n, C)$ over some K-algebra C with $n\geqq 2$. Then as is easily seen from* [8] *Th.* 3.6, *$\tilde{E}(B)$ (resp. $\tilde{E}(B_P)$) can be identified with the elementary subgroup $E(n, C)$ (resp. $E(n, C_P)$) of $B^\times=GL(n, C)$ (resp. $(B_P)^\times=GL(n, C_P)$.) Hence, by* 1.2.1, *B has the "strong approximation."*

1.3. LEMMA. *Let $\mathcal{E}_P$, H_P be subgroups of $B_P^\times$ such that $B_P^\times=\mathcal{E}_P H_P$, and $\mathcal{E}=(B_A)^\times\cap\prod\mathcal{E}_P$. Suppose that $B^\times\cap\mathcal{E}$ is dense in $\mathcal{E}$. Then, for any open subgroup $\mathcal{U}$ of $(B_A)^\times$, we have:*

(i) *The double coset space $B^\times\backslash(B_A)^\times/\mathcal{U}$ admits a set of representatives in the subgroup $\prod' H_P$ (w.r.t. $\{1\}$) of $(B_A)^\times$.*

(ii) *Further, if $\mathcal{E}_P$ is a normal subgroup of $(B_P)^\times$ with the abelian quotient for any P, then $B^\times\mathcal{U}$ is a normal subgroup containing $\mathcal{E}$, and $B^\times\backslash(B_A)^\times/\mathcal{U}$ is in fact the quotient group $(B_A)^\times/B^\times\mathcal{U}$.*

PROOF. (i) For any $g\in(B_A)^\times$, $(B^\times\cap\mathcal{E})g\mathcal{U}=\mathcal{E}g\mathcal{U}$. Hence, $B^\times g\mathcal{U}=B^\times(B^\times\cap\mathcal{E})g\mathcal{U}=B^\times\mathcal{E}g\mathcal{U}$. (ii) Since $\mathcal{E}$ is normal, $B^\times\mathcal{E}$ and $\mathcal{U}\mathcal{E}$ are subgroups. Since $(B_A)^\times/\mathcal{E}$ is abelian, $B^\times\mathcal{E}$ and $\mathcal{U}\mathcal{E}$ are normal in $(B_A)^\times$. By (i), $B^\times\mathcal{U}=B^\times\mathcal{E}\mathcal{U}=B^\times\mathcal{E}\mathcal{U}\mathcal{E}$ is normal, and $B^\times g\mathcal{U}=B^\times\mathcal{E}g\mathcal{U}=gB^\times\mathcal{E}\mathcal{U}=gB^\times\mathcal{U}$.

2. Proof of Theorem 1'.

Put $L=L'\oplus L''$, $V=KL$, $V'=KL'$, $V''=KL''$, $B=\mathrm{End}_A V$, $\Gamma=\mathrm{End}_\Lambda L$. Let e' (resp. e'') be the idempotent of B corresponding to the projection $V\to V'$ (resp. $V\to V''$), and $B'=e'Be'\cong\mathrm{End}_A V'$, $B''=e''Be''\cong\mathrm{End}_A V''$. As is well known (cf.

[1] 31.18 and 31.35 (iv)), the map $x=(x_P)\mapsto\bigcap(x_P(L_P)\cap V)$ induces the bijection between $B^\times\backslash(B_A)^\times/\mathcal{U}(L)$ and $\mathcal{G}(L)$, where $\mathcal{U}(L)=\prod(\Gamma_P)^\times$. The claim of Theorem 1 is clearly equivalent to

(i) $B^\times\backslash(B_A)^\times/\mathcal{U}(L)$ admits a set of representatives in the diagonal subgroup $(B'_A)^\times\times(B''_A)^\times$.

(ii) If $V\gg V'$, one can even reduce the representatives in the subgroup $\{1\}\times(B'_A)^\times$.

To prove the above, in view of 1.3 together with 1.2.1, it suffice to prove

2.0. *There is a set of orthogonal idempotents $\tilde{e}$ of B such that:*

(i) $(B_P)^\times=E_{\tilde{e}}(B_P)((B'_P)^\times\times(B''_P)^\times)$ *for any P.*

(ii) *If $V\gg V'$, $(B_P)^\times=E_{\tilde{e}}(B_P)(\{1\}\times(B''_P)^\times)$ for any P.*

2.1. Let U_i $(1\leqq i\leqq n)$ be the distinct A-indecomposable direct summand of V, and $n_i>0$, $n'_i\geqq 0$, $n''_i\geqq 0$ be the multiplicity of U_i in V, V' and V'', respectively. Note that the condition $V\gg V'$ means $n'_i>0\Rightarrow n''_i>0$. Decompose e', e'' into the orthogonal sum of primitive idempotents $e_{i\alpha}$, choosing the double index (i,α) in the following way:

$$e_{i\alpha}(V)\cong U_i\ (1\leqq i\leqq n);\quad e'=\sum e'_i,\quad e''=\sum e''_i,$$

where e'_i (resp. e''_i) is the sum $\sum e_{i\alpha}$ over $1\leqq\alpha\leqq n'_i$ (resp. $n'_i<\alpha\leqq n_i$). Then put $e_i=e'_i+e''_i$, and $\boldsymbol{e}=\{e_i;\ 1\leqq i\leqq n\}$.

2.1.1. First, we look at the set of idempotents $\boldsymbol{e}$, and put $B_{ij}=e_iBe_j$, $B_i=B_{ii}$. Then each element $b\in B$ is uniquely written as $b=\sum b_{ij}$ with $b_{ij}=e_ibe_j\in B_{ij}$. The multiplication with $b'=\sum b'_{ij}$ is given as $bb'=\sum c_{ij}$ with $c_{ij}=\sum_k b_{ik}b'_{kj}$. Suggestively said, the correspondence $b\mapsto(b_{ij})$ gives B the structure of n by n matrix algebra with entries in B_{ij}. In particular, if the pair $(B,\boldsymbol{e})$ has the property

(a) $b=\sum b_{ij}\in B^\times\Rightarrow b_{ii}\in B_i^\times$,

then $B^\times$ can be diagonalized by $E_{\boldsymbol{e}}(B)$, $B^\times=E_{\boldsymbol{e}}(B)\prod B_i^\times$.

2.1.2. Lemma. *$(B,\boldsymbol{e})$ of 2.1 has the property* (a).

Proof. It obviously suffice to see:

(a′) If $i\neq k$, $B_{ik}B_{ki}\subset\operatorname{rad}B_i=e_i(\operatorname{rad}B)e_i$.

To see this, we first observe:

$$(1)\qquad e_{i\alpha}Be_{k\beta}Be_{i\alpha}\subset\operatorname{rad}(e_{i\alpha}Be_{i\alpha})=e_{i\alpha}(\operatorname{rad}B)e_{i\alpha}.$$

Indeed, if $x\in e_{i\alpha}Be_{k\beta}$, $x'\in e_{k\beta}Be_{i\alpha}$ and $xx'\notin\operatorname{rad}(e_{i\alpha}Be_{i\alpha})$, then since $e_{i\alpha}Be_{i\alpha}\cong\operatorname{End}_AU_i$ is a local ring, $xx'\in(e_{i\alpha}Be_{i\alpha})^\times\cong\operatorname{Aut}_AU_i$. Hence the A-injection

$x'\colon e_{i\alpha}(V)\to e_{k\beta}(V)$ splits, contradicting $U_i\not\cong U_k$. Since $e_{i\gamma}(V)\cong e_{i\alpha}(V)\cong U_i$, there is some $y\in B^\times$ such that $ye_{i\gamma}B=e_{i\alpha}B$. Multiplying (1) by y, we have $e_{i\gamma}Be_{k\beta}Be_{i\alpha}\subset e_{i\gamma}(\mathrm{rad}\,B)e_{i\alpha}$ for any γ. This implies (a′).

2.1.3. $(B_P)^\times=E_{\boldsymbol{e}}(B_P)\cdot\prod(B_{i,P})^\times$.

PROOF. $B_P^\times$ is open in B_P, and B is dense in B_P. Since $(B,\boldsymbol{e})$ has the property (a), $(B_P,\boldsymbol{e})$ also has the property (a).

2.2. Put $\boldsymbol{e}_i=\{e_{i\alpha}\,;\,1\leqq\alpha\leqq n_i\}$, $\tilde{\boldsymbol{e}}=\bigcup_i\boldsymbol{e}_i$. We shall further reduce $\prod(B_{i,P})^\times$ by $E_{\tilde{\boldsymbol{e}}}(B_P)$ to the form of 2.0. Fixing one arbitrarily chosen P, we simplify the notation by dropping the suffix P, so we mean B_P by B. Put $B_i'=e_i'Be_i'=e_i'B_ie_i'$, $B_i''=e_i''Be_i''$, one of which may be $\{0\}$. Put $C_i=\mathrm{End}_AU_i$.

Since $B_i\cong\mathrm{End}_Ae_i(V)\cong\mathrm{End}_A(n_iU_i)\cong M(n_i,C_i)$, there is an isomorphism $f_i\colon B_i\to M(n_i,C_i)$ mapping $e_{i\alpha}$ to ε_α, the matrix with the α-th diagonal entry 1 and other entries 0. Then f_i maps the diagonal subalgebra $B_i'\oplus B_i''$ onto the diagonal subalgebra $M(n_i',C_i)\oplus M(n_i'',C_i)$, $B_i^\times$ to $GL(n_i,C_i)$ and $E_{\boldsymbol{e}_i}(B_i)$ to $E(n_i,C_i)$. Since C_i is semi-local, applying the lemma of Bass in 0.1 to $GL(n_i,C_i)$, then pulling the result back by f_i, we have

$$B_i^\times=\begin{cases}E_{\boldsymbol{e}_i}(B_i)((B_i')^\times\times(B_i'')^\times)\\ E_{\boldsymbol{e}_i}(B_i)(\{1\}\times(B_i'')^\times)\qquad\text{if } n_i''>0.\end{cases}\tag{2}$$

Since $E_{\tilde{\boldsymbol{e}}}(B)\supset E_{\boldsymbol{e}_i}(B)$ and we are identifying as $E_{\boldsymbol{e}_i}(B_i)=E_{\boldsymbol{e}_i}(B)\subset B^\times$, (2) implies that each $B_i^\times$ (considered as a subgroup of $B^\times$) is contained in $E_{\tilde{\boldsymbol{e}}}(B)((B_i')^\times\times(B_i'')^\times)$. Regrouping $(B_i')^\times$'s to $(B')^\times$ and recovering the suffix P, we have established 2.0.

3. Proof of Theorem 2.

Let $V=KL$ and $B=\mathrm{End}_AV$. By the obvious identification $\mathrm{End}_A(nV)\cong M(n,B)$, the property (c′) in 0.3 is equivalent to:

(c″) The map

$$x\longmapsto\begin{pmatrix}x&0\\0&1_{n-1}\end{pmatrix}$$

induces an injection from $B^\times\backslash(B\otimes\boldsymbol{A})^\times/\mathcal{U}(L)$ into $GL(n,B)\backslash GL(n,B\otimes\boldsymbol{A})/\mathcal{U}(nL)$.

By the assumption that B has the "strong approximation", $\tilde{E}(B)$ is dense in $\tilde{E}(B\otimes\boldsymbol{A})$, hence it is also dense in $(B\otimes\boldsymbol{A})^\times\cap\prod\tilde{E}(B_P)$. While $E(n,B)$ is always dense in $GL(n,B\otimes\boldsymbol{A})\cap\prod E(n,B_P)$ by 1.2.1. In view of 1.2 (ii), what we shall prove is:

(c‴) $(B\otimes\boldsymbol{A})^\times\cap GL(n,B)\mathcal{U}(nL)=B^\times\mathcal{U}(L)$.

The left hand side of (c$'''$) obviously contains the right hand side of it. Since Γ_P is semi-local, by the lemma of Bass, $GL(n, \Gamma_P)=E(n, \Gamma_P)(\Gamma_P)^{\times}$ and $\mathcal{U}(nL)=\prod GL(n, \Gamma_P)=(\prod E(n, \Gamma_P))\prod(\Gamma_P)^{\times}\subset(\prod E(n, B_P))\mathcal{U}(L)$. Since B is also semi-local, $GL(n, B)=B^{\times}E(n, B)\subset B^{\times}\prod E(n, B_P)$. Hence left hand side of (c$'''$) is contained in

$$(B\otimes A)^{\times}\cap B^{\times}(\prod E(n, B_P))\mathcal{U}(L)=B^{\times}((B\otimes A)^{\times}\cap\prod E(n, B_P))\mathcal{U}(L).$$

Now, by the theorem of Vaserstein in 0.2, $(B\otimes K_P)^{\times}\cap E(n, B_P)=\tilde{E}(B_P)$ and $(B\otimes A)^{\times}\cap\prod E(n, B_P)\subset(B\otimes A)^{\times}\cap\prod\tilde{E}(B_P)$. The last group is contained in $B^{\times}\mathcal{U}(L)$ by 1.3 (ii). This showed that the left hand side of (c$'''$) is contained in $B^{\times}\mathcal{U}(L)$, completing the proof of Theorem 2.

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