J. Math. Soc. Japan
Vol. 48, No. 3, 1996

Local cohomology modules of indecomposable surjective-Buchsbaum modules over Gorenstein local rings

By Takesi KAWASAKI[1)]

(Received Oct. 13, 1994)

1. Introduction.

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. We assume that $\dim A=d>0$. The local cohomology functor $H_{\mathfrak{m}}^{i}(-)$ was defined by Grothendieck [**11**] and he showed that for any finitely generated A-module M, the i-th local cohomology module $H_{\mathfrak{m}}^{i}(M)$ vanishes unless

$$\operatorname{depth} M \leqq i \leqq \dim_{A} M$$

and that $H_{\mathfrak{m}}^{i}(M)\neq 0$ if $i=\operatorname{depth} M$ or $i=\dim_{A}M$. We refer to the local cohomology modules $H_{\mathfrak{m}}^{i}(M)$ for which $\operatorname{depth} M<i<\dim_{A}M$ as the intermediate local cohomology modules of M. Pathological behaviors of intermediate local cohomology modules for general Noetherian local rings were reported by several authors. Firstly Sharp [**20**] gave examples of Noetherian local rings whose intermediate local cohomology modules either all vanish or are all non-zero. Furthermore Evans and Griffith [**7**] gave a Noetherian local ring with prescribed local cohomology modules, that is, let $d\geqq 2$ and $h_{1}, \cdots, h_{d-1}\geqq 0$ be arbitrary integers. Then there is a Noetherian local domain A of dimension d such that

$$l_{A}(H_{\mathfrak{m}}^{i}(A)) = h_{i} \quad \text{for all } 1 \leqq i \leqq d-1.$$

By modifying their argument, Goto [8] obtained such a ring from among Buchsbaum local rings. Here a finitely generated A-module M is said to be Buchsbaum if the difference $l_{A}(M/\mathfrak{q}M)-e_{\mathfrak{q}}(M)$ is an invariant of M not depending on the choice of the parameter ideal $\mathfrak{q}$ for M. Moreover a Noetherian local ring A is said to be Buchsbaum if it is a Buchsbaum module over itself.

In this paper we are interested in behaviors of local cohomology modules of *finitely generated indecomposable modules.* Goto [**9**] gave a structure theorem for maximal Buchsbaum modules over regular local rings, that is, if A is a regular local ring of dimension $d>0$ and M is an indecomposable maximal

---

[1)] The author is partially supported by Grant-Aid for Co-operative Research.

Buchsbaum A-module of depth $t<d$, then its intermediate local cohomology modules all vanish and t-th local cohomology module $H_{\mathfrak{m}}^{t}(M)$ is isomorphic to the residue class field, where an A-module is said to be maximal if its dimension is equal to d. The author [**15**] improved this structure theorem for maximal surjective-Buchsbaum modules of finite injective dimension; see the next section for details. Furthermore Amasaki [**1**], Cipu-Herzog-Popescu [**4**], and Yoshino [**23**] observed behaviors of local cohomology modules of maximal Buchsbaum modules or of maximal quasi-Buchsbaum modules.

The aim of this paper is to give indecomposable maximal surjective-Buchsbaum modules with prescribed local cohomology modules.

THEOREM 1.1. *Let A be a Gorenstein local ring of dimension $d>0$. We assume that its multiplicity is greater than 2. Let $h_0, \cdots, h_{d-1} \geqq 0$ be arbitrary integers. Then there exists an indecomposable maximal surjective-Buchsbaum module M such that*

$$l_A(H_{\mathfrak{m}}^{i}(M)) = h_i \quad \text{for all } 0 \leqq i \leqq d-1.$$

A concept of minimal finite injective hull, which was introduced by Auslander and Buchweitz [**2**], plays a key role in this paper. In the preceding studies on local cohomology modules of finitely generated modules, we assumed that the modules have finite injective dimension implicitly or explicitly. We, however, need to consider modules of infinite injective dimension for the theorem. By the minimal finite injective hull, we are able to separate a general finitely generated module into a pair of a maximal Cohen-Macaulay module and a finitely generated module of finite injective dimension. We have known the structure theorem for maximal surjective-Buchsbaum modules of finite injective dimension and many authors studied indecomposable maximal Cohen-Macaulay modules. We will combine them.

It should be noted here that the assumption of the theorem on multiplicity is not superfluous. In fact, Goto [**10**, Corollary 1.2] showed that there exist only finitely many isomorphism classes of indecomposable maximal surjective-Buchsbaum modules over a Gorenstein local ring of dimension 1 and of multiplicity 2.

2. Preliminaries.

Throughout this paper, A denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$ and with residue class field k. We assume that $d=\dim A>0$. For each A-module M, $l_A(M)$ denotes the length of M.

Firstly we recall that the local cohomology functor $H_{\mathfrak{m}}^{i}(-)$ with respect to $\mathfrak{m}$ is naturally equivalent to the functor $\varinjlim_n \operatorname{Ext}_A^i(A/\mathfrak{m}^n, -)$. And so there exists a natural map

$$\phi_M^i : \mathrm{Ext}_A^i(k, M) \longrightarrow H_{\mathfrak{m}}^i(M)$$

for all i.

DEFINITION 2.1 ([**22**], Definition 1.1). A finitely generated A-module M is said to be *surjective-Buchsbaum* if the natural map ϕ_M^i is surjective for all $i \neq \dim_A M$.

A surjective-Buchsbaum module M is Buchsbaum [**21**, Theorem 1] and the converse holds if A is a regular local ring. Naturally, every Cohen-Macaulay module is surjective-Buchsbaum because $H_{\mathfrak{m}}^i(M)=0$ for all $i \neq \dim_A M$.

We will state the structure theorem for maximal surjective-Buchsbaum modules of finite injective dimension. From now on, we assume that A is a Gorenstein local ring, that is, A has finite injective dimension over itself. Then there exists a natural isomorphism

$$H_{\mathfrak{m}}^i(M) \cong \mathrm{Hom}_A(\mathrm{Ext}_A^{d-i}(M, A), E)$$

for any finitely generated A-module M, where E denotes the injective envelope of the residue field k. Refer to [**13**, Vortrag 5] for details.

Let $(F_\bullet, d_\bullet)$ be the minimal free resolution of k and $(-)^* = \mathrm{Hom}_A(-, A)$. We put

$$Y_i = \mathrm{Coker}\, d_{d-i}^* \tag{2.2}$$

for all $0 \leqq i \leqq d$. In particular $Y_d = A$. Then for all $0 \leqq i \leqq d$, there exists an exact sequence

$$0 \longrightarrow (F_0)^* \longrightarrow \cdots \longrightarrow (F_{d-i})^* \longrightarrow Y_i \longrightarrow 0 \tag{2.3}$$

because $\mathrm{Ext}_A^j(k, A)=0$ for all $j<d-i$. Therefore Y_i has finite injective dimension and

$$H_{\mathfrak{m}}^j(Y_i) = \begin{cases} k, & j = i\,; \\ 0, & j \neq i,\, d \end{cases}$$

for all $0 \leqq i \leqq d$.

THEOREM 2.4 ([**15**], Theorem 3.1). *Assume that A is not regular. Then Y_i is an indecomposable maximal surjective-Buchsbaum module for all $0 \leqq i \leqq d$. Furthermore any maximal surjective-Buchsbaum module of finite injective dimension is isomorphic to a unique direct sum of finite copies of $Y_0, Y_1, \cdots, Y_d$.*

In particular a maximal Cohen-Macaulay module of finite injective dimension is a free module.

Next we state on the finite injective hull.

DEFINITION 2.5. Let M be a finitely generated A-module. An exact sequence of A-modules

$$(2.6)\qquad 0 \longrightarrow M \longrightarrow Y \xrightarrow{\phi} X \longrightarrow 0$$

is said to be a *finite injective hull* of M if Y is of finite injective dimension and X is a maximal Cohen-Macaulay module or a zero module. A finite injective hull (2.6) is said to be minimal if X and Y have no common direct summand under ϕ.

Auslander and Buchweitz [**2**] showed that, over a Cohen-Macaulay local ring possessing the canonical module, there exists a minimal finite injective hull of arbitrary finitely generated module M and that it is determined by M up to isomorphisms, that is, let $0\to M\to Y\to X\to 0$ and $0\to M\to Y'\to X'\to 0$ be two minimal finite injective hulls of M. Then there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & 0 \\ & & \| & & \downarrow\wr & & \downarrow\wr & & \\ 0 & \longrightarrow & M & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & 0. \end{array}$$

For the sake of completeness, we give a brief proof of them.

THEOREM 2.7. *There exists a unique minimal finite injective hull of arbitrary finitely generated A-module M.*

PROOF. We will use a dualizing complex; refer the reader to [**18**, Chapter 2] for a notation of complexes and for a dualizing complex. Let $D_A^{\boldsymbol{\cdot}}$ be a dualizing complex of A, which is the minimal injective resolution of A. Let $D(-)=\mathrm{Hom}_A(-, D_A^{\boldsymbol{\cdot}})$ and $(F_{\bullet}, d)$ be the minimal free resolution of $D(M)$. Since there exist quisms

$$M \longrightarrow DD(M) \longrightarrow D(F_{\boldsymbol{\cdot}}) \longleftarrow F_{\boldsymbol{\cdot}}^*(-d),$$

the homology modules of $F_{\boldsymbol{\cdot}}^*$ all vanish except $-d$-th one and $H_{-d}(F_{\boldsymbol{\cdot}}^*)\cong M$. Hence there are three exact sequences

$$(2.8)\qquad 0 \longrightarrow M \longrightarrow \mathrm{Coker}\, d_d^* \longrightarrow \mathrm{Im}\, d_{d+1}^* \longrightarrow 0\,;$$

$$(2.9)\qquad 0 \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \cdots \longrightarrow F_d^* \longrightarrow \mathrm{Coker}\, d_d^* \longrightarrow 0\,;$$

$$(2.10)\qquad 0 \longrightarrow \mathrm{Im}\, d_{d+1}^* \longrightarrow F_{d+1}^* \longrightarrow F_{d+2}^* \longrightarrow \cdots.$$

The exact sequence (2.9) implies that $Y=\mathrm{Coker}\, d_d^*$ is of finite injective dimension. If M is not of finite injective dimension, then $F_i\neq 0$ for all $i>d$ because the rank of F_i is equal to the i-th Bass number of M. And so (2.10) implies that $X=\mathrm{Im}\, d_{d+1}^*$ is a maximal Cohen-Macaulay module or a zero module. Therefore (2.8) is a finite injective hull of M. If X and Y have a common direct summand Z, then Z is a free module. By taking $(-)^*$ of (2.10), we find that

Z is also a direct summand of F^*_{d+1}. However there exists a commutative diagram

$$\begin{array}{ccc} F^*_d & \xrightarrow{d^*_{d+1}} & F^*_{d+1} \\ \downarrow & & \uparrow \\ Y & \longrightarrow & X \end{array}$$

which contradicts the fact that $\operatorname{Im} d^*_{d+1} \subseteqq \mathfrak{m} F^*_{d+1}$. Thus the finite injective hull (2.8) is minimal.

Nextly we show that the uniqueness of minimal finite injective hull. Take a minimal finite injective full of M:

$$0 \longrightarrow M \longrightarrow Y \xrightarrow{\phi} X \longrightarrow 0. \tag{2.11}$$

Let $F_{\bullet}$ and $G_{\bullet}$ be the minimal injective resolutions of $D(X)$ and $D(Y)$, respectively. Then

$$F_i = 0 \quad \text{for all } i < d$$

and

$$G_i = 0 \quad \text{for all } i > d.$$

Furthermore $D(\phi)$ induces a homomorphism $\phi_{\bullet}: F_{\bullet} \to G_{\bullet}$ which makes the following diagram

$$\begin{array}{ccc} D(X) & \xrightarrow{D(\phi)} & D(Y) \\ \uparrow & & \uparrow \\ F_{\bullet} & \xrightarrow[\phi_{\bullet}]{} & G_{\bullet} \end{array}$$

commutative up to homotopy. We remark that the minimality of (2.11) implies that $\phi_d \otimes k = 0$. Since there exist quisms

$$Mc(\phi_{\bullet}) \longrightarrow Mc(D(\phi)) \longrightarrow D(M),$$

where $Mc(-)$ denotes the mapping cone, $Mc(\phi_{\bullet})$ is the minimal free resolution of $D(M)$. The uniqueness of minimal free resolution implies the uniqueness of minimal finite injective hull. □

By the above proof, we find that the finite injective hull (2.6) is minimal if and only if X has no free summand. Furthermore the above proof gives an effective construction of the minimal finite injective hull. In fact, let $G_{\bullet}$ be the minimal free resolution of a finitely generated A-module M. Then there exists the minimal free resolution $F_{\bullet}$ of $G^*_{\bullet}(d)$ because $H_i(G^*_{\bullet}) = \operatorname{Ext}_A^{-i}(M, A) = 0$ for all $i < -d$. Since there are quisms

$$D(M) \longrightarrow D(G_{\bullet}) \longleftarrow G^*_{\bullet}(d),$$

the free complex $F_\bullet$ is also the minimal free resolution of $D(M)$. Of course, when A is a Cohen-Macaulay local ring possessing the canonical module K_A, we can similarly prove the theorem by letting $(-)^*=\mathrm{Hom}_A(-, K_A)$.

Finally we state on matrix factorizations. A Noetherian local ring A is said to be hypersurface if the $\mathfrak{m}$-adic completion of A is a residue class ring of a regular local ring with respect to a principal ideal. Matrix factorizations describe maximal Cohen-Macaulay modules over a hypersurface.

Let B be a regular local ring with maximal ideal $\mathfrak{n}$ and $0\neq f\in\mathfrak{n}^2$.

DEFINITION 2.12. A matrix factorization of f is a pair $(\phi: F\to F', \phi': F'\to F)$ of homomorphisms between two finitely generated free B-modules such that

$$\phi\cdot\phi' = f\cdot\mathrm{id}_{F'} \quad\text{and}\quad \phi'\cdot\phi = f\cdot\mathrm{id}_F.$$

When this is the case, the rank of F' is equal to the one of F. A morphism between two matrix factorizations $(\phi: F\to F', \phi': F'\to F)$ and $(\psi: G\to G', \psi': G'\to G)$ is a pair $(\alpha: F\to G, \alpha': F'\to G')$ of homomorphisms which makes the following diagram

$$\begin{array}{ccccc}
F & \xrightarrow{\phi} & F' & \xrightarrow{\phi'} & F \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\alpha'}\downarrow & & {\scriptstyle\alpha}\downarrow \\
G & \xrightarrow{\psi} & G' & \xrightarrow{\psi'} & G
\end{array}$$

commutative. A matrix factorization (ϕ, ϕ') is called reduced if $\mathrm{Im}\,\phi\subset\mathfrak{m}F'$ and $\mathrm{Im}\,\phi'\subset\mathfrak{m}F$. The matrix factorizations of f form an additive category, denoted by $MF_B(f)$.

Let (ϕ, ϕ') be a matrix factorization of f. Then it is easy to check that $\mathrm{Coker}\,\phi$ is a maximal Cohen-Macaulay module over a hypersurface $A=B/fB$. Furthermore if (ϕ, ϕ') is reduced, then

$$F'\otimes A \xleftarrow{\phi\otimes A} F\otimes A \xleftarrow{\phi'\otimes A} F'\otimes A \xleftarrow{\phi\otimes A} \cdots$$

is the minimal free resolution of $\mathrm{Coker}\,\phi$ as an A-module. This correspondence is an additive functor from $MF_B(f)$ to the category of the maximal Cohen-Macaulay A-modules. Eisenbud [**6**, Corollary 6.3] showed that the functor induces a one-to-one correspondence between reduced matrix factorizations of f and maximal Cohen-Macaulay A-modules having no free summand. By the correspondence, we often identify them.

3. Proof of Theorem 1.1.

We will prove Theorem 1.1 in this section. Let A be a Gorenstein local ring of dimension d and of multiplicity $e_{\mathfrak{m}}(A)>2$. We divide the proof to two parts.

CASE 1. *When A is a hypersurface of dimension* 1.

In this case, we will refine Nishida's argument [**16**]. Let B be a 2-dimensional regular local ring with maximal ideal $\mathfrak{n}=(x, y)$ and $0\neq f\in\mathfrak{n}$ such that $B/fB\cong\hat{A}$, where $\hat{A}$ denotes the completion of A. Since $e_{\mathfrak{m}}(A)>2$, there exist elements $a, b, c\in\mathfrak{n}$ such that $f=ax^2+bxy+cy^2$. Let X be the second syzygy of k. Then X is a maximal Cohen-Macaulay module and $X\otimes_A\hat{A}$ is associated to the matrix factorization of f:

$$(\phi, \phi')=\left(\begin{pmatrix} cy & ax+by \\ x & -y \end{pmatrix}, \begin{pmatrix} y & ax+by \\ x & -cy \end{pmatrix}\right).$$

In the other word, there exists an exact sequence

$$0 \longleftarrow X\otimes_A\hat{A} \longleftarrow \hat{A}^2 \xleftarrow{\phi\otimes_A\hat{A}} \hat{A}^2 \xleftarrow{\phi'\otimes_A\hat{A}} \hat{A}^2 \longleftarrow \cdots.$$

Therefore $\mathrm{Ext}_A^1(X, k)=\mathrm{Ext}_{\hat{A}}^1(X\otimes_A\hat{A}, k)$ is a k-vector space of dimension 2. The endomorphism ring $\mathrm{End}_A(X)$ of X acts on $\mathrm{Ext}_A^1(X, k)$ as k. In fact, an endomorphism of (ϕ, ϕ') is a linear combination of following four morphisms:

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \quad \left(\begin{pmatrix} y & 0 \\ x & 0 \end{pmatrix}, \begin{pmatrix} y & ax+by \\ 0 & 0 \end{pmatrix}\right),$$

$$\left(\begin{pmatrix} 0 & a \\ -c & -b \end{pmatrix}, \begin{pmatrix} -b & -ac \\ 1 & 0 \end{pmatrix}\right), \quad \left(\begin{pmatrix} 0 & y \\ 0 & x \end{pmatrix}, \begin{pmatrix} x & -cy \\ 0 & 0 \end{pmatrix}\right).$$

The later three morphism act on $\mathrm{Ext}_{\hat{A}}^1(X\otimes_A\hat{A}, k)$ as a zero map. In particular, X is indecomposable.

Let $\{e_1, e_2\}$ be a basis of $\mathrm{Ext}_A^1(X, k)$ and $h>0$ an integer. Take an extension of k^h by X^h:

$$0 \longrightarrow k^h \longrightarrow M \longrightarrow X^h \longrightarrow 0$$

which corresponds to

$$\begin{pmatrix} e_1 & e_2 & & & 0 \\ & e_1 & e_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & e_2 \\ 0 & & & & e_1 \end{pmatrix} \in \mathrm{Ext}_A^1(X^h, k^h)=\mathrm{Ext}_A^1(X, k)^{h^2}.$$

Then M is a maximal surjective-Buchsbaum module with $H_{\mathfrak{m}}^0(M)=k^h$ by [**21**, Corollary 4.1]. We will show that M is indecomposable. We may assume that A is complete. If there were non-trivial decomposition $M=M'\oplus M''$, then $[M'/H_{\mathfrak{m}}^0(M')]\oplus[M''/H_{\mathfrak{m}}^0(M'')]=X^h$. By the uniqueness of direct sum decomposition, we obtain

$$H_{\mathfrak{m}}^0(M') = k^{h'}\,; \quad M'/H_{\mathfrak{m}}^0(M') = X^{m'}\,;$$

$$H_{\mathfrak{m}}^0(M'') = k^{h''}\,; \quad M''/H_{\mathfrak{m}}^0(M'') = X^{m''}\,;$$

and the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k^h & \longrightarrow & M & \longrightarrow & X^h & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & k^{h'}\oplus k^{h''} & \longrightarrow & M'\oplus M'' & \longrightarrow & X^{m'}\oplus X^{m''} & \longrightarrow & 0.
\end{array}$$

It means that there are invertible matrix P and Q with entries in k such that

$$P\begin{pmatrix} e_1 & e_2 & & & 0 \\ & & & & \\ & e_1 & e_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & e_2 \\ 0 & & & & e_1 \end{pmatrix} Q = \begin{pmatrix} \Pi' & 0 \\ 0 & \Pi'' \end{pmatrix},$$

where Π' (resp. Π'') is $h'\times m'$ (resp. $h''\times m''$) matrix with entries in $\mathrm{Ext}_A^1(X, k)$. It is impossible by [**16**, Lemma 2.2]. $\square$

CASE 2. *When A is not a hypersurface or A is a hypersurface of dimension $d\geqq 2$.*

We may assume that $(h_0, \cdots, h_{d-1})\neq(1, 0, \cdots, 0)$. Let Y_i be an indecomposable maximal surjective-Buchsbaum module of finite injective dimension defined as (2.2).

When this is the case, there exists an indecomposable maximal Cohen-Macaulay module X such that

$$\beta_{d-i}(X) \geqq \sum_{j=0}^{i} \beta_j(k) h_{i-j} \quad \text{for all } 0 \leqq i \leqq d-1, \tag{3.1}$$

where $\beta_i^A(-)$ denotes the i-th Betti number. In fact, when A is not a hypersurface, Herzog showed that for arbitrary integer $n>0$, there exists an indecomposable maximal Cohen-Macaulay module X such that $\beta_0^A(X)\geqq n$, in the proof of [**12**, Satz 1.2]. Furthermore Ramras [**17**, Corollary 4] gave an inequality for any maximal Cohen-Macaulay module X over arbitrary Cohen-Macaulay ring A

(3.2) $$\beta_{n+1}(X) > \frac{r}{e_{\mathfrak{m}}(A)}\beta_n(X) \quad \text{for all } n > 0,$$

where r is the Cohen-Macaulay type of A. Now, since A is Gorenstein, a maximal Cohen-Macaulay module X having no free summand is the first syzygy of another maximal Cohen-Macaulay module. And so the inequality (3.2) holds for $n=0$ if X is maximal Cohen-Macaulay module having no free summand. Therefore we can take an indecomposable maximal Cohen-Macaulay module X which satisfies (3.1). When A is a hypersurface, we leave to show that such a module exists for the next section.

Let $G_\bullet$ be the minimal free resolution of X and $\varepsilon: G_0 \to X$ the canonical epimorphism. We will give a epimorphism from $Y=(\bigoplus_{i=0}^{d-1} Y_i)\oplus G_0$ to X. Let $F_\bullet^i$ be the minimal free resolution of $Y_i^{h_i}$. Then $(F_\bullet^i)^*$ is a finite copy of a part of the minimal free resolution of k; see (2.3). First we determine $\phi_\bullet^i: F_\bullet^i \to G_\bullet$ for all $0 \leqq i \leqq d-1$. Assume that $h_0 = \cdots = h_{t-1} = 0$ and $h_t \neq 0$. Then let $\phi_\bullet^0, \cdots, \phi_\bullet^{t-1}$ be zero maps and take a split monomorphism $\phi_{d-t}^t: F_{d-t}^t \to G_{d-t}$. It induces a commutative diagram

(3.3)
$$\begin{array}{ccccccccccccc}
0 & \longleftarrow & Y_t^{h_t} & \longleftarrow & F_0^t & \longleftarrow & \cdots & \longleftarrow & F_{d-t-1}^t & \longleftarrow & F_{d-t}^t & \longleftarrow & 0 \\
 & & \Big\downarrow{\scriptstyle H_0(\phi_\bullet^t)} & & \Big\downarrow{\scriptstyle \phi_0^t} & & & & \Big\downarrow{\scriptstyle \phi_{d-t-1}^t} & & \Big\downarrow{\scriptstyle \phi_{d-t}^t} & & \Big\downarrow \\
0 & \longleftarrow & X & \longleftarrow & G_0 & \longleftarrow & \cdots & \longleftarrow & G_{d-t-1} & \longleftarrow & G_{d-t} & \longleftarrow & G_{d-t+1} \longleftarrow \cdots
\end{array}$$

by considering $(F_\bullet^t)^*$ and $G_\bullet^*$. When $\phi_\bullet^0, \cdots, \phi_\bullet^{i-1}$ are given, take a homomorphism

$$\phi_{d-i}^i : F_{d-i}^i \longrightarrow G_{d-i}$$

such that $\dim_k \operatorname{Im} \phi_{d-i}^i \otimes k = h_i$ and

(3.4) $$\operatorname{Im} \phi_{d-i}^i \otimes k \cap \Big[\sum_{j=0}^{i-1} \operatorname{Im} \phi_{d-i}^j \otimes k\Big] = 0$$

as a subspace of $G_{d-i}\otimes k$; we can take such a homomorphism by inequality (3.1). In the same way as (3.3), we get $\phi_\bullet^i$ and $H_0(\phi_\bullet^i)$.

Let

$$\phi = \Big(\bigoplus_{i=0}^{d-1} H_0(\phi_\bullet^i)\Big) \oplus \varepsilon : Y \to X$$

and $M = \operatorname{Ker} \phi$. Then ϕ is epimorphism and $0 \to M \to Y \to X \to 0$ is the minimal injective hull of M. We will show that M is indecomposable except a maximal Cohen-Macaulay module and that its non-Cohen-Macaulay component is the required module. If there were an decomposition $M = M' \oplus M''$ where neither component is a maximal Cohen-Macaulay module. Take the minimal finite injective hull of M' and M'':

$$0 \longrightarrow M' \longrightarrow Y' \longrightarrow X' \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow M'' \longrightarrow Y'' \longrightarrow X'' \longrightarrow 0.$$

Then there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & Y & \xrightarrow{\phi} & X & \longrightarrow & 0 \\ & & \| & & \wr\uparrow\varphi & & \wr\uparrow & & \\ 0 & \longrightarrow & M'\oplus M'' & \longrightarrow & Y'\oplus Y'' & \longrightarrow & X'\oplus X'' & \longrightarrow & 0 \end{array} \tag{3.5}$$

because the lower row is the minimal finite injective hull of $M'\oplus M''$. Without loss of generality, we may assume that $X''=0$ because X is indecomposable. Let

$$Y' = \left(\bigoplus_{i=0}^{d-1} Y_i^{h'_i}\right)\oplus G' \quad \text{and} \quad Y'' = \left(\bigoplus_{i=0}^{d-1} Y_i^{h''_i}\right)\oplus G'',$$

where G' and G'' are free and $h_i=h'_i+h''_i$ for all $0\leqq i\leqq d-1$. Recall that Y'' is not Cohen-Macaulay, and so $h''_s\neq 0$ for some $t\leqq s\leqq d-1$. We regard G_0 as a complex concentrated at degree 0. Then G_0 is the minimal free resolution of itself. The automorphism φ of Y induces an automorphism

$$\left(\begin{array}{ccc|c} \varphi_\bullet^{00} & \varphi_\bullet^{01} & \cdots & \dot{\varphi}_\bullet^{0} \\ \varphi_\bullet^{10} & \varphi_\bullet^{11} & \cdots & \dot{\varphi}_\bullet^{1} \\ \vdots & & \ddots & \vdots \\ \hline \ddot{\varphi}_\bullet^{0} & \ddot{\varphi}_\bullet^{1} & \cdots & \dddot{\varphi}_\bullet \end{array}\right)$$

of $(\oplus_{i=0}^{d-1} F_\bullet^i)\oplus G_0$, where $\varphi_\bullet^{ij}: F_\bullet^j \to F_\bullet^i$, $\dot{\varphi}_\bullet^i: G_0\to F_\bullet^i$, $\ddot{\varphi}_\bullet^j: F_\bullet^j\to G_0$ and $\dddot{\varphi}_\bullet: G_0\to G_0$. We find that $\varphi_k^{ij}\otimes k=0$ if $i>j$ and $k>0$ by considering $(\varphi_\bullet^{ij})^*: (F_\bullet^i)^*\to(F_\bullet^j)^*$. And so φ_{d-s}^{ii} must be an automorphism of F_{d-s}^i for all $0\leqq i\leqq s$. The commutative diagram (3.5) means that

$$\dim_k \sum_{i=0}^{s} \operatorname{Im}(\phi_{d-s}^i\cdot\varphi_{d-s}^{is})\otimes k \leqq h'_s < h_s,$$

which contradicts to (3.4). Hence M is indecomposable except for Cohen-Macaulay summands.

Let $M=M'\oplus M''$ where M' is indecomposable and M'' is a maximal Cohen-Macaulay module or a zero module. If $\dim_A M'=d$, then M' is surjective-Buchsbaum by the commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}_A^i(k, M') & = & \operatorname{Ext}_A^i(k, Y) \\ \phi_{M'}^i\downarrow & & \phi_Y^i\downarrow \\ H_{\mathfrak{m}}^i(M') & = & H_{\mathfrak{m}}^i(Y) = k^{h_i} \end{array} \qquad \text{for all } i<d.$$

Thus the proof of Case 2 is completed. We assume that $s=\dim_A M'<d$. It is easy to show that $H_{\mathfrak{m}}^s(M')$ is not finitely generated if $s\neq 0$. Therefore s must

be equal to 0 and $h_0=\cdots=h_{d-1}=0$, however, $M'=H^0_{\mathfrak{m}}(M')$ is a k-vector space of dimension $h_0>1$, which is a contradiction. □

4. Maximal Cohen-Macaulay module of high Betti numbers.

This section is devoted to the proof of the following theorem, by which we can take an indecomposable maximal Cohen-Macaulay module satisfying (3.1) if $d\geqq 2$.

THEOREM 4.1. *Let A be a hypersurface with maximal ideal $\mathfrak{m}$ of dimension d. We assume that the multiplicity e is greater than 2. Then for any integer $n>e$, a maximal Cohen-Macaulay module* $\mathrm{Syz}^A_{d+1} A/\mathfrak{m}^n$ *is indecomposable and*

$$\beta_0^A(\mathrm{Syz}^A_{d+1} A/\mathfrak{m}^n)\geqq\binom{d+n-1}{d-1}.$$

This theorem was firstly proved by Herzog and Sanders [**14**] in the graded case. We will modify their proof to the local case.

We may assume that A is complete without loss of generality. Hence there exists a regular local ring B with maximal ideal $\mathfrak{n}$ and $0\neq f\in\mathfrak{n}^3$ such that $A=B/fB$. First we construct the minimal free resolution of $A/\mathfrak{m}^n$. There is an exact sequence

$$0\longrightarrow B/\mathfrak{n}^{n-e}\xrightarrow{f}B/\mathfrak{n}^n\longrightarrow A/\mathfrak{m}^n\longrightarrow 0.$$

Let $(F_\cdot, d_\cdot)$ and $(F'_\cdot, d'_\cdot)$ be the minimal free resolution of $B/\mathfrak{n}^{n-e}$ and of $B/\mathfrak{n}^n$ as a B-module, respectively. Then $F_\cdot$ is an Eagon-Northcott complex with respect to an $(n-e)\times(n-e+d)$ matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots\cdots & x_d & & & 0 \\ & x_0 & x_1 & & & x_d & \\ & & \ddots & \ddots & & & \ddots \\ 0 & & & x_0 & x_1 & \cdots\cdots & x_d \end{pmatrix}$$

where $x_0, x_1, \cdots, x_d$ is a minimal basis of $\mathfrak{n}$, see [**5**] and [3, p. 15], and $F'_\cdot$ is also an Eagon-Northcott complex. The homomorphism $f: B/\mathfrak{n}^{n-e}\to B/\mathfrak{n}^n$ lift to a homomorphism $\phi_\cdot: F_\cdot\to F'_\cdot$.

LEMMA 4.2. *For all $i>0$, ϕ_i is a split monomorphism.*

Before the proof, we state a notation on graded rings. Since B is regular, the associated graded ring $\mathrm{gr}_{\mathfrak{n}}(B)=\bigoplus_{i=0}^{\infty}\mathfrak{n}^i/\mathfrak{n}^{i+1}$, denoted by R, is isomorphic to a polynomial ring. Let $\mathfrak{M}$ be the homogeneous maximal ideal of R and f^* the leading term of f. For any B-module M, $\mathrm{gr}_{\mathfrak{n}}(M)=\bigoplus_{i=0}^{\infty}\mathfrak{n}^iM/\mathfrak{n}^{i+1}M$ is a graded R-module. Let $\alpha: M\to N$ be a B-homomorphism such that $\mathrm{Im}\,\alpha\subset\mathfrak{n}^nN$. Then it induces an R-homomorphism $\mathrm{gr}_{\mathfrak{n}}^{n}(\alpha): \mathrm{gr}_{\mathfrak{n}}(M)\to \mathrm{gr}_{\mathfrak{n}}(N)(n)$ in a natural way.

In this notation, the following diagram

$$(4.3)\qquad \begin{array}{ccccccc}
\mathrm{gr}_{\mathfrak{n}}(F_0)(-e) & \xleftarrow{\mathrm{gr}_{\mathfrak{n}}^{n-e}(d_1)} & \mathrm{gr}_{\mathfrak{n}}(F_1)(-n) & \xleftarrow{\mathrm{gr}_{\mathfrak{n}}^{1}(d_2)} & \mathrm{gr}_{\mathfrak{n}}(F_2)(-n-1) & \xleftarrow{\mathrm{gr}_{\mathfrak{n}}^{1}(d_3)} & \cdots \\
\downarrow f^* & & \downarrow \mathrm{gr}_{\mathfrak{n}}^{0}(\phi_1) & & \downarrow \mathrm{gr}_{\mathfrak{n}}^{0}(\phi_2) & & \\
\mathrm{gr}_{\mathfrak{n}}(F_0') & \xleftarrow[\mathrm{gr}_{\mathfrak{n}}^{n}(d_1')]{} & \mathrm{gr}_{\mathfrak{n}}(F_1')(-n) & \xleftarrow[\mathrm{gr}_{\mathfrak{n}}^{1}(d_2')]{} & \mathrm{gr}_{\mathfrak{n}}(F_2')(-n-1) & \xleftarrow[\mathrm{gr}_{\mathfrak{n}}^{1}(d_3')]{} & \cdots
\end{array}$$

is commutative and each row is the minimal free resolution of $R/\mathfrak{M}^{n-e}$ and $R/\mathfrak{M}^{n}$, respectively, because it is an Eagon-Northcott complex.

PROOF OF LEMMA 4.2. We work by induction on i. Firstly we take the n-th homogeneous component of (4.3)

$$\begin{array}{ccccc}
\mathfrak{n}^{n-e}/\mathfrak{n}^{n-e+1} & \longleftarrow & F_1\otimes k & \longleftarrow & 0 \\
f\downarrow & & \downarrow \phi_1\otimes k & & \\
\mathfrak{n}^{n}/\mathfrak{n}^{n+1} & \longleftarrow & F_1'\otimes k & \longleftarrow & 0,
\end{array}$$

which is a commutative diagram of k-vector spaces with exact rows. Since the composite homomorphism

$$F_1\otimes k \longrightarrow \mathfrak{n}^{n-e}/\mathfrak{n}^{n-e+1} \longrightarrow \mathfrak{n}^{n}/\mathfrak{n}^{n+1}$$

is a monomorphism, $\phi_1\otimes k$ is a split monomorphism. Therefore ϕ_1 is also. If ϕ_i is a split monomorphism, then we have a monomorphism

$$F_{i+1}\otimes k \longrightarrow \mathrm{gr}_{\mathfrak{n}}(F_i)_1 \xrightarrow{\mathrm{gr}_{\mathfrak{n}}^{0}(\phi_i)} \mathrm{gr}_{\mathfrak{n}}(F_i')_1$$

by taking the $(n-i)$-th homogeneous component of (4.3). And so ϕ_{i+1} is a split monomorphism by the same way. □

Therefore we can write $\phi_\bullet: F_\bullet\to F_\bullet'$ as the following form

$$\begin{array}{ccccccc}
F_0 & \longleftarrow & F_1 & \longleftarrow & F_2 & \longleftarrow & \cdots \\
f\downarrow & & \downarrow \begin{pmatrix}\mathrm{id}_{F_1}\\ 0\end{pmatrix} & & \downarrow \begin{pmatrix}\mathrm{id}_{F_2}\\ 0\end{pmatrix} & & \\
F_0 & \xleftarrow[(fd_1\ \ \alpha_1)]{} & F_1\oplus G_1 & \xleftarrow[\begin{pmatrix} d_2 & \alpha_2\\ 0 & g_2\end{pmatrix}]{} & F_2\oplus G_2 & \xleftarrow[\begin{pmatrix} d_3 & \alpha_3\\ 0 & g_3\end{pmatrix}]{} & \cdots,
\end{array}$$

where α_i is a homomorphism from G_i to F_{i-1} and g_i is a homomorphism from G_i to G_{i-1}. Taking the mapping cone of $\phi_\bullet$, we have the minimal free resolution of $A/\mathfrak{m}^n$ as a B-module

$$F_0 \xleftarrow{(f\ \alpha_1)} F_0 \oplus G_1 \xleftarrow{\begin{pmatrix} d_1\alpha_2 \\ g_2 \end{pmatrix}} G_2 \xleftarrow{g_3} G_3 \xleftarrow{g_4} \cdots$$

and denote it by $(H_., \partial_.)$.

In the same way, we obtain the minimal free resolution of $R/(\mathfrak{M}^n+f^*R)$ as an R-module

$$\mathrm{gr}_{\mathfrak{n}}(F_0) \xleftarrow{(f^*\ \mathrm{gr}_{\mathfrak{n}}^n(\alpha_1))} \mathrm{gr}_{\mathfrak{n}}(F_0)(-e)\oplus \mathrm{gr}_{\mathfrak{n}}(G_1)(-n) \xleftarrow{\begin{pmatrix} \mathrm{gr}_{\mathfrak{n}}^{n-e+1}(d_1\alpha_2) \\ \mathrm{gr}_{\mathfrak{n}}^1 g_2 \end{pmatrix}}$$

$$\longleftarrow \mathrm{gr}_{\mathfrak{n}}(G_2)(-n-1) \xleftarrow{\mathrm{gr}_{\mathfrak{n}}^1(g_3)} \mathrm{gr}_{\mathfrak{n}}(G_3)(-n-2) \xleftarrow{\mathrm{gr}_{\mathfrak{n}}^1(g_4)} \cdots$$

and denote it by $(\underline{H}_., \underline{\partial}_.)$.

We will construct the minimal free resolution of $A/\mathfrak{m}^n$ as an A-module after Shamash [**19**] and Eisenbud [**6**, Section 7].

LEMMA 4.4. *There exists a family of homomorphisms*

$$\{s_j^i : H_j \to H_{j+2i-1} \mid i,\ j \geqq 0\}$$

such that

(1) $s_j^0=\partial_j$ *for all* j;

(2) *for all* m, $\sum_{i+j=k} s_{m+2j-1}^i s_m^j = \begin{cases} f\cdot \mathrm{id}_{H_m}, & k=1; \\ 0, & k\geqq 2; \end{cases}$

(3) $s_0^1=\begin{pmatrix} \mathrm{id}_{F_0} \\ 0 \end{pmatrix}$ *and* $s_0^i=0$ *for* $i\geqq 2$;

(4) $s_1^i(F_0)=0$ *for all* $i>0$;

(5) $s_j^i\equiv 0$ *modulo* $\mathfrak{n}^2$ *for all* $i,\ j\geqq 1$.

A family satisfying (1) and (2) was given by Shamash [**19**]. But to show (3)-(5), we review his argument.

PROOF OF LEMMA 4.4. We will construct the family by induction on i. Let $s_j^0=\partial_j$ for all j and $s_0^1=\begin{pmatrix} \mathrm{id}_{F_0} \\ 0 \end{pmatrix}$. Since $H_.$ is acyclic and $f\cdot \mathrm{id}_{H_1}-s_0^1 s_1^0$ is concentrated in G_1, there is a homomorphism $\tau: G_1\to G_2$ such that

$$f\cdot \mathrm{id}_{H_1}-s_0^1 s_1^0 = \begin{pmatrix} d_1\alpha_2 \\ g_2 \end{pmatrix}(0\ \ \tau).$$

Consider the following commutative diagram with exact row

$$\begin{tikzcd} \mathrm{gr}_{\mathfrak{n}}(G_1)_0 \arrow[d, "\begin{pmatrix} -\mathrm{gr}_{\mathfrak{n}}^{n-e+1}(\alpha_1) \\ \mathrm{gr}_{\mathfrak{n}}^1(f) \end{pmatrix}"'] \arrow[dr, "\mathrm{gr}_{\mathfrak{n}}^0(\tau)"] & & \\ \mathrm{gr}_{\mathfrak{n}}(F_0)_{n-e+1}\oplus \mathrm{gr}_{\mathfrak{n}}(G_1)_1 & \mathrm{gr}_{\mathfrak{n}}(G_2)_0 \arrow[l] & 0. \arrow[l] \end{tikzcd}$$

Since $\operatorname{Im}\alpha_1 \subset \mathfrak{n}^e F_0$, the vertical homomorphism is a zero morphism. Therefore $\operatorname{Im}\tau$ is contained in $\mathfrak{n}G_2$. Furthermore by the following commutative diagram with exact row

$$\begin{tikzcd}
& \mathrm{gr}_{\mathfrak{n}}(G_1)_0 \arrow[d, "\begin{pmatrix} -\mathrm{gr}_{\mathfrak{n}}^{n-e+2}(\alpha_1) \\ \mathrm{gr}_{\mathfrak{n}}^{2}(f) \end{pmatrix}"'] \arrow[dr, "\mathrm{gr}_{\mathfrak{n}}^{1}(\tau)"] & & \\
& \mathrm{gr}_{\mathfrak{n}}(F_0)_{n-e+2} \oplus \mathrm{gr}_{\mathfrak{n}}(G_1)_2 & \mathrm{gr}_{\mathfrak{n}}(G_2)_1 \arrow[l] & \mathrm{gr}_{\mathfrak{n}}(G_3)_0, \arrow[l]
\end{tikzcd}$$

we find a homomorphism $\tau' : G_1 \to G_3$ such that $\operatorname{Im}(\tau + g_3\tau') \subset \mathfrak{n}^2 G_2$. Let $s_1^1 = (0\ \tau + g_3\tau')$. Then $f \cdot \mathrm{id}_{H_1} = s_0^1 s_1^0 + s_2^0 s_1^1$. In the same way, we obtain $\{s_i^1 \mid i > 1\}$ which satisfy (2), (3) and (5).

When the family $\{s_k^j \mid k \geqq 0,\ 1 \leqq j \leqq i\}$ is given, let

$$\sigma_k = \sum_{j=1}^{i} s_{k+2j-1}^{i-j+1} s_k^j : H_k \to H_{k+2i}.$$

Then $\sigma_\cdot : H_\cdot \to H_\cdot(2i)$ is a chain homomorphism, which is homotopic to the zero map. Furthermore $\sigma_0 = 0$, σ_1 is concentrated in G_1 and $\sigma_i \equiv 0$ modulo $\mathfrak{n}^3$ for $i > 1$ by the induction hypothesis. And so in the same way, we obtain $\{s_j^{i+1} \mid j \geqq 0\}$ satisfying (2)-(5). $\square$

Let $H_i' = (H_i \oplus H_{i-2} \oplus \cdots) \otimes_B A$ and

$$\partial_i' = \begin{pmatrix} s_i^0 & s_{i-2}^1 & \cdots\cdots \\ & s_{i-2}^0 & \\ 0 & & \ddots \end{pmatrix} \otimes_B A.$$

Then $(H_\cdot', \partial_\cdot')$ is a free resolution of $A/\mathfrak{m}^n$ as an A-module and we obtain the reduced matrix factorization of f:

$$(\phi : G_1 \oplus G_3 \cdots \to G_2 \oplus G_4 \cdots,\ \phi' : G_2 \oplus G_4 \cdots \to G_1 \oplus G_3 \cdots)$$
$$= \left(\begin{pmatrix} * & g_3 & & 0 \\ & * & g_5 & \\ & & \ddots & \ddots \\ * & & & * \end{pmatrix}, \begin{pmatrix} g_2 & & 0 \\ & g_4 & \\ & & \ddots \\ * & & \end{pmatrix} \right),$$

which corresponds to $\mathrm{Syz}_{d+1}^A A/\mathfrak{m}^n$ if d is odd or to $\mathrm{Syz}_d^A A/\mathfrak{m}^n$ if d is even. Here $(*)$ parts of ϕ and ϕ' are equal to zero modulo $\mathfrak{n}^2$. Since the rank of G_{d+1} is equal to $\binom{d+n}{d} - \binom{d+n-e}{d}$, we obtain

$$\beta_0^A(\mathrm{Syz}_{d+1}^A A/\mathfrak{m}^n) \geqq \binom{d+n-1}{d-1}.$$

Finally we will prove that $\mathrm{Syz}_{d+1}^A A/\mathfrak{m}^n$ is indecomposable. Let

$$(\alpha, \alpha') = \left(\begin{pmatrix} \alpha_{11} & \alpha_{13} & \cdots \\ \alpha_{31} & \alpha_{33} & \\ \vdots & & \ddots \end{pmatrix}, \begin{pmatrix} \alpha_{22} & \alpha_{24} & \cdots \\ \alpha_{42} & \alpha_{44} & \\ \vdots & & \ddots \end{pmatrix} \right)$$

be an endomorphism of (ϕ, ϕ'), where α_{ij} is a homomorphism from G_j to G_i. Then the diagram

$$\begin{array}{ccccccc} \mathrm{gr}_{\mathfrak{n}}(G_1)(-n) & \longleftarrow & \mathrm{gr}_{\mathfrak{n}}(G_2)(-n-1) & \longleftarrow & \mathrm{gr}_{\mathfrak{n}}(G_3)(-n-2) & \longleftarrow & \cdots \\ \Big\downarrow \mathrm{gr}_{\mathfrak{n}}^0(\alpha_{i1}) & & \Big\downarrow \mathrm{gr}_{\mathfrak{n}}^0(\alpha_{i+1,2}) & & \Big\downarrow \mathrm{gr}_{\mathfrak{n}}^0(\alpha_{i+2,3}) & & \\ \mathrm{gr}_{\mathfrak{n}}(G_i)(-n) & \longleftarrow & \mathrm{gr}_{\mathfrak{n}}(G_{i+1})(-n-1) & \longleftarrow & \mathrm{gr}_{\mathfrak{n}}(G_{i+2})(-n-2) & \longleftarrow & \cdots \end{array} \tag{4.5}$$

is commutative for all $i \geqq 1$. Since $\mathrm{Hom}_R(\underline{H}_\bullet, R)$ is acyclic, (4.5) induces a chain homomorphism $\beta_\bullet : \underline{H}_\bullet \to \underline{H}_\bullet(i-1)$. If $i>1$, then $\beta_\bullet$ is homotopic to zero. Hence we have $\alpha_{jk} \equiv 0$ modulo $\mathfrak{n}$ for $j>k$. If $i=1$, then $\beta_\bullet$ induces an endomorphism of $H_0(\underline{H}_\bullet) = R/(\mathfrak{M}^n + f^*R)$ which is homotopic to the multiplication of a homogeneous element of R. Therefore there is an element $c \in A$ such that

$$\alpha_{ii} \equiv \begin{pmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{pmatrix} \pmod{\mathfrak{n}} \quad \text{for all } i$$

and

$$(\alpha, \alpha') \equiv \left(\begin{pmatrix} c & & * \\ & \ddots & \\ 0 & & c \end{pmatrix}, \begin{pmatrix} c & & * \\ & \ddots & \\ 0 & & c \end{pmatrix} \right) \pmod{\mathfrak{n}}.$$

Thus $\mathrm{End}(\mathrm{Syz}_{d+1} A/\mathfrak{m}^n)$ is local, that is, a sum of non-units is not unit. Therefore $\mathrm{Syz}_{d+1} A/\mathfrak{m}^n$ is indecomposable. The proof of Theorem 4.1 is completed. □

References

[1] M. Amasaki, Free complexes defining maximal quasi-Buchsbaum graded modules over polynomial rings, J. Math. Kyoto Univ., **33** (1993), 143-170.

[2] M. Auslander and R.-O. Buchweitz, The homological theory of maximal Cohen-Macaulay approximations, Soc. Math. France Mem., **38** (1989), 5-37.

[3] W. Bruns and U. Vetter, Determinantal Rings, Lecture Notes in Math., **1327**, Springer-Verlag, Berlin, Heidelberg, New-York, Paris, Tokyo, 1988.

[4] M. Cipu, J. Herzog and D. Popescu, Indecomposable generalized Cohen-Macaulay modules, Trans. Amer. Math. Soc., **342** (1994), 107-136.

[5] J. A. Eagon and D. G. Northcott, Ideals defined by matrices and a certain complex associated with them, Proc. Roy. Soc. London Ser. A, **269** (1962), 188-204.

[6] D. Eisenbud, Homological algebra on a complete intersection, with an application to group representations, Trans. Amer. Math. Soc., **260** (1980), 35-64.

[7] E.G. Evans jr. and P.A. Griffith, Local cohomology modules for normal domains, J. London Math. Soc. (2), **19** (1979), 277-284.

[8] S. Goto, On Buchsbaum Rings, J. Algebra, **67** (1980), 272-279.

[9] S. Goto, Maximal Buchsbaum Modules over Regular Local Rings and a Structure Theorem for Generalized Cohen-Macaulay Modules, Commutative Algebra and Combinatrics, (eds. M. Nagata and H. Matsumura), Adv. Stud. Pure Math., **11**, Kinokuniya, Tokyo, 1987, pp. 39-64.

[10] S. Goto, Curve Singularities of Finite Buchsbaum Representation Type, J. Algebra, **163** (1994), 447-480.

[11] A. Grothendieck, Local Cohomology, Lecture Notes in Math., **41**, Springer-Verlag, Berlin, Heiderberg, New-York, 1967.

[12] J. Herzog, Ringe mit nur endlich vielen Isomorphieklassen von maximalen, unzerlegbaren Cohen-Macaulay-Moduln, Math. Ann., **233** (1978), 21-34.

[13] J. Herzog and E. Kunz, Der kanonische Modul eines Cohen-Macaulay-Rings, Lecture Notes in Math., **238**, Springer-Verlag, Berlin, Heiderberg, New-York, 1971.

[14] J. Herzog and H. Sanders, Indecomposable syzygy-modules of high rank over hypersurface rings, J. Pure Appl. Algebra, **51** (1988), 161-168.

[15] T. Kawasaki, Surjective-Buchsbaum modules over Cohen-Macaulay local rings, Math. Z., **218** (1995), 195-205.

[16] K. Nishida, On a construction of indecomposable modules and applications, Tsukuba J. Math., **13** (1989), 147-155.

[17] M. Ramras, Bounds on Betti numbers, Canad. J. Math., **34** (1982), 589-592.

[18] P. Roberts, Homological invariants of modules over commutative rings, Séminaire de Math., **72**, Les Presses de l'Université de Montréal, 1980.

[19] J. Shamash, The Poincaré Series of a Local Ring, J. Algebra, **12** (1969), 453-470.

[20] R.Y. Sharp, Some results on the vanishing of local cohomology modules, Proc. London Math. Soc. (3), **30** (1975), 177-195.

[21] J. Stückrad and W. Vogel, Toward a theory of Buchsbaum singularities, Amer. J. Math., **100** (1978), 727-746.

[22] K. Yamagishi, Bass number characterization of surjective-Buchsbaum modules, Math. Proc. Cambridge Philos. Soc., **110** (1991), 261-279.

[23] Y. Yoshino, Maximal Buchsbaum Modules of Finite Projective Dimension, J. Algebra, **159** (1993), 240-264.

Takesi KAWASAKI
Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji-shi
Tokyo 192-03
Japan
(E-mail: kawasaki@math.metro-u.ac.jp)