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A note on the structure of the ring of symmetric Hermitian modular forms of degree 2 over the Gaussian field

Dedicated to Professor Hideo Shimizu on his sixtieth birthday

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Introduction.

In Introduction of [**14**], H. L. Resnikoff and Y.-S. Tai summarized known results about the structure of the graded ring of modular forms. They stated there as follows: Freitag [**4**] studied the Hermitian modular group of genus 2 (i.e., acting on the complex 4-dimensional Hermitian tube domain) associated with the ring $\boldsymbol{Z}[i]$ of Gaussian integers and constructed the 6 generators of the graded ring of symmetric Hermitian modular forms of even weight in terms of theta nullwerte, but the relation they satisfy is not yet known ([**14**], p. 98). The main purpose of this note is to give the explicit relation. Let $\boldsymbol{H}_2$ be the Hermitian upper half space of degree 2. The theta constant on $\boldsymbol{H}_2$ with characteristic $\boldsymbol{m}$ is defined by

$$\Theta_{\boldsymbol{m}}(Z)=\Theta(Z\,;\boldsymbol{a},\boldsymbol{b})=\sum_{\boldsymbol{g}\in M_{2\times 1}(\boldsymbol{Z}[i])}\boldsymbol{e}\left[\frac{1}{2}\left(Z\left\{\boldsymbol{g}+\frac{1+i}{2}\boldsymbol{a}\right\}+2\operatorname{Re}\frac{1+i}{2}{}^t\boldsymbol{b}\boldsymbol{g}\right)\right],\quad Z\in\boldsymbol{H}_2,$$

where $\boldsymbol{m}=\binom{\boldsymbol{a}}{\boldsymbol{b}}$, $\boldsymbol{a}, \boldsymbol{b}\in M_{2\times 1}(\boldsymbol{Z})$, $A\{B\}={}^t\bar{B}AB$ and $\boldsymbol{e}[s]=e^{2\pi is}$ for $s\in\boldsymbol{C}$. Denote by $\mathcal{E}$ the set of even characteristics of degree 2 mod 2 (cf. § 1.2). Define

$$\phi_{4k}(Z):=\frac{1}{4}\sum_{\boldsymbol{m}\in\mathcal{E}}\theta_{\boldsymbol{m}}^{4k}(Z),$$

$$\chi_8(Z):=\frac{1}{3072}(\phi_4^2(Z)-\phi_8(Z)),$$

$$\chi_{10}(Z):=2^{-12}\prod_{\boldsymbol{m}\in\mathcal{E}}\theta_{\boldsymbol{m}}(Z),$$

$$\chi_{12}(Z):=2^{-15}\sum_{\text{fifteen}}(\theta_{\boldsymbol{m}_1}(Z)\cdot\theta_{\boldsymbol{m}_2}(Z)\cdots\theta_{\boldsymbol{m}_6}(Z))^2,$$

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$$\chi_{16}(Z) := 2^{-18} \sum_{\text{fifteen}} (\theta_{m_1}(Z)\theta_{m_2}(Z)\theta_{m_3}(Z)\theta_{m_4}(Z))^4, \quad Z \in \boldsymbol{H}_2,$$

where the summation in the definition of χ_{12} (resp. χ_{16}) is extended over the set of fifteen complements of syzygous quadruples (resp. fifteen azygous quadruples). Moreover, put

$$\xi_{12}(Z) := 11\phi_4^3(Z) - 13824\phi_4(Z)\chi_8(Z) + 608256\chi_{12}(Z) - 9\phi_{12}(Z).$$

Denote by $[\Gamma_2(\boldsymbol{Q}(i)), k]^{(s)}$ the vector space of symmetric Hermitian modular forms for the modular group $\Gamma_2(\boldsymbol{Q}(i))$ (cf. § 1.1).

THEOREM. *ϕ_4, χ_8, χ_{10}, χ_{12}, ξ_{12}, χ_{16} are symmetric Hermitian modular forms for $\Gamma_2(\boldsymbol{Q}(i))$, i.e., $\chi_k \in [\Gamma_2(\boldsymbol{Q}(i)), k]^{(s)}$ ($k=8, 10, 12, 16$), $\phi_4 \in [\Gamma_2(\boldsymbol{Q}(i)), 4]^{(s)}$ and $\xi_{12} \in [\Gamma_2(\boldsymbol{Q}(i)), 12]^{(s)}$. Moreover, they form a set of generators of the graded ring $\bigoplus_{k\in 2\boldsymbol{Z}}[\Gamma_2(\boldsymbol{Q}(i)), k]^{(s)}$ and satisfies the relation*

$$2(\phi_4^2\chi_8 + 6\phi_4\chi_{12} + 4032\chi_8^2 - 72\chi_{16})^2 = (\phi_4\chi_8^2 + 12\chi_8\chi_{12} + 36\chi_{10}^2)\xi_{12}.$$

The method of proof we used here is based on Freitag's argument [**4**]. In addition to this, we need explicit calculation of the Fourier coefficients. But his generators are not convenient to calculate them. So, we rewrite his generators as above and then calculate the Fourier coefficients. The recent progress of the theory of Maass space for $SU(2, 2)$ and the theory of Siegel-Eisenstein series enable us to calculate the Fourier coefficients of Hermitian modular forms of degree 2 (e.g., cf. Kojima [**9**], Gritsenko [**5**], Sugano [**16**], Krieg [**10**], Nagaoka [**12**]).

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NOTATION. For a ring $R \subset \boldsymbol{C}$, we denote by $M_{m\times n}(R)$ the R-module of all $m\times n$ matrices with entries in R. We put $M_m(R) = M_{m\times m}(R)$. If X is a matrix, ${}^tX$, $\det(X)$, and $\mathrm{tr}(X)$ stand for its transpose, determinant, and trace. For a complex matrix X, we denote $\bar{X}$ the conjugate matrix. We let $Sym_m(R)$ (resp. $Her_m(R)$) denote the space of symmetric (resp. Hermitian) matrices in $M_m(R)$. For appropriate size matrices A, B, we write by $A\{B\} = {}^t\bar{B}AB$. The identity and zero elements of $M_m(R)$ are denoted by E_m and O_m (when m needs to be stressed). We write as $\boldsymbol{e}[s] = e^{2\pi i s}$ for $s \in \boldsymbol{C}$.

§ 1. Hermitian modular forms.

1.1. Hermitian modular forms.

The *Hermitian upper half space of degree n* is defined by

$$(1.1) \qquad \boldsymbol{H}_n := \{Z \in M_n(\boldsymbol{C}) \mid (2i)^{-1}(Z - {}^t\bar{Z}) > 0\}.$$

The subset

$$S_n := Sym_n(\boldsymbol{C}) \cap \boldsymbol{H}_n \tag{1.2}$$

is called the *Siegel upper half space of degree n*. Define

$$\tilde{\Omega}_n := \left\{ M \in M_{2n}(\boldsymbol{C}) \mid {}^t\overline{M} J_n M = J_n,\ J_n = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \right\}, \tag{1.3}$$

$$\Omega_n := \tilde{\Omega}_n \cap M_{2n}(\boldsymbol{R}). \tag{1.4}$$

The group $\tilde{\Omega}_n$ (resp. Ω_n) acts on $\boldsymbol{H}_n$ (resp. $\boldsymbol{S}_n$) by

$$Z \mapsto M\langle Z\rangle := (AZ+B)(CZ+D)^{-1},\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Let Γ be a subgroup of $\tilde{\Omega}_n$ (resp. Ω_n) and v is an abelian character of Γ. A complex valued function F on $\boldsymbol{H}_n$ (resp. $\boldsymbol{S}_n$) is called an automorphic form of weight k for Γ with multiplier system v if F satisfies the following properties:

(i) F is holomorphic on $\boldsymbol{H}_n$ (resp. $\boldsymbol{S}_n$),

(ii) $$F|[M]_k = v(M)\cdot F \quad \text{for } M \in \Gamma,$$

where

$$F|[M]_k(Z) := \det(CZ+D)^{-k} F(M\langle Z\rangle),\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma. \tag{1.5}$$

Let $\boldsymbol{K}$ be an imaginary quadratic number field with discriminant $d_{\boldsymbol{K}}$. The *Hermitian modular group of degree n over* $\boldsymbol{K}$ is defined by

$$\Gamma_n(\boldsymbol{K}) := \tilde{\Omega}_n \cap M_{2n}(\mathcal{O}_{\boldsymbol{K}}), \tag{1.6}$$

where $\mathcal{O}_{\boldsymbol{K}}$ is the ring of integers of $\boldsymbol{K}$. The *Siegel modular group of degree n* is defined by

$$\Gamma_n := \Omega_n \cap M_{2n}(\boldsymbol{Z}). \tag{1.7}$$

In the rest of this subsection, Γ means a subgroup of $\tilde{\Omega}_n$ (resp. Ω_n) which is commensurable with $\Gamma_n(\boldsymbol{K})$ (resp. Γ_n). The Siegel level group $\Gamma(T)$ is an example of group which is commensurable with Γ_n. For the later purpose, we shall introduce this. Let T be an elementary divisor matrix of the type

$$T = \begin{pmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_n \end{pmatrix},\quad t_i \in \boldsymbol{Z},\ t_j | t_{j+1},$$

for all i, j. Then $\Gamma(T)$ is defined as

$$(1.8)\qquad \begin{aligned}\Gamma(T) &:= \begin{pmatrix} E & 0 \\ 0 & T\end{pmatrix}\Gamma_0(T)\begin{pmatrix} E & 0 \\ 0 & T\end{pmatrix}^{-1},\\ \Gamma_0(T) &:= \left\{M \in M_{2n}(\boldsymbol{Z}) \,|\, {}^tM\begin{pmatrix} 0 & T \\ -T & 0\end{pmatrix}M = \begin{pmatrix} 0 & T \\ -T & 0\end{pmatrix}\right\}.\end{aligned}$$

The *Hermitian* (resp. *Siegel*) *modular form of weight* k *for* $\Gamma\subset\tilde{\Omega}_n$ (resp. $\Gamma\subset\Omega_n$) means an automorphic form of weight k for Γ with the ordinary additional condition at infinity if $n=1$. Denote by $[\Gamma, k, v]$ the space of these modular forms. If $v\equiv 1$, we simply denote as $[\Gamma, k]$. $F\in[\Gamma, k, v]$ is called *symmetric* if $F(Z)=F({}^tZ)$. The subspace of symmetric modular forms in $[\Gamma, k, v]$ is denoted by $[\Gamma, k, v]^{(s)}$.

Each modular form $F\in[\Gamma_n(\boldsymbol{K}), k]$ admits a Fourier expansion of the form

$$F(Z) = \sum_{0\leq H\in\Lambda_n(\boldsymbol{K})} \boldsymbol{a}_F(H)\boldsymbol{e}[\mathrm{tr}(HZ)]$$

where

$$(1.9)\qquad \Lambda_n(\boldsymbol{K}) := \{H=(h_{ij})\in Her_n(\boldsymbol{K}) \,|\, h_{ii}\in\boldsymbol{Z},\ h_{ij}\in\vartheta_{\boldsymbol{K}}^{-1}\ (i\neq j)\}$$

and $\vartheta_{\boldsymbol{K}}$ is the different ideal of $\boldsymbol{K}$. $F\in[\Gamma_n, k]$ has a Fourier expansion of the form

$$F(Z) = \sum_{0\leq T\in\Lambda_n} \boldsymbol{a}_F(T)\boldsymbol{e}[\mathrm{tr}(TZ)]$$

where

$$(1.10)\qquad \Lambda_n := \{T=(t_{ij})\in Sym_n(\boldsymbol{Q}) \,|\, t_{ii}\in\boldsymbol{Z},\ 2t_{ij}\in\boldsymbol{Z}\ (i\neq j)\}.$$

Given $F\in[\Gamma_n(\boldsymbol{K}), k]$ put

$$(1.11)\qquad \Phi(F)(Z_1) := \lim_{\lambda\to\infty} F\begin{pmatrix} Z_1 & 0 \\ 0 & i\lambda\end{pmatrix},\quad Z_1\in\boldsymbol{H}_{n-1}.$$

It is known that this induces a linear map $\Phi: [\Gamma_n(\boldsymbol{K}), k]\to[\Gamma_{n-1}(\boldsymbol{K}), k]$. We call a cusp form by the element of $\mathrm{Ker}\,\Phi$ in the class number one case.

Here we introduce notation and results in the case $n=2$ and $\boldsymbol{K}=\boldsymbol{Q}(i)$ for the later purpose. Given $F\in[\Gamma_2(\boldsymbol{Q}(i)), k]$ define a function F_0 on $\boldsymbol{S}_2$ by

$$(1.12)\qquad F_0(Z) := F\left(Z\left\{\begin{matrix} 1 & 0 \\ 0 & 1+i\end{matrix}\right\}\right),\quad Z\in\boldsymbol{S}_2,$$

(cf. [4], (44)). Put $T=\begin{pmatrix} 1 & 0 \\ 0 & 2\end{pmatrix}$ and define

$$\hat{\Gamma}(T) := \Gamma(T)\cup\Gamma(T)\hat{M},\ \hat{M}=\begin{pmatrix} {}^t\hat{U} & 0 \\ 0 & \hat{U}^{-1}\end{pmatrix},\quad \hat{U}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 2 & 0\end{pmatrix}.$$

PROPOSITION 1.1.1 (Freitag [**4**], Lemma 2). *If* $F\in[\Gamma_2(\boldsymbol{Q}(i)), 4k]$, *then* $F_0\in\left[\hat{\Gamma}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}, 4k\right]$.

Given a function F on $\boldsymbol{S}_2$, define a function $F^{\vee}$ on $\boldsymbol{S}_1\times\boldsymbol{S}_1$ by

$$(1.13)\qquad F^{\vee}(z_1, z_2):=F\begin{pmatrix} z_1 & 0\\ 0 & \frac{1}{2}z_2\end{pmatrix},\quad (z_1, z_2)\in\boldsymbol{S}_1\times\boldsymbol{S}_1$$

(cf. [**4**], (58)).

PROPOSITION 1.1.2. *Assume that* $F\in\left[\hat{\Gamma}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}, 2k\right]$. *Then we have*

(1) $F^{\vee}(z_1, z_2)=F^{\vee}(z_2, z_1)$ *for any* $(z_1, z_2)\in\boldsymbol{S}_1\times\boldsymbol{S}_1$.

(2) $F^{\vee}(z_1, z_2)\in[\Gamma_1, 2k]\otimes[\Gamma_1, 2k]$.

1.2. Theta constants with characteristic.

For column vectors $\boldsymbol{a}, \boldsymbol{b}\in M_{2\times1}(\boldsymbol{Z})$, the *theta constant with characteristic* $\boldsymbol{m}=\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}$ is defined as

$$(1.14)\qquad \theta_{\boldsymbol{m}}(Z)=\theta(Z\,;\boldsymbol{a}, \boldsymbol{b}):=\sum_{\boldsymbol{g}\in M_{2\times1}(\boldsymbol{Z}[i])}\boldsymbol{e}\left[\frac{1}{2}\left(Z\left\{\boldsymbol{g}+\frac{1+i}{2}\boldsymbol{a}\right\}+2\,\mathrm{Re}\frac{1+i}{2}{}^t\boldsymbol{b}\boldsymbol{g}\right)\right].$$

The characteristic $\boldsymbol{m}=\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}$ is called *even* if ${}^t\boldsymbol{a}\boldsymbol{b}\equiv0 \bmod 2$. There are ten even characteristics mod 2.

$$\mathcal{E}:=\left\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\}$$

is the set of those characteristics. Using this, put

$$(1.15)\qquad \varphi_{4k}(Z):=\sum_{\boldsymbol{m}\in\mathcal{E}}\theta_{\boldsymbol{m}}^{4k}(Z),\quad Z\in\boldsymbol{H}_2,\ k\geqq1.$$

$$(1.16)\qquad \Theta(Z):=\prod_{\boldsymbol{m}\in\mathcal{E}}\theta_{\boldsymbol{m}}(Z),\quad Z\in\boldsymbol{H}_2.$$

THEOREM 1.2.1 (Freitag [**4**]). (1) φ_{4k} *is a symmetric Hermitian modular form of weight* $4k$ *for* $\Gamma_2(\boldsymbol{Q}(i))$, *namely*, $\varphi_{4k}\in[\Gamma_2(\boldsymbol{Q}(i)), 4k]^{(s)}$ $(k\geqq1)$.

(2) $\Theta\in[\Gamma_2(\boldsymbol{Q}(i)), 10]^{(s)}$. *Moreover*, $\Theta(Z)$ *has the zeros on the manifold*

$$(1.17)\qquad \boldsymbol{N}:=\left\{Z=\begin{pmatrix}z_1 & w_1\\ w_2 & z_2\end{pmatrix}\in\boldsymbol{H}_2\,|\,w_1=iw_2\right\}$$

of first order and each zero of $\Theta(Z)$ *is* $\Gamma_2(\boldsymbol{Q}(i))$*-equivalent to a point in* $\boldsymbol{N}$.

Since $\boldsymbol{N}=\boldsymbol{S}_2\left\{\begin{pmatrix}1 & 0\\ 0 & 1+i\end{pmatrix}\right\}$, Theorem 1.2.1, (2) asserts that the kernel of the homomorphism $\oplus[\Gamma_2(\boldsymbol{Q}(i)), 4k] \to \oplus\left[\hat{\Gamma}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}, 4k\right]$ induced by (1.12) is a principal ideal generated by $\Theta(Z)$. The following theorem is a main result of [**4**].

THEOREM 1.2.2 (Freitag [**4**], Satz 5). *Every symmetric Hermitian modular form for $\Gamma_2(\boldsymbol{Q}(i))$ with trivial multiplier system is expressed as an isobaric polynomial of*

$$\varphi_4,\ \varphi_8,\ \Theta,\ \eta_6^2,\ \varphi_{12},\ \varphi_{16}.$$

Namely, the graded ring $\oplus_{k\in 2\boldsymbol{Z}}[\Gamma_2(\boldsymbol{Q}(i)), k]^{(s)}$ is generated by the above six forms. Here

$$\eta_6(Z) := \sum \pm \prod_{i=1}^{3} \theta^2(Z\,;\boldsymbol{a}_i, \boldsymbol{b}_i), \tag{1.18}$$

the summation is extended over the set of sixty syzygous triples.

REMARK 1.2.3. The basic terminology on the characteristics, for example, "syzygous" and "azygous", should be referred to [7]. In the degree two case, there are sixty syzygous triples, sixty azygous triples, fifteen syzygous quadruples, and fifteen azygous quadruples. One can find their tables in [7], p. 158.

In order to find a connection with the Siegel modular case, we shall introduce Igusa's structure theorem on the graded ring $\oplus_{k\in 2\boldsymbol{Z}}[\Gamma_2, k]$. We first define the Eisenstein series on $\boldsymbol{S}_n$ by

$$G_k^{(n)}(Z) := \sum_{\left(\begin{smallmatrix}* & *\\ C & D\end{smallmatrix}\right)\in\Gamma_{n,0}\backslash\Gamma_n} \det(CZ+D)^{-k}, \quad Z\in\boldsymbol{S}_n, \tag{1.19}$$

where $\Gamma_{n,0} := \left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_n \,|\, C=0\right\}$. If $k>n+1$ is even, the series converges to an element of $[\Gamma_n, k]$. An explicit formula for the Fourier coefficients of $G_k^{(2)}$ is obtained by Kaufhold [**8**], Maass [**11**]. For simplicity, we put $G_k^{(2)}=G_k$. Next, we define the theta constant on $\boldsymbol{S}_2$ with characteristic similar to (1.14). Put

$$\vartheta_{\boldsymbol{m}}(Z) = \vartheta(Z\,;\boldsymbol{a}, \boldsymbol{b}) := \sum_{g\in M_{2\times 1}(\boldsymbol{Z})} \boldsymbol{e}\left[\frac{1}{2}\left(Z\left\{\boldsymbol{g}+\frac{1}{2}\boldsymbol{a}\right\}+{}^t\boldsymbol{b}\boldsymbol{g}\right)\right], \quad Z\in\boldsymbol{S}_2, \tag{1.20}$$

$\boldsymbol{m}=\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}$. The relation between $\theta(Z\,;\boldsymbol{a},\boldsymbol{b})$ and $\vartheta(Z\,;\boldsymbol{a},\boldsymbol{b})$ is as follows.

LEMMA 1.2.4 ([**4**], p. 7, c). *$\theta(Z\,;\boldsymbol{a},\boldsymbol{b})=\vartheta^2(Z\,;\boldsymbol{a},\boldsymbol{b})$ for $Z\in\boldsymbol{S}_2$.*

According to [7], we define

$$X_{10}(Z) := 2^{-12}\prod_{\boldsymbol{m}\in\mathcal{E}} \vartheta_{\boldsymbol{m}}^2(Z), \tag{1.21}$$

$$X_{12}(Z) := 2^{-15}\sum_{\text{fifteen}} (\vartheta_{\boldsymbol{m}_1}(Z)\cdots\vartheta_{\boldsymbol{m}_6}(Z))^4, \tag{1.22}$$

$Z\in S_2$ and the summation is extended over the set of complements of syzygous quadruples (cf. Remark 1.2.3).

THEOREM 1.2.5 (Igusa [6], [7]). *X_{10} and X_{12} are cusp forms respective weight* 10, 12 *and G_4, G_6, X_{10}, X_{12} form a set of generators of the graded ring* $\oplus_{k\in 2Z}[\Gamma_2, k]$.

REMARK 1.2.6. The constant factors of the definition of X_{10}, X_{12} are selected as the Fourier expansions satisfy

$$a_{X_{10}}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = a_{X_{12}}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = 1 .$$

Some of Fourier coefficients of X_{10}, X_{12} have been calculated by Resnikoff and Saldaña [13].

REMARK 1.2.7. It is known classically that $\oplus_{k\in Z}[\Gamma_1(K), k]=\oplus_{k\in Z}[\Gamma_1, k]$ is generated by the Eisenstein series $G_4^{(1)}$, $G_6^{(1)}$. Here we express the image $\Phi(\varphi_k)$ (k=4, 8, 12, 16) as a polynomial of $G_4^{(1)}$, $G_6^{(1)}$.

$$\Phi(\varphi_4) = 4G_4^{(1)}, \quad \Phi(\varphi_8) = 4(G_4^{(1)})^2 = 4G_8^{(1)}.$$

$$(1.23) \qquad \Phi(\varphi_{12}) = \frac{44}{9}(G_4^{(1)})^3 - \frac{8}{9}(G_6^{(1)})^2.$$

$$\Phi(\varphi_{16}) = -\frac{64}{27}G_4^{(1)}(G_6^{(1)})^2 + \frac{172}{27}(G_4^{(1)})^4.$$

1.3. Eisenstein series.

In this subsection, we define an Eisenstein series for $\Gamma_n(K)$, which becomes a Hermitian modular form. We also introduce an explicit formula for the Fourier coefficients. Put

$$\Gamma_n(K)_0 := \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n(K) \mid C = 0\right\}.$$

Let k be an integer such that $k\equiv 0 \bmod w_K$, where w_K is the order of the unit group of K. We define a kind of Eisenstein series:

$$(1.24) \qquad E_k^{(n)}(Z, s) = \sum_{\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right)\in\Gamma_n(K)_0\backslash\Gamma_n(K)} \det(CZ+D)^{-k}|\det(CZ+D)|^{-s},$$

where $(Z, s)\in H_n\times C$. It is known that this series is absolutely convergent if $\mathrm{Re}(s)+k>2n$ (cf. [1], [15]). Moreover, it follows from Shimura's result [15] that $E_k^{(n)}(Z, s)$ is holomorphic in s at $s=0$ if $k\geqq n$ and $E_k^{(n)}(Z, 0)$ is holomorphic in Z if $k>n+1$ or $k=n$. This shows

$$E_k^{(n)}(Z) := E_k^{(n)}(Z, 0) \in [\Gamma_n(\boldsymbol{K}), k] \tag{1.25}$$

if at least $k \geqq n+2$. We mainly treat the case $n=2$. So we write as $E_k(Z) = E_k^{(2)}(Z)$ simply. An explicit formula for Fourier coefficients of E_k has been obtained by Krieg [**10**] (also, cf. Nagaoka [**12**]). Now we introduce this formula in the case that the class number is one. Let $\chi_{\boldsymbol{K}}$ be the Kronecker symbol of $\boldsymbol{K}$. Define, for $s \in \boldsymbol{C}$, $x \in \boldsymbol{Z}$,

$$\sigma_{s, \chi_{\boldsymbol{K}}}(x) := \sum_{\substack{d>0 \\ d \mid x}} \chi_{\boldsymbol{K}}(d) d^s, \tag{1.26}$$

$$\sigma^*_{s, \chi_{\boldsymbol{K}}}(x) := \sum_{\substack{d>0 \\ d \mid x}} \chi_{\boldsymbol{K}}(x/d) d^s. \tag{1.27}$$

These functions appear in the Fourier coefficients of Hecke's Eisenstein series of neben-type $(\Gamma_0(|d_{\boldsymbol{K}}|), \chi_{\boldsymbol{K}})$.

Introduce

$$G_{\boldsymbol{K}}(s, N) := \frac{1}{1+|\chi_{\boldsymbol{K}}(N)|}(\sigma_{s, \chi_{\boldsymbol{K}}}(N) + \sigma^*_{s, \chi_{\boldsymbol{K}}}(-N)), \tag{1.28}$$

where $(s, N) \in \boldsymbol{C} \times \boldsymbol{Z}$. Given $H \in \Lambda_2(\boldsymbol{K})$ with $\det H \neq 0$ put

$$\gamma(H) := \det(\sqrt{|d_{\boldsymbol{K}}|} H) \in \boldsymbol{Z}. \tag{1.29}$$

Given non-zero $H \in \Lambda_2(\boldsymbol{K})$, also put

$$\varepsilon(H) := \max\{m \in \boldsymbol{Z} \mid m^{-1} H \in \Lambda_2(\boldsymbol{K})\} \in \boldsymbol{Z}_+. \tag{1.30}$$

THEOREM 1.3.1 ([**10**], [**12**]). *Assume that the class number of* $\boldsymbol{K}$ *is one. Let* k *be an integer such that* $k \geqq 4$ *and* $k \in w_{\boldsymbol{K}} \boldsymbol{Z}$. *If*

$$E_k(Z) = \sum_{0 \leq H \in \Lambda_2(\boldsymbol{K})} \boldsymbol{a}_k(H) \boldsymbol{e}[\mathrm{tr}(HZ)]$$

is the Fourier expansion of $E_k(Z) = E_k^{(2)}(Z)$, *then*

$$\boldsymbol{a}_k(H) = \begin{cases} 1 \quad \text{if } H = O_2. \\ -\dfrac{2k}{B_k} \sigma_{k-1}(\varepsilon(H)) \quad \text{if } \operatorname{rank} H = 1, \\ \dfrac{4k(k-1)}{B_k \cdot B_{k-1, \chi_{\boldsymbol{K}}}} \displaystyle\sum_{\substack{d>0 \\ d \mid \varepsilon(H)}} d^{k-1} G_{\boldsymbol{K}}(k-2, \gamma(H)/d^2) \quad \text{if } \operatorname{rank} H = 2, \end{cases} \tag{1.31}$$

where B_m (*resp.* $B_{m, \chi}$) *is the m-th Bernoulli* (*resp. generalized Bernoulli*) *number and* $\sigma_s(x) = \sum_{0<d \mid x} d^s$.

REMARK 1.3.2. The above formula asserts that $\boldsymbol{a}_k(H)$ $(H \neq 0)$ satisfies so-called "the Maass relation" and $E_k \in [\Gamma_2(\boldsymbol{K}), k]^{(S)}$. Numerical examples of $\boldsymbol{a}_k(H)$ will be given in § 1.5.

1.4. New generators.

In this subsection, we introduce new generators for the graded ring $\bigoplus_{k\in 2Z}[\Gamma_2(\boldsymbol{Q}(i)), k]^{(s)}$ (cf. Theorem 1.2.2) and study their properties. Numerical examples of the Fourier coefficients of these generators will be given in § 1.5.

Construction of ϕ_4: By the definition of φ_{4k} (cf. (1.15)), the constant term of the Fourier expansion of φ_{4k} is 4. Normalizing this, we put

$$\phi_{4k} := \frac{1}{4}\varphi_{4k} \in [\Gamma_2(\boldsymbol{Q}(i)), 4k]^{(s)}, \quad k \geqq 1. \tag{1.32}$$

Here we introduce some property of ϕ_4. Iyanaga's matrix

$$I = \begin{pmatrix} 2 & -i & -i & 1 \\ i & 2 & 1 & i \\ i & 1 & 2 & 1 \\ 1 & -i & 1 & 2 \end{pmatrix} \tag{1.33}$$

is a representative of the unique class of unimodular positive Hermitian matrices of degree 4 which are even integral over $\boldsymbol{Z}[i]$. Define a theta series associated with I:

$$\theta(Z\,; I) := \sum_{X\in M_{4\times 2}(\boldsymbol{Z}[i])} \boldsymbol{e}\left[\frac{1}{2}\mathrm{tr}\, I\{X\}\, Z\right], \quad Z \in \boldsymbol{H}_2. \tag{1.34}$$

Then $\theta(Z\,; I)\in[\Gamma_2(\boldsymbol{Q}(i)), 4]^{(s)}$ (e.g., cf. [2]). Moreover, the Fourier expansion is given as follows.

$$\begin{aligned} \theta(Z\,; I) &= \sum_{0\leq H\in\Lambda_2(\boldsymbol{Q}(i))} A(H\,; I)\boldsymbol{e}[\mathrm{tr}(HZ)], \\ A(H; I) &= \#\{X \in M_{4\times 2}(\boldsymbol{Z}[i]) \,|\, I\{X\} = 2H\}. \end{aligned} \tag{1.35}$$

Since $\dim_{\boldsymbol{C}}[\Gamma_2(\boldsymbol{Q}(i)), 4]^{(s)}=1$, we have the following result.

PROPOSITION 1.4.1. (1) *ϕ_4 satisfies*

$$\phi_4 = E_4 = \theta(Z\,; I) \in [\Gamma_2(\boldsymbol{Q}(i)), 4]^{(s)}. \tag{1.36}$$

(2) *In particular, the Fourier coefficient $A(H; I)$ is given by*

$$A(H; I) = \boldsymbol{a}_4(H)$$

for any $H\in\Lambda_2(\boldsymbol{Q}(i))$, $H\geqq 0$ (cf. (1.31)).

Construction of χ_8: Since $\Phi(E_8)=G_8^{(1)}=(G_4^{(1)})^2=\Phi(E_4^2)$, $E_4^2-E_8$ is a cusp form in $[\Gamma_2(\boldsymbol{Q}(i)), 8]^{(s)}$. Normalizing this form, we put

$$\chi_8 := \frac{61}{230400}(E_4^2-E_8) \in [\Gamma_2(\boldsymbol{Q}(i)), 8]^{(s)}. \tag{1.37}$$

PROPOSITION 1.4.2. (1) χ_8 *is a cusp form in* $[\Gamma_2(\boldsymbol{Q}(i)), 8]^{(s)}$.
(2) χ_8 *satisfies*

$$E_4^2 - 3072\chi_8 = \phi_8 . \tag{1.38}$$

(3) χ_8 *is vanishing on* S_2, *namely* $\chi_8 | S_2 \equiv 0$.

PROOF. The identity (1.38) is obtained by comparing the Fourier coefficients of the both sides. Some examples of the Fourier coefficients of χ_8, ϕ_8 are given in § 1.5, Table II, IV. The fact in (3) was already stated in [4], p 30. q.e.d.

REMARK 1.4.3. Proposition 1.4.2, (3) is equivalent to the following fact. For any fixed $a, b, c \in \boldsymbol{Z}$ $(a, b \geqq 0)$,

$$\sum_{\substack{d \in Z \\ 4ab \geq c^2+d^2}} \boldsymbol{a}_{\chi_8}\begin{pmatrix} a & \dfrac{c+di}{2} \\ \dfrac{c-di}{2} & b \end{pmatrix} = 0 . \tag{1.39}$$

Construction of χ_{10}: All the Fourier coefficients of Θ (cf. (1.16)) are rational integral and divisible by 2^{12}. Define

$$\chi_{10} := 2^{-12}\Theta \in [\Gamma_2(\boldsymbol{Q}(i)), 10]^{(s)}. \tag{1.40}$$

PROPOSITION 1.4.4. (1) χ_{10} *is a cusp form in* $[\Gamma_2(\boldsymbol{Q}(i)), 10]^{(s)}$.
(2) *The restriction* $\chi_{10} | S_2$ *coincides with Igusa's Siegel cusp form* X_{10} *introduced in* (1.21). *Namely* $\chi_{10} | S_2 = X_{10}$.
(3) $(\chi_{10})_0 \equiv 0$ (*cf.* (1.12)).

PROOF. By the definition of $\theta(Z\,; \boldsymbol{a}, \boldsymbol{b})$ (cf. (1.14)), for example, $\Phi\Big(\theta\Big(Z\,; \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\Big)\Big) = 0$. Hence $\Phi(\chi_{10}) = 0$. This shows (1). The statement of (2) is a consequence of Lemma 1.2.4. The vanishing property (3) is nothing but Theorem 1.2.1, (2). q.e.d.

REMARK 1.4.5. As we stated in Remark 1.2.6, some of the Fourier coefficients of X_{10} were calculated by Resnikoff and Saldaña [13]. By Proposition 1.4.4, (2), we have

$$\boldsymbol{a}_{X_{10}}\begin{pmatrix} a & \dfrac{c}{2} \\ \dfrac{c}{2} & b \end{pmatrix} = \sum_{\substack{d \in Z \\ \left(\begin{smallmatrix} a & (c+di)/2 \\ (c-di)/2 & b \end{smallmatrix}\right) \geq 0}} \boldsymbol{a}_{\chi_{10}}\begin{pmatrix} a & \dfrac{c+di}{2} \\ \dfrac{c-di}{2} & b \end{pmatrix}.$$

Numerical examples of $\boldsymbol{a}_{\chi_{10}}(H)$ (cf. § 1.5, Table II) and a routine calculation of the above formula also give the values $\boldsymbol{a}_{X_{10}}(T)$. For example,

$$\boldsymbol{a}_{\chi_{10}}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} = \sum_{d=-2}^{2} \boldsymbol{a}_{\chi_{10}}\begin{pmatrix} 1 & \frac{1+di}{2} \\ \frac{1-di}{2} & 2 \end{pmatrix} = 1+0-18+0+1 = -16.$$

REMARK 1.4.6. For $a, b, m \in \boldsymbol{Z}$ and $F \in [\Gamma_2(\boldsymbol{K}), k]$, define

$$S(a, b, m; F) := \sum_{\substack{c,d \\ c+d=m}} \boldsymbol{a}_F\begin{pmatrix} a & \frac{c+di}{2} \\ \frac{c-di}{2} & b \end{pmatrix}, \tag{1.41}$$

where $\boldsymbol{a}_F(H)$ is the Fourier coefficient of F. The Fourier coefficients of F_0 can be expressed by those of F, namely,

$$F_0(Z) = \sum_{a,b,m} S(a, b, m; F)\boldsymbol{e}\left[\mathrm{tr}\begin{pmatrix} a & m/2 \\ m/2 & 2b \end{pmatrix} Z\right]. \tag{1.42}$$

Therefore $(\chi_{10})_0 \equiv 0$ implies $S(a, b, m; \chi_{10}) = 0$ for all possible $a, b, m \in \boldsymbol{Z}$. For example,

$$S(1, 2, 1; \chi_{10}) = \boldsymbol{a}_{\chi_{10}}\begin{pmatrix} 1 & \frac{-1+2i}{2} \\ \frac{-1-2i}{2} & 2 \end{pmatrix} + \boldsymbol{a}_{\chi_{10}}\begin{pmatrix} 1 & \frac{i}{2} \\ -\frac{i}{2} & 2 \end{pmatrix} + \boldsymbol{a}_{\chi_{10}}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix}$$
$$+ \boldsymbol{a}_{\chi_{10}}\begin{pmatrix} 1 & \frac{2-i}{2} \\ \frac{2+i}{2} & 2 \end{pmatrix} = 1+18-18-1 = 0.$$

Construction of ψ_{12} and χ_{12}: ψ_{12} has been already defined as $\psi_{12} = (1/4)\varphi_{12}$ (cf. (1.32)). On the other hand, χ_{12} is defined by

$$\chi_{12} := 2^{-15} \sum_{\text{fifteen}} (\theta_{m_1}\theta_{m_2} \cdots \theta_{m_6})^2, \tag{1.43}$$

where the summation is extended over the set of fifteen complements of syzygous quadruples.

PROPOSITION 1.4.7. (1) $\psi_{12}, \chi_{12} \in [\Gamma_2(\boldsymbol{Q}(i)), 12]^{(s)}$. *In particular,* χ_{12} *is a cusp form and*

$$\Phi(\psi_{12}) = \frac{11}{9}(G_4^{(1)})^3 - \frac{2}{9}(G_6^{(1)})^2. \tag{1.44}$$

(2) χ_{12} *has an expression*:

$$\chi_{12} = AE_{12} + B\psi_{12} + C\psi_4^3 + D\psi_4\chi_8, \tag{1.45}$$

$$A=\frac{34910011}{1838550528000},\quad B=\frac{50521}{1634267136},$$

$$C=-\frac{11468267}{229818816000},\quad D=\frac{625441}{2659940}.$$

(3) *The following identities hold.*

$$(\chi_{12})_0^{\vee}(z_1, z_2)=12\Delta(z_1)\Delta(z_2), \tag{1.46}$$

$$\psi_{12}|S_2=\frac{11}{9}G_4^3-\frac{2}{9}G_6^2+67584X_{12}, \tag{1.47}$$

$$\chi_{12}|S_2=X_{12}, \tag{1.48}$$

where $\Delta(z)=q\prod_{n=1}^{\infty}(1-q^n)^{24}$, $q=\boldsymbol{e}[z]$, *is a cusp form in* $[\Gamma_1, 12]$.

PROOF. The fact $\psi_{12}, \chi_{12}\in[\Gamma_2(\boldsymbol{Q}(i)), 12]^{(s)}$ follows from the "transformation formula" of the theta constants. The identity $\chi_{12}|S_2=X_{12}$ is derived from the definition of χ_{12}, X_{12} and Lemma 1.2.4. This also shows $\Phi(\chi_{12})=\Phi(X_{12})=0$. (1.47) is obtained by a direct calculation. (1.44) is nothing but one of (1.23). The expression (1.45) is obtained by comparing the Fourier coefficients of five modular forms χ_{12}, E_{12}, ψ_{12}, E_4^3, $E_4\chi_8$ (cf. § 1.5, Table I, II, III). Finally, we shall prove (1.46). If $F\in[\Gamma_2(\boldsymbol{Q}(i)), k]$, then $F_0^{\vee}(z_1, z_2)$ is expressed by a power series in $\boldsymbol{C}[[q_1, q_2]]$, where $q_j=\boldsymbol{e}[z_j]$ $(j=1, 2)$. More precisely, we have

$$(1.49)\qquad \begin{aligned} F_0^{\vee}(z_1, z_2) &= \sum_{0\le a, b\in \boldsymbol{Z}} c_{a,b}\, q_1^a q_2^b, \\ c_{a,b} &= \sum \boldsymbol{a}_F\begin{pmatrix} a & * \\ * & b\end{pmatrix}, \end{aligned}$$

where the last summation is extended over all possible $\begin{pmatrix} a & * \\ * & b\end{pmatrix}\in\Lambda_2(\boldsymbol{Q}(i))$ and $\boldsymbol{a}_F(H)$ is the Fourier coefficient of F. By Proposition 1.1.2, (2), $(\chi_{12})_0^{\vee}$ is an element of $[\Gamma_1, 12]\otimes[\Gamma_1, 12]$. The structure theorem for $\bigoplus_{k\in\boldsymbol{Z}}[\Gamma_1, k]$ asserts that $(\chi_{12})_0^{\vee}$ can be expressed by an isobaric polynomial in $G_4^{(1)}(z_1)G_4^{(1)}(z_2)$, $\Delta(z_1)\Delta(z_2)$. By (1.49), we have

$$(\chi_{12})_0^{\vee}(z_1, z_2)=12q_1q_2-288q_1q_2^2-288q_1^2q_2+\cdots.$$

This shows $(\chi_{12})_0^{\vee}(z_1, z_2)=12\Delta(z_1)\Delta(z_2)$. q.e.d.

Construction of χ_{16}: Put

$$\chi_{16}:=2^{-18}\sum_{\text{fifteen}}(\theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4})^4, \tag{1.50}$$

where the summation is extended over the set of fifteen azygous quadruples.

PROPOSITION 1.4.8. (1) *χ_{16} is a cusp form in* $[\Gamma_2(\boldsymbol{Q}(i)), 16]^{(s)}$.
(2) *χ_{16} has an expression*:

(1.51) $$\chi_{16}=A\phi_{16}+B\phi_4\phi_{12}+C\phi_4^4+D\phi_4^2\chi_8+E\phi_4\chi_{12}+F\chi_8^2,$$

$$A=\frac{1}{262144},\quad B=-\frac{1}{98304},\quad C=\frac{5}{786432},$$

$$D=-\frac{3}{128},\quad E=-\frac{1}{8},\quad F=-12.$$

(3) *The following identities hold*:

(1.52) $$(\chi_{16})_0^{\vee}(z_1, z_2)=G_4^{(1)}(z_1)\Delta(z_1)G_4^{(1)}(z_2)\Delta(z_2),$$

(1.53) $$\chi_{16}|\boldsymbol{S}_2=2^{-2}\cdot 3^{-1}(G_4X_{12}-G_6X_{10}).$$

PROOF. We first show that χ_{16} is a cusp form. Put

(1.54) $$X_{16}:=2^{-18}\sum_{\text{fifteen}}(\vartheta_{m_1}\vartheta_{m_2}\vartheta_{m_3}\vartheta_{m_4})^8,$$

where the summation is extended over the set of fifteen azygous quadruples. It is known that X_{16} is a Siegel cusp form of weight 16 and it satisfies

(1.55) $$X_{16}=2^{-3}\cdot 3^{-1}(G_4X_{12}-G_6X_{10}),$$

(e.g., cf. [7], p. 153). The identity $\chi_{16}|\boldsymbol{S}_2=X_{16}$ implies (1.53). Moreover, we have $\Phi(\chi_{16})=\Phi(X_{16})=0$. The identities (1.51) and (1.52) are obtained by direct calculations (cf. § 1.5, Table I, II, III). q.e.d.

1.5. Numerical examples of Fourier coefficients.

In this subsection we shall give numerical examples of the Fourier coefficients of Hermitian modular forms defined in 1.2–1.4. We use the following abbreviation.

(1.56) $$\left(a, \frac{c+di}{2}, b\right)=\begin{pmatrix} a & \frac{c+di}{2} \\ \frac{c-di}{2} & b \end{pmatrix}\in\Lambda_2(\boldsymbol{Q}(i)).$$

Table I: Fourier coefficients of Eisenstein series.

	$c_4\cdot\boldsymbol{a}_{E_4}(H)$	$c_8\cdot\boldsymbol{a}_{E_8}(H)$	$c_{12}\cdot\boldsymbol{a}_{E_{12}}(H)$	$c_{16}\cdot\boldsymbol{a}_{E_{16}}(H)$
$\left(1, \frac{1+i}{2}, 1\right)$	3	63	1023	16383
$\left(1, \frac{1}{2}, 1\right)$	8	728	59048	4782968
$(1, 0, 1)$	15	4095	1048575	268435455

$\left(1, \frac{1+i}{2}, 2\right)$	40	47320	60524200	78368930680
$\left(1, \frac{1}{2}, 2\right)$	48	117648	282475248	678223072848
(1, 0, 2)	63	262143	1073741823	4398046511103
(2, 1+i, 2)	87	270207	1075836927	4398583349247
$\left(1, \frac{1+i}{2}, 3\right)$	78	984438	9990235398	99993896500758
$\left(1, \frac{1}{2}, 3\right)$	120	1771560	25937424600	379749833583240
(1, 0, 3)	136	2982616	61916374696	1283918200896376
(2, 1, 2)	200	3075800	62037305000	1284074929191800
$\left(1, \frac{1+i}{2}, 4\right)$	240	7647120	289537129200	11112685048614480
$\left(1, \frac{1}{2}, 4\right)$	208	11375728	576640684048	29192919926657968
(1, 0, 4)	255	16777215	1099511627775	72057594037927935
(2, 0, 2)	375	17301375	1101659109375	72066390130917375

$$c_4=\frac{1}{960}, \quad c_8=\frac{61}{1920}, \quad c_{12}=\frac{691\cdot 50521}{262080}, \quad c_{16}=\frac{3617\cdot 199360981}{65280}$$

Table II: Fourier coefficients of cusp forms.

	$a_{\chi_8}(H)$	$a_{\chi_{10}}(H)$	$a_{\chi_{12}}(H)$	$a_{\chi_{16}}(H)$
$\left(1, \frac{1+i}{2}, 1\right)$	1	0	0	0
$\left(1, \frac{1}{2}, 1\right)$	−2	1	1	0
(1, 0, 1)	4	0	8	1
$\left(1, \frac{1+i}{2}, 2\right)$	−8	0	−32	12
$\left(1, \frac{1}{2}, 2\right)$	20	−18	−26	32
(1, 0, 2)	−48	0	−96	36
(2, 1+i, 2)	80	0	−96	36
$\left(1, \frac{1+i}{2}, 3\right)$	10	0	512	−128

$\left(1, \frac{1}{2}, 3\right)$	−62	135	303	−576
(1, 0, 3)	224	0	320	−936
(2, 1, 2)	−32	512	2368	−936
$\left(1, \frac{1+i}{2}, 4\right)$	80	0	−3264	−504
$\left(1, \frac{1}{2}, 4\right)$	−20	−510	−2054	3936
(1, 0, 4)	−448	0	640	8144
(2, 0, 2)	64	0	17024	40912
$\left(1, \frac{1+i}{2}, 5\right)$	−231	0	9216	16128
$\left(3, \frac{3+3i}{2}, 3\right)$	1956	0	9216	16128
$\left(1, \frac{1}{2}, 5\right)$	486	765	8685	−9408
(1, 0, 5)	40	0	−6192	−32022
$\left(1, \frac{1+i}{2}, 6\right)$	−248	0	−1504	−121100
$\left(1, \frac{1}{2}, 6\right)$	−676	1242	−21918	−25632
(1, 0, 6)	1408	0	6400	26976
(2, 1+i, 4)	384	0	2304	420192
$\left(1, \frac{1+i}{2}, 7\right)$	1466	0	−62976	464256
$\left(1, \frac{1}{2}, 7\right)$	−996	−7038	21906	204480
$\left(3, \frac{3}{2}, 3\right)$	−5370	12645	199053	204480
(1, 0, 7)	−2240	0	40832	258448
(2, 1, 4)	320	−9216	−12416	1307024
$\left(1, \frac{1+i}{2}, 8\right)$	−80	0	151744	−909576
$\left(1, \frac{1}{2}, 8\right)$	2704	8280	49400	−971712
(1, 0, 8)	1280	0	−99840	207936

Table III : Fourier coefficients of forms of weight 12, 16.

	$\boldsymbol{a}_{\psi_4^3}(H)$	$\boldsymbol{a}_{\psi_4\chi_8}(H)$	$\boldsymbol{a}_{\psi_4^4}(H)$	$\boldsymbol{a}_{\psi_4^2\chi_8}(H)$	$\boldsymbol{a}_{\psi_4\psi_{12}}(H)$	$\boldsymbol{a}_{\psi_4\chi_{12}}(H)$
$\left(1, \frac{1+i}{2}, 1\right)$	8640	1	11520	1	9216	0
$\left(1, \frac{1}{2}, 1\right)$	23040	−2	30720	−2	92160	1
(1, 0, 1)	388800	4	748800	4	1395456	8
$\left(1, \frac{1+i}{2}, 2\right)$	4262400	232	8448000	472	8540160	−32
$\left(1, \frac{1}{2}, 2\right)$	11197440	−460	22302720	−940	48230400	214
(1, 0, 2)	65499840	912	213822720	1872	354168576	1824
(2, 1+i, 2)	26170560	2960	52174080	5840	31585536	−96
$\left(1, \frac{1+i}{2}, 3\right)$	590509440	250	2176197120	58090	2380130304	−7168
$\left(1, \frac{1}{2}, 3\right)$	1493337600	418	5640652800	−114302	8500451328	−3777
(1, 0, 3)	3518599680	−2656	26777610240	224864	35655948288	−5440
(2, 1, 2)	372441600	−6752	910387200	−13472	1789102080	15808
$\left(1, \frac{1+i}{2}, 4\right)$	16315776000	−8080	223070515200	559760	310167134208	50496
$\left(1, \frac{1}{2}, 4\right)$	32923837440	14860	554221547520	−891860	783187316736	21226
(1, 0, 4)	62500766400	−23488	1401426259200	1335872	1950823186176	−76160
(2, 0, 2)	7356484800	−15296	46065081600	430144	66810506496	116864

Table IV: Fourier coefficients of ϕ_k.

	$a_{\phi_4}(H)$	$a_{\phi_8}(H)$	$a_{\phi_{12}}(H)$	$a_{\phi_{16}}(H)$
$\left(1, \frac{1+i}{2}, 1\right)$	2880	2688	6336	11520
$\left(1, \frac{1}{2}, 1\right)$	7680	21504	84480	215040
$(1, 0, 1)$	14400	131712	851136	3022080
$\left(1, \frac{1+i}{2}, 2\right)$	38400	1483776	3801600	13690880
$\left(1, \frac{1}{2}, 2\right)$	46080	3717120	19430400	101068800
$(1, 0, 2)$	60480	8217216	90737856	668785920
$(2, 1+i, 2)$	83520	8561280	11664576	42589440
$\left(1, \frac{1+i}{2}, 3\right)$	74880	30992640	921807744	2808488448
$\left(1, \frac{1}{2}, 3\right)$	115200	55716864	2297599488	12289751040
$(1, 0, 3)$	130560	94036992	5424849408	51411118080
$(2, 1, 2)$	192000	96789504	443097600	3462359040
$\left(1, \frac{1+i}{2}, 4\right)$	230400	240752640	24958061568	460289863680
$\left(1, \frac{1}{2}, 4\right)$	199680	358041600	50307941376	1161044676608
$(1, 0, 4)$	244800	527753856	96019467456	2874331618550
$(2, 0, 2)$	360000	544612992	7462124736	118696500480

It should be remarked that Sugano has already obtained the extended table of Table II. But our tables are enough to derive the relation that our generators satisfy.

Now we refer to some phenomena appearing on the above table. Firstly, it is very likely that each cusp form χ_8, χ_{10}, χ_{12} and χ_{16} satisfies the Maass relation. For example, by Table II,

$$a_{\chi_{16}}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 40912,$$

$$a_{\chi_{16}}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} + 2^{15} \cdot a_{\chi_{16}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 8144 + 2^{15} = 40912.$$

Secondly, χ_{10} has many zero Fourier coefficients. In § 3, we will show that these facts are consequences of Sugano's result on the Maass space for $SU(2, 2)$.

§ 2. Determination of an algebraic relation satisfied by our generators.

In this section, we shall determine an algebraic relation our generators satisfy. We start from the following result on the graded ring $\bigoplus_{k\in 2\mathbf{Z}}\left[\hat{\Gamma}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}, k\right]$.

THEOREM 2.1 (Freitag [**4**], Satz 3). *Every modular form in* $\left[\hat{\Gamma}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}, k\right]$ *of even weight k can be written as an isobaric polynomial of modular forms*

$$f_4,\ f_6,\ h_4^2,\ h_4h_6,\ h_6^2,$$

respective weight 4, 6, 8, 10, 12. (*The definition of these modular forms should be referred to* [**4**], pp. 31-33).

REMARK 2.2. Both h_4 and h_6 are modular forms for $\hat{\Gamma}(T)$ $\left(T=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\right)$ with non-trivial multiplier system v_0 (for the definition of v_0, see [**4**], (57)). Moreover, they have the following properties ([**4**], p. 33):

(i) $h_4^\vee(z_1, z_2)\equiv 0,\quad h_6^\vee(z_1, z_2)=c\cdot\Delta^{1/2}(z_1)\Delta^{1/2}(z_2)\quad (c\neq 0).$

(ii) If $f\in[\hat{\Gamma}(T), 2k]$ satisfies $f^\vee=0$, then

$$f\cdot h_4^{-1}\in[\hat{\Gamma}(T), 2k-4, v_0].$$

(iii) If $g\in[\hat{\Gamma}(T), 2k, v_0]$, then g has the following expression:

$$g=p_1h_4+p_2h_6,\quad p_i\in[\hat{\Gamma}(T), 2k-\nu_i]\quad (\nu_1=4,\ \nu_2=6).$$

Here we quote the following formulas from [**4**], p. 36.

$$(2.1)\qquad \begin{cases} \text{(a)}\quad f_4=c_1(\varphi_4)_0, \\ \text{(b)}\quad f_6=c_2(\eta_6)_0,\ \text{(cf. (1.18))}, \\ \text{(c)}\quad h_4^2=c_3(4\varphi_8-\varphi_4^2)_0, \\ \text{(d)}\quad h_6^2=(c_4\varphi_{12}+P(\varphi_4, \eta_6, \varphi_8))_0, \end{cases}$$

where P is an isobaric polynomial and $c_4\neq 0$.

The following is a key lemma of our main result.

LEMMA 2.3. *Let* $F\mapsto F_0$ *be the mapping defined in* (1.12).

(1) *We have the following identities.*

(i) $$(\phi_4^2\chi_8+6\phi_4\chi_{12}+4032\chi_8^2-72\chi_{16})_0=c_1'h_4h_6f_6,$$

(ii) $$(\phi_4\chi_8+12\chi_{12})_0=c_2'h_6^2,$$

(iii) $$(\chi_8)_0=c_3'h_4^2,$$

(iv) $$\left(\frac{11}{2}\phi_4^3-6912\phi_4\chi_8+304128\chi_{12}-\frac{9}{2}\phi_{12}\right)_0=c_4'f_6^2$$

for non-zero constants c_j' $(j=1, 2, 3, 4)$.

(2) *In particular,*

$$F:=2(\phi_4^2\chi_8+6\phi_4\chi_{12}+4032\chi_8^2-72\chi_{16})^2$$
$$-\chi_8(\phi_4\chi_8+12\chi_{12})(11\phi_4^3-13824\phi_4\chi_8+608256\chi_{12}-9\phi_{12})$$

is a symmetric Hermitian modular form of weight 32 *which is vanishing on the manifold* $\boldsymbol{N}$ *defined in* (1.17).

PROOF. Since the linear map $[\Gamma_2(\boldsymbol{Q}(i)), 4k]^{(s)}\to\left[\hat{\Gamma}\begin{pmatrix}1&0\\0&2\end{pmatrix}, 4k\right]$ defined by $F\mapsto F_0$ is isomorphic when $4k=8, 12, 16$, we can define modular forms ρ_{16}, ρ_{12}, ρ_8, ρ_{12}' satisfying

$$(\rho_{16})_0=h_4h_6f_6,\quad (\rho_{12})_0=h_6^2,$$
$$(\rho_8)_0=h_4^2,\qquad (\rho_{12}')_0=f_6^2, \tag{2.2}$$

up to constants. By (1.38), $4\varphi_8-\varphi_4^2=-49152\chi_8$. Combining this and (2.1), (c), we can take as $\rho_8=\chi_8$. Since

$$h_6^\vee(z_1, z_2)=c\Delta^{1/2}(z_1)\Delta^{1/2}(z_2),\quad c\neq 0 \text{ (cf. Lemma 2.2, (i))},$$
$$(\chi_{12})_0^\vee(z_1, z_2)=12\Delta(z_1)\Delta(z_2)\quad \text{(cf. (1.46))},$$

there exists a constant c' such that

$$f:=(\chi_{12})_0-c'h_6^2\in\left[\hat{\Gamma}\begin{pmatrix}1&0\\0&2\end{pmatrix}, 12\right]$$

satisfies $f^\vee\equiv 0$. It follows from Remark 2.2 that f is constant multiple of $h_4^2f_4$. By (2.1), (a), we can write as $\rho_{12}=\eta\phi_4\chi_8+\delta\chi_{12}$ with some constants η, δ. The constant δ is non-zero. In fact, if $\delta=0$, then $(\rho_{12})_0^\vee=\eta(\phi_4\chi_8)_0^\vee=0$. This is a contradiction. Since $(\rho_{12}')_0=(\eta_6^2)_0$, we may assume that $\rho_{12}'=\eta_6^2$. We write ρ_{12}' as a polynomial of our generators:

$$\rho_{12}'=\eta_6^2=A\phi_4^3+B\phi_4\chi_8+C\chi_{12}+D\phi_{12}. \tag{2.3}$$

Since $\eta_6|\boldsymbol{S}_2=G_6$, $\phi_4|\boldsymbol{S}_2=G_4$, $\chi_{12}|\boldsymbol{S}_2=X_{12}$ (cf. [**4**], p. 37), (2.3) implies

$$G_6^2=A(\phi_4|\boldsymbol{S}_2)^3+C(\chi_{12}|\boldsymbol{S}_2)+D(\phi_{12}|\boldsymbol{S}_2)$$
$$=AG_4^3+CX_{12}+DY_{12}, \tag{2.4}$$

where we put $Y_{12}=\phi_{12}|\boldsymbol{S}_2$. Some of Fourier coefficients of G_6^2, G_4^3, X_{12}, Y_{12} are given as follows:

T	$a_{G_6^2}(T)$	$a_{G_4^3}(T)$	$a_{X_{12}}(T)$	$a_{Y_{12}}(T)$
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	1	1	0	1
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	-1008	720	0	1104
$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$	88704	40320	1	97152

By (2.4) and this table, we have $A=11/2$, $C=304128$, $D=-9/2$. Hence we can take as

$$\rho'_{12} = \frac{11}{2}\phi_4^3+\omega\phi_4\chi_8+304128\chi_{12}-\frac{9}{2}\phi_{12}.$$

Since $(h_4h_6f_6)^\vee\equiv 0$, ρ_{16} is a cusp form. So, we can write as a linear combination of $\phi_4^2\chi_8$, $\phi_4\chi_{12}$, χ_8^2 and χ_{16} (note that the terms ϕ_4^4 and $\phi_4\phi_{12}$ do not appear). Consequently, we can write as

$$(2.5)\qquad \begin{cases} \rho_{16} = \phi_4^2\chi_8+\alpha\phi_4\chi_{12}+\beta\chi_8^2+\gamma\chi_{16}, \\ \rho_{12} = \eta\phi_4\chi_8+\delta\chi_{12}, \\ \rho_8 = \chi_8, \\ \rho'_{12} = \dfrac{11}{2}\phi_4^3+\omega\phi_4\chi_8+304128\chi_{12}-\dfrac{9}{2}\phi_{12}. \end{cases}$$

Put

$$(2.6)\qquad J_1 := \rho_{16}^2, \quad J_2 := \rho_8\rho_{12}\rho'_{12}.$$

We can determine α, β, γ, η, δ, ω as $J:=J_1-J_2$ satisfies $J_0\equiv 0$. Here we recall the definition of $S(a, b, m; F)$ (cf. (1.41)). We have already known that $F_0\equiv 0$ implies $S(a, b, m; F)=0$ for all possible a, b, m. The examples of Fourier coefficients given in 1.5 yield $S(2, 2, 4; J_1)=1$ and $S(2, 2, 4; J_2)=\eta$. This shows $\eta=1$. Moreover, we have the following formulas:

$$(2.7)\qquad \begin{cases} S(2, 2, 3; J_1) = 2(\alpha-2), \\ S(2, 2, 3; J_2) = \delta-4, \end{cases}$$
$$\begin{cases} S(2, 2, 2; J_1) = 2\{(\alpha-2)^2+2\}+2(8\alpha+\gamma+4)+2(\alpha-2)^2, \\ S(2, 2, 2; J_2) = 2(6-2\delta)+(4\delta+16) = 28, \end{cases}$$
$$\begin{cases} S(2, 2, 1; J_1) = 6(\alpha-2)+2(\alpha-2)(8\alpha+\gamma+4), \\ S(2, 2, 1; J_2) = -9\delta-28. \end{cases}$$

Since δ is non-zero, we have $\alpha=6$, $\gamma=-72$, $\delta=12$. We substitute these values into (2.5). We also get

$$(2.8)\quad \begin{cases} S(3,3,6;J_1)=-1104\cdot 2+(5472+2\beta), \\ S(3,3,6;J_2)=-1248\cdot 2+(20736+\omega), \end{cases} \quad \begin{cases} S(3,3,5;J_1)=2(-8904+138464), \\ S(3,3,5;J_2)=2\{-8904+(6\omega+179936)\}. \end{cases}$$

These identities imply $\beta=4032$, $\omega=-6912$. Substituting all these values into (2.5), we get (1). Simultaneously, this shows (2). q.e.d.

EXAMPLE 2.4. Set $F_i=2J_i$ $(i=1, 2)$. Then $F=F_1-F_2$ becomes the modular form defined in the above lemma, (2). Of course, F satisfies $F_0\equiv 0$. We give some examples of the Fourier coefficients of F, F_1, F_2, which cite $S(a, b, m; F)=0$.

H	$a_{F_1}(H)$	$a_{F_2}(H)$	$a_F(H)$	
$\left(2, \frac{-1+3i}{2}, 2\right)$	0	0	0	
$(2, i, 2)$	36	-36	72	$S(2, 2, 2; F)=0$
$\left(2, \frac{1+i}{2}, 2\right)$	-16	128	-144	

H	$a_{F_1}(H)$	$a_{F_2}(H)$	$a_F(H)$	
$(2, -1+2i, 3)$	0	0	0	
$\left(2, \frac{-1+3i}{2}, 3\right)$	-16	128	-144	$S(2, 3, 2; F)=0$
$(2, i, 3)$	-34304	40576	-74880	
$\left(2, \frac{1+i}{2}, 3\right)$	24864	-125184	150048	

H	$a_{F_1}(H)$	$a_{F_2}(H)$	$a_F(H)$	
$\left(3, \frac{-1+5i}{2}, 3\right)$	0	0	0	
$(3, 2i, 3)$	-34304	40576	-74880	$S(3, 3, 4; F)=0$
$\left(3, \frac{1+3i}{2}, 3\right)$	1559488	265216	1294272	
$(3, 1+i, 3)$	3563904	6002688	-2438784	

H	$a_{F_1}(H)$	$a_{F_2}(H)$	$a_F(H)$
$\left(3, \frac{-2+5i}{2}, 3\right)$	0	0	0
$\left(3, \frac{-1+4i}{2}, 3\right)$	-17808	-17808	0
$\left(3, \frac{3i}{2}, 3\right)$	3578976	-1139616	4718592
$\left(3, \frac{1+2i}{2}, 3\right)$	13248576	17967168	-4718592

$S(3, 3, 3; F)=0.$

Now we note that the function F defined in Lemma 2.3, (2) is divisible by χ_{10}^2. In fact, by Theorem 1.2.1, (2), F is divisible by χ_{10}. Since the weight of $F\cdot\chi_{10}^{-1}$ is 22, the structure theorem for $\oplus[\Gamma_2(\boldsymbol{Q}(i)), k]^{(s)}$ asserts that $F\cdot\chi_{10}^{-1}$ is also divisible by χ_{10}. So we can write as

$$F=\chi_{10}^2(A\phi_4^3+B\phi_4\chi_8+C\chi_{12}+D\phi_{12}) \tag{2.9}$$

for some constants A, B, C, D. By using the above tables, we can determine these values as follows. Let G be the modular form defined by the right-hand side of (2.9). Then, we have

$$\begin{aligned} &a_F\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}=72, && a_G\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}=A+D, \\ &a_F\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}=-74880, && a_G\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}=688A+1072D. \end{aligned} \tag{2.10}$$

This shows $A=396$, $D=-324$. Next, the identities

$$a_F\begin{pmatrix} 3 & 1+i \\ 1-i & 3 \end{pmatrix}=-2438784, \quad a_G\begin{pmatrix} 3 & 1+i \\ 1-i & 3 \end{pmatrix}=-2B-3434112, \tag{2.11}$$

implies $B=-497664$. Finally, we have

$$a_F\begin{pmatrix} 3 & \frac{3i}{2} \\ -\frac{3i}{2} & 3 \end{pmatrix}=4718592, \quad a_G\begin{pmatrix} 3 & \frac{3i}{2} \\ -\frac{3i}{2} & 3 \end{pmatrix}=D-17178624. \tag{2.12}$$

Hence we get $D=21897216$. Consequently, we obtain

$$\begin{aligned} G &= \chi_{10}^2(396\phi_4^3-497664\phi_4\chi_8+21897216\chi_{12}-324\phi_{12}) \\ &= 36\chi_{10}^2(11\phi_4^3-13824\phi_4\chi_8+608256\chi_{12}-9\phi_{12}). \end{aligned}$$

It should be noted that the last factor already appeared in the definition of F (Lemma 2.3, (2)). Consequently, we have the following theorem.

THEOREM 2.5. *Define*

$$\xi_{12} := 11\phi_4^3 - 13824\phi_4\chi_8 + 608256\chi_{12} - 9\phi_{12} \in [\Gamma_2(\boldsymbol{Q}(i)), 12]^{(s)}.$$

Then ϕ_4, χ_8, χ_{10}, χ_{12}, ξ_{12}, χ_{16} *form a set of generators of the graded ring* $\bigoplus_{k\in 2\boldsymbol{Z}}[\Gamma_2(\boldsymbol{Q}(i)), k]^{(s)}$. *Moreover, they satisfies the relation*

$$2(\phi_4^2\chi_8 + 6\phi_4\chi_{12} + 4032\chi_8^2 - 72\chi_{16})^2 = (\phi_4\chi_8^2 + 12\chi_8\chi_{12} + 36\chi_{10}^2)\xi_{12}.$$

REMARK 2.6. By several reasons, it is likely that the above generators have essentially single relation, i.e.,

$$\bigoplus[\Gamma_2(\boldsymbol{Q}(i)), k]^{(s)} \cong \boldsymbol{C}[X_1, X_2, X_3, X_4, X_5, X_6]/(\Xi)$$

where

$$\begin{aligned}&\Xi(X_1, X_2, X_3, X_4, X_5, X_6)\\ &= 2(X_1^2X_2 + 6X_1X_4 + 4032X_2^2 - 72X_6)^2 - (X_1X_2^2 + 12X_2X_4 + 36X_3^2)X_5.\end{aligned}$$

§ 3. Some properties of Fourier coefficients of our generators.

As we stated at the end of § 1, our generators have interesting properties. In this section, we summarize some results on the Fourier coefficients of our generators without proof.

Denote the Maass space on $\boldsymbol{H}_2$ for $\boldsymbol{Q}(i)$ in the sense of Sugano [**16**] (resp. Krieg [**10**]) by $\mathfrak{M}_k^S$ (resp. $\mathfrak{M}_k^K$). In [**16**], Sugano constructed an isomorphism $\Psi_S : \mathfrak{S}_{1,k}(\Gamma^J) \hookrightarrow \mathfrak{M}_k^S$, where $\mathfrak{S}_{1,k}(\Gamma^J)$ is the vector space of Jacobi cusp forms of index 1 and weight k for the Jacobi modular group Γ^J over $\boldsymbol{Q}(i)$. On the other hand, Krieg [**10**] constructed an isomorphism $\Psi_K : \mathfrak{M}_k^K \to G_{k-1}^*(\Gamma_0(4), \left(\frac{-4}{*}\right))$, where $G_{k-1}^*(\Gamma_0(4), \left(\frac{-4}{*}\right))$ is the vector space of elliptic modular forms of nebentype of weight $k-1$ and character $\chi_K = \left(\frac{-4}{*}\right)$ on $\Gamma_0(4)$ such that the Fourier coefficients $\boldsymbol{a}_f(n)$ vanish if $\left(\frac{-4}{n}\right) = 1$.

PROPOSITION 3.1. *Let* $E_k^J(\boldsymbol{z})$ $(\boldsymbol{z} \in \mathfrak{D} := \boldsymbol{H}_1 \times \boldsymbol{C}^2)$ *be the Jacobi Eisenstein series of index* 1 *and weight* k (*cf.* [**16**], [**12**]) *and* $\theta_\mu(\boldsymbol{z})$ *be the theta series defined by*

$$\theta_\mu(z, w_1, w_2) = \sum_{l\in\mathcal{O}_K} \boldsymbol{e}[\boldsymbol{N}_K(l+\mu)z + (\overline{l+\mu})w_1 + (l+\mu)w_2],$$

$(z, w_1, w_2) \in \mathfrak{D}$. *We put*

$$f_8(\boldsymbol{z}) := \frac{61}{480}(G_4^{(1)}(z)E_4^J(\boldsymbol{z}) - E_8^J(\boldsymbol{z})),$$

$$f_{10}(\boldsymbol{z}) := \eta^{18}(z)(\theta_{1/2}(\boldsymbol{z}) - \theta_{i/2}(\boldsymbol{z})),$$

$$f_{12}(\boldsymbol{z}) := \frac{2509}{204\cdot 504}(G_6^{(1)}(z)E_6^J(\boldsymbol{z})-E_{12}^J(\boldsymbol{z}))-\frac{1211}{204\cdot 240}(G_8^{(1)}(z)E_4^J(\boldsymbol{z})-E_{12}^J(\boldsymbol{z})),$$

$$f_{16}(\boldsymbol{z}) := \frac{1}{12}G_4^{(1)}(z)f_{12}(\boldsymbol{z})-\frac{277}{72\cdot 3528}G_6^{(1)}(z)(G_6^{(1)}(z)E_4^J(\boldsymbol{z})-E_{10}^J(\boldsymbol{z}))$$
$$+\frac{1}{72}G_8^{(1)}(z)f_8(\boldsymbol{z}), \quad \boldsymbol{z}=(z, w_1, w_2)\in\mathfrak{D},$$

where $\eta(z)$ *is the Dedekind eta function. Then* $f_k\in\mathfrak{S}_{1,k}(\Gamma^J)$ *and* $\Psi_S(f_k)=\chi_k$ ($k=8, 10, 12, 16$).

This result is due to Sugano.

COROLLARY 3.2. *The Fourier coefficients* $\boldsymbol{a}_{\chi_{10}}(H)$ *of* χ_{10} *have the following property*:

$$\boldsymbol{a}_{\chi_{10}}(H)=0 \quad \text{if } \chi_{\boldsymbol{K}}(\gamma(H)/\varepsilon(H)^2)=0,$$

where $\gamma(H)$ *and* $\varepsilon(H)$ *were defined in* (1.29), (1.30).

PROPOSITION 3.3. *Set*

$$\theta(z) := 1+2\sum_{n=1}^{\infty}q^{n^2}, \quad F_2(z) := \sum_{\substack{n\geq 1\\ n:\text{ odd}}}\sigma_1(n)q^n,$$

where $\sigma_1(n)=\sum_{0<d\mid n}d$, $q=\boldsymbol{e}[z]$, $z\in\boldsymbol{H}_1$. *We also put*

$$g_3(z) := \theta^6(z)-12\theta^2(z)F_2(z),$$

$$g_7(z) := \theta^6(z)F_2^2(z)-16\theta^2(z)F_2^3(z),$$

$$g_{11}(z) := 2\theta^{10}(z)F_2^3(z)-32\theta^6(z)F_2^4(z),$$

$$g_{15}(z) := \theta^{14}(z)F_2^4(z)-28\theta^{10}(z)F_2^5(z)+192\theta^6(z)F_2^6(z).$$

Then $g_{k-1}\in G_{k-1}^*(\Gamma_0(4), \left(\frac{-4}{*}\right))$ *and* $\Psi_K(\phi_4)=120i\cdot g_3$, $\Psi_K(\chi_k)=(i/2)g_{k-1}$ ($k=8, 12, 16$).

COROLLARY 3.4. *Denote by* $\boldsymbol{a}_{g_{k-1}}(n)$ *the Fourier coefficient of* g_{k-1}. *If* $4ab-\boldsymbol{N}_{\boldsymbol{K}}(\alpha)\neq 0$, *then we have*

(1) $$\boldsymbol{a}_{\phi_4}\begin{pmatrix} a & \alpha/2\\ \bar{\alpha}/2 & b\end{pmatrix}=\frac{240}{1+\chi_{\boldsymbol{K}}(\boldsymbol{N}_{\boldsymbol{K}}(\alpha))}\sum_{\substack{d>0\\ d\mid(a,b,\alpha)}}d^3\boldsymbol{a}_{g_3}((4ab-\boldsymbol{N}_{\boldsymbol{K}}(\alpha))/d^2),$$

(2) $$\boldsymbol{a}_{\chi_k}\begin{pmatrix} a & \alpha/2\\ \bar{\alpha}/2 & b\end{pmatrix}=\frac{1}{1+\chi_{\boldsymbol{K}}(\boldsymbol{N}_{\boldsymbol{K}}(\alpha))}\sum_{\substack{d>0\\ d\mid(a,b,\alpha)}}d^{k-1}\boldsymbol{a}_{g_{k-1}}((4ab-\boldsymbol{N}_{\boldsymbol{K}}(\alpha))/d^2),$$

($k=8, 12, 16$), $\chi_{\boldsymbol{K}}=\left(\frac{-4}{*}\right)$.

References

[1] H. Braun, Hermitian modular functions, Ann. of Math., **50** (1949), 827-855.

[2] D.M. Cohen and H.L. Resnikoff, Hermitian quadratic forms and Hermitian modular forms, Pacific J. Math., **76** (1978), 329-337.

[3] M. Eichler and D. Zagier, The theory of Jacobi forms, Progr. Math., **55**, Birkhäuser, Boston-Basel-Stuttgart, 1985.

[4] E. Freitag, Modulformen zweiten Grades zum rationalen und Gaußschen Zahlkörper, Sitzungsber. Heidelb. Akad. Wiss., (1967).

[5] V.A. Gritsenko, The Maass space for $SU(2,2)$, Hecke operators, and zeta functions, (Russian), LOMI R-7-85, Leningrad, 1985.

[6] J. Igusa, On Siegel modular forms of genus two, Amer. J. Math., **84** (1962), 175-200; (II), ibid., **88** (1966), 221-236.

[7] J. Igusa, On the ring of modular forms of degree two over $\boldsymbol{Z}$, Amer. J. Math., **101** (1979), 149-183.

[8] G. Kaufhold, Dirichletsche Reihe mit Funktionalgleichung in der Theorie der Modulfunktion 2. Grades, Math. Ann., **137** (1959), 454-476.

[9] H. Kojima, An arithmetic of Hermitian modular forms of degree two, Invent. Math., **69** (1982), 217-227.

[10] A. Krieg, The Maass space on the Hermitian half-space of degree 2, Math. Ann., **289** (1991), 663-681.

[11] H. Maass, Die Fourierkoeffizienten der Eisensteinreihen zweiten Grades, Mat.-Fys. Medd. Danske Vid. Selsk., **34** (1964), Nr. 7.

[12] S. Nagaoka, On Eisenstein series for the Hermitian modular groups and the Jacobi groups, Abh. Math. Sem. Univ. Hamburg, **62** (1992), 117-146.

[13] H.L. Resnikoff and R.L. Saldaña, Some properties of Fourier coefficients of Eisenstein series of degree two, J. Reine Angew. Math., **265** (1974), 90-109.

[14] H.L. Resnikoff and Y.-S. Tai, On the structure of a graded ring of automorphic forms on the 2-dimensional complex ball, Math. Ann., **238** (1978), 97-117.

[15] G. Shimura, On Eisenstein series, Duke Math. J., **50** (1983), 417-476.

[16] T. Sugano, The Maass space for $SU(2,2)$, (Japanese), Res. Inst. Math. Sci. Kokyuroku, **546** (1985), 1-16.

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