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Brelot spaces of Schrödinger equations

By Mitsuru NAKAI

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Consider a Radon measure μ of not necessarily constant sign on a subregion W of the Euclidean space $\boldsymbol{R}^d$ of dimension $d \geqq 2$. A function u on an open subset U of W is said to be *μ-harmonic* on U if u is continuous on U and satisfies the Schrödinger equation $(-\Delta+\mu)u=0$ on U in the sense of distributions. The family of μ-harmonic functions on open subsets of W determines a *sheaf* H_μ of functions on W (cf. § 1.1 below), i.e., $H_\mu(U)$ is the set of μ-harmonic functions on U. In order for us to be able to effectively discuss various global structures such as the Martin boundary related to the equation $(-\Delta+\mu)u=0$ on W, it is the least requirement for the sheaf H_μ to give rise to a Brelot harmonic space, or simply *Brelot space*, (W, H_μ) (cf. § 1.2). This paper concerns the question under what condition on μ the sheaf H_μ generates a Brelot space (W, H_μ). It was shown by Boukricha [**3**] for a positive measure μ and by Boukricha-Hansen-Hueber [**4**] for a signed measure μ that (W, H_μ) is a Brelot space if μ is of *Kato class* (cf. § 2.2). It is a natural question to ask whether for μ to be of Kato class is the widest possible condition for (W, H_μ) to be a Brelot space; specifically we ask whether μ is of Kato class if (W, H_μ) is a Brelot space. The answer to this question is given as follows:

MAIN THEOREM. *Although a Radon measure μ of constant sign being of Kato class is necessary and sufficient for the pair (W, H_μ) to be a Brelot space, a Radon measure μ of nonconstant sign being of Kato class is sufficient but not necessary in general for (W, H_μ) to be a Brelot space.*

We will give a self contained complete proof to the above assertion and actually more than described in the above statement as follows. We introduce a new notion of, what we call, a Radon measure of *quasi Kato class* (cf. § 3.2). We then have the following result:

THEOREM 1. *If μ is a Radon measure of quasi Kato class, then the pair (W, H_μ) is a Brelot space.*

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Since it is easily seen, by examining the very definitions of both classes, that a Radon measure μ on W is of quasi Kato class on W if it is of Kato class on W, the above theorem 1 is, at least superficially, a generalization of the above cited results of Boukricha [**3**] and Boukricha-Hansen-Hueber [**4**] (cf. also Strum [**11**]). That it is a strict and essential generalization is seen by the following result:

THEOREM 2. *On any Euclidean subregion W there always exists a Radon measure μ which is of quasi Kato class on W but not of Kato class on W.*

From theorems 1 and 2 the main theorem follows at once except for the part that a Radon measure μ of constant sign is of Kato class if (W, H_μ) is a Brelot space. The proof of this fact is quite easy and will briefly be given in § 2.2 among other things. Thus we only have to concentrate ourselves upon the proofs of theorems 1 and 2.

The paper consists of six sections. Brelot spaces are explained in § 1. Here a simple example of (W, H_μ) which is not a Brelot space is stated. In § 2 measures of Kato class are considered. A central fact treated in this section concerns the Brelot spaces (W, H_μ) with positive or negative measures μ. A new notion of measures of quasi Kato class is introduced in § 3 and Green potentials of measures of quasi Kato class are discussed in § 4. Based upon the results in the preceding section, the proof of Theorem 1 is given in § 5. In the last § 6, Theorem 2 is proved. The flat cone criterion for Dirichlet regularity is used in § 6 and thus a proof for this fact is given in Appendix at the end of this paper.

1. Brelot spaces.

1.1. We denote by $\boldsymbol{R}^d$ the Euclidean space of dimension $d \geqq 2$ and $\lambda = \lambda^d$ the Lebesgue measure on $\boldsymbol{R}^d$. We sometimes use the notation $|X|$ to mean the volume $\lambda(X)$ of a measurable subset X of $\boldsymbol{R}^d$. We also denote the volume element $d\lambda(x)$ by $dx = dx_1 \cdots dx_d$ where $x = (x_1, \cdots, x_d)$ is a point of $\boldsymbol{R}^d$. The length of x is denoted by $|x|$. A subregion or region W of $\boldsymbol{R}^d$ is an open and connected set. A typical example of regions is an open ball $B(a, r)$ of radius $r > 0$ centered at $a \in \boldsymbol{R}^d$. We also denote by $\bar{B}(a, r)$ the closed ball $\overline{B(a, r)} = B(a, r) \cup \partial B(a, r)$. A *Radon measure* μ on a region W is a difference of two regular positive Borel measures on W (i.e., defined for Borel subsets of W) so that the total variation $|\mu|$ of μ and the positive (negative, resp.) part $\mu^+ = (|\mu| + \mu)/2$ $(\mu^- = (|\mu| - \mu)/2$, resp.) of μ are positive regular Borel measures on W. If a Radon measure μ on W takes only nonnegative (nonpositive, resp.) values, then μ is said to be positive (negative, resp.), $\mu \geqq 0$ $(\mu \leqq 0$, resp.) in notation. Positive or negative Radon measures are said to be of constant sign.

Otherwise they are said to be of nonconstant sign. To stress that μ is not necessarily positive or negative we sometimes say that μ is a signed Radon measure.

Using a Radon measure μ on a region W in $\boldsymbol{R}^d$ $(d\geqq 2)$ as its potential we consider a stationary (i.e., time independent) Schrödinger operator $-\Delta+\mu$ on W. By a *solution* u on an open subset U of W of the Schrödinger equation

$$(1.1) \qquad (-\Delta+\mu)u=0 \quad (\Delta=\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_d^2)$$

we mean that $u\in L_{1,\mathrm{loc}}(U, \lambda+|\mu|)$ and u satisfies (1.1) on U in the sense of distributions, i.e.,

$$(1.2) \qquad -\int_U u(x)\Delta\varphi(x)dx+\int_U u(x)\varphi(x)d\mu(x)=0$$

for every test function $\varphi\in C_0^\infty(U)$. A solution u of (1.1) on U may not be continuous (i.e., may not have a continuous representative as an element of $L_{1,\mathrm{loc}}(U, \lambda+|\mu|)$) even if μ is of Kato class defined later (cf. [**10**] and also [**11**]) unless μ is absolutely continuous with respect to λ (cf. [**1**]) and thus we have to assume it if we wish to have the continuity of a solution u. A function u defined on an open subset U of W is said to be *μ-harmonic* on U if $u\in C(U)$ and u is a solution of (1.1) on U. Thus we may say that u is a μ-harmonic function on U if and only if $u\in C(U)$ and satisfies (1.2).

We denote by $H_\mu(U)$ the set of all μ-harmonic functions on an open subset U of W. Then we can define a *sheaf* H_μ of functions in W, i.e., H_μ gives rise to a mapping $U\mapsto H_\mu(U)$ defined on the family of all open sets U of W satisfying the following three sheaf axioms:

(S.1) For any open set U in W, $H_\mu(U)$ is a family of functions on U;

(S.2) For any two open sets U and V in W such that $U\subset V$, the restriction to U of a function in $H_\mu(V)$ belongs to $H_\mu(U)$, i.e., $H_\mu(V)|U\subset H_\mu(U)$;

(S.3) For any family $\{U_\iota\}_{\iota\in I}$ of open sets U_ι in W and any function u on $\bigcup_{\iota\in I}U_\iota$, $u\in H_\mu(\bigcup_{\iota\in I}U_\iota)$ if $u|U_\iota\in H_\mu(U_\iota)$ for every $\iota\in I$.

It is entirely obvious that H_μ certainly satisfies (S.1) and (S.2). It may be less obvious that H_μ satisfies (S.3). Suppose a function u on $\bigcup_{\iota\in I}U_\iota$ satisfies $u|U_\iota\in H_\mu(U_\iota)$ for every $\iota\in I$. In particular $u|U_\iota\in C(U_\iota)$ implies that $u\in C(\bigcup_{\iota\in I}U_\iota)$. Fix a partition $\{\phi_\alpha\}_{\alpha\in A}$ of unity subordinate to a locally finite refinement of $\{U_\iota\}_{\iota\in I}$. Choose an arbitrary $\varphi\in C_0^\infty(\bigcup_{\iota\in I}U_\iota)$. Since $\operatorname{supp}\varphi$ is compact, $\{\alpha\in A: \varphi\phi_\alpha\not\equiv 0\}$ is a finite set $\{\alpha(k): 1\leqq k\leqq n\}$. Let $\varphi_k=\varphi\phi_{\alpha(k)}$ and $\iota(k)\in I$ be such that $\operatorname{supp}\varphi_k\subset U_{\iota(k)}$. From $u|U_{\iota(k)}\in H_\mu(U_{\iota(k)})$ it follows that

$$-\int_{U_{\iota(k)}} u(x)\Delta\varphi_k(x)dx+\int_{U_{\iota(k)}} u(x)\varphi_k(x)d\mu(x)=0 \quad (k=1, \cdots, n).$$

Adding the above identities for $k=1, \cdots, n$ and then observing that $\varphi=\sum_{k=1}^{n}\varphi_k$, we deduce (1.2) for $U=\bigcup_{\iota\in I}U_\iota$.

1.2. An open set U in W is said to be *regular* for H_μ if it is relatively compact in W and $\partial U\neq\emptyset$ and for every continuous function f defined on ∂U there is a unique continuous function u on $\bar{U}$ such that

$$u|\partial U = f,\quad u|U \in H_\mu(U) \quad\text{and}\quad u \geqq 0 \text{ if } f \geqq 0.$$

We say that a pair (W, H_μ) forms a Brelot harmonic space or simply *Brelot space* if the following three axioms are satisfied:

AXIOM 1 (*Linearity*). For any open set U of W, $H_\mu(U)$ is a linear subspace of the space $C(U)$;

AXIOM 2 (*Local solvability of Dirichlet problem*). There is a base for the topology of W such that each set in the base is a regular region for H_μ;

AXIOM 3 (*The Harnack principle*). If U is a region in W and $\{u_n\}$ is any increasing sequence in $H_\mu(U)$, then $u=\sup_n u_n$ belongs to $H_\mu(U)$ unless u is identically $+\infty$.

For a general theory of harmonic spaces including Brelot spaces, see e.g., Maeda [**9**] and Constantinescu-Cornea [**5**], among others. Under Axioms 1 and 2, Axiom 3 is seen to be equivalent to the following property (cf. e.g., Loeb-Walsh [**8**]): For each region U in W and each compact subset K of U there exists a constant $c>0$ such that for any $u\in H_\mu^+(U)$ (where $\mathcal{F}^+$ always indicates the subfamily of a family $\mathcal{F}$ of functions consisting of all nonnegative members in $\mathcal{F}$)

$$\sup_{x\in K} u(x) \leqq c\cdot\inf_{x\in K} u(x) \quad (\textit{The Harnack inequality}).$$

As an example consider the Radon measure 0 on $\boldsymbol{R}^d$, i.e., the Radon measure whose values at every Borel sets are zero. The corresponding equation is the Laplace equation $-\Delta u=0$. For any distributional solution $u\in L_{1,\mathrm{loc}}(U, \lambda)$ of $-\Delta u=0$ on an open set U, there exists a classical harmonic function $u^\sim\in C^\infty(U)$ satisfying $-\Delta u^\sim=0$ on U in the genuine sense such that $u^\sim=u$ λ-a.e. on U. This is known as the *Weyl lemma* which is an easy consequence of the standard mollifier method. In this case, hence, there is no essentially discontinuous solutions of $-\Delta u=0$ other than 0-harmonic functions. Thus in this case the sheaf H_0 is determined by $H_0(U)=\{u\in C_0^\infty(U): -\Delta u=0 \text{ on } U\}$ for each open subset U of $\boldsymbol{R}^d$. Then it is a well known classical result that $(\boldsymbol{R}^d, H_0)$ is a Brelot space. It is one of traditional ways to treat the equation $(-\Delta+\mu)u=0$ by reducing it to $-\Delta u=0$ through harmonic Green potentials.

1.3. Needless to say a sheaf H_μ on W need not generate a Brelot space (W, H_μ) in general. For example, take W as any subregion of $\boldsymbol{R}^d$ containing the origin 0 of $\boldsymbol{R}^d$ and δ the Dirac measure at 0. Then δ is a positive Radon measure on W and we can form the sheaf H_δ of δ-harmonic functions on open sets of W. We maintain that (W, H_δ) does *not* form a Brelot space. Of course Axiom 1 is always satisfied by any sheaf of functions on W as far as it comes from a *linear* equation like the one $(-\Delta+\delta)u=0$ for H_δ. Thus if we assume (W, H_δ) forms, contrary to our assertion, a Brelot space, then it simply means that (W, H_δ) satisfies both of Axioms 2 and 3. By Axiom 2 there is a regular subregion U of W for H_δ containing the origin 0. We can find a $u\in C(\bar{U})\cap H_\delta(U)$ with $u|\partial U=1$ so that (1.2) with μ replaced by δ is satisfied. Hence we have

$$\int_U u(x)\Delta\varphi(x)dx = u(0)\varphi(0)$$

for every $\varphi\in C_0^\infty(U)$. By considering φ with $\operatorname{supp}\varphi\subset U\setminus\{0\}$ we see that u is harmonic in $U\setminus\{0\}$. The Riemann removability theorem (cf. e.g., [**2**], p. 32, or [**12**], p. 67) assures that $u\in H_0(U)$ and therefore the left hand side of the above identity must be zero for every $\varphi\in C_0^\infty(U)$. A fortiori $u(0)\varphi(0)=0$ for every $\varphi\in C_0^\infty(U)$ which means that $u(0)=0$. Since $u|\partial U=1\geqq 0$, Axiom 2 implies that $u|U\geqq 0$. Observe that $\{nu\}_{n\geqq 1}$ is an increasing sequence in $H_\delta(U)$. Again by $u|\partial U=1$, there exists a point $a\in U$ such that $u(a)>0$. Hence, if we set $v=\sup_n nu$ on U, then $v(a)=+\infty$ and $v(0)=0$, contradicting Axiom 3. Thus we have shown that (W, H_δ) is not a Brelot space.

2. Measures of Kato class.

2.1. As before we fix a subregion W of $\boldsymbol{R}^d$. A *kernel* k on W is a continuous mapping k of $W\times W$ to $(-\infty, +\infty]$ such that $k(x, y)$ is finitely continuous on $W\times W$ outside its diagonal set and bounded from below on $K\times K$ for any compact subset K of W. The *k-potential* $k\mu$ of a Radon measure μ on W is defined by

$$k\mu(x) = \int_W k(x, y)d\mu(y)$$

as far as it is meaningful, which is the case, for example, if $\mu\geqq 0$ and has a compact support in W. Clearly $k\mu\in C(W\setminus\operatorname{supp}\mu)$ if μ has a compact support in W and $k\mu$ is well defined. If $\mu\geqq 0$ has a compact support in W, then $k\mu$ is lower semicontinuous on W. If μ and ν are positive and have compact supports in W, then $k(\mu+\nu)\in C(W)$ implies $k\mu$, $k\nu\in C(W)$ since $k\mu=k(\mu+\nu)-k\nu$ is also upper semicontinuous.

To talk about a certain kind of regularity of μ and $k\mu$ we introduce the following quantity

$$\gamma(a, \mu, k) = \lim_{\varepsilon \downarrow 0}\Big(\sup_{x\in B(a,\varepsilon)}\int_{B(a,\varepsilon)} k(x, y)d|\mu|(y)\Big)$$

for each point $a\in W$. Note that the quantity γ concerns the potential $k|\mu|$ and not $k\mu$ and in fact $\gamma(a, \mu, k)=\gamma(a, |\mu|, k)$. If $k(a, a)<+\infty$, then $\gamma(a, \mu, k)=k(a, a)|\mu|(\{a\})$ and, in particular, $\gamma(a, \mu, k)=0$ if and only if $|\mu|(\{a\})=0$. If $k(a, a)=+\infty$, then $\gamma(a, \mu, k)\geqq k(a, a)|\mu|(\{a\})$. Hence in this case of $k(a, a)=+\infty$ we see that $|\mu|(\{a\})=0$ if $\gamma(a, \mu, k)<+\infty$.

LEMMA 2.1. *Suppose $k=+\infty$ on the diagonal set of $W\times W$ and μ (and hence $|\mu|$) has a compact support in W. Then $k|\mu|\in C(W)$ if and only if $\gamma(a, \mu, k)=0$ for every $a\in W$.*

PROOF. Take an arbitrary point $a\in W$ and assume $\gamma(a, \mu, k)=0$. For each $\varepsilon>0$ let μ_ε be the restriction of μ to $\bar{B}(a, \varepsilon)$ and $\nu_\varepsilon=\mu-\mu_\varepsilon$. For any $\delta>0$ there exists an $\varepsilon>0$ such that $\bar{B}(a, \varepsilon)\subset W$ and $|k|\mu_\varepsilon||<\delta/2$ on $B(a, \varepsilon)$. Then $k|\mu|=k|\mu_\varepsilon|+k|\nu_\varepsilon|$ and

$$|k|\mu|(x)-k|\mu|(a)| \leqq |k|\nu_\varepsilon|(x)-k|\nu_\varepsilon|(a)|+\delta$$

for every $x\in B(a, \varepsilon)$. Since $k|\nu_\varepsilon|\in C(B(a, \varepsilon))$, we have

$$\limsup_{x\to a}|k|\mu|(x)-k|\mu|(a)| \leqq \delta$$

so that $k|\mu|$ is continuous at a and therefore $k|\mu|\in C(W)$.

Assume $k|\mu|\in C(W)$ and again take an arbitrary $a\in W$. Let μ_ε and ν_ε be as above. Since $k|\mu_\varepsilon|$ and $k|\nu_\varepsilon|$ are lower semicontinuous on W, the fact that $k|\mu_\varepsilon|+k|\nu_\varepsilon|=k|\mu|\in C(W)$ implies that $k|\mu_\varepsilon|$ is continuous (and so is $k|\nu_\varepsilon|$) on W. From

$$k(a, a)|\mu_\varepsilon|(\{a\}) \leqq k|\mu_\varepsilon|(a) < +\infty$$

and $k(a, a)=+\infty$ it follows that $|\mu_\varepsilon|(\{a\})=|\mu|(\{a\})=0$. Hence $k|\mu_\varepsilon|(x)\downarrow k(x, a)|\mu|(\{a\})=0$ ($\varepsilon\downarrow 0$) at each point $x\in W$ and thus the Dini theorem assures that the convergence is uniform on each compact subset of W. Thus $\gamma(a, \mu, k)=0$. □

Let $N(x, y)$ be the *Newtonean kernel* on $\boldsymbol{R}^d$, i.e., $N(x, y)=1/|x-y|^{d-2}$ for $d\geqq 3$ and $N(x, y)=\log(1/|x-y|)$ for $d=2$. It is a kernel on $\boldsymbol{R}^d$ and hence on any subregion W of $\boldsymbol{R}^d$ in the sense of this section. We say that a kernel k on W is an *N-kernel* if there exists a constant $c>0$ such that $k-cN\in C(W\times W)$.

LEMMA 2.2. *Let k be an N-kernel on W with the associated constant c on W and $a\in W$. Then $\gamma(a, \mu, k)<+\infty$ if and only if $\gamma(a, \mu, N)<+\infty$ and in this* *case* $\gamma(a, \mu, k)=c\gamma(a, \mu, N)$.

PROOF. By the above remark, $|\mu|(\{a\})=0$ if either $\gamma(a, \mu, k)$ or $\gamma(a, \mu, N)$ is finite. Then $\gamma(a, \mu, k-cN)=\gamma(a, \mu, cN-k)=0$. Hence $\gamma(a, \mu, k)=\gamma(a, \mu, cN)$ $=c\gamma(a, \mu, N)$ assures the assertion. $\square$

2.2. A Radon measure μ on an Euclidean subregion W is said to be *of Kato class* on W if

$$\gamma(a, \mu, N)=\lim_{\varepsilon\downarrow 0}\Big(\sup_{x\in B(a,\varepsilon)}\int_{B(a,\varepsilon)}N(x, y)d|\mu|(y)\Big)=0 \tag{2.1}$$

for every a in W. By Lemma 2.1, the condition (2.1) is equivalent to that the potential $N|\mu_B|\in C(W)$ (or equivalently $N|\mu_B|\in C(\boldsymbol{R}^d)$ in this case) for every open ball B with $\bar{B}\subset W$, where $\mu_B=\mu|B$ (cf. [4], [11]). That $N|\mu_B|\in C(W)$ is equivalent to $N\mu_B^{\pm}\in C(W)$ and, in particular, $N\mu_B\in C(W)$ is deduced. It is extremely important to keep it in mind that $N\mu_B\in C(W)$ need not imply $N|\mu_B|\in C(W)$ and actually we will give such an example in § 6. Originally the Kato class is considered for functions f on W (cf. e.g., [1]): f is a function of Kato class on W if and only if, in our present terminology, $f\lambda$ (i.e., $d(f\lambda)=fd\lambda$) is a Radon measure of Kato class. Here recall λ is the Lebesgue measure on $\boldsymbol{R}^d$.

We will prove a fact (i.e., Theorem 1) which contains a result of Boukricha-Hansen-Hueber [4]: If μ is a Radon measure of Kato class on W, then (W, H_μ) is a Brelot space. We will also prove that the converse of the above is not true in general (cf. Theorem 2). However we have the following result:

PROPOSITION 2.1. *Suppose μ is a Radon measure of constant sign on a subregion W so that μ is positive or negative on W. In this case the fact that the pair (W, H_μ) forms a Brelot space implies that μ is of Kato class on W.*

PROOF. We only consider the case $\mu\geqq 0$. (The case of $\mu\leqq 0$ can be treated similarly.) We only have to show that $\gamma(a, \mu, N)=0$ for any fixed $a\in W$. Axiom 2 assures that there is a regular region V for H_μ such that $a\in V\subset B(a, 1/2)$. We choose a function $u\in C(\bar{V})\cap H_\mu(V)$ such that $u|\partial V=1$. Since $u|\partial V=1\geqq 0$, we have $u\geqq 0$ on V. We maintain that actually $u>0$ on V and in particular $u(a)>0$. Contrary to the assertion suppose there is a $b\in V$ such that $u(b)=0$. By continuity of u on $\bar{V}$, $u|\partial V=1$ assures the existence of a $c\in V$ with $u(c)>0$. The sequence $\{nu\}_{n\geq 1}$ is an increasing sequence in $H_\mu(V)$ and hence $v=\sup_n nu\in H_\mu(V)$ or $v\equiv+\infty$ on V in view of Axiom 3. However $v(b)=0$ and $v(c)=+\infty$, a contradiction. Therefore $u(a)>0$.

For simplicity we set $\nu=u\mu$ (i.e., $d\nu=ud\mu$) which is a Radon measure on W with compact support in W by defining $u=0$ on $W\setminus\bar{V}$. Consider the function

$$U(x)=(1/\kappa_d)N\nu(x)=(1/\kappa_d)\int_V N(x,\ y)u(y)d\mu(y)$$

for $x\in \boldsymbol{R}^d$, where the space constant $\kappa_d=2\pi$ for $d=2$ and $\kappa_d=(d-2)\sigma_d$ for $d\geqq 3$ with σ_d the surface area of the unit sphere S^{d-1} in $\boldsymbol{R}^d$. Since $V\subset B(a,\ 1/2)$ and $N>0$ on $B(a,\ 1/2)\times B(a,\ 1/2)$ for every dimension $d\geqq 2$, we see that $0\leqq U(x)\leqq+\infty$ on V. (In the case of $\mu\leqq 0$, consider $-U$ instead of U.) By the Fubini theorem we see that

$$\kappa_d\int_V U(x)dx=\int_V\left(\int_V N(x,\ y)dx\right)u(y)d\mu(y)\leqq K\cdot(\sup_V u)\mu(\bar{V})<+\infty$$

so that $U\in L_1(V,\ \lambda)$ where

$$\int_V N(x,\ y)dx\leqq\int_{B(y,1)}N(x,\ y)dx=\int_{B(0,1)}N(x,\ 0)dx=K<+\infty$$

for every $y\in V$. Using the well known identity

$$\varphi(y)=-(1/\kappa_d)\int_V N(x,\ y)\Delta\varphi(x)dx\quad(y\in V)$$

for every $\varphi\in C_0^\infty(V)$ (cf. e.g., [**12**], p. 13), the Fubini theorem again assures that

$$\int_V U(x)\Delta\varphi(x)dx=\int_V\frac{1}{\kappa_d}\left(\int_V N(x,\ y)\Delta\varphi(x)dx\right)u(y)d\mu(y)=-\int_V\varphi(y)u(y)d\mu(y)$$

so that we have $\Delta U=-u\mu$ on V in the sense of distributions. The μ-harmonicity of u of course implies that $\Delta u=u\mu$ in the sense of distributions. We set $h=u+U$ on V. Then $\Delta h=\Delta u+\Delta U=u\mu-u\mu=0$ on V in the distributional sense. Hence by the Weyl lemma there is a classical harmonic function (i.e., a 0-harmonic function) $h^\sim\in C_0^\infty(V)$ such that $h=h^\sim$ λ-a.e. on V, λ being the d-dimensional Lebesgue measure.

Let M_ε be an averaging operator so that for any function $f\in L_{1,\mathrm{loc}}(V,\ \lambda)$

$$M_\varepsilon f(x)=\frac{1}{|B(0,\ \varepsilon)|}\int_{B(0,\varepsilon)}f(x+y)dy\quad(x\in V)$$

for any $\varepsilon>0$ with $\bar{B}(x,\ \varepsilon)\subset V$, where $|B(0,\ \varepsilon)|=\lambda(B(0,\ \varepsilon))$ is the volume of ε-ball $B(0,\ \varepsilon)$. From the identity $h=u+U$ valid in $L_1(V,\ \lambda)$ and hence valid only λ-a.e. on V, we deduce a numerical identity

$$M_\varepsilon h(x)=M_\varepsilon u(x)+M_\varepsilon U(x)$$

valid for every $x\in V$. Since $h=h^\sim$ λ-a.e. on V we see that $M_\varepsilon h(x)=M_\varepsilon h^\sim(x)$ for every $x\in V$ and then by the mean value property for 0-harmonic functions we see $M_\varepsilon h^\sim(x)=h^\sim(x)$ for every $x\in V$ so that

$$h^{\sim}(x)=M_{\varepsilon}u(x)+M_{\varepsilon}U(x)$$

for every $x\in V$. The continuity of u on V, and of course at x, implies that $M_{\varepsilon}u(x)\rightarrow u(x)$ $(\varepsilon\downarrow 0)$. It is an elementary knowledge that the superharmonicity (i.e., 0-superharmonicity) of U on V assures that $M_{\varepsilon}U(x)\uparrow U(x)$ $(\varepsilon\downarrow 0)$ for every $x\in V$ (cf. e.g., [**6**], p. 71). (In the case of $\mu\leqq 0$, consider $-U$ instead of U.) Hence on letting $\varepsilon\downarrow 0$ in the above identity we see that

$$h^{\sim}(x)=u(x)+U(x)$$

for every $x\in V$. Hence $U=h^{\sim}-u\in C(V)$ or $N\nu\in C(V)$. By Lemma 2.1, $\gamma(a, \nu, N)=0$. If we choose $\varepsilon>0$ sufficiently small so that $\bar{B}(a, \varepsilon)\subset V$ and $u>u(a)/2$ on $\bar{B}(a, \varepsilon)$, then

$$\int_{B(a,\varepsilon)}N(x, y)d\nu(y)\geqq\frac{u(a)}{2}\int_{B(a,\varepsilon)}N(x, y)d\mu(y)$$

which in turn implies that $\gamma(a, \nu, N)\geqq(u(a)/2)\gamma(a, \mu, N)$. (In the case of $\mu\leqq 0$, consider $-\mu$ instead of μ.) This proves that $\gamma(a, \mu, N)=0$ along with $\gamma(a, \nu, N)=0$. $\square$

3. Measures of quasi Kato class.

3.1. We will make the essential use of the harmonic Green function $G_0^{B(a,\varepsilon)}(x, y)$ of the open ball $B(a, \varepsilon)$. We denote by x^* the inversion of $x\in\boldsymbol{R}^d\setminus\{a\}$ with respect to the boundary sphere $\partial B(a, \varepsilon)$ of $B(a, \varepsilon)$: $x^*=a+\varepsilon^2|x-a|^{-2}(x-a)$. Recall that (cf. e.g., [**6**], p. 77), for $d=2$

$$\kappa_d G_0^{B(a,\varepsilon)}(x, y)=\log\Big(\frac{|a-x|}{\varepsilon}\frac{|y-x^*|}{|y-x|}\Big)\quad (y\in B(a, \varepsilon)\setminus\{x\},\ x\neq a), \tag{3.1}$$

$\log(\varepsilon/|y-a|)$ $(y\in B(a, \varepsilon)\setminus\{a\},\ x=a)$, and $+\infty$ $(y=x)$; for $d\geqq 3$

$$\kappa_d G_0^{B(a,\varepsilon)}(x, y)=\frac{1}{|y-x|^{d-2}}-\Big(\frac{\varepsilon}{|x-a|}\Big)^{d-2}\frac{1}{|y-x^*|^{d-2}} \tag{3.2}$$

$(y\in B(a, \varepsilon)\setminus\{x\},\ x\neq a)$, $1/|y-a|^{d-2}-1/\varepsilon^{d-2}$ $(y\in B(a, \varepsilon)\setminus\{a\},\ x=a)$, and $+\infty$ $(y=x)$. Here κ_d is the space constant already considered in § 2.2, i.e., $\kappa_d=2\pi$ for $d=2$ and $\kappa_d=(d-2)\sigma_d$ for $d\geqq 3$ where σ_d is the surface area of the unit sphere $S^{d-1}=\partial B(0, 1)$ of $\boldsymbol{R}^d$.

We consider another space constant τ_d given by

$$\tau_d=\sup_{x,y,z\in B(0,1)}\Big(\frac{G_0^{B(0,1)}(x, z)G_0^{B(0,1)}(z, y)}{G_0^{B(0,1)}(x, y)\cdot\max(G_0^{B(0,2)}(x, z),\ G_0^{B(0,2)}(z, y))}\Big). \tag{3.3}$$

It is far from being trivial to see that $\tau_d<+\infty$ (cf. e.g., [**4**], [**13**] among others) but $\tau_d>1$ can be easily seen by considering the value of the ratio under the supremum sign at e.g., $x=-y=(1/2, 0, \cdots, 0)$ and $z=(0, \cdots, 0)$:

$$1<\tau_d<+\infty. \tag{3.4}$$

We also remark that in the definition of τ_d we may replace $B(0, 1)$ and $B(0, 2)$ by $B(a, \rho)$ and $B(a, 2\rho)$, respectively, where a is any point in $\boldsymbol{R}^d$ and ρ is any positive number. Although the value itself is changed but the finiteness is unchanged in the right hand side of (3.3) if we replace $B(0, 1)$ and $B(0, 2)$ by $B(a, r)$ and $B(a, \rho)$, respectively, with $0<r<\rho<+\infty$. Here, if $d\geqq 3$, then we may take $0<r<\rho\leqq+\infty$ or even $r=\rho=+\infty$.

3.2. The condition $\gamma(a, \mu, N)=0$ $(a\in W)$ for a Radon measure μ on a subregion W to be of Kato class implies the following two properties: $\gamma(a, \mu, N)$ is less than any fixed positive constant on W; $N\mu_B\in C(\boldsymbol{R}^d)$ for any open ball B with $\bar{B}\subset W$ where $\mu_B=\mu|B$. The latter is a consequence of $N|\mu_B|\in C(\boldsymbol{R}^d)$ (cf. Lemma 2.1). We will show that to ensure for (W, H_μ) to be a Brelot space the full powers of $\gamma(a, \mu, N)=0$ $(a\in W)$ are not needed but only weak forms of the above two consequences suffice.

We say that a Radon measure μ on a subregion W of $\boldsymbol{R}^d$ is *of quasi Kato class* if the following two conditions are fulfilled: Firstly, μ satisfies

$$\gamma(a, \mu, N)=\lim_{\varepsilon\downarrow 0}\left(\sup_{x\in B(a,\varepsilon)}\int_{B(a,\varepsilon)}N(x, y)d|\mu|(y)\right)<\frac{\kappa_d}{4\tau_d} \tag{3.5}$$

for every $a\in W$; Secondary, there is a base of neighborhood system at any point $a\in W$ such that each set in the base is an N-regular ball for μ centered at a. Here an open ball B is said to be *N-regular* for μ if $\bar{B}\subset W$ and

$$N\mu_B=\int_B N(\cdot, y)d\mu(y)\in C(\boldsymbol{R}^d). \tag{3.6}$$

As we have observed at the beginning of this § 3.2, a Radon measure μ on W of Kato class is automatically a Radon measure of quasi Kato class.

For simplicity we write $\nu=\mu_B=\mu|B$ for a Radon measure μ of quasi Kato class on a region W and an N-regular ball B for μ in W. In view of (3.5) $N|\nu|$ is locally bounded on $\boldsymbol{R}^d$ and by (3.6) $N\nu\in C(\boldsymbol{R}^d)$. For such a measure we have the following result.

LEMMA 3.1. *Let ν be a Radon measure on $\boldsymbol{R}^d$ with compact support such that $N|\nu|$ is locally bounded and $N\nu\in C(\boldsymbol{R}^d)$. Then for any $f\in C(\mathrm{supp}\,\nu)$*

$$N(f\nu)=\int_{\mathrm{supp}\,\nu}N(\cdot, y)f(y)d\nu(y)\in C(\boldsymbol{R}^d).$$

PROOF. We fix a ball $B=B(0, \rho)\supset K=\operatorname{supp}\nu$ and set

$$M=\sup_{x\in B}\int_K |N(x, y)|\,d|\nu|(y)<+\infty .$$

Clearly $N(f\nu)\in C(\boldsymbol{R}^d\setminus K)$ and hence we only have to prove the continuity of $N(f\nu)$ at an arbitrary point $a\in K$. For any positive number $\varepsilon>0$ there is a ball $V=B(a, \eta)$ $(\eta>0)$ with $\bar{V}\subset B$ such that $N>0$ on $V\times V$ and

$$\sup_{y\in V\cap K}|f(y)-f(a)|<\varepsilon/2M .$$

In terms of $\alpha=\nu|V$ and $\beta=\nu|(\boldsymbol{R}^d\setminus V)$ we have

$$N(f\nu)(x)-N(f\nu)(a)=(N(f\alpha)(x)-N(f\alpha)(a))+(N(f\beta)(x)-N(f\beta)(a))$$

for any $x\in V$ and the first term on the right hand side of the above is expressed as

$$(N(f\alpha)(x)-N(f(a)\alpha)(x))+(N(f(a)\alpha)(x)-N(f(a)\alpha)(a))$$
$$+(N(f(a)\alpha)(a)-N(f\alpha)(a)) .$$

The first term of the above in the absolute value is dominated by

$$\Big(\sup_{y\in V\cap K}|f(y)-f(a)|\Big)N|\nu|(x)\leqq(\varepsilon/2M)\cdot M=\varepsilon/2$$

for every $x\in V$ and similarly the last term of the above in the absolute value is dominated by

$$\Big(\sup_{y\in V\cap K}|f(y)-f(a)|\Big)N|\nu|(a)\leqq(\varepsilon/2M)\cdot M=\varepsilon/2 .$$

The second term of the above in the absolute value is $|f(a)|\,|N\alpha(x)-N\alpha(a)|$. Thus we deduce that

$$|N(f\nu)(x)-N(f\nu)(a)|\leqq|f(a)|\,|N\alpha(x)-N\alpha(a)|+|N(f\beta)(x)-N(f\beta)(a)|+\varepsilon .$$

Observe that $N(f\beta)$ and $N\beta$ are continuous at a since $a\notin(\operatorname{supp}\beta)\cup(\operatorname{supp}(f\beta))$. In view of $N\alpha=N\nu-N\beta$ and $N\nu\in C(\boldsymbol{R}^d)$, $N\alpha$ is also continuous at a along with $N\beta$. Therefore, taking the superior limits of both sides of the above inequality as $x\to a$, we see that

$$\limsup_{x\to a}|N(f\nu)(x)-N(f\nu)(a)|\leqq\varepsilon . \qquad \square$$

3.3. Take a Radon measure μ of quasi Kato class on a region $W\subset\boldsymbol{R}^d$ so that $\gamma(a, \mu, N)<\kappa_d/4\tau_d$ $(a\in W)$ and there exists a sequence of N-regular balls B for μ centered at any given point $a\in W$ and shrinking to a. Recall that $N\mu_B\in C(\boldsymbol{R}^d)$ for N-regular balls B for μ. Since $\gamma(a, \mu, N)$ is upper semicontinuous on W as a function of $a\in W$, there is an $a_1\in K$ for any compact subset $K\subset W$ such that

$$\sup_{a\in K}\gamma(a, \mu, N)=\gamma(a_1, \mu, N)<\kappa_d/4\tau_d .$$

Therefore we can find a positive number $q=q(K, \mu)$ such that

$$\frac{2\tau_d}{\kappa_d}\cdot\sup_{a\in K}\gamma(a, \mu, N)<q<1/2 .$$

It is convenient to call $q=q(K, \mu)$ a μ-constant for K, and in particular, a *μ-constant* at a when $K=\{a\}$. For any μ-constant $q\in((2\tau_d/\kappa_d)\gamma(a, \mu, N), 1/2)$ at $a\in W$ there is a ball $B(a, \varepsilon)$ of radius $\varepsilon\in(0, 1/2)$ centered at a such that $B(a, \varepsilon)$ is N-regular for μ and

$$\frac{2\tau_d}{\kappa_d}\sup_{x\in B(a,\varepsilon)}\int_{B(a,\varepsilon)}N(x, y)d|\mu|(y)<q<\frac{1}{2}. \tag{3.7}$$

Such a ball $B(a, \varepsilon)$ is said to be a *μ-ball* at a associated with a μ-constant q at a.

We denote by $G(x, y)=G_0^{B(a,\varepsilon)}(x, y)$ the harmonic Green function on $B(a, \varepsilon)$ (cf. §3.1). Since $(1/\kappa_d)N(x, y)-G(x, y)$ is nonnegative and finitely continuous for $(x, y)\in B(a, \varepsilon)\times B(a, \varepsilon)$ as a consequence of $\varepsilon\in(0, 1/2)$, we have

$$\sup_{x\in B(a,\varepsilon)}\int_{B(a,\varepsilon)}G(x, y)d|\mu|(y)<q/2\tau_d<q \tag{3.8}$$

where $q\in(0, 1/2)$ is a μ-constant at a and $B(a, \varepsilon)$ is a μ-ball at a associated with q. Here we must recall (3.4): $1<\tau_d<+\infty$.

4. Potential operator.

4.1. Let μ be a Radon measure of quasi Kato class on a subregion W of $\boldsymbol{R}^d$. We fix an arbitrary point $a\in W$, a μ-constant $q\in(0, 1/2)$ at a, and a μ-ball $V=B(a, \varepsilon)$ at a associated with q. We consider the Banach space $C(\bar{V})$ of continuous functions f on $\bar{V}$ equipped with the norm $\|f\|=\sup_{\bar{V}}|f|$. We denote by $G(x, y)=G_0^V(x, y)$ the harmonic Green function on V. First we prove the following result.

LEMMA 4.1. *For any $f\in C(\bar{V})$ the Green potential*

$$G(f\mu_V)=\int_V G(\cdot, y)f(y)d\mu(y)\in C(\bar{V})$$

and $G(f\mu_V)|\partial V=0$ where $\mu_V=\mu|V$.

PROOF. To begin with we consider the behavior of $G(f\mu_V)$ on V. Since

$N-\kappa_d G \in C(V \times V)$ and $|f\mu|(V) < +\infty$, we see that

$$N(f\mu_V) - \kappa_d G(f\mu_V) = (N - \kappa_d G)(f\mu_V) \in C(V).$$

By virtue of the N-regularity of V for μ, Lemma 3.1 can be applied to μ_V to conclude that $N(f\mu_V) \in C(\boldsymbol{R}^d)$. Thus we can see that $G(f\mu_V) \in C(V)$.

Next we examine the behavior of $G(f\mu_V)$ on $\overline{V} \setminus \{a\}$. We need to consider cases of $d=2$ and $d \geqq 3$ separately. If $d=2$, then by (3.1) we have

$$\kappa_d G(f\mu_V)(x) = N(f\mu_V)(x) - N(f\mu_V)(x^*) + \left(\log \frac{|a-x|}{\varepsilon}\right) \int_V f \, d\mu$$

for $x \in \overline{V} \setminus \{a\}$. By Lemma 3.1, $N(f\mu_V) \in C(\boldsymbol{R}^d)$ so that $N(f\mu_V)(x)$ and $N(f\mu_V)(x^*)$ are continuous functions of x on $\overline{V} \setminus \{a\}$. Hence we see that $G(f\mu_V) \in C(\overline{V} \setminus \{a\})$. If $x \in \partial V$, then $|a-x| = \varepsilon$ and $x = x^*$ assure that $G(f\mu_V)(x) = 0$. If $d \geqq 3$, then (3.2) implies that

$$\kappa_d G(f\mu_V)(x) = N(f\mu_V)(x) - \left(\frac{\varepsilon}{|x-a|}\right)^{d-2} N(f\mu_V)(x^*)$$

for $x \in \overline{V} \setminus \{a\}$. By the same fashion as in the case of $d=2$, we see that $G(f\mu_V) \in C(\overline{V} \setminus \{a\})$ and $G(f\mu_V)|\partial V = 0$. □

4.2. We now define a linear operator T of $C(\overline{V})$ into itself by

$$Tf(x) = \int_V G(x, y) f(y) d\mu(y) \quad (x \in \overline{V}) \tag{4.1}$$

for each $f \in C(\overline{V})$. Lemma 4.1 assures that $Tf = G(f\mu_V) \in C(\overline{V})$ and

$$Tf|\partial V = 0. \tag{4.2}$$

We also consider an auxiliary linear operator $|T|$ of $C(\overline{V})$ into $L_\infty(\overline{V}, \lambda)$ defined by

$$|T|f(x) = \int_V G(x, y) f(y) d|\mu|(y) \quad (x \in \overline{V})$$

for every $f \in C(\overline{V})$. By (3.8) we see that

$$|Tf(x)|,\ ||T|f(x)| \leqq |T||f|(x) \leqq \|f\| |T|1(x) \leqq (q/2\tau_d)\|f\| \leqq q\|f\|$$

for every $x \in \overline{V}$ and for every $f \in C(\overline{V})$. Hence

$$\|T\| \leqq q/2\tau_d < q/2 < q < 1/2 < 1 \tag{4.3}$$

which assures the existence of the inverse linear operator $(I+T)^{-1}$ of $C(\overline{V})$ onto itself of the operator $I+T$ where I is the identity operator of $C(\overline{V})$ onto itself. As is well known, $(I+T)^{-1}$ is given by the C. Neumann series:

$$(I+T)^{-1} = \sum_{n=0}^{\infty} (-1)^n T^n. \tag{4.4}$$

4.3. Recall that we denoted by $\mathcal{F}^+$ the class of nonnegative members of a class $\mathcal{F}$ of functions. Hence $H_0^+(V)$ is the class of nonnegative classical harmonic (i.e., 0-harmonic) functions on V. The following is the crucial property of the potential operator T in the proof of Theorem 1:

LEMMA 4.2. *For any $h \in C(\bar{V}) \cap H_0^+(V)$, the inequalities*

$$|T^n h| \leqq q^n h \quad (n = 1, 2, \cdots) \tag{4.5}$$

hold on V.

PROOF. Fix an arbitrary $h \in C(\bar{V}) \cap H_0^+(V)$. For each $m = 1, 2, \cdots$, let $h_m \in C(\bar{V}) \cap H_0(V \setminus \bar{B}(a, \varepsilon - \varepsilon/2m))$ such that $h_m | \bar{B}(a, \varepsilon - \varepsilon/2m) = h$ and $h_m | \partial V = 0$. Then h_m is a potential on $V = B(a, \varepsilon)$, i.e., a nonnegative superharmonic function with vanishing greatest harmonic minorant on V. By the Riesz decomposition theorem (cf. e.g., [6], pp. 116-117) there is a unique positive Radon measure ν_m on V with $\operatorname{supp} \nu_m \subset \partial B(a, \varepsilon - \varepsilon/2m)$ such that

$$h_m(x) = \int G(x, y) d\nu_m(y) \quad (x \in V).$$

By the Fubini theorem, (3.3) and (3.7), we see that

$$\left| \int_V G(x, z) h_m(z) d\mu(z) \right| \leqq \int_V G(x, z) h_m(z) d|\mu|(z)$$

$$= \int_V G(x, z) \left(\int_V G(z, y) d\nu_m(y) \right) d|\mu|(z)$$

$$= \int_V \left(\int_V G(x, z) G(z, y) d|\mu|(z) \right) d\nu_m(y)$$

$$\leqq \int_V \left(\int_V \tau_d G(x, y) \max\left(G_0^{B(a, 2\varepsilon)}(x, z), G_0^{B(a, 2\varepsilon)}(z, y) \right) d|\mu|(z) \right) d\nu_m(y)$$

$$\leqq \frac{\tau_d}{\kappa_d} \int_V G(x, y) \left(\int_{B(a, \varepsilon)} \max(N(x, z), N(z, y)) d|\mu|(z) \right) d\nu_m(y)$$

$$\leqq \frac{\tau_d}{\kappa_d} \int_V G(x, y) \left(\int_{B(a, \varepsilon)} N(x, z) d|\mu|(z) + \int_{B(a, \varepsilon)} N(y, z) d|\mu|(z) \right) d\nu_m(y)$$

$$\leqq q \int_V G(x, y) d\nu_m(y) = q h_m(x)$$

for every $x \in V$, i.e., we have shown that

$$\left| \int_V G(x, y) h_m(y) d\mu(y) \right| \leqq \int_V G(x, y) h_m(y) d|\mu|(y) \leqq q h_m(x) \quad (x \in V).$$

Since $h_m \uparrow h$ $(m \uparrow \infty)$ and h is $(G(x, \cdot) d\mu^{\pm})$- and $(G(x, \cdot) d|\mu|)$-integrable over V, by the Lebesgue dominated convergence theorem, we deduce, on making $m \uparrow \infty$

in the above identity, that

$$\left|\int_V G(x, y)h(y)d\mu(y)\right| \leqq \int_V G(x, y)h(y)d|\mu|(y) \leqq qh(x) \quad (x \in V).$$

In terms of the operator T and $|T|$ we can restate the above as

$$|Th| \leqq |T|h \leqq qh \tag{4.6}$$

on V. We now show (4.5) inductively. It is true for $n=1$ by (4.6). Suppose $|T^n h| \leqq q^n h$ on V. Then, since $|T|$ is order preserving, we see, by (4.6), that

$$|T^{n+1}h| = |T(T^n h)| \leqq |T||T^n h| \leqq |T|(q^n h) = q^n|T|h \leqq q^n(qh) = q^{n+1}h.$$

The induction is herewith complete. □

5. Proof of Theorem 1.

5.1. Let μ be a Radon measure of quasi Kato class on a Euclidean region W. We wish to show that (W, H_μ) satisfies Axioms 1, 2 and 3. Since the Schrödinger operator $-\Delta+\mu$ is linear, the class $H_\mu(U)$ of μ-harmonic functions on an open set $U \subset W$ forms a linear subspace of $C(U)$ and thus Axiom 1 is trivially satisfied.

We proceed to the proof for that (W, H_μ) satisfies Axiom 2. For the purpose choose any point $a \in W$ and an open set U containing a. We only have to show the existence of a regular region for H_μ contained in U and containing a. Take a μ-constant $q \in (0, 1/2)$ at a and a μ-ball $V=B(a, \varepsilon)$ at a associated with q. We maintain that V is a required regular region for H_μ. We take the potential operator associated with V (cf. (4.1)).

Choose an arbitrary $f \in C(\partial V)$. There is an $h \in C(\bar{V}) \cap H_0(V)$ such that $h|\partial V=f$. Set $u=(I+T)^{-1}h \in C(\bar{V})$, i.e., $h=u+Tu$. By using the well known identity

$$\int_V G(x, y)\Delta\varphi(y)dy = -\varphi(x)$$

for every $\varphi \in C_0^\infty(V)$ (cf. e.g., [**6**], p. 71), we see, by the Fubini theorem, that

$$\int_V Tu(x)\Delta\varphi(x)dx = \int_V\left(\int_V G(x, y)u(y)d\mu(y)\right)\Delta\varphi(x)dx$$
$$= \int_V\left(\int_V G(x, y)\Delta\varphi(x)dx\right)u(y)d\mu(y) = \int_V(-\varphi(y)u(y))d\mu(y)$$

so that $\Delta Tu=-u\mu$ on V and $\Delta u=\Delta h-\Delta Tu=0-(-u\mu)=u\mu$ on V in the sense of distributions, i.e., $u \in C(\bar{V}) \cap H_\mu(V)$. Since $Tu|\partial V=0$, we have $u|\partial V= h|\partial V-Tu|\partial V=f$.

Suppose $v\in C(\bar{V})\cap H_\mu(V)$ such that $v|\partial V=f$. Then $w=u-v\in C(\bar{V})\cap H_\mu(V)$ by Axiom 1 and $w|\partial V=u|\partial V-v|\partial V=f-f=0$. Let $k=w+Tw$ on $\bar{V}$. By the same method as above we see that $\Delta Tw=-w\mu$. Thus $\Delta k=\Delta w+\Delta Tw=w\mu-w\mu=0$. A fortiori $k\in C(\bar{V})\cap H_0(V)$ and $k|\partial V=w|\partial V+Tw|\partial V=0$ and therefore $k=0$ on $\bar{V}$, or $w=-Tw$ on $\bar{V}$. The inequality $\|w\|=\|Tw\|\leqq q\|w\|$ with $q\in(0, 1/2)$ yields that $w=0$ on V and thus we have seen the uniqueness of u with $u\in C(\bar{V})\cap H_\mu(V)$ and $u|\partial V=f$.

To complete the proof concerning Axiom 2 we need to show that $f\geqq 0$ on ∂V implies $u\geqq 0$ on V. Set $h=u+Tu\in C(\bar{V})\cap H_0(V)$. Since $h|\partial V=u|\partial V=f\geqq 0$, we see that $h\geqq 0$ on $\bar{V}$. By (4.4) we see that

$$u=(I+T)^{-1}h=\sum_{n=0}^{\infty}(-1)^n T^n h=h+\sum_{n=1}^{\infty}(-1)^n T^n h\geqq h-\sum_{n=1}^{\infty}|T^n h|$$

on V. By (4.5) and $q\in(0, 1/2)$, we then deduce

$$u\geqq h-\sum_{n=1}^{\infty}q^n h=\frac{1-2q}{1-q}h\geqq 0$$

so that we have shown $u\geqq 0$ on V.

5.2. Before proceeding to the proof for that (W, H_μ) satisfies Axiom 3, we prove a form of the Harnack inequality. For an arbitrary $a\in W$, choose a μ-constant $q\in(0, 1/2)$ at a and a μ-ball $V=B(a, \varepsilon)$ at a associated with q. We prove the following Harnack inequality:

$$C^{-1}u(y)\leqq u(x)\leqq Cu(y)\quad (C=4\cdot 3^d/(1-2q)) \tag{5.1}$$

for any pair of points x and y in $\bar{B}(a, \varepsilon/2)$ and for every $u\in C(\bar{B}(a, \varepsilon))\cap H_\mu^+(B(a, \varepsilon))$, where $H_\mu^+(B(a, \varepsilon))$ is the family of nonnegative μ-harmonic functions u on $V=B(a, \varepsilon)$. Set $h=(I+T)u$. Because of the fact that $h|\partial V=u|\partial V+Tu|\partial V=u\geqq 0$ on $\partial B(a, \varepsilon)$, we see that $h\in H_0^+(B(a, \varepsilon))$. As is well known

$$(1/4\cdot 3^d)h(y)\leqq h(x)\leqq 4\cdot 3^d h(y) \tag{5.2}$$

for every pair of points x and y in $\bar{B}(a, \varepsilon/2)$ (cf. e.g., [**6**], p. 29 or [**2**], p. 47, etc.). Similar to the proof of $u\geqq((1-2q)/(1-q))h$ on V given in §5.1, we can show that $u\leqq(1/(1-q))h$ on V. In fact, by (4.4) and (4.5), we see that

$$u=(I+T)^{-1}h=\sum_{n=0}^{\infty}(-1)^n T^n h=h+\sum_{n=1}^{\infty}(-1)^n T^n h$$

$$\leqq h+\sum_{n=1}^{\infty}|T^n h|\leqq h+\sum_{n=1}^{\infty}q^n h=\frac{1}{1-q}h$$

on V. Hence we have

$$\frac{1-2q}{1-q}h(z) \leqq u(z) \leqq \frac{1}{1-q}h(z) \tag{5.3}$$

for every $z \in V$. Combining inequalities (5.2) and (5.3) with $z=x$ and $z=y$, we deduce (5.1).

5.3. We now complete the proof of Theorem 1 by showing that (W, H_μ) satisfies Axiom 3. For the purpose, fix an arbitrary region U in W and choose any increasing sequence $\{u_n\}$ in $H_\mu(U)$ and set $u=\sup_n u_n$. We have to show that $u \in H_\mu(U)$ unless $u \equiv +\infty$. Replacing $\{u_n\}$ by $\{u_n - u_1\}$ if necessary, we may assume that $\{u_n\}$ is an increasing sequence in $H_\mu^+(U)$. Put

$$E = \{x \in U : u(x) = \sup_n u_n(x) < +\infty\}.$$

If $E=\emptyset$, then $u \equiv +\infty$ on U and the proof is over. Thus we assume that $E \neq \emptyset$. For any $a \in W$, let $q \in (0, 1/2)$ be a μ-constant at a, $B(a, \varepsilon)$ a μ-ball at a associated with q and $C=4\cdot 3^d/(1-2q)$. If $a \in E$, then by (5.1)

$$u_n(x) \leqq C u_n(a) \quad (x \in \bar{B}(a, \varepsilon/2))$$

for every $n=1, 2, \cdots$. Hence $u(x) \leqq Cu(a) < +\infty$, i.e., $B(a, \varepsilon/2) \subset E$. This proves that E is open. If $a \in \bar{E}$, then there is a $b \in E \cap B(a, \varepsilon/2)$. Thus again by (5.1) we see that $u_n(a) \leqq C u_n(b)$ for every $n=1, 2, \cdots$. Hence $u(a) \leqq Cu(b) < +\infty$, i.e., $a \in E$. This proves that E is closed. Therefore $E=U$ and $u(x) = \sup_n u_n(x) = \lim_n u_n(x)$ defines a numerical function on U. Again by (5.1)

$$0 \leqq u_{n+p}(x) - u_n(x) \leqq C(u_{n+p}(a) - u_n(a)) \quad (x \in \bar{B}(a, \varepsilon/2))$$

for every n and $p=1, 2, \cdots$. On letting $p \uparrow \infty$ we see that

$$0 \leqq u(x) - u_n(x) \leqq C(u(a) - u_n(a)) \quad (x \in \bar{B}(a, \varepsilon/2))$$

for every $n=1, 2, \cdots$. Since $a \in W$ is arbitrary, the above proves that $\{u_n\}$ converges to u locally uniformly on W so that $u \in C(U)$. On each $V=B(a, \varepsilon)$ above, set $h_n = u_n + Tu_n$, which belongs to $C(\bar{V}) \cap H_0^+(V)$. Since $\|u_n - u\| \to 0$ $(n \uparrow \infty)$ in $C(\bar{V})$, we see that $h = \lim_n h_n = \lim_n (u_n + Tu_n) = u + Tu$. As a uniform limit of the sequence $\{h_n\}$ of harmonic functions, $h = u + Tu \in C(\bar{V})$ is harmonic on V. Thus

$$\Delta u = \Delta h - \Delta T u = 0 - (-u\mu) = u\mu$$

(cf. §5.1 for $\Delta Tu = -u\mu$) shows that $u \in H_\mu(V)$ for every admissible V so that $u \in H_\mu(U)$.

The proof of Theorem 1 is herewith complete. □

6. Proof of Theorem 2.

6.1. It may be convenient to say that a Radon measure μ on a Euclidean region W is *of Brelot class* if (W, H_μ) forms a Brelot space. Then we have seen, as consequences of Theorem 1 and Proposition 2.1 that

$$\{\text{Kato class}\} \subset \{\text{quasi Kato class}\} \subset \{\text{Brelot class}\}$$

and

$$\{\text{Kato class}\}^{\pm} = \{\text{quasi Kato class}\}^{\pm} = \{\text{Brelot class}\}^{\pm},$$

where, e.g., {Kato class} mean the set of all Radon measures of Kato class on an arbitrarily fixed region and $\{\text{Kato class}\}^{+}$ ($\{\text{Kato class}\}^{-}$, resp.) is the subfamily of positive (negative, resp.) measures in {Kato class}. We now wish to show that the first inclusion relation in the above displayed diagram is *strict* or equivalently there is a measure μ in

$$\{\text{quasi Kato class}\} \setminus \{\text{Kato class}\} \neq \emptyset$$

on any region W. Thus the required μ must be of nonconstant sign.

Hence for any Euclidean region W we will construct a signed measure μ on W which is of quasi Kato class but not of Kato class. Fixing an arbitrary point $a \in W$ and an arbitrary ball $B(a, r) \subset W$ we only have to construct a required μ with compact support in $B(a, r)$. By translation and dilation we may suppose that $a=0$ and $r=1$. Thus all we have to do is to construct a signed Radon measure μ of compact support on the open unit ball $R=B(0, 1)$ which is of quasi Kato class on R but not of Kato class on R. The measure μ we are going to construct satisfies $\gamma(a, \mu, N)=0$ for every $a \in R \setminus \{0\}$ and $\gamma(0, \mu, N)>0$ so that μ is certainly not of Kato class on R but of Kato class on R except for a miserable meager set consisting of only one point 0. It is of quasi Kato class if $\gamma(0, \mu, N)<\kappa_d/4\tau_d$ which is achieved by multiplying a small constant to μ as far as $\gamma(0, \mu, N)<+\infty$.

6.2. Let $R=B(0, 1)$ in $\boldsymbol{R}^d$ $(d \geqq 2)$. Fix a sequence $\{a_n\}$ of points a_n contained in the x_1-axis such that

$$0 < \hat{a_{n+1}} < \hat{a_n} < 1 \quad \text{and} \quad \lim_{n \to \infty} \hat{a_n} = 0$$

where $a_n = (\hat{a_n}, 0, \cdots, 0)$. Fix a sequence $\{r_n\}$ in $(0, 1)$ so small that $\bar{B}(a_n, r_n) \subset R \setminus \{0\}$ and $\bar{B}(a_n, r_n) \cap \bar{B}(a_{n+1}, r_{n+1}) = \emptyset$ $(n=1, 2, \cdots)$. Choose one more sequence $\{s_n\}$ with $0<s_n<r_n$ $(n=1, 2, \cdots)$ which will be determined below. Since every boundary point of $R \setminus \bar{B}(a_n, s_n)$ satisfies the cone condition (or even ball condition), it is regular for H_0 by the Zaremba theorem (cf. e.g., [**6**], p. 173). Take a $w_n \in C(\bar{R}) \cap H_0(R \setminus \bar{B}(a_n, s_n))$ such that $w_n | \bar{B}(a_n, s_n)=1$ and $w_n | \partial R=0$ for each $n=1, 2, \cdots$. For each fixed n, $w_n \downarrow 0$ $(s_n \downarrow 0)$ on $\bar{R} \setminus B(a_n, r_n)$. We can thus

determine $s_n \in (0, r_n)$ so small that

$$(6.1) \qquad w_n | (\bar{R} \setminus B(a_n, r_n)) < 1/5^n \quad (n = 1, 2, \cdots).$$

We put $P = \{x = (x_1, \cdots, x_d) \in \boldsymbol{R}^d : x_d = 0\}$, the $(d-1)$-dimensional hyperplane perpendicular to x_d-axis. Consider the compact set $K_n = P \cap \bar{B}(a_n, s_n/2)$ contained in $B(a_n, s_n)$ $(n=1, 2, \cdots)$. Choose and fix an $\varepsilon_n \in (0, (1-\hat{a_n}-r_n)/4) \cap (0, s_n/2)$ so small that

$$(6.2) \qquad w_n | (\bar{R} \setminus B(0, 1-4\varepsilon_n)) < 1/3 \cdot 2^n \quad (n = 1, 2, \cdots).$$

Choose the third sequence $\{t_n\}$ with $t_n \in (0, \varepsilon_n)$ which will be again determined below. Take the vector $e_d = (0, \cdots, 0, 1)$ and set $K_n^{\pm} = K_n \pm t_n e_d$ which is contained in $B(a_n, s_n)$ by the choice of t_n: $0 < t_n < \varepsilon_n < s_n/2$. Since every boundary point of the region $R \setminus K_n^{\pm}$ satisfies the flat cone condition, it is regular for H_0 (see Appendix at the end of this paper). Thus we can construct functions $u_n^{\pm} \in C(\bar{R}) \cap H_0(R \setminus K_n^{\pm})$ such that $u_n^{\pm} | K_n^{\pm} = 1$ and $u_n^{\pm} | \partial R = 0$ for all $n = 1, 2, \cdots$, where double signs on shoulders are taken in the same order. Since $K_n^{\pm} \subset B(a_n, s_n)$, by the maximum principle, (6.1) assures that

$$(6.3) \qquad u_n^{\pm} | (\bar{R} \setminus B(a_n, r_n)) < 1/5^n \quad (n = 1, 2, \cdots).$$

For each fixed n, we choose and then fix a $t_n \in (0, \varepsilon_n)$ so small that

$$(6.4) \qquad \sup_{x \in \bar{R}} | u_n^{+}(x) - u_n^{-}(x) | < 1/2^n \quad (n = 1, 2, \cdots).$$

We need a proof for the possibility of choosing such a t_n. For the purpose we take an auxiliary function $v_n \in C(\boldsymbol{R}^d) \cap H_0(B(0, 1-2\varepsilon_n) \setminus K_n)$ such that $v_n | K_n = 1$ and $v_n | (\boldsymbol{R}^d \setminus B(0, 1-2\varepsilon_n)) = 0$ for every $n = 1, 2, \cdots$. We then set $v_n^{\pm}(x) = v_n(x \pm t_n e_d)$. By the uniform continuity of v_n, there exists a $t_n \in (0, \varepsilon_n)$ such that

$$| v_n^{+}(x) - v_n^{-}(x) | < 1/3 \cdot 2^n \quad (x \in \boldsymbol{R}^d).$$

Consider the function $u_n^{\pm} - v_n^{\pm}$ on R. In view of (6.2) and $u_n^{\pm} \leqq w_n$ on R, the maximum principle yields

$$| u_n^{\pm}(x) - v_n^{\pm}(x) | < 1/3 \cdot 2^n \quad (x \in \bar{R}).$$

Using these two inequalities we deduce

$$| u_n^{+} - u_n^{-} | \leqq | u_n^{+} - v_n^{+} | + | v_n^{+} - v_n^{-} | + | v_n^{-} - u_n^{-} | < 1/2^n$$

on $\bar{R}$, i.e., we have chosen $t_n \in (0, \varepsilon_n)$ such that (6.4) is valid.

6.3. We denote by $G(x, y) = G_0^R(x, y)$ the harmonic Green function on R. Judging from the boundary values of $u_n^{\pm}$, we see that $u_n^{\pm}$ is the capacitary potential of $K_n^{\pm}$ relative to R. Hence $u_n^{\pm}$ is represented as a Green potential

$$u_n^{\pm}(x)=\int G(x, y)d\nu_n^{\pm}(y) \quad (x \in \bar{R})$$

by using the capacitary distribution $\nu_n^{\pm}$ for $K_n^{\pm}$ which is a positive Radon measure with support in $K_n^{\pm}$ (cf. e.g., [6], p. 128). We set

$$\nu=\sum_{n=1}^{\infty}(\nu_n^{+}-\nu_n^{-}),$$

which is easily seen to define a Radon measure on $\boldsymbol{R}^d$ with support in the compact set

$$K=\Big(\bigcup_{n=1}^{\infty}K_n^{+}\Big)\cup\Big(\bigcup_{n=1}^{\infty}K_n^{-}\Big)\cup\{0\}\subset R.$$

Then the total variation $|\nu|$ of ν is

$$|\nu|=\sum_{n=1}^{\infty}(\nu^{+}+\nu^{-}).$$

We set

$$u(x)=\sum_{n=1}^{\infty}(u_n^{+}(x)-u_n^{-}(x))=\int G(x, y)d\nu(y) \quad (x \in \bar{R}).$$

By (6.4), the Weierstrass M-test assures that the series converges uniformly on $\bar{R}$. Since $u_n^{+}-u_n^{-}\in C(\bar{R})$, we conclude that $u\in C(\bar{R})$. Finally we set

$$U(x)=\sum_{n=1}^{\infty}(u_n^{+}(x)+u_n^{-}(x))=\int G(x, y)d|\nu|(y) \quad (x \in \bar{R}).$$

6.4. We maintain that $U\in C(\bar{R}\setminus\{0\})$, U is *discontinuous* at $x=0$, and U is bounded on $\bar{R}$: $U(x)\leqq 5/2$ $(x\in\bar{R})$.

First choose an arbitrary $x\in K$. Then either there is an m such that $x\in K_m^{+}\cup K_m^{-}$ or $x=0$. In the former case, by (6.3), we see that

$$U(x)=\sum_{n\geqq 1,\, n\neq m}(u_n^{+}(x)+u_n^{-}(x))+(u_m^{+}(x)+u_m^{-}(x))\leqq\sum_{n=1}^{\infty}2/5^n+2=5/2.$$

In the latter case we also see by (6.3) that

$$U(x)=U(0)=\sum_{n=1}^{\infty}(u_n^{+}(0)+u_n^{-}(0))\leqq\sum_{n=1}^{\infty}2/5^n=1/2<5/2.$$

We have thus seen that $U\leqq 5/2$ on the support of the measure $|\nu|$ of the Green potential U. By the Maria-Frostman domination principle (cf. e.g., [6], p. 134), we conclude that $U\leqq 5/2$ on $\bar{R}$.

We set $R^+=\{x\in R: x^\wedge>0\}$ and $R^-=\{x\in R: x^\wedge<0\}$ where, as before, $x^\wedge$ is the first component of $x=(x_1, \cdots, x_d)$ so that $x^\wedge=x_1$. If $x\in R^-$, then (6.3) assures that $u_n^{\pm}(x)<1/5^n$ and thus $U(x)<1/2$. Hence

$$\liminf_{x\to 0} U(x) \leqq 1/2 .$$

On the other hand, observe that 0 is an accumulation point of $K\setminus\{0\}$ so that there exists a sequence $\{x_m\}$ in $K\setminus\{0\}$ converging to 0. For each x_m there is an n such that $x_m\in K_n^+\cup K_n^-$. Hence $U(x_m)>u_n^+(x_m)+u_n^-(x_m)\geqq 1$ and thus

$$\limsup_{x\to 0} U(x) \geqq \limsup_{m\to\infty} U(x_m) \geqq 1 .$$

Therefore U is *not* continuous at $x=0$.

Finally, there is an $\bar{n}$ for any $\eta\in(0, 1)$ such that $\bar{B}(a_n, r_n)\cap\{\eta\leqq|x|\leqq 1\}=\emptyset$ for all $n\geqq\bar{n}$. By (6.3), the Weierstrass M-test assures that $\sum_{n\geqq\bar{n}}(u_n^++u_n^-)$ is uniformly convergent on $\{\eta\leqq|x|\leqq 1\}$. Since $u_n^++u_n^-\in C(\bar{R})$ for any n, U is continuous on $\bar{R}\setminus\{0\}$.

6.5. By $U(x)\leqq 5/2$ $(x\in R)$, we have $\gamma(a, \nu, G)\leqq 5/2$ $(a\in R)$. Since G is an N-kernel, i.e., $G-\kappa_d^{-1}N\in C(R\times R)$, Lemma 2.2 assures that $\gamma(a, \nu, N)=\kappa_d\gamma(a, \nu, G)$ $\leqq 5\kappa_d/2$ $(a\in R)$. By the fact that $U\in C(\bar{R}\setminus\{0\})$, Lemma 2.1 assures that $\gamma(a, \nu, N)=\kappa_d\gamma(a, \nu, G)=0$ for every $a\in R\setminus\{0\}$.

Fix an arbitrary $\alpha\in(0, 1/10\tau_d)$ and set $\mu=\alpha\nu$. Then $\gamma(a, \mu, N)=\alpha\gamma(a, \nu, N)$ $=0$ $(a\in R\setminus\{0\})$ and $\gamma(0, \mu, N)=\alpha\gamma(0, \nu, N)\leqq\alpha\cdot 5\kappa_d/2<\kappa_d/4\tau_d$. Thus μ satisfies the condition (3.5) on R.

Take an arbitrary $a\in R\setminus\{0\}$ and an arbitrary ball $B=B(a, \varepsilon)$ with $\bar{B}\subset R\setminus\{0\}$. Let $\mu_B=\mu|B$. Since $\alpha U=G|\mu|=G|\mu_B|+G|\mu-\mu_B|$ is continuous on $R\setminus\{0\}$, we see that $G|\mu_B|$ is continuous on $R\setminus\{0\}$. Clearly $G|\mu_B|$ is continuous at 0 and thus $G|\mu_B|$ is continuous on R. Clearly $(N-\kappa_d G)|\mu_B|=$ $N|\mu_B|-\kappa_d G|\mu_B|$ is continuous on R and hence $N|\mu_B|$ is continuous on R. Clearly $N|\mu_B|$ is continuous on $\boldsymbol{R}^d\setminus B$ and a fortiori $N|\mu_B|$ is continuous on $\boldsymbol{R}^d$. Thus $N\mu_B\in C(\boldsymbol{R}^d)$. Thus the family of N-regular balls for μ centered at a forms a base of neighborhood system at $a\in R\setminus\{0\}$.

Take any ball $B=B(0, \varepsilon)$ with $\bar{B}\subset R$. Clearly $G\mu_B=G\mu-G(\mu-\mu_B)=$ $\alpha u-G(\mu-\mu_B)\in C(B)$. Since $G|\mu|=G|\mu_B|+G|\mu-\mu_B|\in C(R\setminus\{0\})$, we see that $G|\mu_B|\in C(R\setminus\{0\})$ and thus $G\mu_B\in C(R\setminus\{0\})$. Hence $G\mu_B\in C(R)$ and a fortiori $N\mu_B\in C(R)$. It is clear that $N\mu_B\in C(\boldsymbol{R}^d\setminus B)$ and finally we have $N\mu_B\in C(\boldsymbol{R}^d)$. Thus the family of N-regular balls for μ centered at 0 forms a base of neighborhood system at 0. Therefore we have seen that the Radon measure μ constructed above is of quasi Kato class.

Finally we maintain that $\gamma(0, \mu, N)=\kappa_d\gamma(0, \mu, G)>0$. Otherwise, since $\gamma(a, \mu, N)=\kappa_d\gamma(a, \mu, G)=\alpha\kappa_d\gamma(a, \nu, G)=0$ $(a\in R\setminus\{0\})$, we have $\gamma(a, \mu, G)=0$

($a \in R$) and therefore by Lemma 2.1, $G|\mu| = \alpha G|\nu| = \alpha U \in C(R)$, which contradicts the discontinuity of U at 0. Thus μ is not of Kato class.

The proof of Theorem 2 is herewith complete. □

Appendix: Flat cone condition.

A.1. Let D be a bounded region in the Euclidean space $\boldsymbol{R}^d$ ($d \geqq 2$). We denote by H_f^D the harmonic Dirichlet solution on D for a boundary function f in $C(\partial D)$ obtained by the Perron-Wiener-Brelot method (cf. e.g., [**6**], pp. 156-162). A point $x \in \partial D$ is *Dirichlet regular* if $H_f^D(y)$ approaches to $f(x)$ as $y \in D$ tends to x for every $f \in C(\partial D)$. A cone $\Lambda(x, a; \theta)$ with x as its vertex and θ as its half of the opening angle and containing a on its axis of symmetry is given by

$$\Lambda(x, a; \theta) = \{y \in \boldsymbol{R}^d : (x-a)\cdot(x-y) \geqq |x-a||x-y| \cos\theta\}$$

where $(x-a)\cdot(x-y)$ denotes the inner product of $x-a$ and $x-y$. A *truncated flat cone* with vertex x is the set of the form $\Lambda(x, a; \theta) \cap P \cap \bar{B}(x, r)$ ($r>0$) where P is a $(d-1)$-dimensional hyperplane containing the axis of symmetry of $\Lambda(x, a; \theta)$.

THEOREM A. *A boundary point x of a bounded region D in $\boldsymbol{R}^d$ ($d \geqq 2$) is Dirichlet regular if there is a truncated flat cone with vertex x in the complement $\sim D = \boldsymbol{R}^d \setminus D$ of D.*

An interesting but unique proof is found in Kuran [**7**]. For the convenience of the reader we give here a proof to the above theorem simply by combining the standard common knowledge: Bouligand barrier criterion, monotoneity and subadditivity of the capacity, and Wiener test.

A.2. A function w is a *barrier* at $x \in \partial D$ if w is defined on $B \cap D$ for some open ball B centered at x and possesses the following properties: (i) w is superharmonic on $B \cap D$; (ii) $w>0$ on $B \cap D$; (iii) $w(y) \to 0$ as $y \in B \cap D$ tends to x. The *Bouligand criterion* then states that $x \in \partial D$ is Dirichlet regular if and only if there is a barrier at x (cf. e.g., [**6**], p. 171).

Suppose S is a region in $\boldsymbol{R}^d$ with a harmonic Green function G. The *capacity* of any compact subset K of S relative to S is given by $C(K) = \sup\{\mu(K) : G\mu \leqq 1$ on S, μ a positive Radon measure with support in $K\}$. Then we have the *monotoneity*: $K_1 \subset K_2$ implies $C(K_1) \leqq C(K_2)$, and the *subadditivity*: $C(K_1 \cup K_2) + C(K_1 \cap K_2) \leqq C(K_1) + C(K_2)$ (cf. e.g., [**6**], p. 141).

Fix a point $x \in \partial D$ and consider the capacity C relative to the open ball S of radius 1/2 centered at x. For $\lambda > 1$ we consider spherical rings

$$A(\lambda, n) = \{y \in \boldsymbol{R}^d : \lambda^n \leqq N(x, y) \leqq \lambda^{n+1}\} \quad (n = 1, 2, \cdots)$$

where $N(x, y)$ is the Newtonean kernel on $\boldsymbol{R}^d$ $(d \geqq 2)$. Let $\bar{n}$ be the least positive integer such that $A(\lambda, n) \subset S$ for every $n \geqq \bar{n}$. The *Wiener test* maintains (cf. e.g., [**6**], p. 220) that $x \in \partial D$ is Dirichlet regular if and only if

$$\sum_{n \geqq \bar{n}} \lambda^n C((\sim D) \cap A(\lambda, n)) = +\infty .$$

A.3. *Proof of Theorem* A. By translation we may assume that $x=0$ is the boundary point of D in question. We assume that a truncated flat cone T with vertex 0 is contained in $\sim D$. By rotation about the origin we can assume that T is contained in the hyperplane $P = \{y = (y_1, \cdots, y_d) \in \boldsymbol{R}^d : y_d = 0\}$ so that T is expressed as follows:

$$T = \Lambda(0, a ; \theta) \cap P \cap \bar{B}(0, \rho) \subset \sim D$$

where $\rho \in (0, 1/2)$ and $\theta \in (0, \pi/2)$. We can also take $|a| = \rho$. Observe that there is a finite number of points $a_1 = a, a_2, \cdots, a_m$ in $P \cap \partial \bar{B}(0, \rho)$ such that

$$K = P \cap \bar{B}(0, \rho) \subset \bigcup_{j=1}^{m} T_j, \quad T_j = \Lambda(0, a_j ; \theta) \cap P \cap \bar{B}(0, \rho) .$$

We consider the capacity C relative to the open ball $S = B(0, 1/2)$. Since C is clearly invariant under rotation of S around the origin and all $T_j \cap A(\lambda, n)$ are congruent to $T \cap A(\lambda, n)$ by suitable rotations of S about the origin, we see that

$$C(T_j \cap A(\lambda, n)) = C(T \cap A(\lambda, n)) \quad (j = 1, \cdots, m ; n = 1, 2, \cdots) .$$

By the monotoneity and the subadditivity of C we see that

$$C(K \cap A(\lambda, n)) \leqq C\Big(\Big(\bigcup_{j=1}^{m} T_j\Big) \cap A(\lambda, n)\Big) \leqq \sum_{j=1}^{m} C(T_j \cap A(\lambda, n)) = mC(T \cap A(\lambda, n)) .$$

Observe that $w(y) = w(y_1, \cdots, y_d) = |y_d|$ is a barrier at $0 \in \partial(S \setminus K)$ since it is superharmonic (and actually harmonic) on $B(0, \rho) \cap (S \setminus K)$ and has vanishing boundary values on $B(0, \rho) \cap \partial(S \setminus K) = B(0, \rho) \cap K$ and in particular at $x=0$. A fortiori $x = 0 \in \partial(S \setminus K)$ is Dirichlet regular for the region $S \setminus K$. Hence by the Wiener criterion

$$\begin{aligned} +\infty &= \sum_{n \geqq \bar{n}} \lambda^n C((\sim(S \setminus K)) \cap A(\lambda, n)) = \sum_{n \geqq \bar{n}} \lambda^n C(K \cap A(\lambda, n)) \\ &\leqq m \sum_{n \geqq \bar{n}} \lambda^n C(T \cap A(\lambda, n)) \leqq m \sum_{n \geqq \bar{n}} \lambda^n C((\sim D) \cap A(\lambda, n)) \end{aligned}$$

and, again by the Wiener criterion, $x = 0 \in \partial D$ is Dirichlet regular for the region D. □

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Mitsuru NAKAI
Department of Mathematics
Nagoya Institute of Technology
Gokiso, Showa, Nagoya 466
Japan