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N-body resolvent estimates

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1. Introduction.

This paper concerns micro-local resolvent estimates for a large class of N-body Schrödinger operators. It seems that further progress in our understanding of some basic problems in many-body scattering theory relies on such estimates. In any case recent success in the study of scattering amplitudes and eigenfunction expansion for some specific cases (channels), cf. [**B**], [**I2**], [**I3**], [**S1**], [**HSk**], is based heavily on results of this type. The main purpose of our paper is to generalize known micro-local resolvent estimates as far as possible by a new method that we find elementary and easy to handle. Basically (as for previous proofs) the problem boils down to the so called Mourre estimate. Further progress would probably need additional new tools.

We consider N ($\geqq 2$) ν-dimensional particles labelled $1, \cdots, N$ in the configuration space

$$\mathcal{X}=\left\{x=(x_1, \cdots, x_N) \mid x_i \in \boldsymbol{R}^{\nu}, \sum_{i=1}^{N} m_i x_i=0\right\}.$$

Here x_i and m_i denote the position vectors and the masses, respectively.

As usual we order the set of all cluster decompositions $a=(C_1, \cdots, C_{\# a})$ by inclusion of respective clusters. The $N-1$ cluster decomposition defined by letting particle i and particle j form a cluster is denoted by (ij).

Throughout the paper the potential $V=\sum_{(ij)} v_{ij}(x_i-x_j)$ obeys the following.

CONDITION. *There exists* $1>\varepsilon_1>0$ *such that for all pairs* (ij), $v_{ij}(y)=v_{ij}^{(1)}(y)+v_{ij}^{(2)}(y)$, *where*

1) $v_{ij}^{(1)}(y)$ *is smooth and for any multiindex* α

$$|\partial_y^{\alpha} v_{ij}^{(1)}(y)| \leqq C_{\alpha}(1+|y|)^{-|\alpha|-\varepsilon_1},$$

2) $v_{ij}^{(2)}(y)$ *is compactly supported and* $v_{ij}^{(2)}(-\Delta_y+1)^{-1}$ *is compact on* $L^2(\boldsymbol{R}_y^{\nu})$.

Here and in the sequel the notation C is used for various constants.

Equipping $\mathcal{X}$ with the metric $x \cdot y=\sum_{i=1}^{N} 2m_i x_i \cdot y_i$, the Hamiltonian is given by $H=-\Delta+V$ on $\mathcal{H}:=L^2(\mathcal{X})$. Let $\mathcal{T}$ be the set of thresholds (i.e., eigenvalues for subsystems), and $\sigma_c(H)$ and $\sigma_{pp}(H)$ the continuous respective the pure point spectrum. Let $\Lambda=\mathcal{T}\cup\sigma_{pp}(H)$. The resolvent of H is denoted by $R(z)=(H-z)^{-1}$; $z\notin\boldsymbol{R}$.

Let X be the (maximal) operator of multiplication by $X(x)=\langle x\rangle=(1+|x|^2)^{1/2}$ on $\mathcal{H}$.

Now one of our results (see Definition 2.11 and Theorem 2.12 (1) for the precise formulation) can be stated as follows:

Let $\lambda\in\sigma_c(H)\setminus\Lambda$, $l\in\boldsymbol{N}$, $s'>s>l-1/2$ and P_- be a bounded pseudodifferential operator supported in the region $x/\langle x\rangle\cdot\xi\leqq C$ where C is a certain positive constant depending on the distance from λ to the nearest smaller threshold. Then there exists a constant C' such that

$$\|X^{s-l}P_-R(z)^l X^{-s'}\|\leqq C',$$

uniformly in $\mathrm{Re}\, z\in N_\lambda$, *a neighborhood of* λ, *and* $\mathrm{Im}\, z>0$; $\|\cdot\|$ *being the operator norm on* $L^2(\mathcal{X})$.

Different types of estimates that accommodate the N-body geometry are also discussed. More specifically we shall prove estimates involving only pseudodifferential localization for the intercluster motion when localizing to some geometrically determined regions of the configuration space. We have results for energy-dependent symbols (Theorem 3.5), but also one for low energies for energy-independent symbols (see Definition 3.6 and Theorem 3.7). As for the latter type of symbols there is a complete result for $N=3$ and partial results in the general case for high energies (Theorem 3.8). It is clear from the context (cf. Definition 3.6) what the conjecture would be for the general case.

While we shall not give an account for the large literature on micro-localization for two-body operators (see though [**I1**], [**J**]), let us mention that for the free channel the above mentioned result for energy-independent symbols was first proved by a time-dependent method (based among others on [**SS**]) by one of the authors [**S2**]. It was implicitly suggested in [**S1**] that an application of a "conjugate" operator with "symbol" $x\cdot\xi-C\langle x\rangle$ and Mourre theory ([**J**], [**JMP**], [**M1**], [**PSS**]) would provide another proof. This line was pursued by Wang [**W2**] independently with whom we have many overlapping results. The present paper presents a third method. In common are the technical tools of positive commutators and calculi for associated (functions of) selfadjoint operators and pseudodifferential operators. These tools date back to the pioneering works [**M1**], [**M2**]. In this paper we (roughly speaking) invent an appropriate calculus not for the "symbol" mentioned above but rather for the "symbol" obtained by dividing it by $\langle x\rangle$. The method facilitates calculations and allows a simple stationary approach that in its basic form is well known (see for example the proof of Theorem 30.2.6 in [**H1**]). Except for the Mourre estimate and the limiting absorption principle the paper is completely selfcontained.

We complete this section by a piece of notations. Given a cluster decomposition a we introduce the following classes of Ps.D.Op.'s on the corresponding subspace

$$\mathcal{X}_a = \{x \in \mathcal{X} \mid x_i = x_j \text{ if } i, j \in C \text{ for some } C \in a\}$$

of intercluster motion.

For $r, l \in \boldsymbol{R}$ the class $S_l^r(\mathcal{X}_a)$ consists of the smooth functions $p(x_a, \xi_a)$ on $\mathcal{X}_a \times \mathcal{X}_a^*$ such that

$$|\partial_{x_a}^\alpha \partial_{\xi_a}^\beta p(x_a, \xi_a)| \leqq C_{\alpha,\beta} \langle x_a \rangle^{l-|\alpha|} \langle \xi_a \rangle^r$$

for all multiindices α and β ; $\langle x_a \rangle = (1+|x_a|^2)^{1/2}$.

We put $S(\mathcal{X}_a) = S_0^0(\mathcal{X}_a)$ and $S_{comp}(\mathcal{X}_a) = \{p \in S(\mathcal{X}_a) \mid \exists C>0 : p(x_a, \xi_a)=0$ for $|\xi_a|^2 > C\}$.

We quantize according to the standard formula

$$(P(x_a, D_a)\Psi)(x_a) = (2\pi)^{-\dim \mathcal{X}_a/2} \int e^{ix_a \cdot \xi_a} p(x_a, \xi_a) \hat{\Psi}(\xi_a) d\xi_a .$$

Let $\mathcal{X}^a$ be the orthogonal complement of $\mathcal{X}_a$. Then the notation for the corresponding decomposition of any $x \in \mathcal{X}$ reads $x = x^a \oplus x_a \in \mathcal{X}^a \oplus \mathcal{X}_a$.

We shall frequently use the subscript prime to indicate objects of same type as those discussed in a given context. In case of functions this notation should not be confused with derivatives (indicated by a different notation).

2. Commutator calculus and "global" estimates.

In this section we develop a calculus that consecutively is used for proving resolvent estimates. The first type of such estimates (Theorem 2.10) involves localization in terms of some operators defined by the functional calculus for selfadjoint operators. The other type (Theorem 2.12) involves pseudodifferential localization in terms of operators with symbols in $S_{comp}(\mathcal{X})$ satisfying a certain energy-dependent support condition (see Definition 2.11).

We introduce

DEFINITIONS.

(1) Let $\mathcal{V}$ be the algebra of C^∞-functions v on $\mathcal{X}$ such that

$$|\partial_x^\alpha (x \cdot \nabla)^k v(x)| \leqq C_{\alpha,k} ; \forall \alpha, k .$$

(2) Let $\mathcal{V}_+^1$ be the class of positive C^∞-functions r on $\mathcal{X}$ such

$$r(x)^2 - |x|^2 \in \mathcal{V}.$$

For $\lambda \in \sigma_c(H) \backslash \Lambda$ we denote by

$$a(\lambda)=\inf\{\lambda-\mu \mid \mu\in\mathcal{T},\ \mu<\lambda\}\ ;$$

i.e., the distance to the nearest threshold to the left of λ. We want to study the evolution at some fixed $\lambda\in\sigma_c(H)\setminus\Lambda$. Our calculus is based on the commutators of H and certain selfadjoint operators. In order to accommodate local singularities of the potentials, instead of using the usual generator of dilations, we adopt the vector field of Graf [**Gr**]. The following lemma is proved in [**S2**].

LEMMA 2.1. *Let $\lambda\in\sigma_c(H)\setminus\Lambda$ and $\varepsilon>0$ be given. Then there exist an open neighborhood N_λ of λ and $r\in\mathcal{V}_+^1$ such that with A given as the selfadjoint operator on $\mathcal{H}$*

$$A=(\omega\cdot D+D\cdot\omega)/2\,;\ \omega=r\nabla r,\quad D=-i\partial_x,$$

(1) *$i[H, A]$ defined as a form on $\mathcal{D}(H)\cap\mathcal{D}(A)$ extends to a symmetric operator on $\mathcal{D}(H)$, and in fact*

$$i[H, A]=\sum_{|\alpha|\leq 2} v_\alpha D^\alpha\,;\quad v_\alpha\in\mathcal{V}.$$

(2) *$\varphi(H)i[H, A]\varphi(H)\geqq 2a(\lambda)(1-\varepsilon)\varphi(H)^2$ for all real-valued $\varphi\in C_0^\infty(N_\lambda)$.*

Let in the following $\lambda\in\sigma_c(H)\setminus\Lambda$ and $\varepsilon>0$ be fixed. Then we choose (and fix) N_λ and $r\in\mathcal{V}_+^1$ accordingly. (Note that the assertion (1) follows from the fact that $i[v_{ij}^{(2)}, A]=0$.) We can assume that $N_\lambda\cap\Lambda=\emptyset$.

DEFINITIONS. With B given as the selfadjoint operator on $\mathcal{H}$

$$B=r^{-1/2}Ar^{-1/2}=(\nabla r\cdot D+D\cdot\nabla r)/2,$$

we let $\mathcal{D}$ be the (dense) domain

$$\mathcal{D}=\cap\mathcal{D}(Q),$$

where the intersection is over all polynomials Q in X and B.

Then we define for any $m\in\boldsymbol{R}$ the class $\mathcal{O}p^m(X)$ of operators P with the properties

1) $\mathcal{D}(P)$ and $\mathcal{D}(P^*)$ contain $\mathcal{D}$ and P and P^* restricted to $\mathcal{D}$ map into itself.
2) $\forall n\in\boldsymbol{N}\cup\{0\}\,\forall\alpha,\ \beta\in\boldsymbol{R}$ such that $\alpha+\beta=n-m: X^\alpha ad_n(P, B)X^\beta$ extends to a bounded operator on $\mathcal{H}$.

Here P^* is the adjoint of P, and the iterated commutator $ad_n(P, B)$ is given by $ad_0(P, B)=P$ and $ad_n(P, B)=[ad_{n-1}(P, B), B]\,;\ n\geqq 1$.

REMARK. It is readily verified that $[X, B]\in\mathcal{V}$, $[B, v]\in X^{-1}\mathcal{V}$ $(\subset\mathcal{V})$ for any $v\in\mathcal{V}$, and in fact that $\mathcal{V}\subset\mathcal{O}p^0(X)$ and $X^l\in\mathcal{O}p^l(X)$ for any $l\in\boldsymbol{R}$. Also it is remarked that these properties as well as the definition above are independent of the particular choice of $X(\cdot)\in\mathcal{V}_+^1$. In particular we could take $X(\cdot)=r(\cdot)$.

We omit the straightforward proof of the following result:

LEMMA 2.2 (Algebraic properties).
(1) $P\in\mathcal{O}p^m(X)\Rightarrow[P, B]\in\mathcal{O}p^{m-1}(X)$.
(2) $P\in\mathcal{O}p^m(X)\Rightarrow P^*\in\mathcal{O}p^m(X)$.
(3) $P\in\mathcal{O}p^m(X),\ Q\in\mathcal{O}p^{m_1}(X)\Rightarrow PQ\in\mathcal{O}p^{m+m_1}(X)$.

We will give some examples of operators in $\mathcal{O}p^0(X)$ and discuss associated commutator properties. For that we will use the following general facts.

Let for any $m\in\boldsymbol{R}$, $\mathcal{F}^m$ be the class of C^∞-functions on $\boldsymbol{R}$ such that

$$|f^{(k)}(t)|\leqq C_k(1+|t|)^{m-k},\quad \forall k\geqq 0.$$

Let $f\in\mathcal{F}^m$. Then one can choose (cf. [**Ge**]) an almost analytic extension $\tilde{f}\in C^\infty(\boldsymbol{C})$ of f with (more precisely) the properties:

$$\begin{aligned}&\tilde{f}(t)=f(t)\quad\text{for } t\in\boldsymbol{R},\\ &|\partial_{\bar z}\tilde{f}(z)|\leqq C_N\langle z\rangle^{m-1-N}|\mathrm{Im}\, z|^N\quad\text{for all } N\in\boldsymbol{N};\ \langle z\rangle=(1+|z|^2)^{1/2},\\ &\mathrm{supp}\,\tilde{f}(\cdot)\subset\{z\in\boldsymbol{C}\,|\,|\mathrm{Im}\, z|\leqq 1+|\mathrm{Re}\, z|\}.\end{aligned}\tag{2.1}$$

Furthermore $\partial_t^k\tilde{f}(t+i\,\mathrm{Im}\, z)$ is an extension of $f^{(k)}(t)$ with the same properties (with m replaced by $m-k$).

For $f\in\mathcal{F}^m$ with $m<0$ and $\tilde{f}$ as above the following formula holds for any selfadjoint operator S (cf. [**H2**, p. 63], [**HSj**]).

$$f(S)=\frac{1}{\pi}\int_{\boldsymbol{C}}\partial_{\bar z}\tilde{f}(z)(S-z)^{-1}du\,dv\,;\ z=u+iv.\tag{2.2}$$

Moreover if $f\in\mathcal{F}^m$ for arbitrarily given m and S and T are linear operators on the same Hilbert space, S selfadjoint and T bounded, then for any positive integer $N>m$

$$[T,\ f(S)]=\sum_{k=1}^{N-1}\frac{(-1)^{k-1}}{k!}ad_k(T,\ S)f^{(k)}(S)+R_N,\tag{2.3}$$

$$R_N=-\frac{1}{\pi}\int_{\boldsymbol{C}}\partial_{\bar z}\tilde{f}(z)(S-z)^{-1}ad_N(T,\ S)(S-z)^{-N}du\,dv\,;\ z=u+iv,\tag{2.4}$$

provided that all the commutators $ad_k(T, S)$ up to order $k=N$ are given as bounded operators (defined iteratively as extensions of forms on $\mathcal{D}(S)$).

This statement follows readily from (2.2).

LEMMA 2.3.
(1) *$P_{2,0}R(z)\in\mathcal{O}p^0(X)$ for* $\mathrm{Im}\, z\neq 0$ *and $P_{2,0}=\sum_{|\alpha|\leqq 2}v_\alpha D^\alpha$; $v_\alpha\in\mathcal{V}$.*
(2) *$f(H),\ f(B)\in\mathcal{O}p^0(X)$ if $f\in\mathcal{F}^m$ for some $m<0$.*
(3) *For $f\in\mathcal{F}^m$ for some $m\in\boldsymbol{R}$ and $\varphi\in C_0^\infty(\boldsymbol{R})$, $f(B)\varphi(H)\in\mathcal{O}p^0(X)$.*

PROOF. We first show (1). We notice that $[B, P_{2,0}]=X^{-1}P'_{2,0}$, where $P'_{2,0}$ has the same form as $P_{2,0}$. Let in the following P_j and P'_i denote operators of this type. By induction one can show that $ad_n(P_{2,0}R(z), B)$ is a finite sum, each term being a product

$$A_1 \cdots A_k,$$

where $k \leqq n+1$ and

$$A_j = X^{-\alpha_j} P_j R(z), \quad \alpha_j \geqq 0,$$

and

$$\sum_{j=1}^{k} \alpha_j = n.$$

For any real α and β with $\alpha+\beta=n$ we can write

$$X^{\alpha} A_1 \cdots A_k X^{\beta} = (X^{\alpha} A_1 X^{\alpha_1-\alpha})(X^{\alpha-\alpha_1} A_2 X^{\alpha_1+\alpha_2-\alpha}) \cdots (X^{\alpha_k-\beta} A_k X^{\beta}).$$

But for any real γ, $X^{\gamma} R(z) X^{-\gamma}$ is a finite sum of products of the form

$$A'_1 \cdots A'_{k'},$$

where $k' \leqq |\gamma|+2$ and

$$A'_i = P'_i R(z).$$

Putting together we see that

$$X^{\alpha} ad_n(P_{2,0}R(z), B) X^{\beta}$$

is a finite sum of operators of the product type

$$P_1 R(z) P_2 R(z) \cdots P_k R(z),$$

where

$$1 \leqq k \leqq (n+1)(\max(|\alpha|, |\beta|)+2) =: \gamma.$$

Since a similar argumentation works for the adjoint we have proved (1).

As for the statement of (2), that $f(H) \in \mathcal{O}p^0(X)$ if $f \in \mathcal{F}^m$ for $m<0$, we use (2.2) with $S=H$ and the arguments above which for $P_{2,0}=I$ leads to a similar expansion with $P_1=I$. Hence we obtain the estimate

$$\|X^{\alpha} ad_n(R(z), B) X^{\beta}\| \leqq C \langle z \rangle^{\gamma-1} |\operatorname{Im} z|^{-\gamma},$$

which together with the bound

$$|\partial_{\bar{z}} F(z)| \leqq C_{\gamma} \langle z \rangle^{m-\gamma-1} |\operatorname{Im} z|^{\gamma}$$

gives a representation in terms of a norm-convergent integral.

As for the second statement in (2) we need only to show that

$$\|X^{\alpha}(B-z)^{-1} X^{-\alpha}\| \leqq C_{\alpha} |\operatorname{Im} z|^{-|\alpha|-2} \langle z \rangle^{|\alpha|+1}.$$

But this bound follows from the fact that the operator is represented as a sum of products

$$v_1(B-z)^{-1}v_2(B-z)^{-1}\cdots v_k(B-z)^{-1}v_{k+1},$$

where $v_j\in\mathcal{V}$ and $1\leqq k\leqq|\alpha|+2$.

It remains to prove (3). This is done by writing (for $m\in\boldsymbol{N}$)

$$f(B)\varphi(H)=\{f(B)(B-i)^{-m-1}\}(B-i)^{m+1}R(i)^{m+1}\{(H-i)^{m+1}\varphi(H)\}$$

and then using (1) and (2). □

LEMMA 2.4. *Let $f\in\mathcal{F}^0$ and $P\in\mathcal{O}p^m(X)$; $m\in\boldsymbol{R}$. Then for any positive integer $N>m$ there exists $P_{m-N}\in\mathcal{O}p^{m-N}(X)$ such that, as an identity on $\mathcal{D}$,*

$$[P,\ f(B)]-\sum_{k=1}^{N-1}\frac{(-1)^{k-1}}{k!}ad_k(P,\ B)f^{(k)}(B)=P_{m-N}.$$

Moreover for any positive integer $j<N$, $B^jP_{m-N}\in\mathcal{O}p^{m-N}(X)$.

PROOF. We replace P by $T=(1+\kappa|x|^2)^{-m}P$ for $\kappa>0$, use (2.3) and (2.4) in which formulas we let $\kappa\downarrow 0$. The latter statement follows also by inspection of (2.4). □

COROLLARY 2.5. *Let I be an open interval, $f_0\in\mathcal{F}^0$ and $P\in\mathcal{O}p^m(X)$; $m\in\boldsymbol{R}$. Suppose* $\operatorname{supp} f_0\subset I$. *Then for any $M\in\boldsymbol{R}$ the operators $f_0(B)P$ and $Pf_0(B)$ can be written, modulo terms in $\mathcal{O}p^M(X)$, as finite sums of terms*

$$f(B)P_mf'(B),$$

where $f,\ f'\in\mathcal{F}^0$, $\operatorname{supp} f$, $\operatorname{supp} f'\subset I$ *and $P_m\in\mathcal{O}p^m(X)$.*

PROOF. We can write $f_0=f\cdot f'$ for some f and f' as in the statement. Then we write

$$f_0(B)P=f(B)Pf'(B)-f(B)[P,\ f'(B)],$$

and use Lemmas 2.3 (2) and 2.4 for the second term.

As for $Pf_0(B)$ we use the result just proven for the adjoint. □

We shall use Corollary 2.5 with $I=I_-=(-\infty,\ \sqrt{C_0})$, where we put

$$C_0=C_0(\lambda,\ \varepsilon)=a(\lambda)(1-\varepsilon).$$

(Here ε is the fixed constant.)

So by assumption (cf. Lemma 2.1 (2)), the Mourre estimate reads

$$\varphi(H)i[H,\ A]\varphi(H)\geqq 2C_0\varphi(H)^2$$

for all real-valued $\varphi\in C_0^\infty(N_\lambda)$.

DEFINITION. For $m\in\boldsymbol{R}$ we let $\mathcal{F}_-^m$ denote the class of $f\in\mathcal{F}^m$ such that $\operatorname{supp} f\subset I_-$. $\mathcal{F}_-:=\mathcal{F}_-^0$.

We shall now consider the following special functions of this type.

For fixed $0<\varepsilon_0<C_0/3$ we use the notation F_0 for any function in $\mathcal{F}_-$, such that

$$\operatorname{supp} F_0 \subset (-\infty, \sqrt{C_0-2\varepsilon_0}),$$
$$F_0(t) \geqq 0, \ \sqrt{F_0(t)} \in C^\infty(\boldsymbol{R}),$$
$$\frac{d}{dt}F_0(t) \leqq 0, \ \sqrt{-\frac{d}{dt}F_0(t)} \in C^\infty(\boldsymbol{R}),$$
$$F_0(t) = 1, \quad \text{if } t < \sqrt{C_0-3\varepsilon_0}.$$

We put $C_1=C_1(\varepsilon_0)=\sqrt{C_0-\varepsilon_0}$, and introduce for $m>-1/2$

$$F_m(t) = (C_1-t)^m F_0(t)$$

and

$$\tilde{F}_{2m+1}(t) = (C_1-t)F_m(t)^2.$$

The crucial ingredient of our approach is the following positivity statement.

LEMMA 2.6. *Let $m>-1/2$, $C_0/3>\varepsilon_0>0$ and F_0, F_m and $\tilde{F}_{2m+1}$ be as above. Let for real-valued $\varphi\in C_0^\infty(N_\lambda)$*

$$P_m = r^m F_m(B)\varphi(H).$$

Then for any $M\in\boldsymbol{R}$ (and as forms on $\mathcal{D}$)

$$-\operatorname{Re}\varphi(H)i[H, r^{2m+1}\tilde{F}_{2m+1}(B)]\varphi(H)$$
$$\geqq 2(2m+1)\varepsilon_0 P_m^* P_m + \sum_{\text{finite}} f(B)P'_{2m-1}f'(B)+P'_M,$$

where $f, f'\in\mathcal{F}_-$, $P'_{2m-1}\in\mathcal{O}p^{2m-1}(X)$, and $P'_M\in\mathcal{O}p^M(X)$.

PROOF. The proof relies on all the machinery developed so far; i.e., Lemmas 2.1-2.4 and Corollary 2.5. Instead of giving a full formal proof we start by explaining the idea. Afterwards we indicate how to perform the computation rigorously.

We "compute" the (leading) term in $\mathcal{O}p^{2m}(X)$ modulo $\mathcal{O}p^{2m-1}(X)$. By use of the identities

$$i[H, r] = 2B$$
$$i[H, B] = r^{-1/2}(i[H, A]-2B^2)r^{-1/2}+P_{-2}$$

with $P_{-2}=\nabla r\cdot(\partial^2 r/\partial x^2)\nabla r/2r^2 \in \mathcal{O}p^{-2}(X)$, we get

$$i[H, r^{2m+1}] \cong (2m+1)r^{2m}2B,$$
$$i[H, \tilde{F}_{2m+1}(B)] \cong \left\{-(2m+1)F_m(B)^2+(C_1-B)^{2m+1}\left(\frac{d}{dt}F_0^2\right)(B)\right\},$$
$$r^{-1}\{i[H, A]-2B^2\},$$

and hence that

$$\begin{aligned}
&-\operatorname{Re}\varphi(H)i[H,\ r^{2m+1}\tilde{F}_{2m+1}(B)]\varphi(H)\\
&\cong -\varphi(H)(2m+1)r^{2m}\{2B(C_1-B)-i[H,\ A]+2B^2\}F_m(B)^2\varphi(H)\\
&\quad -\varphi(H)r^{2m}(C_1-B)^{2m+1}\Big(\frac{d}{dt}F_0^2\Big)(B)\{i[H,\ A]-2B^2\}\varphi(H)\\
&\gtrsim (2m+1)\{2C_0-2\sqrt{C_0-2\varepsilon_0}C_1\}P_m^*P_m\\
&\quad -\varphi(H)^2r^{2m}(C_1-B)^{2m+1}\Big(\frac{d}{dt}F_0^2\Big)(B)\{2C_0-2(C_0-2\varepsilon_0)\}\\
&\gtrsim 2(2m+1)\varepsilon_0 P_m^*P_m\,.
\end{aligned}$$

In the first estimate we used the Mourre estimate and the support property of F_m. In the second we removed the term containing $-(dF_0^2/dt)(B)$ as can be done by the non-negativity of the latter operator.

This was the idea. In order to justify the computations one can proceed as follows. At first we notice that for any $\varphi_1\in C_0^\infty(N_\lambda)$ and real m_1,

$$i[\varphi_1(H),\ r^{m_1}]-\varphi_1^{(1)}(H)2m_1r^{m_1-1}B\in \mathcal{O}p^{m_1-2}(X)\,. \tag{2.5}$$

This identity follows by the same method that was used in the proof of Lemma 2.3.

We choose $\varphi_1\in C_0^\infty(N_\lambda)$ such that the function $\varphi_1(t)=t$ in a neighborhood of the support of φ. Then

$$\begin{aligned}
&\varphi(H)i[H,\ r^{2m+1}\tilde{F}_{2m+1}(B)]\varphi(H)\\
&=\varphi(H)\{i[\varphi_1(H),\ r^{2m+1}]\tilde{F}_{2m+1}(B)+r^{2m+1}i[\varphi_1(H),\ \tilde{F}_{2m+1}(B)]\}\varphi(H)\,.
\end{aligned}$$

For the first term we use the identity (2.5). For the second term we use Lemma 2.4. We need to look at the term

$$\varphi(H)r^{2m+1}i[\varphi_1(H),\ B]\tilde{F}_{2m+1}^{(1)}(B)\varphi(H)\in\mathcal{O}p^{2m}(X)\,.$$

Up to remainders in $\mathcal{O}p^{2m-1}(X)$ it is equal to

$$r^{2m+1}\varphi(H)i[H,\ B]\varphi(H)\tilde{F}_{2m+1}^{(1)}(B)\,.$$

Then we insert the expression for the commutator and use (2.5) again to pull $r^{-1/2}$ through $\varphi(H)$. After symmetrizing we can then estimate by the assumption of Lemma 2.1 (2). To treat the contribution from $-2B^2$ we apply Lemma 2.4 (again). We omit the straightforward details. □

LEMMA 2.7. *Let $m>-1/2$, $t>1$ and $F\in\mathcal{F}_-$. Then for any $\varphi\in C_0^\infty(N_\lambda)$*

$$r^mF(B)\varphi(H)R(z)r^{-m-t}$$

is bounded uniformly in $\operatorname{Im}z>0$.

PROOF. By Mourre theory and Lemma 2.1 one gets for all $l \in \boldsymbol{N}$, $s' \in \boldsymbol{R}$ with $s' > l - 1/2$, and $\varphi \in C_0^\infty(N_\lambda)$ the bound

$$\|X^{-s'}\varphi(H)R(z)^l X^{-s'}\| \leqq C \tag{2.6}$$

uniformly in $\operatorname{Im} z \neq 0$ (cf. [J], [JMP], [M1], [PSS]).

We will use (2.6) with $l=1$ and Lemma 2.6. Let δ be fixed such that $0 < \delta < \min((t-1)/2, 1/2)$. We will show by induction in $k \in \boldsymbol{N}$ that

$q(k)$: Let $m = -1/2 + k\delta$. Then for all $F \in \mathcal{F}_-$ and all real-valued $\varphi \in C_0^\infty(N_\lambda)$

$$\|r^m F(B)\varphi(H)R(z)r^{-m-t}\| \leqq C,$$

uniformly in $\operatorname{Im} z > 0$.

If we know $q(k)$, then we are done by (2.6) and a simple interpolation.

In the following we give preliminary steps for $q(k)$. We use the notation $P'_{m'}$ for operators in $\mathcal{O}p^{m'}(X)$. Let $m > -1/2$, $F \in \mathcal{F}_-$ and $\varphi \in C_0^\infty(N_\lambda)$ be given. Let φ' be any real function in $C_0^\infty(N_\lambda)$ such that $\varphi'\varphi = \varphi$.

At first we claim that we can find $F' \in \mathcal{F}_-$, and P_m as in Lemma 2.6 for ε_0 sufficiently small, such that

$$r^m F(B)\varphi(H) = P'_0 P_m \varphi'(H) + P'_{m-1}F'(B)\varphi(H) + P'_{-1}\varphi(H). \tag{2.7}$$

To see this we write for ε_0 small enough

$$r^m F(B)\varphi(H) = P'_0 P_m \varphi'(H) + r^m(I - \varphi'(H))F(B)\varphi(H),$$

where $P'_0 = r^m\varphi'(H)F(B)(C_1 - B)^{-m}r^{-m}$. As for the second term we apply Lemma 2.4 to the commutator $[\varphi'(H), F(B)]$.

We use the notation

$$u = R(z)r^{-m-t}v \quad \text{for } v \in \mathcal{D} \text{ and } \operatorname{Im} z > 0.$$

By Lemma 2.6

$$2(2m+1)\varepsilon_0\|P_m\varphi'(H)u\|^2 \leqq A_1 + \cdots + A_5;$$

$$A_1 = -2\operatorname{Im} z \operatorname{Re}\langle u, \varphi(H)r^{2m+1}\tilde{F}_{2m+1}(B)\varphi(H)u\rangle,$$

$$A_2 = |\langle \varphi(H)r^{-m-t}v, r^{2m+1}\tilde{F}_{2m+1}(B)\varphi(H)u\rangle|,$$

$$A_3 = |\langle \varphi(H)u, r^{2m+1}\tilde{F}_{2m+1}(B)\varphi(H)r^{-m-t}v\rangle|,$$

$$A_4 = \sum_{\text{finite}} |\langle \varphi'(H)u, f(B)P'_{2m-1}f'(B)\varphi'(H)u\rangle|,$$

$$A_5 = |\langle \varphi'(H)u, P'_M\varphi'(H)u\rangle|.$$

Here we choose $M = -2$. We want $A_1 + \cdots + A_5 \leqq C\|v\|^2$, uniformly in $\operatorname{Im} z > 0$.

As for A_1 an application of Lemma 2.4 and Corollary 2.5 gives

$$\operatorname{Re}\varphi(H)r^{2m+1}\tilde{F}_{2m+1}(B)\varphi(H) \geqq \varphi(H)\{f(B)P'_{2m-1}f(B) + P'_{-2}\}\varphi(H)$$

for some real $f \in \mathcal{F}_-$.

By this estimate and (2.6) it suffices to look at terms of the form A_2, A_3 and A_4.

We can now prove $q(1)$:

By (2.6) and (2.7) it suffices to bound A_2, A_3 and A_4. Since $m+1-t<-t/2$, we have that

$$r^{-m-t}\varphi(H)r^{2m+1}\tilde{F}_{2m+1}(B),\ r^{-m-t}\varphi(H)\tilde{F}_{2m+1}(B)r^{2m+1}\in \mathcal{O}p^{-t/2}(X),$$

and hence by (2.6) that

$$A_2,\ A_3 \leqq C\|v\|^2.$$

Clearly the same bound holds for terms of the form A_4 (since $2m-1<-1$).

Suppose we have shown $q(k-1)$; $k>1$. We need to prove $q(k)$. By (2.6), (2.7) and $q(k-1)$ it suffices to bound A_2, A_3 and A_4.

Writing

$$r^{-m-t}\varphi(H)r^{2m+1}\tilde{F}_{2m+1}(B)=P'_{1+\delta-t}r^{m-\delta}f(B),$$
$$r^{-m-t}\varphi(H)\tilde{F}_{2m+1}(B)r^{2m+1}=P''_{1+\delta-t}r^{m-\delta}f(B)+P'_{-1},$$

the bounds for A_2 and A_3 follow from (2.6) and $q(k-1)$. (Notice that $1+\delta-t\leqq 0$.)

Since by $q(k-1)$

$$\|r^{m-1/2}f(B)^*\varphi'(H)u\|,\ \|r^{m-1/2}f'(B)\varphi'(H)u\|\leqq C\|v\|,$$

we get the bound for A_4 also. This proves $q(k)$. □

DEFINITION. Let $I_+=(-\sqrt{C_0},\infty)$ and $\mathcal{F}_+$ be the class of functions $f\in\mathcal{F}^0$ such that $\operatorname{supp} f\subset I_+$.

By the same method one can prove the following analogue of Lemma 2.7.

LEMMA 2.8. *Let $m>-1/2$, $t>1$ and $F\in\mathcal{F}_+$. Then for any $\varphi\in C_0^\infty(N_\lambda)$*

$$r^mF(B)\varphi(H)R(z)r^{-m-t}$$

is bounded uniformly in $\operatorname{Im} z<0$.

Also, by a simple interpolation and by changing "the weights" to the right in the proofs of Lemmas 2.7 and 2.8 one can prove following two sided estimate.

LEMMA 2.9. *Let $m\in\boldsymbol{R}$, $F_-\in\mathcal{F}_-$ and $F_+\in\mathcal{F}_+$. Suppose there exists $\sigma\in\boldsymbol{R}$ such that* $\operatorname{supp} F_-\subset(-\infty,\sigma)$ *and* $\operatorname{supp} F_+\subset(\sigma,\infty)$. *Then for any $\varphi\in C_0^\infty(N_\lambda)$*

$$r^mF_-(B)\varphi(H)R(z)F_+(B)r^m$$

is bounded uniformly in $\operatorname{Im} z>0$.

By combining Lemmas (2.7)-(2.9) and (2.6) (actually only the statement for $l=1$ is needed) one can get bounds for powers of the resolvent. This is done by introducing repeatedly suitable splittings $I=F_-+F_+$ (cf. Jensen [**J**] and Isozaki [**I1**]).

The result is

THEOREM 2.10.

(1) *Let* $l\in\boldsymbol{N}$, $s'>s>l-1/2$, $F_-\in\mathcal{F}_-$ *and* $\varphi\in C_0^\infty(N_\lambda)$. *Then*

$$\|X^{s-l}F_-(B)\varphi(H)R(z)^lX^{-s'}\|\leqq C\,;\ \mathrm{Im}\,z>0\,.$$

(2) *Let* $l\in\boldsymbol{N}$, $s'>s>l-1/2$, $F_+\in\mathcal{F}_+$ *and* $\varphi\in C_0^\infty(N_\lambda)$. *Then*

$$\|X^{s-l}F_+(B)\varphi(H)R(z)^lX^{-s'}\|\leqq C\,;\ \mathrm{Im}\,z<0\,.$$

(3) *Let* $l\in\boldsymbol{N}$, $s\in\boldsymbol{R}$, $F_-\in\mathcal{F}_-$, $F_+\in\mathcal{F}_+$ *and* $\varphi\in C_0^\infty(N_\lambda)$. *Suppose there exists* $\sigma\in\boldsymbol{R}$ *such that* $\mathrm{supp}\,F_-\subset(-\infty,\sigma)$ *and* $\mathrm{supp}\,F_+\subset(\sigma,\infty)$. *Then*

$$\|X^sF_-(B)\varphi(H)R(z)^lF_+(B)X^s\|\leqq C\,;\ \mathrm{Im}\,z>0\,.$$

The last issue of this section is to convert these estimates into pseudodifferential analogues. For that we need

DEFINITION 2.11. For $-1<\rho<1$ the notation $S_-^\lambda(\rho)$ stands for the class of symbols $p_-\in S_{\mathrm{comp}}(\mathcal{X})$ such that

$$\mathrm{supp}\,p_-\subset\left\{(x,\xi)\,|\,\frac{x}{\langle x\rangle}\cdot\xi\leqq\rho\sqrt{a(\lambda)}\right\},$$

and similarly $S_+^\lambda(\rho)$ for the class with

$$\mathrm{supp}\,p_+\subset\left\{(x,\xi)\,|\,\frac{x}{\langle x\rangle}\cdot\xi\geqq\rho\sqrt{a(\lambda)}\right\}.$$

For a symbol $p_-\in S_-^\lambda(\rho)$ the corresponding Ps.D.Op. is denoted by P_-. Similarly for $p_+\in S_+^\lambda(\rho)$.

We shall show (for fixed $\lambda\in\sigma_c(H)\backslash\Lambda$)

THEOREM 2.12.

(1) *Let* $-1<\rho<1$. *Then there exists a neighborhood* N'_λ *of* λ *such that for any* $l\in\boldsymbol{N}$, $s'>s>l-1/2$, $p_-\in S_-^\lambda(\rho)$

$$\|X^{s-l}P_-R(z)^lX^{-s'}\|\leqq C\,,$$

uniformly in $\mathrm{Re}\,z\in N'_\lambda$ *and* $\mathrm{Im}\,z>0$.

(2) *Let* $-1<\rho<1$. *Then there exists a neighborhood* N'_λ *of* λ: $\forall l\in\boldsymbol{N}$, $s'>s>l-1/2$, $p_+\in S_+^\lambda(\rho)$

$$\|X^{s-l}P_+R(z)^lX^{-s'}\|\leqq C\,,$$

uniformly in $\mathrm{Re}\,z\in N'_\lambda$ *and* $\mathrm{Im}\,z<0$.

(3) *Let* $-1<\rho_-<\rho_+<1$. *Then there exists a neighborhood* N_λ' *of* λ: $\forall l\in\boldsymbol{N}$, $s\in\boldsymbol{R}$, $p_-\in S_-^\lambda(\rho_-)$, $p_+\in S_+^\lambda(\rho_+)$

$$\|X^s P_- R(z)^l P_+ X^s\| \leqq C,$$

uniformly in $\operatorname{Re} z\in N_\lambda'$ *and* $\operatorname{Im} z>0$.

To show Theorem 2.12 we notice that for given $-1<\rho<1$ we can choose $\varepsilon>0$ such that $\sqrt{1-\varepsilon}>|\rho|$. With such ε taken as input in the definition of B in the beginning of this section we can write for a suitable splitting $I=F_-+F_+$

$$X^{s-l}P_- = A_1 X^{s-l} F_-(B) + A_2 X^{-s'},$$

where

$$A_1 = X^{s-l} P_- X^{l-s}$$

and

$$A_2 = X^{s-l} P_- F_+(B) X^{s'}.$$

Since A_1 is bounded we see that Theorem 2.12 (1) follows from Theorem 2.10 (1) and (2.6) provided we can show that A_2 is bounded. For that it is most convenient to assume that F_- is of the form $F_-=F_0$ for some $\varepsilon_0>0$.

We claim that indeed for ε_0 small enough, A_2 is bounded. To see this we use the calculus of Ps.D.Op.'s and Lemma 2.4 to reduce the problem to the following statements for fixed $\rho_1\in(\rho, \sqrt{1-\varepsilon})$ and $f_+\in\mathcal{F}^0$ with supp $f_+\subset(\rho_1\sqrt{a(\lambda)}, \infty)$.

$q(k)$: For any $\rho_2<\rho_1$, $p_-\in S_-^\lambda(\rho_2)$ and $P_{k/2}\in\mathcal{O}p^{k/2}(X)$, $P_-P_{k/2}f_+(B)$ is bounded.

So suppose $q(k-1)$. Then we need to verify $q(k)$. To the operator $B_1=B-\rho_2\sqrt{a(\lambda)}$ we associate the "symbol" $(x/\langle x\rangle)\cdot\xi-\rho_2\sqrt{a(\lambda)}$. This is motivated by the following formula valid on $\mathcal{D}(D)$,

$$B_1 = \frac{x}{\langle x\rangle}\cdot D - \rho_2\sqrt{a(\lambda)} + \frac{1}{\langle x\rangle} P_{1,0},$$

where $P_{1,0}=\sum_{|\alpha|\leq 1} v_\alpha D^\alpha$; $v_\alpha\in\mathcal{V}$.

Obviously the "symbol" is non-positive on supp p_- for $p_-\in S_-^\lambda(\rho_2)$. Furthermore $e^{-tB_1}f_+(B)$ is exponentially decreasing for $t\to+\infty$.

Let for given $p_-\in S_-^\lambda(\rho_2)$ and $P_{k/2}\in\mathcal{O}p^{k/2}(X)$

$$P(t) = Q(t)^*Q(t)\,;\; Q(t) = P_- P_{k/2} f_+(B) e^{-tB_1}.$$

We need to show that $P(0)$ is bounded.

By the identity $P(0)=-\int_0^\infty (d/dt)P(t)dt$ it is enough to show that

$$-\frac{d}{dt}P(t) \leqq P'(t),$$

where $\|P'(t)\|\leqq Ce^{-\varepsilon' t}$ for some $\varepsilon'>0$.

By $q(k-1)$ this will hold if

$$\mathrm{Re}\{P_{k/2}^{*}P_{-}^{*}P_{-}P_{k/2}B_{1}\} \leqq \sum_{\text{finite}} \mathrm{Re}\{P_{(k-1)/2}^{*}P_{-}^{\prime}P_{(k-1)/2}^{\prime}\}+C,$$

where $P_{(k-1)/2}$, $P_{(k-1)/2}^{\prime}$ and $P_{-}^{\prime}$ satisfy the conditions of $q(k-1)$.

Since $[P_{k/2}, B_{1}] \in \mathcal{O}p^{(k-2)/2}(X)$ it suffices to look at

$$\mathrm{Re}\{P_{k/2}^{*}P_{-}^{*}P_{-}B_{1}P_{k/2}\},$$

or by the same argument,

$$\mathrm{Re}\Big\{P_{k/2}^{*}P_{-}^{*}P_{-}\Big(\frac{x}{\langle x\rangle}\cdot D-\rho_{2}\sqrt{a(\lambda)}\Big)P_{k/2}\Big\}.$$

But by the Gårding inequality and the sign property mentioned, we can estimate

$$\mathrm{Re}\Big\{P_{-}^{*}P_{-}\Big(\frac{x}{\langle x\rangle}\cdot D-\rho_{2}\sqrt{a(\lambda)}\Big)\Big\} \leqq \langle x\rangle^{-1/2}P_{-}^{\prime}\langle x\rangle^{-1/2}+C\langle x\rangle^{-k}.$$

This completes the proof of $q(k)$, and hence of Theorem 2.12 (1).

The proof of the other statements of the theorem goes along the same line. □

REMARK. The proof of Theorem 2.12 can be extended to a more general symbol class than $S_{\mathrm{comp}}(\mathcal{X})$ in Definition 2.11 (see [**GIS**]). This fact plays a role in [**I4**] but not for the application given in the next section.

3. Geometrical resolvent estimates.

In this section we convert the statements of Theorem 2.12 into some more natural geometrical ones. By this we mean estimates that (converted to time decay estimates) reflect our expectation for disintegration of the motion into stable clusters moving freely in the remote future (cf. asymptotic completeness [**D**], [**Gr**]). As is natural these estimates involve pseudodifferential localization for the intercluster motion only, which of course put on restrictions on what regions in the configuration space should be considered. In addition to the Ps.P.Op.'s introduced in Section 1 we need the following notations for any cluster decomposition a.

$$\mathcal{S}=\{x \in \mathcal{X} \mid |x|=1\},$$

$$\mathcal{Y}_{a}=\mathcal{X}\setminus\bigcup_{b\not\subset a}\mathcal{X}_{b},\ \mathcal{Y}_{a}^{1}=\mathcal{Y}_{a}\cap\mathcal{S},$$

and for $\varepsilon>0$

$$\mathcal{Y}_{a,\varepsilon}=\mathcal{Y}_{a}\cap\{x \in \mathcal{X} \mid |x^{a}|<\varepsilon|x|\},\ \mathcal{Y}_{a,\varepsilon}^{1}=\mathcal{Y}_{a,\varepsilon}\cap\mathcal{S}.$$

For later use we state

LEMMA 3.1. *Let $\mathcal{K}\subset\mathcal{Y}_a^1$ be compact. Then for any $\varepsilon>0$ we can find compact sets $\mathcal{K}_b\subset\mathcal{Y}_{b,\varepsilon}^1$ for $b\subsetneqq a$ such that*

$$\mathcal{K}=(\mathcal{K}\cap\mathcal{Y}_{a,\varepsilon}^1)\cup\bigcup_{b\subsetneqq a}\mathcal{K}_b\,.$$

PROOF. Notice that the compact set $\mathcal{K}\cap\{x\,|\,|x^a|\geqq\varepsilon|x|\}$ is contained in the open covering $\bigcup_{b\subsetneqq a}\mathcal{Y}_{b,\varepsilon}^1$. Then the result follows from a simple compactness argument. □

We also need the following notations:

Smooth functions: $\mathcal{X}\to\boldsymbol{C}$ are denoted by $J_{a,\varepsilon}$ respectively J_a, if they are homogeneous of degree zero outside $\mathcal{S}$ and if

$$\operatorname{supp} J_{a,\varepsilon}\subset\mathcal{Y}_{a,\varepsilon}\quad\text{or}\quad\operatorname{supp} J_a\subset\mathcal{Y}_a\,.$$

Smooth functions: $\boldsymbol{R}\to\boldsymbol{R}$ are denoted by $F(t>C)$ or $F(t<C)$, if they are locally constant outside some compact set and if

$$\operatorname{supp} F(\cdot>C)\subset(C,\infty)\quad\text{or}\quad\operatorname{supp} F(\cdot<C)\subset(-\infty,C)\,.$$

Similarly to Definition 2.11 we introduce for $-1<\rho<1$, the class $S_-^\lambda(\rho,a)$ of symbols $p_-\in S(\mathcal{X}_a)$ such that

$$\operatorname{supp} p_-\subset\left\{(x_a,\xi_a)\,\middle|\,\frac{x_a}{\langle x_a\rangle}\cdot\xi_a\leqq\rho\sqrt{a(\lambda)}\right\},$$

and the class $S_+^\lambda(\rho,a)$ of symbols $p_+\in S(\mathcal{X}_a)$ such that

$$\operatorname{supp} p_+\subset\left\{(x_a,\xi_a)\,\middle|\,\frac{x_a}{\langle x_a\rangle}\cdot\xi_a\geqq\rho\sqrt{a(\lambda)}\right\}.$$

We introduce the sub-Hamiltonian

$$H^a=(D^a)^2+V^a\,;\ V^a=\sum_{(ij)\subset a}v_{ij}(x_i-x_j),$$

and

$$H_a=D^2+V^a=(D_a)^2+H^a,$$

$$R_a(z)=(H_a-z)^{-1}\,;\ z\notin\boldsymbol{R}\,.$$

Let $E_a=\inf\sigma(H_a)=\inf\sigma(H^a)$.

The following kind of result is well known.

LEMMA 3.2. *For given $\lambda\in\boldsymbol{R}$ and $F(\cdot>\lambda-E_a)$ there exists a neighborhood N_λ, such that for any $m\in\boldsymbol{R}$, J_a (as defined above) and $\varphi\in C_0^\infty(N_\lambda)$*

$$X^mJ_aF(D_a^2>\lambda-E_a)\varphi(H)X^m$$

is bounded.

PROOF. For fixed $\varepsilon>0$ and $m\geqq 0$ we will show the following statement by induction.

$q(k)$: For any $F(\cdot>\lambda-E_a+\varepsilon)$, J_a and $\varphi\in C_0^\infty((-\varepsilon+\lambda, \lambda+\varepsilon))$

$$X^{k\varepsilon_1-m}F(D_a^2>\lambda-E_a+\varepsilon)J_a\varphi(H)X^m$$

is bounded. (Here ε_1 is the constant of the Condition in Section 1.)

Given $q(k-1)$ we need to show $q(k)$. So let F, J_a and φ be given accordingly. Then we choose similar functions F', J'_a and φ' which are equal to one on the supports of their respective counterparts. Let $\tilde{\varphi}'\in C_0^\infty(\boldsymbol{C})$ be an almost analytic extension of φ'. For bounded operators B_1, B_2 we write $B_1\cong B_2$ if $X^{k\varepsilon_1-m}(B_1-B_2)X^m$ is bounded. Then, by $q(k-1)$ and the calculus of Ps.D.Op.'s,

$$\begin{aligned}
&F(D_a^2>\lambda-E_a+\varepsilon)J_a\varphi(H)\\
&\cong J_aFJ'_a\varphi = J_aFF'(H_a-E_a>\lambda-E_a+\varepsilon)J'_a\varphi\\
&=: J_aFF'J'_a\varphi\\
&\cong J_aFF'J'_aF_R\varphi'\varphi \text{ (with } F_R:=F(|x|>R)\text{; } R>0)\\
&=\frac{-1}{\pi}J_aFF'\int_C\partial_{\bar z}\tilde{\varphi}'(z)R_a(z)\\
&\quad\cdot\{J'_aF_R(V-V^a)+[J'_aF_R, D^2]\}R(z)\varphi dudv \text{ (with } z=u+iv)\\
&\cong\frac{-1}{\pi}J_a\int_C\partial_{\bar z}\tilde{\varphi}'(z)R_a(z)F(V-V^a)F_RJ'_a\varphi R(z)dudv\\
&\cong 0 \text{ (for } R \text{ large and by a commutation).}
\end{aligned}$$

□

The key for applying Theorem 2.12 is the following technical result.

LEMMA 3.3. *Let $\rho_a<\rho<1$ and $\delta>0$ be given. There exists $\varepsilon>0$ such that the following statement holds.*

For any (fixed) $\lambda>E_a$ with $a(\lambda)>\delta\langle\lambda\rangle$ and function $F(t<2\langle\lambda\rangle)$, we let $F_\lambda=F(H<2\langle\lambda\rangle)$ and introduce the notation

$$Q_m = X^mP_a^-J_{a,\varepsilon}F_\lambda$$

for any operator of this form for $m\in\boldsymbol{R}$, $p_a^-\in S_-^\lambda(\rho_a, a)$ and $J_{a,\varepsilon}$ as given above.

For any $N>0$ we can write (any) Q_m as a finite sum of terms, each one being of one of the following three forms:

$$\begin{aligned}
A_1 &= TX^mP_-F_\lambda\,;\ p_-\in S_-^\lambda(\rho),\\
A_2 &= TX^{-N}F_\lambda\,,\\
A_3 &= \int_C T(z)Q_{m-\varepsilon_1}R(z)dudv\,;\ z=u+iv\,.
\end{aligned}$$

The T's are bounded operators and for A_3, $T(\cdot)\in C_0^0(\boldsymbol{C}, \mathcal{B}(\mathcal{H}))$ such that moreover for all $M>0$

$$\|T(z)\| \leqq C_M |\operatorname{Im} z|^M.$$

REMARK. By some more work one can put $A_3=0$.

PROOF. We have to explain how to choose ε. Let F_1 denote a function $F_1(t)=F(t<1)$ with $F_1(t)=1$ for $t<1/2$. We claim that for $C>0$ large enough, and independently of λ, the operator

$$T = T(C, \lambda) = \Big(I-F_1\Big(\frac{D^2}{C\langle\lambda\rangle}\Big)\Big)F_1\Big(\frac{H_a}{5\langle\lambda\rangle}\Big)$$

satisfies the bound $\|T\|<1/2$.

To see this we let $f\in C_0^\infty(\boldsymbol{R})$ be given such that $f(t)=1$ for $t\in[E_a, 5]$ and $\tilde{f}\in C_0^\infty(\boldsymbol{C})$ be an almost analytic extension. Then for C large we can insert $f(H_a/\langle\lambda\rangle)-f(D^2/\langle\lambda\rangle)$ in the middle of the two factors in the definition of T and apply (2.2). This leads to

$$T = \frac{-1}{\pi}\int_C \partial_{\bar z}\tilde{f}(z)\Big(I-F_1\Big(\frac{D^2}{C\langle\lambda\rangle}\Big)\Big)\Big(\frac{D^2}{\langle\lambda\rangle}-z\Big)^{-1}\frac{V^a}{\langle\lambda\rangle}\Big(\frac{H_a}{\langle\lambda\rangle}-z\Big)^{-1}F_1\Big(\frac{H_a}{5\langle\lambda\rangle}\Big)du\,dv\,.$$

By the relative boundedness of the potential we have uniformly in $\langle\lambda\rangle$ and $z\in\operatorname{supp}\partial_{\bar z}\tilde{f}$

$$\Big\|\frac{V^a}{\langle\lambda\rangle}\Big(\frac{H_a}{\langle\lambda\rangle}-z\Big)^{-1}\Big\| \leqq C_1|\operatorname{Im} z|^{-1}$$

and

$$\Big\|\Big(I-F_1\Big(\frac{D^2}{C\langle\lambda\rangle}\Big)\Big)\Big(\frac{D^2}{\langle\lambda\rangle}-z\Big)^{-1}\Big\| \leqq \frac{C_1}{C}\,.$$

So $T=O(1/C)$ uniformly in $\langle\lambda\rangle$. We fix C such that $\|T\|<1/2$ and choose $\varepsilon>0$ such that uniformly in λ (with $a(\lambda)>\delta\langle\lambda\rangle$)

$$2\varepsilon\sqrt{C\langle\lambda\rangle} \leqq (\rho-\rho_a)\sqrt{a(\lambda)}\,.$$

Under this condition it follows from the calculus of Ps.D.Op.'s that for any $m\in\boldsymbol{R}$ and $p_{\bar a}\in S_-^\lambda(\rho_a, a)$, the operator

$$S = F_1\Big(\frac{D^2}{C\langle\lambda\rangle}\Big)X^m P_{\bar a} F_1\Big(\frac{|x^a|}{|x|2\varepsilon}\Big)$$

has the form $S=X^m P_- + T'X^{-N}$ for any given N and with $p_-\in S_-^\lambda(\rho)$ and T' bounded.

Now, let any N and Q_m be given. Then using the operators S and T introduced above we can write

$$(I-T)Q_m = SJ_{a,\varepsilon}F_\lambda+\Big(I-F_1\Big(\frac{D^2}{C\langle\lambda\rangle}\Big)\Big)\tilde{Q}_m,$$

where

$$\tilde{Q}_m = \Big(I-F_1\Big(\frac{H_a}{5\langle\lambda\rangle}\Big)\Big)Q_m.$$

By applying the inverse of $(I-T)$ on both sides it suffices to look at $\tilde{Q}_m$. We choose $f\in C_0^\infty((-\infty, 5\langle\lambda\rangle/2))$ such that $f(t)=1$ for $t\in[\inf\sigma(H), 2\langle\lambda\rangle]$ and let $\tilde{f}\in C_0^\infty(\boldsymbol{C})$ be an almost analytic extension.

Then (cf. a similar formula in the first part of the proof)

$$\tilde{Q}_m = \frac{-1}{\pi}\int_C \partial_{\bar{z}}\tilde{f}(z)\Big(I-F_1\Big(\frac{H_a}{5\langle\lambda\rangle}\Big)\Big)R_a(z)(S_1+S_2)R(z)F_\lambda\,du\,dv, \tag{3.1}$$

where

$$S_1 = X^m P_a^- J_{a,\varepsilon}(V-V^a),$$

and

$$S_2 = [X^m P_a^- J_{a,\varepsilon},\ D^2].$$

We can write S_1+S_2 as a finite sum of operators of the form

$$P_{2,0}(X^{m-\varepsilon_1}P_a^{-\prime}J'_{a,\varepsilon}+T'X^{-N}), \tag{3.2}$$

where $P_{2,0}=\sum_{|\alpha|\leq 2}v_\alpha D^\alpha$ for $v_\alpha\in\mathcal{V}$ (i.e., the class of functions introduced in the beginning of Section 2) and the prime indicates an operator of same type. (For T', just a bounded operator.)

By using the bounds

$$\|R_a(z)P_{2,0}\| \leqq \frac{C_1}{|\mathrm{Im}\, z|}$$

and

$$\|X^{-N}R(z)X^N\| \leqq \frac{C_1}{|\mathrm{Im}\, z|^{N+2}}$$

(cf. the proof of Lemma 2.3), we are done by inserting the expressions (3.2) into the integrand on the right hand side of (3.1). □

REMARK 3.4. A similar result holds for P_a^- (and P_- in the conclusion) replaced by P_a^+ (and P_+ in the conclusion) upon replacing the condition $\rho_a<\rho<1$ by $-1<\rho<\rho_a$.

THEOREM 3.5. *Let a be an arbitrary cluster decomposition, and $\lambda\in(E_a, \infty)\setminus\Lambda$. Then*

(1) *For any $\rho_a<1$ there exist $\varepsilon>0$ and a neighborhood N_λ of λ: $\forall l\in\boldsymbol{N}$, $s'>s>l-1/2$, $p_-\in S_-^\lambda(\rho_a, a)$ and $J_{a,\varepsilon}$*

$$\|X^{s-l}P_-J_{a,\varepsilon}R(z)^l X^{-s'}\| \leqq C,$$

uniformly in $\operatorname{Re} z \in N_\lambda$ *and* $\operatorname{Im} z > 0$.

(2) *For any* $-1 < \rho_a$ *there exist* $\varepsilon > 0$ *and a neighborhood* N_λ : $\forall l \in \boldsymbol{N}$, $s' > s > l - 1/2$, $p_+ \in S_+^\lambda(\rho_a, a)$ *and* $J_{a,\varepsilon}$

$$\|X^{s-l} P_+ J_{a,\varepsilon} R(z)^l X^{-s'}\| \leqq C ,$$

uniformly in $\operatorname{Re} z \in N_\lambda$ *and* $\operatorname{Im} z < 0$.

(3) *Suppose* b *is another cluster decomposition and that* $\lambda > E_b$. *Then for any* $-1 < \rho_a < \rho_b < 1$ *there exist* $\varepsilon > 0$ *and a neighborhood* N_λ : $\forall l \in \boldsymbol{N}$, $s \in \boldsymbol{R}$, $p_- \in S_-^\lambda(\rho_a, a)$, $p_+ \in S_+^\lambda(\rho_b, b)$, $J_{a,\varepsilon}$, $J_{b,\varepsilon}$ *and* $F_\lambda := F(H < 2\langle\lambda\rangle)$

$$\|X^s P_- J_{a,\varepsilon} F_\lambda R(z)^l J_{b,\varepsilon} P_+ X^s\| \leqq C$$

uniformly in $\operatorname{Re} z \in N_\lambda$ *and* $\operatorname{Im} z > 0$.

PROOF. The idea is to reduce to Theorem 2.12 by use of Lemma 3.3 (and Remark 3.4).

To see how (1) follows from Theorem 2.12 (1) and Lemma 3.3 we choose for fixed $\rho_a < 1$ a ρ such that $\rho_a < \rho < 1$. Also we choose $\delta > 0$ small enough to assure $a(\lambda) > \delta\langle\lambda\rangle$. We determine ε by using these numbers as input in Lemma 3.3. We let $F(t < 2\langle\lambda\rangle)$ be equal to one in a neighborhood of λ, say N_λ'', and let N_λ' be the neighborhood in Theorem 2.12 (1). Then we claim that the statement (1) holds for any neighborhood N_λ with the property that its closure is a subset of the interior of $N_\lambda' \cap N_\lambda'' \cap \Lambda^c$. To see this we fix any such set, and insert (for given l, s', s, p_- and $J_{a,\varepsilon}$) $I = F_\lambda + (I - F_\lambda)$ in front of the power of the resolvent; $F_\lambda := F(H < 2\langle\lambda\rangle)$. Only the contribution from the first term requires attention. With notation of Lemma 3.3 we have to bound

$$\|Q_{s-1} R(z)^l X^{-s'}\| \leqq C ,$$

uniformly in $\operatorname{Re} z \in N_\lambda$ and $\operatorname{Im} z > 0$. For that we iterate the statement of Lemma 3.3 with $N \geqq s'$ to obtain, using the notation A for operators of the form A_1 or A_2, that Q_{s-l} is a finite sum of operators of the form A or (for some $J \in \boldsymbol{N}$)

$$\int_C \cdots \int_C T_1(z_1) \cdots T_j(z_j) A R(z_1) \cdots R(z_j) du_1 dv_1 \cdots du_j dv_j ; \; 1 \leqq j \leqq J .$$

We can then bound each of these when applied to the power of the resolvent (times weight) by removing F_λ in the definition of A_1 and A_2. Here we use the above argument in reverse order. We are then left with operators that can easily by estimated on $N_\lambda' \cap \Lambda^c$ (and hence on N_λ) by Theorem 2.12 (1) (and the limiting absorption type bound (2.6)). This completes the proof of (1).

The proofs of (2) and (3) are similar, although as for the proof of (3) one needs (for some terms) a slight extension of Theorem 2.12 (3) which easily follows by its proof. □

REMARKS.

(1) In the stated form Theorem 3.5 is useful in the study of regularity of scattering amplitudes (cf. recent papers by Bommier [**B**], Isozaki [**I2**] and Skibsted [**S1**]). For some related problems we refer to [**I3**], [**HSk**].

(2) Theorem 3.5 might not be the best general result of this type. For the applications mentioned above and other purposes (including conceptual ones) it would be interesting if one could improve the constant $\sqrt{a(\lambda)}$ in the definition of p_- and p_+. In particular it would be a very powerful result (in our opinion) if this constant could be replaced by the optimal one, $|\xi_a|$. We discuss partial results below.

For $\mathcal{M}\subset\mathcal{S}$ open and with $\hat{x}:=x/|x|$ we use the notation $\chi_{\mathcal{M}}$ for operators of multiplication by the characteristic function $1_{\mathcal{M}}(\hat{x})$, and introduce the following strong notion of non-propagation. For simplicity we restrict ourselves to one-sided estimates in the upper half plane only (similar to those of Theorem 3.5 (1)).

DEFINITION 3.6. For $\sigma>0$ the notation $S_-(\sigma, a)$ stands for the class of symbols $p_-\in S(\mathcal{X}_a)$ such that

$$\operatorname{supp} p_- \subset \left\{(x_a, \xi_a)\left|\frac{x_a}{\langle x_a\rangle}\cdot\xi_a \leqq (1-\sigma)|\xi_a|,\ \sigma \leqq |\xi_a|\right.\right\}.$$

A pair $(\lambda, \theta)\in((E_a, \infty)\setminus\Lambda)\times\mathcal{Y}_a^1$ is said to be a-regular if for any $\sigma>0$ there exist open neighborhoods $N\subset\boldsymbol{R}$ and $\mathcal{M}\subset\mathcal{S}$ of λ and θ, respectively:

$$\forall l\in\boldsymbol{N},\ s'>s>l-\frac{1}{2} \quad\text{and}\quad p_-\in S_-(\sigma, a)$$
$$\|X^{s-l}\chi_{\mathcal{M}}P_-R(z)^lX^{-s'}\|\leqq C,$$

uniformly in $\operatorname{Re} z\in N$ and $\operatorname{Im} z>0$.

The set of a-regular points is denoted by $\mathcal{R}_a$. If $((E_a, \infty)\setminus\Lambda)\times\mathcal{Y}_a^1=\mathcal{R}_a$ the cluster decomposition a is regular.

REMARK. There is a time-dependent equivalent to the introduced notion of "a-regularity". In particular (by velocity estimates cf. [**SS**], [**S2**]) one obtains an equivalent notion by fixing $s=l$ (but not l and $s'>l-1/2$).

As shown by Perry [**P**], the set $\{E>E_a|(E_a, E)\cap\mathcal{T}=\emptyset\}$ is non-empty. We denote its supremum by E_a'.

THEOREM 3.7.

$$((E_a, E_a')\setminus\sigma_{pp}(H))\times\mathcal{Y}_a^1\subset\mathcal{R}_a.$$

PROOF. We proceed by induction in $\#a$ starting from $\#a=N$ in which case the result follows easily from Lemma 3.2, Theorem 3.5 (1) and (2.6). Now, suppose the result is known for $\#b\geqq N-k+1$, then we have to show it for an arbitrarily given a with $\#a=N-k$. So let $(\lambda, \theta)\in(E_a, E_a')\times\mathcal{Y}_a^1$ and $(2>)\sigma>0$ be given. Suppose $\theta\in\mathcal{Y}_a^1\cap\mathcal{X}_a$ then we are done by the same arguments as for the case $\#a=N$. If $\theta\in\mathcal{Y}_a^1\backslash\mathcal{X}_a$ we need to specify the neighborhoods N and $\mathcal{M}$. As for $\mathcal{M}$ we choose an arbitrary compact neighborhood $\mathcal{K}\subset\mathcal{Y}_a^1\backslash\mathcal{X}_a$ and let $\mathcal{M}$ be the interior of $\mathcal{K}$. As for N we choose for $(1>)\varepsilon>0$ small and for $b\subsetneqq a$, a compact set $\mathcal{K}_b\subset\mathcal{Y}_{b,\varepsilon}^1$, such that $\mathcal{K}\subset\bigcup_{b\subsetneqq a}\mathcal{K}_b$. Here we used Lemma 3.1. In the sequel these sets are fixed. The idea is now to exploit the induction hypothesis on $\mathcal{K}_b$ if $\lambda>E_b$. (In this case $E_b=E_a$.) Notice that in conjunction with a compactness argument it gives the following bounds for any given $\sigma_b>0$ and for some neighborhoods N_b' and $\mathcal{M}_b$ of λ and $\mathcal{K}_b$, respectively:

$$\begin{aligned}&\forall l\in \boldsymbol{N},\ s'>s>l-\frac{1}{2}\quad\text{and}\quad p_b^-\in S_-(\sigma_b, b)\\ &\|X^{s-l}\chi_{\mathcal{M}_b}P_b^-R(z)^lX^{-s'}\|\leqq C,\end{aligned}\tag{3.3}$$

uniformly in $\operatorname{Re} z\in N_b'$ and $\operatorname{Im} z>0$. We can assume that $\mathcal{M}_b\subset\mathcal{Y}_{b,\varepsilon}^1$.

In order to choose σ_b we first prove that on the support of any $p_-\in S_-(\sigma, a)$

$$\frac{x_b}{\langle x_b\rangle}\cdot\xi_b\leqq|\xi_b|\Big(1-\sigma^3\Big(1-\frac{\sigma}{2}\Big)|\xi_b|^{-2}\Big)\quad\text{and}\quad\sigma\leqq|\xi_b|.\tag{3.4}$$

To prove (3.4), since we are going to treat vectors x_b, we regard $\mathcal{X}_b$ as $\mathcal{X}$ and drop the subscript b. Then on the support of any $p_-\in S_-(\sigma, a)$

$$\sigma\,(\leqq|\xi_a|)\leqq|\xi|$$

and

$$\begin{aligned}x\cdot\xi&=x_a\cdot\xi_a+x^a\cdot\xi^a\\ &\leqq(1-\sigma)\langle x_a\rangle|\xi_a|+|x^a||\xi^a|\ \text{(by the support property)}\\ &\leqq\langle x\rangle((1-\sigma)^2|\xi_a|^2+|\xi^a|^2)^{1/2}\ \text{(by the Cauchy Schwarz inequality)}\\ &=\langle x\rangle|\xi|\Big(1-(1-(1-\sigma)^2)\frac{|\xi_a|^2}{|\xi|^2}\Big)^{1/2}\ \text{(by orthogonality)}\\ &\leqq\langle x\rangle|\xi|(1-2\eta)^{1/2};\ \eta=\sigma^3\Big(1-\frac{\sigma}{2}\Big)|\xi|^{-2}\ \text{(by the support property)}\\ &\leqq\langle x\rangle|\xi|(1-\eta).\end{aligned}$$

We have proved (3.4).

Motivated by (3.4) we choose some $\sigma_b<\min(\sigma^3(1-\sigma/2)(\lambda-E_b)^{-1}, \sigma)$. This choice and the calculus allows the construction of $p_b^-\in S_-(\sigma_b, b)$ and $F(\cdot>\lambda-E_b)$ such that for any $p_-\in S_-(\sigma, a)$

$$P_{-\infty}:=P_-(I-P_b^-)(I-F(D_b^2>\lambda-E_b))\in S^0_{-\infty}(\mathcal{X}_b)=\bigcap_l S^0_l(\mathcal{X}_b).$$

According to Lemma 3.2 there is associated to λ and this $F(\cdot>\lambda-E_b)$ an open neighborhood of λ which we denote by N_b''.

Now to the choice of the neighborhood N of λ. We claim it can be taken as any neighborhood with the property that its closure is contained in

$$(\bigcap_{\substack{b\subsetneq a:\\ \lambda>E_b}} N_b'\cap N_b'')\cap\Lambda^c\cap\bigcap_{\substack{b\subsetneq a:\\ \lambda<E_b}}(-\infty,\ E_b).$$

To see this we fix such N, and let $l\in\boldsymbol{N}$, $s'>s>l-1/2$ and $p_-\in S_-(\sigma,\ a)$ be given. Then with the notations above it suffices to estimate

$$\|X^{s-l}1_{\mathcal{K}_b}(\hat{x})P_-R(z)^lX^{-s'}\|\leqq C$$

for $\operatorname{Re} z\in N$ and $\operatorname{Im} z>0$.

By an application of Lemma 3.2 we only need to deal with the cases $\lambda>E_b$. But for such b we can decompose $P_-=P_-P_b^-+P_-(I-P_b^-)F(D_b^2>\lambda-E_b)+P_{-\infty}$, and estimate separately. Only the first two terms requires attention. For that we pick $J_{b,\varepsilon}$ such that $1_{\mathcal{K}_b}(\hat{x})=1_{\mathcal{K}_b}(\hat{x})J_{b,\varepsilon}(x)$ outside $\mathcal{S}$ and $\mathcal{S}\cap\operatorname{supp}J_{b,\varepsilon}\subset\mathcal{M}_b$. Then by inserting we can write

$$X^{s-l}1_{\mathcal{K}_b}(\hat{x})P_-P_b^-=T_1X^{s-l}\chi_{\mathcal{M}_b}P_b^-+T_2X^{-s'}\ ;\ T_j \text{ bounded.}$$

By (3.3) this gives the estimate for the first term. To deal with the second term we pick $\varphi\in C_0^\infty(N_b'')$ such that $\varphi=1$ on a neighborhood of N. Then it is enough to estimate

$$X^{s-l}J_{b,\varepsilon}P_-(I-P_b^-)F(D_b^2>\lambda-E_b)\varphi(H)R(z)^lX^{-s'}.$$

But by the calculus $X^{s-l}J_{b,\varepsilon}P_-(I-P_b^-)$ can be written as a finite sum of terms of the form $T_1X^{s-l}J_{b,\varepsilon}'$ plus $T_2X^{-s'}$, where $J_{b,\varepsilon}'$ is of the same type and the T's are bounded. By Lemma 3.2 $X^{s-l}J_{b,\varepsilon}'F(D_b^2>\lambda-E_b)\varphi(H)X^{s'}$ is bounded. So (again) we have reduced to (2.6). $\square$

It is clear that unless $H^a\geqq 0$ (a condition that always holds for the free channel $a=(1)\cdots(N)$), then the statement of Theorem 3.7 is a statement for very low energies only. For high energies there are partial results as it follows from the following theorem (cf. [**Ge**], [**I2**] and [**W1**]).

THEOREM 3.8. *Suppose* $(0,\ \infty)\cap\Lambda=\emptyset$. *Then*

(1) *For* $N=3$ *all* a *are regular.*

(2) *If* $\#a=N-1$ *and* $d_a(\cdot)$ *denotes the distance function to* $\bigcup_{b\not\subset a}\mathcal{X}_b$, *then*

$$\left\{(\lambda,\ \theta)\in((E_a,\ \infty)\backslash\Lambda)\times\mathcal{S}_a^1\,\middle|\,\left(1-\frac{d_a(\theta)^2}{8}\right)\left(\frac{\lambda-E_a}{a(\lambda)}\right)^{1/2}<1\right\}\subset\mathcal{R}_a.$$

(3) *For any compact* $\mathcal{K}\subset\mathcal{Y}_a^1$ *there exists* $E>0$ *such that*

$$[E, \infty)\times\mathcal{K} \subset \mathcal{R}_a.$$

In particular if $\#a=2$, *then* $[E, \infty)\times\mathcal{Y}_{a,\varepsilon}^1 \subset \mathcal{R}_a$ *for some* E, $\varepsilon>0$.

We shall give a brief outline of a proof of Theorem 3.8. The idea is in all cases the same namely to use Theorem 3.5 (1) to localize the following "two-body observable" to the region $\mathcal{Y}_{a,2\varepsilon}$ for $\varepsilon>0$ small. We consider the real part of Ps.D.Op.'s on $\mathcal{X}_a$ that have symbols of the form

$$-(\langle x_a\rangle|\xi_a|-x_a\cdot\xi_a)^m F\Big(\frac{x_a}{\langle x_a\rangle}\cdot\frac{\xi_a}{|\xi_a|}<1-\frac{\sigma}{2}\Big)F\Big(|\xi_a|>\frac{\sigma}{2}\Big);\ m>0.$$

By a somewhat similar computation to the one in the proof of Lemma 2.6 we see that to "first order" these observables are non-positive with a non-negative Heisenberg derivative (when localized to $\mathcal{Y}_a$).

To explain how to localize to the region $\mathcal{Y}_{a,2\varepsilon}$ we introduce the following notations:

$$\mathcal{S}_a=\mathcal{S}\cap\mathcal{X}_a,\quad \mathcal{Z}_a^1=\mathcal{Y}_a\cap\mathcal{S}_a.$$

For given compact $\mathcal{K}_a\subset\mathcal{Z}_a^1$

$$j_{a,1},\ j_{a,2}:\mathcal{S}_a\to[0,1]$$

denote smooth functions supported in $\mathcal{Z}_a^1$ and with the properties that $j_{a,2}=1$ on a neighborhood of $\mathcal{K}_a$ and $j_{a,1}=1$ on a neighborhood of $\operatorname{supp} j_{a,2}$.

Now we multiply the previous symbols by the "localization factor"

$$j_{a,1}\Big(\frac{x_a}{|x_a|}\Big)j_{a,2}\Big(\frac{\xi_a}{|\xi_a|}\Big)F\Big(\frac{|x^a|}{|x|}<2\varepsilon\Big)F(|x|>R).$$

In this way we obtain symbols on $\mathcal{X}$ that as a function of x is supported in $\mathcal{Y}_{a,2\varepsilon}$. The parameter R is a large positive constant introduced only for accomodating local singularities of the potentials.

After symmetrizing the above symbols we go through the same scheme as in the proof of Lemma 2.7 starting with small m. For that we need to control terms containing derivatives of the "localization factor" with respect to x. One term is supported in the free channel region (in case of (1) or (2)) so that Theorem 3.7 can be applied, or in a region where an induction hypothesis (only for (3), see below) can be applied. Another term comes from differentiating $j_{a,1}$. However by the properties of $j_{a,1}$ and $j_{a,2}$ the resulting symbol has the form needed for applying Theorem 3.5 (1).

Given the estimates resulting from the above described procedure we can obtain the statement of the theorem for points θ near $\mathcal{K}_a$ by removing the factor $j_{a,2}$. This is possible by another application of Theorem 3.5 (1).

We now briefly discuss the three statements of the theorem separately.

Since $\mathcal{Z}_a^1=\mathcal{S}_a$ if $\#a=2$ we can choose $j_{a,1}=j_{a,2}=1$ in this case. So the statement (1) follows from the known result in the free channel region.

As for (2), the set is found by optimizing the choice of $j_{a,1}$ and $j_{a,2}$ (given a one-point set $\mathcal{K}_a$). The distance function d_a should for that purpose most conveniently be replaced by the quasi-distance function $\tilde{d}_a(\theta_1, \theta_2)=1-\theta_1\cdot\theta_2=(d_a(\theta_1, \theta_2))^2/2$.

As for (3) we proceed by induction with respect to $\#a$ as in the proof of Theorem 3.7. As was the case for that proof we shall use Lemmas 3.1 and 3.2 and Theorem 3.5 (1). In addition we need the observation that the proof of Theorem 3.5 (1) shows that we can choose ε in the statement independent of λ.

So suppose $\mathcal{K}\subset\mathcal{Y}_a^1$ is compact. Then we pick a compact $\mathcal{K}_a\subset\mathcal{Z}_a^1$ such that for all small enough $\varepsilon>0$

$$\mathcal{K}\cap\mathcal{Y}_{a,\varepsilon}^1\subset\left\{\theta\in\mathcal{S}\,|\,|\theta^a|\leqq\varepsilon,\ \frac{\theta_a}{|\theta_a|}\in\mathcal{K}_a\right\}\subset\mathcal{Y}_a^1.$$

Next we choose functions $j_{a,1}$ and $j_{a,2}$ as discribed above. Clearly for all small enough $\varepsilon>0$

$$\left\{\theta\in\mathcal{S}\,|\,|\theta^a|\leqq 2\varepsilon,\ \frac{\theta_a}{|\theta_a|}\in\mathcal{K}_a'\right\}\subset\mathcal{Y}_a^1\,;\ \mathcal{K}_a'=\operatorname{supp} j_{a,1}.$$

As outlined we need two applications of Theorem 3.5 (1) with some ρ_a's given by the properties of $\mathcal{K}_a$, $j_{a,1}$ and $j_{a,2}$. In accordance with these inputs we fix a small $\varepsilon>0$ (independently of λ). We assume that the function $F(t<2\varepsilon)$ appearing in the "localization factor" obeys $F(t<2\varepsilon)=1$ for $t\leqq\varepsilon$. Using Lemma 3.1 we decompose

$$\mathcal{K}=(\mathcal{K}\cap\mathcal{Y}_{a,\varepsilon}^1)\cup\bigcup_{b\subsetneqq a}\mathcal{K}_b\,;\ \mathcal{K}_b\subset\mathcal{Y}_{b,\varepsilon}^1,$$

and

$$\left\{\theta\in\mathcal{S}\,|\,\varepsilon\leqq|\theta^a|\leqq 2\varepsilon,\ \frac{\theta_a}{|\theta_a|}\in\mathcal{K}_a'\right\}=\bigcup_{b\subsetneqq a}\mathcal{K}_b'\,;\ \mathcal{K}_b'\subset\mathcal{Y}_{b,\varepsilon}^1.$$

For the compact sets $\mathcal{K}_b\cup\mathcal{K}_b'$ $(\subset\mathcal{Y}_b^1)$ the induction hypothesis applies (cf. the proof of Theorem 3.7). For the set $\mathcal{K}\cap\mathcal{Y}_{a,\varepsilon}^1$ we apply the described observables to get the estimates. □

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