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Metastable behaviors of diffusion processes with small parameter

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1. Introduction.

Let $\mathcal{M}$ be an orientable σ-compact d-dimensional Riemannian manifold of class C^{∞} with Riemannian metric $g=(g_{ij})$. Suppose a potential function $U\in C^{\infty}(\mathcal{M})$ is given and consider a second order differential operator $\mathcal{L}^{\varepsilon}$ on $\mathcal{M}$ defined by

$$\mathcal{L}^{\varepsilon}=\frac{\varepsilon^{2}}{2}\Delta-\frac{1}{2}\operatorname{grad}U, \qquad \varepsilon>0, \tag{1.1}$$

where Δ is the Laplace-Beltrami operator on $\mathcal{M}$ and grad means the Riemannian gradient. This paper is concerned with metastable behaviors of the diffusion process $(x_{t}^{\varepsilon}, P_{x})$ generated by $\mathcal{L}^{\varepsilon}$ on the space $\mathcal{M}$. Namely, we shall show that, for an appropriate choice of α_{ε}, the finite-dimensional distributions of the scaled process $\{x_{t\alpha_{\varepsilon}}^{\varepsilon}\}$ converge as $\varepsilon\downarrow 0$ to those of a Markov jump process living on the bottom $N=\{U=0\}$ of the potential. The results will be stated precisely in Section 2.

The metastable behaviors of diffusion processes have been studied by several authors, while all of them concern the diffusions on the Euclidean space with a double-well patential whose heights of the local minima are different from each other. For the one-dimensional Euclidean space, Kipnis and Newman [**10**] took up this problem and Ogura [**13**] solved it completely. Galves, Olivieri and Vares [**6**] considered the multi-dimensional case and used some smoothing and breaking procedure to obtain weak convergence on the path space, but the convergence of finite-dimensional distributions like this paper does not follow from their results.

There is a problem to be solved before establishing the metastable behaviors: namely, it should be determined the asymptotic behavior as $\varepsilon\downarrow 0$ of the first exit time

$$\tau_{G}^{\varepsilon}=\inf\{t>0\,;\,x_{t}^{\varepsilon}\notin G\} \tag{1.2}$$

of x_{t}^{ε} from a domain $G\subset\mathcal{M}$.

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The asymptotics of probabilities or expectations concerning exit time and exit position in the form e^{C_0/ε^2} as $\varepsilon\downarrow 0$ (rough asymptotics, i.e., up to logarithmic equivalence) are known via the Wentzell-Freidlin theory [4]. One can see that, for diffusions of gradient type, the rate C_0 is expressed by the difference between two values of the potential U at the critical points and the boundary points in our previous papers [17], [18]. For instance, if the domain G contains several components N of the bottom $\{U=0\}$, the rate C_0 of the mean exit time $E_x[\tau_G^\varepsilon]$, $x\in N$, coincides with the following quantity:

$$V_0=\min_{\phi}\max_{t\in[0,1]}U(\phi(t)),$$

where the minimum is taken over all trajectories $\phi\in C([0,1],\bar{G})$ such that $\phi(0)\in N$ and $\phi(1)\in\partial G$. However, the investigation of our model requires sharp asymptotics, i.e., those of the form $C_1\varepsilon^{\mu}e^{C_0/\varepsilon^2}$ up to equivalence. We are particularly interested in those of the mean exit time $E_x[\tau_G^\varepsilon]$ and of the distribution $P_x(x^\varepsilon_{\tau_G^\varepsilon}\in A)$, $A\subset\partial G$, of the exit position from the boundary in case that the bounded domain G contains exactly one component of the bottom. The latter problem has been considered formally by Matkowsky and Schuss [12], [15] and more rigorously by Kamin [9], although the arguments were restricted to the globally attractive cases. In this paper, we shall use a different approach. Namely, we shall consider asymptotics of the principal eigenvalue λ^ε and eigenfunction φ^ε for the Dirichlet boundary value problems in a bounded domain Ω:

$$\mathcal{L}^\varepsilon\varphi+\lambda\varphi=0 \quad \text{in } \Omega \qquad \text{with } \varphi=0 \quad \text{on } \partial\Omega\,. \tag{1.3}$$

Indeed, Friedman [5] gave a basic relation between the principal eigenvalue and the first exit time and it was proved in the papers [19], [1], [18] that $\lambda^\varepsilon E_x[\tau_\Omega^\varepsilon]$ converges to 1 as $\varepsilon\downarrow 0$ uniformly in x belonging to some subdomain of Ω. Hence, the asymptotics of the mean exit time must follow immediately from those of the principal eigenvalue. We shall also obtain sharp asymptotics of the distribution of the exit position from the boundary by using those of the principal eigenfunction together with the Wentzell-Freidlin $\{\partial G\}$-graph in Section 4.

In order to calculate the asymptotic behaviors of the principal eigenvalues and eigenfunctions, we mainly use the Rayleigh-Ritz formula and the Fermi coordinate. To this end, we need to suppose that all connected components of the bottom N and the compacta M in $\{U=V_0\}$ are submanifolds of $\mathcal{M}$. In Section 3, the limit of $\varepsilon^{\mu}e^{V_0/\varepsilon^2}\lambda^\varepsilon$ as $\varepsilon\downarrow 0$ is obtained for a suitable rate μ determined by the dimensions of N and M. In fact, it is represented by means of a variational formula, each term of which is written by the Hessian of the potential U. For the proof, we shall use the fact that φ^ε converges to a constant as $\varepsilon\downarrow 0$ on each valley, i.e., connected components of $\{U<V_0\}$ and see

that limits of φ^{ε} on the valleys attain the minimum of the variational formula. The key is Lemma 3.7, where the fast behaviors of φ^{ε} in a tubular neighborhood of the compactum on $\boldsymbol{M}$ will be estimated from below by the slow ones in the valleys and the Hessian of the potential U.

It is known that $\{\lambda^{\varepsilon}\tau_G^{\varepsilon}\}_{\varepsilon>0}$ or $\{\tau_G^{\varepsilon}/E_x[\tau_G^{\varepsilon}]\}_{\varepsilon>0}$ converges in distribution to an exponential random variable by several authors [**15**], [**19**], [**1**], [**6**], [**11**], [**18**]. One may infer from this that the limit of the scaled diffusion process $\{x_{t\alpha_\varepsilon}^{\varepsilon}\}$ might be a Markov jump process. In addition, it is natural that the limit process should be living on the bottom $\boldsymbol{N}$ among all the local minima of the potential U by rough asymptotics of the first exit time. Actually, there are our main results; precise statements are given in Section 2. Section 5 will be devoted to the proof of the convergence of the scaled process, where we shall use the above properties together with the results in Section 4.

2. Description of the model and statement of the main results.

Assume the potential $U\in C^{\infty}(\mathcal{M})$ satisfies the following conditions:

(C_1) the set $\{x\in\mathcal{M}; U(x)\leqq a\}$ is compact in $\mathcal{M}$ for all $a\geqq 0$ and $\min_{x\in\mathcal{M}} U(x)=0$;

(C_2) for each $a\geqq 0$, the set $\{x\in\boldsymbol{K}; U(x)\leqq a\}$ consists of finite number of connected components $\{K_i\}$ (each of which is called compactum) such that, for any two points $x, y\in K_i$, there is an absolutely continuous function $\phi\in C_{01}^{x,y}(K_i)$ satisfying $\int_0^1\|\dot{\phi}(t)\|^2dt<\infty$.

Here we write $\boldsymbol{K}=\{x\in\mathcal{M}; \operatorname{grad} U(x)=0\}$,

$$C_{0T}^{x,y}(F)=\{\phi\in C([0, T], F); \phi(0)=x, \phi(T)=y\}, \qquad x, y\in F,\ T>0,$$

for an open or closed set F and $\|\cdot\|=\sqrt{g(\cdot,\cdot)}$ denotes the Riemannian norm on $\mathcal{M}$.

REMARK 2.1. (i) One can show that the condition (C_2) implies ($\boldsymbol{A}$) in [**4**, p. 169]. (See [**17**].)

(ii) For arbitrary two points x, y belonging to the same compactum, we have $U(x)=U(y)$. In particular, the set of critical values of U is discrete.

We take a constant V_0 and subsets $\boldsymbol{N}^{(1)}, \cdots, \boldsymbol{N}^{(l)}$ of the bottom $\{U=0\}$ so that

$$V_0>U(x, y) \quad \text{for all } x, y\in\boldsymbol{N}^{(j)},\ 1\leqq j\leqq l,$$

$$V_0=U(x, y) \quad \text{for all } x\in\boldsymbol{N}^{(j)} \text{ and } y\in\boldsymbol{N}^{(j')},\ 1\leqq j, j'\leqq l,\ j\neq j',$$

$$V_0<U(x, y) \quad \text{for all } x\in\bigcup_{j=1}^{l}\boldsymbol{N}^{(j)} \text{ and } y\in\{U=0\}\setminus\bigcup_{j=1}^{l}\boldsymbol{N}^{(j)},$$

where we write

$$U(x, y)=\min_{\phi\in C_{01}^{x,y}(\mathcal{M})}\max_{t\in[0,1]}U(\phi(t)). \tag{2.1}$$

For the sake of Remark 2.1 (ii), one can find a connected component $\boldsymbol{D}$ of $\{x\in\mathcal{M}; U(x)<(V_0+V_{-1})/2\}$ such that it contains the bottoms $\boldsymbol{N}^{(1)}, \cdots, \boldsymbol{N}^{(l)}$, where $V_{-1}=[\min_{x\in\boldsymbol{K}; U(x)>V_0}U(x)]\wedge(V_0+1)$. Here one notices the boundary $\partial\boldsymbol{D}$ of $\boldsymbol{D}$ is non-empty and smooth. From (C_2), the set $\{x\in\boldsymbol{D}; U(x)<V_0\}$ consists of finite number of connected components $\mathcal{V}_1, \cdots, \mathcal{V}_{L'}$ and they are taken in the manner that $\mathcal{V}_j\supset\boldsymbol{N}^{(j)}$, $1\leqq j\leqq l$. We call each of them "valley" and write $\boldsymbol{V}'=\{\mathcal{V}_1, \cdots, \mathcal{V}_{L'}\}$. We shall also use the notation $\boldsymbol{V}_0=\{\mathcal{V}_1, \cdots, \mathcal{V}_l\}$ in order to distinguish the deepest valleys in $\boldsymbol{V}'$. Let $M_1, \cdots, M_{K'}$ be all compacta in $\{x\in\boldsymbol{D}; U(x)=V_0\}$, namely, $\bigcup_{\alpha=1}^{K'}M_\alpha=\{x\in\boldsymbol{D}\cap\boldsymbol{K}; U(x)=V_0\}$. We impose the following restrictions on the relationship between $\{\mathcal{V}_i\}$ and $\{M_\alpha\}$:

(C_4) for each M_α, $1\leqq\alpha\leqq K'$, there exist exactly two valleys $\mathcal{V}_i, \mathcal{V}_{i'}\in\boldsymbol{V}'$, $\mathcal{V}_i\neq\mathcal{V}_{i'}$, such that

$$\bar{\mathcal{V}}_i\cap M_\alpha\neq\emptyset \quad\text{and}\quad \bar{\mathcal{V}}_{i'}\cap M_\alpha\neq\emptyset\,. \tag{2.2}$$

Moreover, for every two points $x\in\mathcal{V}_i$, $y\in\mathcal{V}_{i'}$ in different valleys, if a trajectory $\phi\in C_{01}^{x,y}(\mathcal{M})$ attains the minimum in (2.1), one has $\phi(t)\in M_\alpha$ for some $t\in(0, 1)$ and $1\leqq\alpha\leqq K'$.

DEFINITION 2.2. For two valleys $\mathcal{V}_j, \mathcal{V}_{j'}\in\boldsymbol{V}'$ and a subset $\boldsymbol{W}$ of $\boldsymbol{V}'$, a finite sequence of steps $\mathcal{V}_i\to\mathcal{V}_{i'}$ ($\mathcal{V}_i, \mathcal{V}_{i'}\in\boldsymbol{W}\cup\{\mathcal{V}_j, \mathcal{V}_{j'}\}$, $\mathcal{V}_i\neq\mathcal{V}_{i'}$) is called a $\{\mathcal{V}_j, \mathcal{V}_{j'}\}$-route through $\boldsymbol{W}$, if it satisfies the next conditions:

(1) the valleys of each step $\mathcal{V}_i\to\mathcal{V}_{i'}$ satisfy (2.2) for some $1\leqq\alpha\leqq K'$;

(2) the first step starts from $\mathcal{V}_j$ and the last one ends at $\mathcal{V}_{j'}$;

(3) the end point of each step becomes the starting point of the next step except the last one;

(4) there are no closed cycles in each route.

We denote by $\mathfrak{R}_{\mathcal{V}_j\mathcal{V}_{j'}}(\boldsymbol{W})$ the set of $\{\mathcal{V}_j, \mathcal{V}_{j'}\}$-routes through $\boldsymbol{W}$. One can find a subset $\boldsymbol{V}$ of $\boldsymbol{V}'$ defined by

$$\boldsymbol{V}=\bigcup_{1\leq j,j'\leq l, j\neq j'}\{\mathcal{V}_i\in\boldsymbol{V}'; \text{there exists a route } \mathfrak{r}\in\mathfrak{R}_{\mathcal{V}_j\mathcal{V}_{j'}}(\tilde{\boldsymbol{V}}') \text{ such that } (\mathcal{V}_i\to\mathcal{V}_{i'})\in\mathfrak{r}\},$$

where $\tilde{\boldsymbol{V}}'=\{\mathcal{V}_{l+1}, \cdots, \mathcal{V}_{L'}\}$. We note $\boldsymbol{V}\supset\boldsymbol{V}_0$ and write $\boldsymbol{V}=\{\mathcal{V}_1, \cdots, \mathcal{V}_L\}$ by replacing the indices of $\mathcal{V}_i$'s if necessary. Let us prepare the next lemma.

LEMMA 2.3. (i) *$\mathfrak{R}_{\mathcal{V}_j\mathcal{V}_{j'}}(\tilde{\boldsymbol{V}}')=\mathfrak{R}_{\mathcal{V}_j\mathcal{V}_{j'}}(\tilde{\boldsymbol{V}})$ for all $\mathcal{V}_j, \mathcal{V}_{j'}\in\boldsymbol{V}_0$, $\mathcal{V}_j\neq\mathcal{V}_{j'}$, where $\tilde{\boldsymbol{V}}=\{\mathcal{V}_{l+1}, \cdots, \mathcal{V}_L\}$.*

(ii) *For each* $\mathcal{V}_i \in \boldsymbol{V}' \setminus \boldsymbol{V}$, *there is a unique* $\mathcal{V}_{i'} \in \boldsymbol{V}$ *such that* $\mathfrak{R}_{\mathcal{V}_i \mathcal{V}_{i'}}(\boldsymbol{V}' \setminus \boldsymbol{V}) \neq \emptyset$.

PROOF. The assertion (i) is obvious from the definition. For the claim (ii), fix an arbitrary valley $\mathcal{V}_i \in \boldsymbol{V}' \setminus \boldsymbol{V}$. Since $\mathfrak{R}_{\mathcal{V}_i \mathcal{V}_{i'}}(\boldsymbol{V}') \neq \emptyset$ for all $\mathcal{V}_{i'} \in \boldsymbol{V}$, one can easily find $\mathcal{V}_{i'} \in \boldsymbol{V}$ so that $\mathfrak{R}_{\mathcal{V}_i \mathcal{V}_{i'}}(\boldsymbol{V}' \setminus \boldsymbol{V}) \neq \emptyset$. In order to prove the uniqueness, suppose there are two valleys $\mathcal{V}_{i(1)}, \mathcal{V}_{i(2)} \in \boldsymbol{V}$ such that $\mathfrak{R}_{\mathcal{V}_i \mathcal{V}_{i(1)}}(\boldsymbol{V}' \setminus \boldsymbol{V}) \neq \emptyset$ and $\mathfrak{R}_{\mathcal{V}_i \mathcal{V}_{i(2)}}(\boldsymbol{V}' \setminus \boldsymbol{V}) \neq \emptyset$, say

$$\mathfrak{r}_1 = \{(\mathcal{V}_i \to \mathcal{V}_{i_1}), (\mathcal{V}_{i_1} \to \mathcal{V}_{i_2}), \cdots, (\mathcal{V}_{i_k} \to \mathcal{V}_{i(1)})\} \in \mathfrak{R}_{\mathcal{V}_i \mathcal{V}_{i(1)}}(\boldsymbol{V}' \setminus \boldsymbol{V}),$$

$$\mathfrak{r}_2 = \{(\mathcal{V}_i \to \mathcal{V}_{i'_1}), (\mathcal{V}_{i'_1} \to \mathcal{V}_{i'_2}), \cdots, (\mathcal{V}_{i'_{k'}} \to \mathcal{V}_{i(2)})\} \in \mathfrak{R}_{\mathcal{V}_i \mathcal{V}_{i(2)}}(\boldsymbol{V}' \setminus \boldsymbol{V}),$$

and find $\mathcal{V}_{j(1)}, \mathcal{V}_{j(2)}, \mathcal{V}_{j(3)}, \mathcal{V}_{j(4)} \in \boldsymbol{V}_0$, $\mathcal{V}_{j(1)} \neq \mathcal{V}_{j(2)}$, $\mathcal{V}_{j(3)} \neq \mathcal{V}_{j(4)}$, and routes

$$\mathfrak{r}_3 = \{(\mathcal{V}_{j(1)} \to \mathcal{V}_{j_{-n}}), (\mathcal{V}_{j_{-n}} \to \mathcal{V}_{j_{-n+1}}), \cdots, (\mathcal{V}_{j_{-1}} \to \mathcal{V}_{i(1)}),$$
$$(\mathcal{V}_{i(1)} \to \mathcal{V}_{j_1}), \cdots, (\mathcal{V}_{j_{n'}} \to \mathcal{V}_{j(2)})\} \in \mathfrak{R}_{\mathcal{V}_{j(1)} \mathcal{V}_{j(2)}}(\tilde{\boldsymbol{V}}),$$

$$\mathfrak{r}_4 = \{(\mathcal{V}_{j(3)} \to \mathcal{V}_{j'_{-m}}), (\mathcal{V}_{j'_{-m}} \to \mathcal{V}_{j'_{-m+1}}), \cdots, (\mathcal{V}_{j'_{-1}} \to \mathcal{V}_{i(2)}),$$
$$(\mathcal{V}_{i(2)} \to \mathcal{V}_{j'_1}), \cdots, (\mathcal{V}_{j'_{m'}} \to \mathcal{V}_{j(4)})\} \in \mathfrak{R}_{\mathcal{V}_{j(3)} \mathcal{V}_{j(4)}}(\tilde{\boldsymbol{V}}),$$

passing $\mathcal{V}_{i(1)}$ and $\mathcal{V}_{i(2)}$, respectively. Set

$$n_* = \min\{p \geqq 0\,;\, \mathcal{V}_{j_p} = \mathcal{V}_{j'_q} \text{ for some } q = -m-1, \cdots, m'+1\},$$

$$k_* = \max\{p \geqq 0\,;\, \mathcal{V}_{i_p} = \mathcal{V}_{i'_q} \text{ for some } q = 0, \cdots, k'+1\},$$

where one puts $\mathcal{V}_{j_0} = \mathcal{V}_{i_{k+1}} = \mathcal{V}_{i(1)}$, $\mathcal{V}_{j_{n'+1}} = \mathcal{V}_{j(2)}$, $\mathcal{V}_{j'_{-m-1}} = \mathcal{V}_{j(3)}$, $\mathcal{V}_{j'_0} = \mathcal{V}_{i'_{k'+1}} = \mathcal{V}_{i(2)}$, $\mathcal{V}_{j'_{m'+1}} = \mathcal{V}_{j(4)}$ and $\mathcal{V}_{i_0} = \mathcal{V}_{i'_0} = \mathcal{V}_i$ simply. Note $0 \leqq k_* \leqq k+1$ and assume $n_* < \infty$. We take $m_* \in \{-m-1, \cdots, m'+1\}$ and $k'_* \in \{0, \cdots, k'+1\}$ so that $\mathcal{V}_{j'_{m_*}} = \mathcal{V}_{j_{n_*}}$ and $\mathcal{V}_{i_{k_*}} = \mathcal{V}_{i'_{k'_*}}$, respectively. If $m_* = 0$, i.e., $\mathcal{V}_{j_{n_*}} = \mathcal{V}_{i(2)} (\neq \mathcal{V}_{i(1)})$, one can find a $\{\mathcal{V}_{j(1)}, \mathcal{V}_{j(2)}\}$-route $\mathfrak{r}_5$ passing $\mathcal{V}_{i_{k_*}} \in \boldsymbol{V}' \setminus \boldsymbol{V}$

$$\mathfrak{r}_5 = \{(\mathcal{V}_{j(1)} \to \mathcal{V}_{j_{-n}}), \cdots, (\mathcal{V}_{j_{-1}} \to \mathcal{V}_{i(1)}), (\mathcal{V}_{i(1)} \to \mathcal{V}_{i_k}), \cdots, \mathcal{V}_{i_{k_*-1}} \to \mathcal{V}_{i_{1k_*}}),$$
$$(\mathcal{V}_{i'_{k'_*}} \to \mathcal{V}_{i_{k'_*+1}}), \cdots, (\mathcal{V}_{i'_k} \to \mathcal{V}_{i(2)}), (\mathcal{V}_{j_{n_*}} \to \mathcal{V}_{j_{n_*+1}}), \cdots, (\mathcal{V}_{j_{n'}} \to \mathcal{V}_{j(2)})\}.$$

In case that $m_* > 0$, consider a sequence of steps

$$\mathfrak{r}_6 = \{(\mathcal{V}_{j(3)} \to \mathcal{V}_{j'_{-m}}), \cdots, (\mathcal{V}_{j'_{-1}} \to \mathcal{V}_{i(2)}), (\mathcal{V}_{i(2)} \to \mathcal{V}_{i'_{k'}}), \cdots, (\mathcal{V}_{i'_{k'_*+1}} \to \mathcal{V}_{i'_{k'_*}}),$$
$$(\mathcal{V}_{i_{k_*}} \to \mathcal{V}_{i_{k_*+1}}), \cdots, (\mathcal{V}_{i_k} \to \mathcal{V}_{i(1)}), (\mathcal{V}_{i(1)} \to \mathcal{V}_{j_1}), \cdots, (\mathcal{V}_{j_{n_*-1}} \to \mathcal{V}_{j_{n_*}}),$$
$$(\mathcal{V}_{j'_{m_*}} \to \mathcal{V}_{j'_{m_*+1}}), \cdots, (\mathcal{V}_{j'_{m'}} \to \mathcal{V}_{j(4)})\}.$$

Since $\{\mathcal{V}_{j_1}, \cdots, \mathcal{V}_{j_{n_*-1}}, \mathcal{V}_{j'_{-m}}, \cdots, \mathcal{V}_{j'_{m'}}\} \subset \boldsymbol{V}$ and $\{\mathcal{V}_{i_{k_*+1}}, \cdots, \mathcal{V}_{i_k}, \mathcal{V}_{i'_{k'}}, \cdots,$

$\mathcal{V}_{i'_{k'_*+1}}\}\subset \boldsymbol{V}'\setminus\boldsymbol{V}$, the definitions of n_* and k_* imply that $\{\mathcal{V}_{j_1}, \cdots, \mathcal{V}_{j_{n_*-1}}\}$, $\{\mathcal{V}_{j'_{-m'}}, \cdots, \mathcal{V}_{j'_{-1}}\}$, $\{\mathcal{V}_{j'_{m_*}}, \cdots, \mathcal{V}_{j'_{m'}}\}$, $\{\mathcal{V}_{i_{k_*+1}}, \cdots, \mathcal{V}_{i_k}\}$ and $\{\mathcal{V}_{i'_{k'}}, \cdots, \mathcal{V}_{i'_{k'_*+1}}\}$ are mutually disjoint. Hence we know that $\mathfrak{r}_6$ is a $\{\mathcal{V}_{j(3)}, \mathcal{V}_{j(4)}\}$-route passing $\mathcal{V}_{i_{k_*}}\in \boldsymbol{V}'\setminus\boldsymbol{V}$. One can treat the case of $m_*<0$ in the same manner. Finally, if $n_*=\infty$, a sequence of steps

$$\mathfrak{r}_7 = \{(\mathcal{V}_{j(2)}\to\mathcal{V}_{j_{n'}}), \cdots, (\mathcal{V}_{j_1}\to\mathcal{V}_{i(1)}), (\mathcal{V}_{i(1)}\to\mathcal{V}_{i_k}), \cdots, (\mathcal{V}_{i_{k_*-1}}\to\mathcal{V}_{i_{k_*}}), (\mathcal{V}_{i'_{k'_*}}\to\mathcal{V}_{i'_{k'_*+1}}), \cdots, (\mathcal{V}_{i'_k}\to\mathcal{V}_{i(2)}), (\mathcal{V}_{i(2)}\to\mathcal{V}_{j'_1}), \cdots, (\mathcal{V}_{j'_{m'}}\to\mathcal{V}_{j(4)})\}$$

is a $\{\mathcal{V}_{j(2)}, \mathcal{V}_{j(4)}\}$-route passing $\mathcal{V}_{i_{k_*}}\in \boldsymbol{V}'\setminus\boldsymbol{V}$. And now we reach the contradiction. □

We define

$$\mathcal{V}_i^e = \mathcal{V}_i \cup \bigcup_{\mathcal{V}_{i'}\in \boldsymbol{V}'\setminus \boldsymbol{V}:\, \mathfrak{R}_{\mathcal{V}_{i'},\mathcal{V}_i}(\boldsymbol{V}'\setminus\boldsymbol{V})\neq\emptyset}\Big[\mathcal{V}_{i'}\cup \bigcup_{\alpha:\, M_\alpha\cap \mathcal{V}_{i'}\neq\emptyset} M_\alpha\Big], \qquad 1\leqq i\leqq L,$$

and denote by M_α, $1\leqq\alpha\leqq K$, all compacta on $\{x\in\boldsymbol{D}\,;\,U(x)=V_0\}$ each of which satisfies (2.2) for some $\mathcal{V}_i, \mathcal{V}_{i'}\in\boldsymbol{V}$, $\mathcal{V}_i\neq\mathcal{V}_{i'}$, (not $\boldsymbol{V}'$) by replacing the indices of M_α's if necessary.

The Hessian $H=H(x)$ of U is a symmetric tensor field of type (0, 2). Especially for $x\in\boldsymbol{K}$, it is written by

$$H\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)=\frac{\partial^2 U}{\partial x^i\partial x^j}(x), \qquad 1\leqq i,\ j\leqq d\,,$$

in terms of an arbitrary local coordinate $x=(x^i)$. A tensor $H^*=H^*(x)$ of type (1, 1) is naturally associated with H by means of the metric g:

$$g(H^*X, Y) = H(X, Y), \qquad X, Y\in T_x\mathcal{M}\,,$$

where $T_x\mathcal{M}$ stands for the tangent space to $\mathcal{M}$ at $x\in\mathcal{M}$.

Now we give the conditions on the bottoms $\boldsymbol{N}^{(1)}, \cdots, \boldsymbol{N}^{(l)}$ and the compacta $M_1, \cdots, M_K$:

(C_4) each $\boldsymbol{N}^{(j)}$, $1\leqq j\leqq l$, is a disjoint union of finite number of connected $n_\alpha^{(j)}$-dimensional compact submanifolds $N_\alpha^{(j)}$, $1\leqq\alpha\leqq l_j$, of $\mathcal{M}$, $\boldsymbol{N}^{(j)}=\bigcup_{\alpha=1}^{l_j} N_\alpha^{(j)}$, and if $\partial N_\alpha^{(j)}\neq\emptyset$ there exists a connected $n_\alpha^{(j)}$-dimensional submanifold $\tilde{N}_\alpha^{(j)}$ of $\mathcal{M}$ such that the interior of $\tilde{N}_\alpha^{(j)}$ contains $N_\alpha^{(j)}$;

(C_5) each $N_\alpha^{(j)}$, $1\leqq\alpha\leqq l_j$, $1\leqq j\leqq l$, is non-degenerate in the sense that the Hessian $H^*(x)$ has rank $d-n_\alpha^{(j)}$;

(C_6) each M_α, $1\leqq\alpha\leqq K$, is an m_α-dimensional compact submanifold of $\mathcal{M}$, and if $\partial M_\alpha\neq\emptyset$ there exists a connected m_α-dimensional submanifold $\tilde{M}_\alpha$ of $\mathcal{M}$ such that the interior of $\tilde{M}_\alpha$ contains M_α;

(C_7) each M_α, $1\leqq\alpha\leqq K$, is non-degenerate and index 1, namely, for every

$x \in M_\alpha$, rank $H^*(x)=d-m_\alpha$ and $H^*(x)$ has exactly one negative eigenvalue.

REMARK 2.4. If M_α, $1 \leqq \alpha \leqq K$, and $\mathcal{V}_i, \mathcal{V}_{i'} \in \boldsymbol{V}$ $(\mathcal{V}_i \neq \mathcal{V}_{i'})$ satisfy (2.2), the one notices $M_\alpha \subset \bar{\mathcal{V}}_i \cap \bar{\mathcal{V}}_{i'}$.

Let $H_+(x)$ be the product of all the positive eigenvalues of the Hessian $H^*(x)$ for $x \in \bigcup_{j=1}^{l} N^{(j)} \cup \bigcup_{\alpha=1}^{K} M_\alpha$ and $-H_-(x)$ the negative eigenvalue of $H^*(x)$ for $x \in \bigcup_{\alpha=1}^{K} M_\alpha$. We set, for $1 \leqq j \leqq l$, $n^{(j)} = \max_{1 \leqq \alpha \leqq l_j} n_\alpha^{(j)}$ and

$$\nu^{(j)} = (2\pi)^{(d-n^{(j)})/2} \sum_{\alpha: n_\alpha^{(j)} = n^{(j)}} \int_{N_\alpha^{(j)}} H_+(y)^{-1/2} dy \,. \tag{2.3}$$

For an integer $a \geqq 0$ and $\mathcal{V}_i, \mathcal{V}_{i'} \in \boldsymbol{V}$ $(\mathcal{V}_i \neq \mathcal{V}_{i'})$, we define $H^{(a)}_{\mathcal{V}_i \mathcal{V}_{i'}}$ in the following manner: if $\max_{\alpha: M_\alpha \subset \bar{\mathcal{V}}_i \cap \bar{\mathcal{V}}_{i'}} m_\alpha \leqq a$,

$$H^{(a)}_{\mathcal{V}_i \mathcal{V}_{i'}} = (2\pi)^{(d-a-2)/2} \sum_{\alpha: M_\alpha \subset \bar{\mathcal{V}}_i \cap \bar{\mathcal{V}}_{i'},\, m_\alpha = a} \int_{M_\alpha} \left\{ \frac{H_-(y)}{H_+(y)} \right\}^{1/2} dy \,; \tag{2.4}$$

otherwise $H^{(a)}_{\mathcal{V}_i \mathcal{V}_{i'}} = +\infty$; where the maximum of the empty set is equal to $-\infty$ and, if there are no α's satisfying the condition of $\sum$, then the summation is equal to 0. In (2.3) and (2.4), dy stands for the volume element of $N_\alpha^{(j)}$ or M_α induced from g on $\mathcal{M}$; if $\dim N_\alpha^{(j)}=0$ or $\dim M_\alpha = 0$, then dy should be understood as the δ-mass. We set, for $a \geqq 0$ and $0 \leqq \zeta_1, \cdots, \zeta_L \leqq 1$,

$$H^{(a)}(\zeta_1, \cdots, \zeta_L) = \sum_{1 \leqq i < i' \leqq L} (\zeta_i - \zeta_{i'})^2 H^{(a)}_{\mathcal{V}_i \mathcal{V}_{i'}} \,, \tag{2.5}$$

and, for $1 \leqq j \leqq l$,

$$m^{(j)} = \max_{1 \leqq j' \leqq l,\, j' \neq j} \ \max_{\mathfrak{r} \in \mathfrak{R}_{\mathcal{V}_j \mathcal{V}_{j'}}(\tilde{\boldsymbol{V}})} \ \min_{(\mathcal{V}_i \to \mathcal{V}_{i'}) \in \mathfrak{r}} \ \max_{M_\alpha \subset \bar{\mathcal{V}}_i \cap \bar{\mathcal{V}}_{i'}} m_\alpha \,, \tag{2.6a}$$

$$H_j = \min_{0 \leqq \zeta_{l+1}, \cdots, \zeta_L \leqq 1} H^{(m^{(j)})}(\zeta_1, \cdots, \zeta_L), \tag{2.6b}$$

where one takes $\zeta_j = 1$ and $\zeta_{j'} = 0$ for $1 \leqq j' \leqq l$, $j' \neq j$, in (2.6b). In order to formulate the first result, we notice the following lemma.

LEMMA 2.5. *We have* $0 < H_j < +\infty$ *for every* $1 \leqq j \leqq l$.

PROOF. Set $\bar{\zeta}_j = 1$, $\bar{\zeta}_{j'} = 0$, $1 \leqq j' \leqq l$, $j' \neq j$, and choose $0 \leqq \bar{\zeta}_{l+1}, \cdots, \bar{\zeta}_L \leqq 1$ in the manner that $\bar{\zeta}_i = \bar{\zeta}_{i'}$ if two valleys $\mathcal{V}_i, \mathcal{V}_{i'} \in \boldsymbol{V}$ satisfy $\max_{M_\alpha \subset \bar{\mathcal{V}}_i \cap \bar{\mathcal{V}}_{i'}} m_\alpha > m^{(j)}$. From (2.6a), it is well-defined and we have $H_j \leqq H^{(m^{(j)})}(\bar{\zeta}_1, \cdots, \bar{\zeta}_L) < +\infty$. On the other hand, one can find $1 \leqq j' \leqq l$, $j' \neq j$ and $\mathfrak{r} \in \mathfrak{R}_{\mathcal{V}_j \mathcal{V}_{j'}}(\tilde{\boldsymbol{V}})$ attaining the maximum in (2.6a), say $\mathfrak{r} = \{(\mathcal{V}_j \to \mathcal{V}_{j_1}), (\mathcal{V}_{j_1} \to \mathcal{V}_{j_2}), \cdots, (\mathcal{V}_{j_p} \to \mathcal{V}_{j'})\}$. Since $H^{(m^{(j)})}_{\mathcal{V}_{j_q} \mathcal{V}_{j_{q+1}}} > 0$ for all $0 \leqq q \leqq p$, it is obvious that

$$H_j \geqq \min_{0 \leqq \zeta_{j_1}, \cdots, \zeta_{j_p} \leqq 1} \sum_{q=0}^{p} (\zeta_{j_q} - \zeta_{j_{q+1}})^2 H^{(m^{(j)})}_{\mathcal{V}_{j_q} \mathcal{V}_{j_{q+1}}} > 0,$$

where we write $\zeta_{j_0}=1$, $\zeta_{j_{p+1}}=0$ and $\mathcal{V}_{j_0}=\mathcal{V}_j$, $\mathcal{V}_{j_{p+1}}=\mathcal{V}_{j'}$ simply. □

Let $\{\bar{x}_t(x); t\geqq 0, x\in\mathcal{M}\}$ be the flow determined by $-(1/2)\operatorname{grad} U$, i.e., $\bar{x}_t=\bar{x}_t(x)$ is a unique solution of the ordinary differential equation (ODE):

$$\frac{d\bar{x}_t}{dt}=-\frac{1}{2}\operatorname{grad} U(\bar{x}_t), \qquad \bar{x}_0=x\,. \tag{2.7}$$

We denote the ω-limit set of a point $x\in\mathcal{M}$ and the domain of the attraction of a connected open or closed set F in $\mathcal{M}$ with respect to this flow, respectively, by $\omega(x)$ and $\mathcal{D}(F)$:

$$\omega(x)=\{y\in\mathcal{M}\,;\ \bar{x}_{t_n}(x)\to y \text{ for some sequence } t_n\to\infty\},$$

$$\mathcal{D}(F)=\{x\in\mathcal{M}\,;\ \omega(x)\subset F\}.$$

For $1\leqq j\leqq l$, let B_j be an open neighborhood of $\boldsymbol{N}^{(j)}$ such that $\overline{B_j}$ is contained by $\mathcal{V}_j\cap\mathcal{D}(\boldsymbol{N}^{(j)})$ and that ∂B_j is smooth. We set

$$\boldsymbol{D}_j=\boldsymbol{D}\setminus\bigcup_{1\leq j'\leq l,\, j'\neq j}\overline{B_{j'}}$$

and define the first exit time $\tau^{\varepsilon}_{\boldsymbol{D}_j}$ from $\boldsymbol{D}_j$ by (1.2) where $\boldsymbol{G}$ should be replaced with $\boldsymbol{D}_j$.

Now we formulate our first result.

THEOREM 1. *For $1\leqq j\leqq l$, we have*

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-m^{(j)}+n^{(j)}}e^{-V_0/\varepsilon^2}E_x[\tau^{\varepsilon}_{\boldsymbol{D}_j}]=2\nu^{(j)}/H_j \tag{2.8}$$

uniformly in x belonging to any compact subset in $\boldsymbol{D}_j\cap\mathcal{D}(\mathcal{V}^{\varepsilon}_j)$.

We shall also have the following theorem.

THEOREM 2. *For $1\leqq j_0,\ j_1\leqq l,\ j_0\neq j_1$, we have*

$$\lim_{\varepsilon\downarrow 0}P_x(x^{\varepsilon}_{\tau^{\varepsilon}_{\boldsymbol{D}_{j_0}}}\in\partial B_{j_1})=q_{j_0,j_1}\,, \tag{2.9}$$

uniformly in x belonging to any compact subset of $\boldsymbol{D}\cap\mathcal{D}(\mathcal{U}_{j_0})$, where the constant q_{j_0,j_1} and the domain $\mathcal{U}_{j_0}$ will be defined precisely below in Section 4.

In order to state our main result concerning metastable behaviors, we need some more preparations. Let us denote $\boldsymbol{B}=\{\boldsymbol{N}^{(1)},\cdots,\boldsymbol{N}^{(l)}\}$ and set, for $1\leqq j\leqq l$,

$$c_j=\begin{cases} H_j/2\nu^{(j)}, & \text{if } m^{(j)}-n^{(j)}=\mu\,,\\ 0, & \text{otherwise,}\end{cases} \tag{2.10}$$

where $\mu=\max_{1\leq j\leq l}\{m^{(j)}-n^{(j)}\}$. Then, introduce a bounded operator $\mathcal{G}$ on $B(\boldsymbol{B})$ by

(2.11) $$\mathcal{G} f(\boldsymbol{N}^{(j)}) = c_j \sum_{j'=1}^{l} q_{j,j'} \{f(\boldsymbol{N}^{(j')}) - f(\boldsymbol{N}^{(j)})\}, \qquad f \in B(\boldsymbol{B}),$$

for $\boldsymbol{N}^{(j)} \in \boldsymbol{B}$, where $q_{j,j}=0$ for $1 \leqq j \leqq l$ and $B(\boldsymbol{B})$ stands for the space of all bounded functions on $\boldsymbol{B}$. We write $(X_t, \boldsymbol{P}_{N^{(j)}})$ for the Markov jump process realized on some probability space $(\boldsymbol{\Omega}, \mathcal{F}, \boldsymbol{P})$ generated by $\mathcal{G}$ satisfying $\boldsymbol{P}_{N^{(j)}}(X_0 = \boldsymbol{N}^{(j)}) = 1$ for $\boldsymbol{N}^{(j)} \in \boldsymbol{B}$.

Now we formulate our main result.

THEOREM 3. *Set* $\alpha_\varepsilon = \varepsilon^\mu e^{V_0/\varepsilon^2}$ *and* $y_t^\varepsilon = x_{t\alpha_\varepsilon}^\varepsilon$ *for* $t \geqq 0$. *Then for all* $0 < t_1 < t_2 < \cdots < t_N$, $\boldsymbol{N}^{(j_1)}, \boldsymbol{N}^{(j_2)}, \cdots, \boldsymbol{N}^{(j_N)} \in \boldsymbol{B}$ *and sufficiently small* $\delta > 0$, *we have*

(2.12) $$\lim_{\varepsilon \downarrow 0} P_x(y_{t_1}^\varepsilon \in \boldsymbol{N}_\delta^{(j_1)}, y_{t_2}^\varepsilon \in \boldsymbol{N}_\delta^{(j_2)}, \cdots, y_{t_N}^\varepsilon \in \boldsymbol{N}_\delta^{(j_N)})$$
$$= \boldsymbol{P}_{N^{(j_0)}}(X_{t_1} = \boldsymbol{N}^{(j_1)}, X_{t_2} = \boldsymbol{N}^{(j_2)}, \cdots, X_{t_N} = \boldsymbol{N}^{(j_N)})$$

for all $x \in \mathcal{D}(\mathcal{V}_{j_0})$, *where* $\boldsymbol{N}_\delta^{(j)}$ *stands for the* δ*-neighborhood of* $\boldsymbol{N}^{(j)}$. *In particular, if* $x \in \boldsymbol{N}^{(j_0)}$, *then* (2.12) *also holds in case that* $0 = t_1 < t_2 < \cdots < t_N$.

COROLLARY (Metastable behavior). *Let us assume* $\boldsymbol{N}^{(j)}$ *consists of one point* b_j *and the Hessian* $H^*(b_j)$ *has rank* d *for every* $1 \leqq j \leqq l$. *Then we have*

$$\lim_{\varepsilon \downarrow 0} E_x[f_1(y_{t_1}^\varepsilon) f_2(y_{t_2}^\varepsilon) \cdots f_N(y_{t_N}^\varepsilon)] = \boldsymbol{E}_{b_j}[f_1(X_{t_1}) f_2(X_{t_2}) \cdots f_N(X_{t_N})]$$

for all $x \in \mathcal{D}(\mathcal{V}_j)$, $0 < t_1 < t_2 < \cdots < t_N$ *and bounded continuous functions* $f_1, f_2, \cdots, f_N$ *on* $\mathcal{M}$, *where* $\boldsymbol{E}_{b_j}$ *stands for the expectation with respect to* $\boldsymbol{P}_{b_j}$.

3. Singularly perturbed Dirichlet problems.

Let Ω be a connected open domain in $\mathcal{M}$ with a C^∞-boundary and a compact closure. Recall the elliptic operator $\mathcal{L}^\varepsilon$ defined by (1.1). In this section, we shall mainly study asymptotics of the principal eigenvalue λ^ε and the associated eigenfunction φ^ε for the Dirichlet boundary value problem (1.3).

We denote by $H_0^1(\Omega)$ the completion of a metric space $(C_0^\infty(\Omega), \|\cdot\|_{H^1(\Omega)})$, where $\|\cdot\|_{H^1(\Omega)}$ is a Hilbertian norm defined by

$$\|\varphi\|_{H^1(\Omega)}^2 = \int_\Omega |\varphi|^2 dx + \int_\Omega \|\operatorname{grad} \varphi\|^2 dx,$$

$dx = \sqrt{\mathfrak{g}}\, dx^1 \wedge \cdots \wedge dx^d$ stands for the Riemannian volume element on $\mathcal{M}$ and $\mathfrak{g} = \det(g_{ij})$. Since $\mathcal{M}$ is orientable, one can apply the Stokes' formula [**16**] and get

$$\int_\Omega \varphi_1 \mathcal{L}^\varepsilon \varphi_2 e^{-U/\varepsilon^2} dx = -\frac{\varepsilon^2}{2}\int_\Omega g(\text{grad}\,\varphi_1, \text{grad}\,\varphi_2) e^{-U/\varepsilon^2} dx + \frac{\varepsilon^2}{2}\int_{\partial\Omega} \varphi_1 e^{-U/\varepsilon^2} \iota_{\text{grad}\,\varphi_2} dx \tag{3.1}$$

for $\varphi_1, \varphi_2 \in C^\infty(\bar{\Omega})$, where ι_X denotes the interior product for a vector field X, i.e., $\iota_{\text{grad}\,\varphi} dx$ is a $(d-1)$-form defined by

$$\iota_{\text{grad}\,\varphi} dx = \sum_{i,j=1}^d (-1)^{i-1} g^{ij} \frac{\partial \varphi}{\partial x_j} \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots dx^d,$$

and (g^{ij}) is the inverse matrix of (g_{ij}). Since the second term of the right hand side (RHS) of (3.1) vanishes for $\varphi_1 \in C_0^\infty(\Omega)$, $(\mathcal{L}^\varepsilon, C_0^\infty(\Omega))$ is a semi-bounded symmetric operator on $L^2(\Omega, e^{-U/\varepsilon^2}dx)$. Hence, there exists the Friedrichs extension $(\hat{\mathcal{L}}^\varepsilon, H_0^1(\Omega))$ of $(\mathcal{L}^\varepsilon, C_0^\infty(\Omega))$, (see [**14**, Volume II, p. 177]) and one knows $-\hat{\mathcal{L}}^\varepsilon$ is self-adjoint and non-negative. Then, we arrive at the Rayleigh-Ritz formula [**14**, Volume IV, p. 82]

$$\hat{\lambda}^\varepsilon = \frac{\varepsilon^2}{2} \min_{\varphi \in H_0^1(\Omega)} \frac{J^\varepsilon(\varphi)}{\|\varphi\|_\varepsilon^2}, \tag{3.2}$$

where $\hat{\lambda}^\varepsilon$ is the minimum eigenvalue of $-\hat{\mathcal{L}}^\varepsilon$ and

$$J^\varepsilon(\varphi) = \int_\Omega \|\text{grad}\,\varphi\|^2 e^{-U/\varepsilon^2} dx,$$

$$\|\varphi\|_\varepsilon = \|\varphi\|_{L^2(\Omega, e^{-U/\varepsilon^2}dx)} = \left\{\int_\Omega |\varphi|^2 e^{-U/\varepsilon^2} dx\right\}^{1/2}.$$

On the other hand, $\hat{\lambda}^\varepsilon$ is simple and one can find a non-trivial positive function $\hat{\varphi}^\varepsilon \in H_0^1(\Omega)$ such that $\hat{\mathcal{L}}^\varepsilon \hat{\varphi}^\varepsilon + \hat{\lambda}^\varepsilon \hat{\varphi}^\varepsilon = 0$ in the weak sense. (See Gilbarg and Trudinger [7, pp. 212-214] for details.) But Theorem 8.13 in [7] verifies $\hat{\varphi}^\varepsilon \in C^\infty(\bar{\Omega})$ and $\hat{\varphi}^\varepsilon \equiv 0$ on $\partial\Omega$; namely, $\hat{\varphi}^\varepsilon$ becomes the unique solution of (1.3) in the classical sense and $\lambda^\varepsilon = \hat{\lambda}^\varepsilon$. Here we also notice the formula (3.2) is rewritten into

$$\lambda^\varepsilon = \frac{\varepsilon^2}{2} \frac{J^\varepsilon(\varphi^\varepsilon)}{\|\varphi^\varepsilon\|_\varepsilon^2}. \tag{3.3}$$

In order to observe the asymptotic behavior of φ^ε, we normalize φ^ε as

$$\sup_{x \in \Omega} \varphi^\varepsilon(x) = \sup_{x \in \Omega} |\varphi^\varepsilon(x)| = 1. \tag{3.4}$$

Here, we notice that φ^ε is continuous and positive.

We shall assume the potential U satisfies the conditions (C_1)-(C_7) and employ the same notations as those in Section 2. Moreover we shall suppose an open domain (Ω) satisfies the following conditions:

(Ω_1) $\Omega \subset \boldsymbol{D}$ and $\partial\Omega$ is smooth;

(Ω_2) Ω contains exactly one $\mathcal{V}_{j_0}\in V_0$ and Ω does not contain an open neighborhood of $N^{(j)}$ for $1\leqq j\leqq l$, $j\neq j_0$;

(Ω_3) if $\mathcal{V}_i\subset\Omega$, then $\bar{\mathcal{V}}_i^e\subset\Omega$ for $1\leqq i\leqq L$;

(Ω_4) there are no critical points of U on $\partial\Omega$.

We consider the following three families of indices of $\mathcal{V}_i$'s:

$$\mathcal{N}^\Omega = \{1\leqq i\leqq L\,;\ \mathcal{D}(\mathcal{V}_i)\cap\Omega\neq\emptyset\},$$

$$\mathcal{N}_1^\Omega = \{l+1\leqq i\leqq L\,;\ \bar{\mathcal{V}}_i^e\subset\Omega\},$$

$$\mathcal{N}_0^\Omega = \mathcal{N}^\Omega\setminus(\mathcal{N}_1^\Omega\cup\{j_0\}).$$

Define a function $H_\Omega(\zeta)$ on $\zeta=(\zeta_i)_{i\in\mathcal{N}_1^\Omega}\in[0,1]^{\mathcal{N}_1^\Omega}$ by

$$H_\Omega(\zeta) = H^{(m_\Omega)}(\zeta_1,\cdots,\zeta_L)\,|_{\zeta_{j_0}=1,\,\zeta_j=0,\,j\in\{1,\cdots,L\}\setminus(\mathcal{N}_1^\Omega\cup\{j_0\})}, \tag{3.5}$$

where we set

$$m_\Omega = \max_{j\in\mathcal{N}_0^\Omega}\ \max_{\mathfrak{r}\in\mathfrak{R}_{\mathcal{V}_{j_0}\mathcal{V}_j}(V_\Omega)}\ \min_{(\mathcal{V}_i\to\mathcal{V}_{i'})\in\mathfrak{r}}\ \max_{M_\alpha\subset\bar{\mathcal{V}}_i\cap\bar{\mathcal{V}}_{i'}} m_\alpha$$

and $V_\Omega=\{\mathcal{V}_i\}_{i\in\mathcal{N}_1^\Omega}$. Let H_Ω be the minimum of $H_\Omega(\zeta)$ in $\zeta\in[0,1]^{\mathcal{N}_1^\Omega}$. Then, one has $0<H_\Omega<+\infty$ in a similar manner to Lemma 2.5.

In this section, we shall show the following theorem.

THEOREM 3.1. *We have*

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{m_\Omega-n^{(j_0)}}e^{V_0/\varepsilon^2}\lambda^\varepsilon = H_\Omega/2\nu^{(j_0)}\,. \tag{3.6}$$

If the minimum of $H_\Omega(\zeta)$ is attained by a unique value $\bar{\zeta}=(\bar{\zeta}_i)_{i\in\mathcal{N}_1^\Omega}\in[0,1]^{\mathcal{N}_1^\Omega}$, then we have

$$\lim_{\varepsilon\downarrow 0}\sup_{x\in F_i}|\varphi^\varepsilon(x)-\bar{\zeta}_i| = 0\,,\qquad i\in\mathcal{N}^\Omega\,, \tag{3.7}$$

for every compact subset F_i in $\Omega\cap\mathcal{D}(\mathcal{V}_i^e)$, where $\bar{\zeta}_{j_0}=1$ and $\bar{\zeta}_j=0$, $j\in\mathcal{N}_0^\Omega$.

The proof of the theorem will be divided into three parts.

3-1. Fermi coordinates based on the function U. Recall the condition (C_6) for the manifolds M_α and $\tilde{M}_\alpha$, $1\leqq\alpha\leqq K$, where we set $\tilde{M}_\alpha=M_\alpha$ if $\partial M_\alpha=\emptyset$. In this subsection, we shall introduce a convenient coordinate system on a tubular neighborhood of $\tilde{M}_\alpha$.

Let M be a submanifold of $\mathcal{M}$. For $y\in M$, $T_yM^\perp$ denotes the orthogonal complement of T_yM in $T_y\mathcal{M}$ with respect to the metric g. We write by $\mathcal{N}(M)$ the normal bundle of M in $\mathcal{M}$: namely,

$$\mathcal{N}(M) = \{(y,v)\,;\ y\in M \text{ and } v\in T_yM^\perp\}.$$

The map $\exp:\mathcal{N}(M)\to\mathcal{M}$ is defined by $\exp(y,v)=\exp_y v$ for $(y,v)\in\mathcal{N}(M)$, where

$\exp_x$ denotes the exponential map of $\mathcal{M}$ at $x\in\mathcal{M}$. It is known that $\exp: \mathfrak{N}(\mathcal{M})\to\mathcal{M}$ maps a neighborhood of $M\subset\mathfrak{N}(\mathcal{M})$ diffeomorphically onto a neighborhood of M in $\mathcal{M}$; see [8, p. 17]. Denote $\mathfrak{N}_\delta(M)=\{(y, v)\in\mathfrak{N}(M);\ \|v\|<\delta\}$ and $M(\delta)=\{x\in\mathcal{M};$ there exists a geodesic less that δ from x meeting M orthogonally$\}$ for $\delta>0$. The set $M(\delta)$ is called a δ-tubular neighborhood of M.

The condition (C_6) guarantees us to find $\delta_1>0$ and an open subset M'_α of $\tilde{M}_\alpha$, $1\leqq\alpha\leqq K$, satisfying $M'_\alpha\supset M_\alpha$ and the map $\exp: \mathfrak{N}_{\delta_1}(M'_\alpha)\to M'_\alpha(\delta_1)$ is a diffeomorphism.

In order to define a convenient coordinate system on $M'_\alpha(\delta_1)$, we take an arbitrary coordinate $\xi=(\xi^1, \cdots, \xi^{m_\alpha})$ defined on an open set V of M'_α together with orthonormal C^∞-sections $E_{m_\alpha+1}, \cdots, E_d$ of the restriction of $\mathfrak{N}(M'_\alpha)$ to V. We set

$$V(\delta) = \{\exp(\textstyle\sum_{p=m_\alpha+1}^d \eta^p E_p(y))\in M'_\alpha(\delta_1);\ y\in V,\ |\eta|<\delta\},$$

$$\eta = (\eta^{m_\alpha+1}, \cdots, \eta^d)\in \boldsymbol{R}^{d-m_\alpha},$$

where $\eta^p E_p(y)=(y, \eta^p v_p(y))$ and $v_p(y)$ is determined from E_p by $E_p=(y, v_p(y))$. For each point $x=\exp(\sum\eta^p E_p(y))$ of $V(\delta_1)$, we assign its coordinate $(\xi^1, \cdots, \xi^{m_\alpha}, \eta^{m_\alpha+1}, \cdots, \eta^d)$ by $\xi^r(x)=\xi^r(y)$, $1\leqq r\leqq m_\alpha$, and $\eta^q(x)=\eta^q$, $m_\alpha+1\leqq q\leqq d$. This is called the Fermi coordinate on $M'_\alpha(\delta_1)$ relative to the coordinate system $\{V, \xi\}$ and the orthonormal sections $\{E_{m_\alpha+1}, \cdots, E_d\}$. We shall sometimes write simply $x=(\xi, \eta)$ by identifying $x\in M'_\alpha(\delta_1)$ and its coordinate and denote

$$\nabla_\eta U(\xi, \eta) = \Big(\frac{\partial U}{\partial\eta^{m_\alpha+1}}(\xi, \eta), \cdots, \frac{\partial U}{\partial\eta^d}(\xi, \eta)\Big),$$

$$\nabla_\eta^2 U(\xi, \eta) = \Big(\frac{\partial^2 U}{\partial\eta^q\partial\eta^{q'}}(\xi, \eta)\Big)_{m_\alpha+1\leq q, q'\leq d}.$$

Taking (3.8), below, into account, we shall also sometimes use the notation $(\xi, \eta)=(\xi, \eta', \eta^d)\in\boldsymbol{R}^{m_\alpha}\times\boldsymbol{R}^{d-m_\alpha-1}\times\boldsymbol{R}$ in order to distinguish the d-th coordinate.

Since the coordinate vector fields $\{\partial/(\partial\eta^{m_\alpha+1}), \cdots, \partial/(\partial\eta^d)\}$ are orthonormal and orthogonal to M'_α, we have the following lemma.

LEMMA 3.2. *There are an open subset M''_α of M'_α and $0<\delta_2\leqq\delta_1$ such that $M''_\alpha\supset M_\alpha$, that $\nabla_\eta^2 U(\xi, \eta)$ is non-singular at every $(\xi, \eta)\in M''_\alpha(\delta_2)$ and that the eigenvalues of $\nabla_\eta^2 U(\xi, 0)$ coincide with non-zero ones of $H^*(y)$ for $y=\xi\in M_\alpha$.*

The next lemma is immediately verified from the implicit function theorem without difficulty.

LEMMA 3.3. *There exists a C^∞ map $y\mapsto x_0(y)$ from $M''_\alpha\subset M'_\alpha$ to $M'_\alpha(\delta_2/2)$ such that* Int $M''_\alpha\supset M_\alpha$ *and that, in terms of the Fermi coordinate (ξ, η) on $M'_\alpha(\delta_1)$, $\nabla_\eta U(\xi, \eta_0(\xi))=0$, where we write $x_0(y)=(\xi, \eta_0(\xi))$ for $y=(\xi, 0)$. In particular, $x_0(y)=y$, i.e., $\eta_0\equiv 0$ on M_α.*

We take M'_α and $\delta_1>0$ satisfying the assertions in Lemmas 3.2 and 3.3 without loss of generality and use the notation $\eta_0(\xi)$ defined in Lemma 3.3. Noting that $\nabla_\eta^2 U(\xi, \eta_0(\xi))$ is non-singular and has one negative eigenvalue, say $H_d(\xi)$, one can also find the orthonormal sections $\{E_{m_\alpha+1}, \cdots, E_d\}$ so that

$$\nabla_\eta^2 U(\xi, \eta_0(\xi)) = \begin{pmatrix} \nabla_{\eta'}^2 U(\xi, \eta_0(\xi)) & 0 \\ 0 & H_d(\xi) \end{pmatrix} \tag{3.8}$$

and that $\nabla_{\eta'}^2 U(\xi, \eta_0(\xi))$ is positive definite, where

$$\nabla_{\eta'}^2 U(\xi, \eta) = \left(\frac{\partial U^2}{\partial \eta^p \partial \eta^{p'}}(\xi, \eta)\right)_{m_\alpha+1\leq p, p'\leq d-1}.$$

We call this the Fermi coordinate on $M'_\alpha(\delta_1)$ based on the function U. Remark the determinant of $\nabla_{\eta'}^2 U(\xi, \eta_0(\xi))$ is equal to $H_+(y)$ and $H_d(\xi)=-H_-(y)$ at every $y=\xi\in M_\alpha$.

One can apply the Taylor formula for $U(\xi, \eta)$ with respect to η and get

$$U(\xi, \eta)-U(\xi, \eta_0(\xi)) = \frac{1}{2}\langle \eta'-\eta'_0(\xi), \nabla_{\eta'}^2 U(\xi, \eta_0(\xi))(\eta'-\eta'_0(\xi))\rangle + \frac{1}{2}H_d(\xi)\{\eta^d-\eta_0^d(\xi)\}^2+\frac{1}{6}R(\xi, \bar\eta)(\eta), \tag{3.9}$$

for $\xi\in M'_\alpha$, where

$$\begin{aligned}&\langle \eta'-\eta'_0(\xi), \nabla_{\eta'}^2 U(\xi, \eta_0(\xi))(\eta'-\eta'_0(\xi))\rangle \\ &= \sum_{p,p'=m_\alpha+1}^{d-1} \{\eta^p-\eta_0^p(\xi)\}\{\eta^{p'}-\eta_0^{p'}(\xi)\}\frac{\partial^2 U}{\partial\eta^p\partial\eta^{p'}}(\xi, \eta_0(\xi)),\\ &R(\xi, \bar\eta)(\eta)\\ &= \sum_{q,q',q''=m_\alpha+1}^{d} \{\eta^q-\eta_0^q(\xi)\}\{\eta^{q'}-\eta_0^{q'}(\xi)\}\{\eta^{q''}-\eta_0^{q''}(\xi)\}\frac{\partial^3 U}{\partial\eta^q\partial\eta^{q'}\partial\eta^{q''}}(\xi, \bar\eta),\\ &\qquad \bar\eta = c(\eta-\eta_0(\xi))+\eta_0(\xi), \qquad c\in(0, 1).\end{aligned}$$

Find $h_+>h_->0$ and $\delta_1>\delta_2>0$ so that

$$h_- < |H_d(\xi)| < h_+, \qquad h_-|\zeta'|^2 < \langle\zeta', \nabla_{\eta'}^2 U(\xi, \eta_0(\xi))\zeta'\rangle < h_+|\zeta'|^2,$$

for all $\xi\in M'_\alpha$ and $\zeta'\in \boldsymbol{R}^{d-m_\alpha-1}$ and that

$$\frac{1}{4}h_-|\eta-\eta_0(\xi)|^2 \geqq \frac{1}{3}|R(\xi, \bar\eta)(\eta)| \tag{3.10}$$

for all $|\eta-\eta_0(\xi)|\leqq\delta_2$, $|\bar\eta|\leqq\delta_1$ and $\xi\in M'_\alpha$. Then, one obtains the following estimates:

$$U(\xi, \eta)-U(\xi, \eta_0(\xi)) \leqq \left(\frac{1}{2}h_+ + \frac{1}{4}h_-\right)|\eta'-\eta'_0(\xi)|^2-\frac{1}{4}h_-|\eta^d-\eta_0^d(\xi)|^2, \tag{3.11a}$$

(3.11b) $$U(\xi, \eta)-U(\xi, \eta_0(\xi)) \geqq \frac{1}{4}h_-|\eta'-\eta_0'(\xi)|^2-\left(\frac{1}{2}h_+ +\frac{1}{4}h_-\right)|\eta^d-\eta_0^d(\xi)|^2,$$

for $|\eta-\eta_0(\xi)|\leqq\delta_2$ and $\xi\in M_\alpha'$.

We enclose this subsection with the next remark.

REMARK 3.4. We assume without loss of generality that M_α' has the following properties.

(i) $U(\xi, \eta_0'(\xi), \eta^d+\eta_0^d(\xi))<V_0$ for all $\xi\in M_\alpha'$ and $\delta_2/2<|\eta^d|<\delta_2$.

(ii) $U(\xi, \eta_0(\xi))>V_0$ for all $\xi\in M_\alpha'\setminus M_\alpha$.

3-2. Upper bound estimates. In the subsection we shall prove the upper bound estimate

(3.12) $$\limsup_{\varepsilon\downarrow 0} \varepsilon^{m_\Omega-n^{(j_0)}}e^{V_0/\varepsilon^2}\lambda^\varepsilon \leqq H_\Omega/2\nu^{(j_0)}.$$

But with the help of the Rayleigh-Ritz formula (3.2), it suffices to construct a sequence of functions $\{\psi^\varepsilon\}\subset C_0^\infty(\Omega)$ satisfying that

(3.13) $$\limsup_{\varepsilon\downarrow 0} \varepsilon^{m_\Omega-n^{(j_0)}+2}e^{V_0/\varepsilon^2}\frac{J^\varepsilon(\psi^\varepsilon)}{\|\psi^\varepsilon\|_\varepsilon^2} \leqq \frac{H_\Omega}{\nu^{(j_0)}}.$$

To this end, we consider the tubular neighborhood $M_\alpha'(\delta_1)$ introduced in the previous subsection. Assuming $M_\alpha'(\delta_1)\subset\Omega$ if $M_\alpha\subset\Omega$, we fix $M_\alpha\subset\Omega$ and an open subset W_α of M_α' such that $M_\alpha'\supset\overline{W}_\alpha$ and $W_\alpha\supset M_\alpha$. From Remark 3.4 (ii), there is $0<\delta_3<\delta_2$ such that $U(x)>V_0+\delta_3$ for every $x\in[M_\alpha']_0\setminus[W_\alpha]_0$, where we put

$$[M_\alpha']_0 = \{(\xi, \eta)\in M_\alpha'(\delta_1)\,;\,|\eta-\eta_0(\xi)|<\delta_3,\ \xi\in M_\alpha'\},$$

$$[W_\alpha]_0 = \{(\xi, \eta)\in M_\alpha'(\delta_1)\,;\,|\eta-\eta_0(\xi)|<\delta_3,\ \xi\in W_\alpha\}.$$

Remembering the inequality (3.11a), set $\delta_3'=\sqrt{h_-/(4h_+ +2h_-)}\cdot\delta_3$ and find open subsets O_k', $k=1, 2, 3, 4$, of $\{(x, y)\in \boldsymbol{R}^2\,;\, x^2+y^2<\delta_3^2\}$ satisfying the following conditions:

(1) $O_k'\supset\overline{O_{k+1}'}$ for all $k=1, 2, 3$;

(2) $(x, y)\in O_k'$ if and only if $(-x, y)\in O_k'$ for all $k=1, 2, 3, 4$;

(3) $O_4'\supset[\{y\geqq\delta_3'/4\}\cup\{|x|\leqq\delta_3/4,\ y\geqq-\delta_3'/8\}]$,
$\overline{O_3'}\subset[\{y>\delta_3'/8\}\cup\{|x|\leqq\delta_3/2,\ y\geqq-\delta_3'/4\}]$,
$O_2'\supset[\{|y|\geqq\delta_3'/8\}\cup\{|x|\leqq 3\delta_3/4,\ |y|<\delta_3'/8\}]$,
$\overline{O_1'}\subset[\{|y|>0\}\cup\{|x|\leqq 9\delta_3/10,\ y=0\}]$.

Then, set

$$O_k(y) = \{(\xi, \eta+\eta_0(\xi))\,;\,(|\eta'|, \eta^d)\in O_k'\}, \qquad y=\xi\in M_\alpha',\ k=1, 2, 3, 4,$$

$$[W_\alpha]_k = \bigcup\nolimits_{y\in W_\alpha} O_k(y), \qquad k=1, 2, 3, 4,$$

where we remark each $O_k(y)$, $k=1, 2, 3, 4$, does not depend on any particular choice of the Fermi coordinate based on the function U.

Now we start to construct $\phi\in C_0^\infty(\Omega)$ (which does not depend on $\varepsilon>0$). Let $\tilde{\zeta}=(\tilde{\zeta}_i)_{i\in\mathcal{N}_1^\Omega}$ attain the minimum of $H_\Omega(\zeta)$; recall (2.5) and (3.5) for the definition of $H_\Omega(\zeta)$, and put $\tilde{\zeta}_{j_0}=1$ and $\tilde{\zeta}_j=0$, $j\in\mathcal{N}_1^\Omega$. First we define ϕ so that

$$\operatorname{grad}\phi(x)=0 \quad \text{for } x\in\{x\,;\,U(x)<V_0+\delta_3\}\setminus\cup_\alpha[M'_\alpha]_0,$$

$$\phi(x)=\tilde{\zeta}_i \quad \text{for } x\in\mathcal{V}_i^\varepsilon\setminus\cup_\alpha[W_\alpha]_0,\ i\in\mathcal{N}^\Omega.$$

Next, we construct ϕ on $[W_\alpha]_0$. Assume $M_\alpha\subset\bar{\mathcal{V}}_i\cap\bar{\mathcal{V}}_{i'}$, $i, i'\in\mathcal{N}^\Omega$. If $\tilde{\zeta}_i<\tilde{\zeta}_{i'}$, set

$$\phi=0 \quad \text{on } [W_\alpha]_0\setminus[W_\alpha]_1,\quad 0\leqq\phi\leqq\tilde{\zeta}_i \text{ on } [W_\alpha]_1\setminus[W_\alpha]_2,$$

$$\phi=\tilde{\zeta}_i \quad \text{on } [W_\alpha]_2\setminus[W_\alpha]_3,\ \tilde{\zeta}_i\leqq\phi\leqq\tilde{\zeta}_{i'} \text{ on } [W_\alpha]_3\setminus[W_\alpha]_4,\ \phi=\tilde{\zeta}_{i'} \text{ on } [W_\alpha]_4,$$

and put ϕ similarly in the other cases.

Let $\tau: (-\sqrt{h_-}\delta'_3/4, \sqrt{h_-}\delta'_3/4)\to\boldsymbol{R}^1$ denote a smooth function satisfying $\tau(\mathrm{x})=\mathrm{x}$ on $(-\sqrt{h_-}\delta'_3/8, \sqrt{h_-}\delta'_3/8)$, $\tau'(\mathrm{x})\geqq1$ on $(-\sqrt{h_-}\delta'_3/4, \sqrt{h_-}\delta'_3/4)$ and $|\tau(\mathrm{x})|\to+\infty$, $\tau'(\mathrm{x})e^{-a\tau(\mathrm{x})}\to0$ as $\mathrm{x}\to\pm\sqrt{h_-}\delta'_3/4$ for every $a>0$. Then, we define $\sigma^\varepsilon(\mathrm{x})$ on $(-\sqrt{h_+}\delta'_3, \sqrt{h_+}\delta'_3)$ by

$$\sigma^\varepsilon(\mathrm{x})=\begin{cases}0, & -\sqrt{h_+}\delta'_3<\mathrm{x}\leqq-\sqrt{h_-}\delta'_3/4,\\ \dfrac{1}{\sqrt{2\pi}\cdot\varepsilon}\displaystyle\int_{-\infty}^{\tau(\mathrm{x})}e^{-y^2/2\varepsilon^2}dh, & -\sqrt{h_-}\delta'_3/4<\mathrm{x}<\sqrt{h_-}\delta'_3/4,\\ 1, & \sqrt{h_-}\delta'_3/4\leqq\mathrm{x}<\sqrt{h_+}\delta'_3,\end{cases}$$

and $\theta^\varepsilon(\xi, \eta)=\sigma^\varepsilon(\{\eta^d-\eta_0^d(\xi)\}\sqrt{|H_\alpha(\xi)|})$, $(\xi, \eta', \eta^d)\in[M'_\alpha]_0$. Note that $\theta^\varepsilon(\xi, \eta)$ does not depend on any particular choice of the Fermi coordinates based on the function U and one can be regarded as a smooth function on $[M'_\alpha]_0$.

Finally, we set $\phi^\varepsilon=\phi$ on $\Omega\setminus\cup_\alpha[M'_\alpha]_0$ and, for $x\in[M'_\alpha]_0$,

$$\phi^\varepsilon(x)=\begin{cases}\phi(x), & \text{if } |\eta'(x)-\eta'_0(\xi(x))|\geqq\delta_3/2 \text{ or } |\eta^d(x)-\eta_0^d(\xi(x))|\geqq\delta'_3,\\ (\phi(x)-\tilde{\zeta}_i)\theta^\varepsilon(x)+\tilde{\zeta}_i, & \text{otherwise},\end{cases}$$

in case that $\tilde{\zeta}_i>\tilde{\zeta}_{i'}$, and similarly in the other cases. Here one remarks $\phi^\varepsilon\in C_0^\infty(\Omega)$ and $U(x)>V_0+(h_-\delta_3^2/128)>\delta_3$ for $x\in\{\operatorname{grad}\phi\neq0,\ \theta^\varepsilon\neq0\}$. This follows from the estimate (3.11b) and the property that $\theta^\varepsilon(x)=0$ for $x=(\xi, \eta', \eta^d)\in[M'_\alpha]_0$ satisfying $-\delta'_3\leqq\eta^d-\eta_0^d(\xi)\leqq-\delta'_3/4$.

Let us start to prove (3.13). Note

$$\limsup_{\varepsilon\downarrow0}\varepsilon^{-(d-m_\Omega)+2}\int_{\Omega\setminus\cup_\alpha[W_\alpha]_5}\|\operatorname{grad}\phi^\varepsilon\|^2e^{-(U(x)-V_0)/\varepsilon^2}dx=0,$$

where $[W_\alpha]_5=\{(\xi, \eta+\eta_0(\xi))\in[W_\alpha]_0\,;\,\xi\in W_\alpha,\ |\eta'|<\delta_3,\ |\eta^d|<\delta'_3/4\}$. From the Leibniz rule and the triangle inequality,

$$\int_{[W_\alpha]_5} \|\text{grad}\, \phi^\varepsilon(x)\|^2 e^{-(U(x)-V_0)/\varepsilon^2} dx \leq [\{J_\alpha^\varepsilon\}^{1/2} + e^{-a'/\varepsilon^2}]^2$$

for some $a'>0$, where

$$J_\alpha^\varepsilon = \int_{[W_\alpha]_5} \{\phi(x) - \tilde{\zeta}_i\}^2 \|\text{grad}\, \theta^\varepsilon(x)\|^2 e^{-(U(x)-V_0)/\varepsilon^2} dx .$$

Note that

$$J_\alpha^\varepsilon = \sum_\iota \int \rho_\iota(\xi, 0) d\xi \int_{|\eta'|<\delta_3} d\eta' \int_{|\eta^d|<\delta_3'/4} d\eta^d \Psi^\varepsilon(\xi, \eta + \eta_0(\xi)),$$

where $\{\rho_\iota\}$ is a C^∞ partition of unity on M_α' and

$$\Psi^\varepsilon(\xi, \eta) = \{\phi(\xi, \eta) - \tilde{\zeta}_i\}^2 \sum_{i,j=1}^d g^{ij}(\xi, \eta) \frac{\partial \theta^\varepsilon}{\partial \chi^i}(\xi, \eta) \frac{\partial \theta^\varepsilon}{\partial \chi^j}(\xi, \eta) e^{-(U(\xi,\eta)-V_0)/\varepsilon^2} \sqrt{g(\xi, \eta)},$$

and $\partial/\partial \chi^r = \partial/\partial \xi^r$, $1 \leqq r \leqq m_\alpha$, $\partial/\partial \chi^p = \partial/\partial \eta^p$, $m_\alpha + 1 \leqq p \leqq d$. By the formulae

$$\frac{\partial \theta^\varepsilon}{\partial \xi^r}(\xi, \eta) = \tau'(\{\eta^d - \eta_0^d(\xi)\} \sqrt{|H_d(\xi)|}) \frac{\partial}{\partial \xi^r} \{(\eta^d - \eta_0^d(\xi)) \sqrt{|H_d(\xi)|}\}$$

$$\times \frac{1}{\sqrt{2\pi} \cdot \varepsilon} e^{-\frac{\tau(\{\eta^d - \eta_0^d(\xi)\} \sqrt{|H_d(\xi)|})^2}{2\varepsilon^2}} \qquad 1 \leqq r \leqq m_\alpha ,$$

$$\frac{\partial \theta^\varepsilon}{\partial \eta^p}(\xi, \eta) = 0, \qquad m_\alpha + 1 \leqq p \leqq d-1 ,$$

$$\frac{\partial \theta^\varepsilon}{\partial \eta^d}(\xi, \eta) = \tau'(\{\eta - \eta_0^d(\xi)\} \sqrt{|H_d(\xi)|}) \frac{\sqrt{|H_d(\xi)|}}{\sqrt{2\pi}\, \varepsilon} e^{-\frac{\tau(\{\eta^d - \eta_0^d(\xi)\} \sqrt{|H_d(\xi)|})^2}{2\varepsilon^2}} ,$$

and the property that $\eta_0 \equiv 0$ on M_α, Laplace's methods imply

$$\text{(3.14)} \qquad \lim_{\varepsilon \downarrow 0} \varepsilon^{-(d-m_\alpha)+2} \int_{|\eta'|<\delta_3} d\eta' \int_{|\eta^d|<\delta_3'/4} d\eta^d \Psi^\varepsilon(\xi, \eta + \eta_0(\xi))$$

$$= (2\pi)^{(d-m_\alpha-2)/2} \{\tilde{\zeta}_{i'} - \tilde{\zeta}_i\}^2 \left\{\frac{H_-(\xi, 0)}{H_+(\xi, 0)}\right\}^{1/2} \sqrt{g(\xi, 0)},$$

if $\xi \in M_\alpha$, and otherwise the left hand side (LHS) of (3.14) vanishes. Now, we conclude

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-(d-m_\alpha)+2} J_\alpha^\varepsilon = (2\pi)^{(d-m_\alpha-2)/2} \{\tilde{\zeta}_{i'} - \tilde{\zeta}_i\}^2 \int_{M_\alpha} \left\{\frac{H_-(y)}{H_+(y)}\right\}^{1/2} dy ,$$

and therefore

$$\text{(3.15)} \qquad \limsup_{\varepsilon \downarrow 0} \varepsilon^{-(d-m_\Omega)+2} e^{V_0/\varepsilon^2} J^\varepsilon(\phi^\varepsilon) \leqq H_\Omega .$$

Next, we claim

$$\text{(3.16)} \qquad \lim_{\varepsilon \downarrow 0} \varepsilon^{-(d-n(j_0))} \|\phi^\varepsilon\|_\varepsilon^2 = \nu^{(j_0)} .$$

Take $\delta>0$ and an open subset N'_α of $\tilde{N}_\alpha$, $1\leqq\alpha\leqq l_{j_0}$, so that $N'_\alpha\supset N_\alpha$ and that the map $\exp: \mathcal{N}_\delta(N'_\alpha)\to N'_\alpha(\delta)$ is a diffeomorphism, where we write $N_\alpha=N_\alpha^{(j_0)}$ and $\tilde{N}_\alpha=\tilde{N}_\alpha^{(j_0)}$, $1\leqq\alpha\leqq l_{j_0}$, simply and set $\tilde{N}_\alpha=N_\alpha$ if $\partial N_\alpha=\emptyset$. We shall also write $x=(\xi, \eta)$ by identifying $x\in N'_\alpha(\delta)$ and the Fermi coordinate. One can prove the following lemmas in the same manner as Lemmas 3.2 and 3.3.

LEMMA 3.2′. *There are an open subset N''_α of N'_α and $0<\delta'\leqq\delta$ such that $N''_\alpha\supset N_\alpha$, that $\nabla_\eta^2 U(\xi, \eta)$ is non-singular at every $(\xi, \eta)\in N''_\alpha(\delta')$ and that the eigenvalues of $\nabla_\eta^2 U(\xi, 0)$ coincide with non-zero ones of $H^*(y)$ for $y=\xi\in N_\alpha$.*

LEMMA 3.3′. *There exists a C^∞ map $y\mapsto x_1(y)$ from $N''_\alpha\subset N'_\alpha$ to $N'_\alpha(\delta'/2)$ such that* $\operatorname{Int} N''_\alpha\supset N_\alpha$ *and that, in terms of the Fermi coordinate (ξ, η) on $N'_\alpha(\delta)$, $\nabla_\eta U(\xi, \eta_1(\xi))=0$, where we write $x_1(y)=(\xi, \eta_1(\xi))$ for $y=(\xi, 0)$. In particular, $x_1(y)=y$, i.e., $\eta_1\equiv 0$ on N_α.*

We shall assume that N'_α and $\delta'>0$ satisfy the assertions in Lemmas 3.2′ and 3.3′ without loss of generality and use the notation $\eta_1(\xi)$ defined in Lemma 3.3′. Note $U(x)>0$ at every $x\in N'_\alpha(\delta')\setminus N_\alpha$.

Now we start to show (3.16). Since $\zeta_{j_0}=1$, one notices

$$\|\psi^\delta\|_\varepsilon^2=\sum_{\alpha=1}^{l_{j_0}}\int_{N'_\alpha(\delta')} e^{-U(x)/\varepsilon^2}dx+O(e^{-a''/\varepsilon^2}),\qquad \text{as } \varepsilon\downarrow 0, \tag{3.17}$$

for some $a''>0$. If we fix $1\leqq\alpha\leqq l_{j_0}$ and denote by $\{\rho_\iota\}$ a C^∞ partition of unity on N'_α, we have

$$\int_{N'_\alpha(\delta')} e^{-U(x)/\varepsilon^2}dx=\sum_\iota\int\rho_\iota(\xi, 0)d\xi\int_{|\eta|<\delta'}d\eta\,\sqrt{g(\xi, \eta+\eta_1(\xi))}e^{-U(\xi,\eta+\eta_1(\xi))/\varepsilon^2}.$$

But, from Lemmas 3.2′ and 3.3′, Laplace's methods imply

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\varepsilon^{-(d-n(j_0))}\int_{|\eta|<\delta'}d\eta\,\sqrt{g(\xi, \eta+\eta_1(\xi))}e^{-U(\xi,\eta+\eta_1(\xi))/\varepsilon^2}\\ =(2\pi)^{(d-n(j_0))/2}H_+(\xi, 0)^{-1/2}\sqrt{g(\xi, 0)}\end{aligned} \tag{3.18}$$

for $\xi\in N_\alpha$ and otherwise the LHS of (3.18) vanishes. Hence, we have

$$\lim_{\varepsilon\downarrow 0}\int_{N'_\alpha(\delta')} e^{-U(x)/\varepsilon^2}dx=(2\pi)^{(d-n(j_0))/2}\int_{N_\alpha}H_+(y)^{-1/2}dy$$

and, together with (3.17), we arrive at (3.16).

The estimate (3.13) is obtained immediately from (3.15) and (3.16). □

3-3. Proof of Theorem 3.1.

Recall that φ^ε is the eigenfunction associated with the principal eigenvalue λ^ε and is normalized as (3.4). The following proposition was proved in [**18**, Theorem 3.1], where we also use the Stokes' formula. See also [**2**, Lemma 2.2].

PROPOSITION 3.5. (i) *For every* $i\in\mathcal{N}^{\Omega}$ *and compact subset* F_i *of* $\mathcal{D}(\mathcal{V}_i^{\varepsilon})\cap\Omega$, *there is* $r>0$ *so that, for all sufficiently small* $\varepsilon>0$,

$$\sup_{x,y\in F_i}|\varphi^{\varepsilon}(x)-\varphi^{\varepsilon}(y)|\leqq e^{-r/\varepsilon^2},$$

and in particular, if $i=j_0$,

$$\inf_{x\in F_{j_0}}\varphi^{\varepsilon}(x)\geqq 1-e^{-r/\varepsilon^2}.$$

(ii) *For every* $j\in\mathcal{N}_0^{\Omega}$ *and compact set* $F_j\subset\mathcal{D}(\mathcal{V}_j^{\varepsilon})$, *there is* $r>0$ *so that*

$$\sup_{x\in F_j\cap\Omega}\varphi^{\varepsilon}(x)\leqq e^{-r/\varepsilon^2}$$

for all sufficiently small $\varepsilon>0$.

The next lemma is shown from the above proposition by the same methods as (3.16).

LEMMA 3.6. $\lim_{\varepsilon\downarrow 0}\varepsilon^{-(d-n(j_0))}\|\varphi^{\varepsilon}\|_{\varepsilon}^2=\nu^{(j_0)}$.

Let $b_i\in\Omega$ be a fixed point in $\mathcal{V}_i$ (not $\mathcal{V}_i^{\varepsilon}$) for each $i\in\mathcal{N}^{\Omega}$.

LEMMA 3.7. *Let* M_α *in* Ω *and* $\mathcal{V}_i$, $\mathcal{V}_{i'}$, i, $i'\in\mathcal{N}^{\Omega}$, *satisfy* $M_\alpha\subset\bar{\mathcal{V}}_i\cap\bar{\mathcal{V}}_{i'}$ *and suppose* $\lim_{n\to\infty}\varphi^{\varepsilon_n}(b_i)=\zeta_i$ *and* $\lim_{n\to\infty}\varphi^{\varepsilon_n}(b_{i'})=\zeta_{i'}$ *for some subsequence* $\{\varepsilon_n\}$ *of* $\{\varepsilon\}$. *Then, we have*

$$\liminf_{n\to\infty}\varepsilon_n^{-d+m_\alpha+2}e^{V_0/\varepsilon_n^2}\int_{M_\alpha(\delta_2)}\|\text{grad}\varphi^{\varepsilon_n}(x)\|^2e^{-U(x)/\varepsilon_n^2}dx \tag{3.19}$$
$$\geqq\{\zeta_i-\zeta_{i'}\}^2(2\pi)^{(d-m_\alpha-2)/2}\int_{M_\alpha}\left\{\frac{H_-(y)}{H_+(y)}\right\}^{1/2}dy.$$

PROOF. Remembering (3.11a) and Remark 3.4 (i), in terms of the Fermi coordinate (ξ,η) based on the function U, set $O_6=\{\eta\in\boldsymbol{R}^{d-m_\alpha};\ |\eta'|\leqq\delta_2'/4,\ |\eta^d|\leqq 3\delta_2/4\}$ and $[M_\alpha]_6=\{(\xi,\eta);\ \xi\in M_\alpha,\ \eta\in O_6\}$, where $\delta_2'=\sqrt{h_-/(2h_++h_-)}\cdot\delta_2$, and choose compact sets $F_i\subset\mathcal{D}(\mathcal{V}_i^{\varepsilon})\cap\Omega$, $F_{i'}\subset\mathcal{D}(\mathcal{V}_{i'}^{\varepsilon})\cap\Omega$ such that $b_i\in F_i$, $b_{i'}\in F_{i'}$ and that

$$(\xi,\eta)\in F_i\quad\text{for all}\quad \xi\in M_\alpha\ \text{and}\ |\eta'|\leqq\delta_2'/4,\ \delta_2/2\leqq\eta^d\leqq\delta_2, \tag{3.20a}$$

$$(\xi,\eta)\in F_{i'}\quad\text{for all}\quad \xi\in M_\alpha\ \text{and}\ |\eta'|\leqq\delta_2'/4,\ -\delta_2\leqq\eta^d\leqq-\delta_2/2, \tag{3.20b}$$

respectively. From Proposition 3.5, one can find $r>0$ so that

$$\sup_{x\in F_i}|\varphi^{\varepsilon}(x)-\varphi^{\varepsilon}(b_i)|\leqq e^{-r/\varepsilon^2},\quad \sup_{x\in F_{i'}}|\varphi^{\varepsilon}(x)-\varphi^{\varepsilon}(b_{i'})|\leqq e^{-r/\varepsilon^2}, \tag{3.21}$$

for all sufficiently small $\varepsilon>0$. For a C^∞ partition of unity $\{\rho_l\}$ on $\tilde{M}_\alpha$,

$$I^{\varepsilon}\equiv\int_{[M_\alpha]_6}\|\operatorname{grad}\varphi^{\varepsilon_n}(x)\|^2 e^{-(U(x)-V_0)/\varepsilon^2}dx$$
$$=\sum_\iota\int\rho_\iota(\xi,0)d\xi\int_{O_6}d\eta\sum_{i,j=1}^{d}g^{ij}(\xi,\eta)\frac{\partial\varphi^\varepsilon}{\partial\chi^i}(\xi,\eta)\frac{\partial\varphi^\varepsilon}{\partial\chi^j}(\xi,\eta)e^{-(U(\xi,\eta)-V_0)/\varepsilon^2}\sqrt{g(\xi,\eta)}.$$

Since $g^{ip}(\xi,0)=\delta^{ip}$ for $1\leqq i\leqq d$ and $m_\alpha+1\leqq p\leqq d$, there is a continuous function $\lambda(\xi,\eta)$ satisfying $\lambda(\xi,0)=1$ and

$$\sum_{i,j=1}^{d}g^{ij}(\xi,\eta)\frac{\partial\varphi^\varepsilon}{\partial\chi^i}(\xi,\eta)\frac{\partial\varphi^\varepsilon}{\partial\chi^j}(\xi,\eta)\geqq\lambda(\xi,\eta)\left|\frac{\partial\varphi^\varepsilon}{\partial\eta^d}(\xi,\eta)\right|^2,$$

where we suppose $\lambda(\xi,\eta)>0$ on $[M_\alpha]_6$ without loss of generality. From (3.9), Schwarz's inequality implies

$$I^{\varepsilon}\geqq\sum_\iota\int\rho_\iota(\xi,0)\frac{[I_+^{\varepsilon}(\xi)]^2}{I_-^{\varepsilon}(\xi)}d\xi, \tag{3.22}$$

where

$$I_+^{\varepsilon}(\xi)=\iint_{O_6}d\eta'd\eta^d\left|\frac{\partial\varphi^\varepsilon}{\partial\eta^d}(\xi,\eta)\right|\exp\left\{-\frac{1}{2\varepsilon^2}\langle\eta',\nabla_{\eta'}^2U(\xi,0)\eta'\rangle\right\},$$
$$I_-^{\varepsilon}(\xi)=\iint_{O_6}d\eta'd\eta^d\lambda(\xi,\eta)^{-1}\sqrt{g(\xi,\eta)}^{-1}$$
$$\times\exp\left\{-\frac{1}{2\varepsilon^2}(\langle\eta',\nabla_{\eta'}^2U(\xi,0)\eta'\rangle-H^d(\xi)|\eta^d|^2)-\frac{1}{6\varepsilon^2}R(\xi,\bar\eta)(\eta)\right\}.$$

Noting the estimate

$$\int_{-3\delta_2/4}^{3\delta_2/4}\left|\frac{\partial\varphi^\varepsilon}{\partial\eta^d}(\xi,\eta)\right|d\eta^d\geqq|\varphi^\varepsilon(b_i)-\varphi^\varepsilon(b_{i'})|-2e^{-r/\varepsilon^2},$$

which follows from (3.20) and (3.21), one applies Laplace's methods and gets

$$\liminf_{n\to\infty}\varepsilon^{-(d-m_\alpha-1)}I_+^{\varepsilon_n}\geqq|\zeta_i-\zeta_{i'}|\cdot\frac{(2\pi)^{(d-m_\alpha-1)/2}}{\sqrt{H_+(\xi)}}. \tag{3.23}$$

On the other hand, with the help of (3.10), Laplace's methods also verify

$$\lim_{\varepsilon\to0}\varepsilon^{-(d-m_\alpha)}I_-^{\varepsilon}=\frac{(2\pi)^{(d-m_\alpha)/2}}{\sqrt{H_+(\xi)H_-(\xi)}}\sqrt{g(\xi,0)}^{-1}. \tag{3.24}$$

Therefore (3.19) is obtained from (3.20) by using Fatou's lemma with (3.23) and (3.24). □

COROLLARY 3.8. *If* $\max_{M_\alpha\subset\mathcal{V}_i\cap\mathcal{V}_{i'}}>m_\Omega$, *then* $\zeta_i=\zeta_{i'}$.

PROOF. By virtue of Lemmas 3.6 and 3.7 combined with the formula (3.3), we have

$$c\cdot|\zeta_i-\zeta_{i'}|^2\leqq\frac{1}{2}\liminf_{n\to\infty}\frac{\varepsilon_n^{-d+m_\alpha+2}e^{V_0/\varepsilon_n^2}I_\alpha^{\varepsilon_n}}{\varepsilon_n^{-d+n(j_0)}\|\varphi^{\varepsilon_n}\|_{\varepsilon_n}^2}\leqq\frac{1}{2}\limsup_{\varepsilon\downarrow0}\varepsilon^{m_\alpha-n(j_0)}e^{V_0/\varepsilon^2}\lambda_\varepsilon,$$

for some constant $c>0$, where $I_{\alpha}^{\varepsilon}=\int_{M_{\alpha}(\delta_2)}\|\mathrm{grad}\varphi^{\varepsilon}\|^2 e^{-U/\varepsilon^2}dx$. Hence, if $m_{\alpha}>m_{\Omega}$, the RHS of the above inequality vanishes from (3.12). □

PROOF OF THEOREM 3.1. Let $\{\varepsilon_n\}$ be an arbitrary subsequence of $\{\varepsilon\}$. Since $0\leqq\varphi^{\varepsilon}(b_i)\leqq 1$ for every $\varepsilon>0$ and $i\in\mathcal{N}^{\Omega}$, there exists an subsequence $\{\varepsilon_{n'}\}$ of $\{\varepsilon_n\}$ such that $\lim_{n'\to\infty}\varphi^{\varepsilon_{n'}}(b_i)=\tilde{\zeta}_i$ for some $0\leqq\tilde{\zeta}_i\leqq 1$, $i\in\mathcal{N}^{\Omega}$. From Proposition 3.5, we know $\tilde{\zeta}_{j_0}=1$ and $\tilde{\zeta}_j=0$, $j\in\mathcal{N}_0^{\Omega}$. By using the formula (3.3) with Lemmas 3.6, 3.7 and Corollary 3.8, it holds that

$$\begin{aligned}
&\liminf_{n'\to\infty}\varepsilon_{n'}^{m_{\Omega}-n(j_0)}e^{V_0/\varepsilon_{n'}^2}\lambda^{\varepsilon_{n'}}\\
&\geqq \liminf_{n'\to\infty}\frac{\varepsilon_{n'}^{-d+m_{\Omega}+2}e^{V_0/\varepsilon_{n'}^2}\sum_{i,i'\in\mathcal{N}^{\Omega}:\ \max_{M_{\alpha}\subset\bar{\mathcal{V}}_i\cap\bar{\mathcal{V}}_{i'}}m_{\alpha}\leqq m_{\Omega}}\sum_{\alpha:\ M_{\alpha}\subset\bar{\mathcal{V}}_i\cap\bar{\mathcal{V}}_{i'}}I_{\alpha}^{\varepsilon_{n'}}}{2\varepsilon_{n'}^{-d+n(j_0)}\|\varphi^{\varepsilon_{n'}}\|_{\varepsilon_{n'}}^2}\\
&\geqq H_{\Omega}(\tilde{\zeta})/2\nu^{(j_0)},
\end{aligned}$$

where $I_{\alpha}^{\varepsilon}=\int_{M_{\Omega}(\delta_2)}\|\mathrm{grad}\varphi^{\varepsilon}\|^2 e^{-U/\varepsilon^2}dx$ and $\tilde{\zeta}=(\tilde{\zeta}_i)_{i\in\mathcal{N}_1^{\Omega}}$. Comparing with (3.12), we have $H_{\Omega}(\tilde{\zeta})=H_{\Omega}$ and (3.6) is obtained. However, if the minimum of $H_{\Omega}(\zeta)$ is attained by exactly one value $\bar{\zeta}=(\bar{\zeta}_i)_{i\in\mathcal{N}_1^{\Omega}}$, we have $\tilde{\zeta}_i=\bar{\zeta}_i$, $i\in\mathcal{N}_1^{\Omega}$, which immediately implies (3.7) from Proposition 3.5. □

4. Exit problems.

Let λ_j^{ε} denote the principal eigenvalue of $\mathcal{L}^{\varepsilon}$ in $\boldsymbol{D}$ with the Dirichlet boundary condition, i. e., for the boundary value problem (1.3) in which $\boldsymbol{D}$ should be replaced with Ω, for $1\leqq j\leqq l$. One can find the next theorem in [18].

THEOREM 4.1. *Let F be a compact subset to $\boldsymbol{D}\cap\mathcal{D}(\mathcal{V}_j^{\varepsilon})$. There exists a positive constant r so that*

$$\sup_{x\in F}|\lambda_j^{\varepsilon}E_x[\tau_{Dj}^{\varepsilon}]-1|\leqq e^{-r/\varepsilon^2}$$

for all sufficiently small $\varepsilon>0$.

By combining with Theorem 3.1, the above theorem immediately verifies Theorem 1. (See also Remark 4.9, below.)

We move to the proof of Theorem 2.

For a non-negative integer a, we introduce an equivalence relation $\sim_a$ on the set of valleys $\boldsymbol{V}\equiv\{\mathcal{V}_1,\cdots,\mathcal{V}_L\}$ in the following manner:

(1) $\mathcal{V}_i\sim_a\mathcal{V}_i$;

(2) $\mathcal{V}_i\sim_a\mathcal{V}_{i'}$, $\mathcal{V}_i\neq\mathcal{V}_{i'}$, if $\max_{M_{\Omega}\subset\bar{\mathcal{V}}_{i_q}\cap\bar{\mathcal{V}}_{i_{q+1}}}m_{\alpha}>a$, $0\leqq q\leqq p$, for some $\mathcal{V}_{i_1},\cdots,\mathcal{V}_{i_p}\in\boldsymbol{V}$, where $\mathcal{V}_{i_0}=\mathcal{V}_i$ and $\mathcal{V}_{i_{p+1}}=\mathcal{V}_{i'}$.

We denote the equivalence class of $\mathcal{V}_i\in\boldsymbol{V}$ by $C_a(\mathcal{V}_i)=\{\mathcal{V}_{i'}\in\boldsymbol{V};\ \mathcal{V}_{i'}\sim_a\mathcal{V}_i\}$.

Let us decompose $\boldsymbol{V}$ into $\{\boldsymbol{V}_i\}_{i=1,\dots,L_0}$. First, set $\boldsymbol{V}_j = \mathcal{C}_{m(j)}(\mathcal{V}_j)$, $1 \leqq j \leqq l$; recall (2.6) for $m^{(j)}$. Next, $\boldsymbol{V}_{l+1}, \cdots, \boldsymbol{V}_{L_1}$ denote $\mathcal{C}_0(\mathcal{V}_i)$'s satisfying $\mathcal{C}_0(\mathcal{V}_i) \subset \tilde{\boldsymbol{V}} (= \{\mathcal{V}_{l+1}, \cdots, \mathcal{V}_L\})$. Finally, for $1 \leqq a \leqq \bar{m} (= \max_{1 \leq j \leq l} m^{(j)})$, define $\boldsymbol{V}_{L_a+1}, \cdots, \boldsymbol{V}_{L_{a+1}}$ as $\mathcal{C}_a(\mathcal{V}_i)$'s such that $\#[\mathcal{C}_{a-1}(\mathcal{V}_i) \cap \boldsymbol{V}_0] \geqq 2$ and $\mathcal{C}_a(\mathcal{V}_i) \subset \tilde{\boldsymbol{V}}$, and put $L_0 = L_{\bar{m}+1}$. Recalling (2.4) for $H^{(a)}_{\mathcal{V}_i \mathcal{V}_{i'}}$, we set,

$$\mu_{k_1} = \max_{\mathcal{V}_{i_1} \in \boldsymbol{V}_k, \mathcal{V}_{i_2} \notin \boldsymbol{V}_k} \max_{M_\alpha \subset \bar{\mathcal{V}}_{i_1} \cap \bar{\mathcal{V}}_{i_2}} m_\alpha, \qquad 1 \leqq k \leqq L_0,$$

$$\kappa_{k_1, k_2} = \sum_{\mathcal{V}_{i_1} \in \boldsymbol{V}_{k_1}, \mathcal{V}_{i_2} \in \boldsymbol{V}_{k_2}} H^{(\mu_{k_1})}_{\mathcal{V}_{i_1} \mathcal{V}_{i_2}},$$

$$p_{k_1, k_2} = \frac{\kappa_{k_1, k_2}}{\sum_{k \neq k_1} \kappa_{k_1, k}}, \qquad 1 \leqq k_1, k_2 \leqq L_0, \ k_1 \neq k_2,$$

and $\kappa_{k,k} = p_{k,k} = 0$, $1 \leqq k \leqq L_0$. Notice that $0 < \max_{1 \leq k' \leq L_0} \kappa_{k,k'} < \infty$ for every $1 \leqq k \leqq L_0$. For $l+1 \leqq k \leqq L_0$, B_k denotes a neighborhood with a C^∞-boundary of some stable compactum in $\bigcup_{\mathcal{V}_i \in \boldsymbol{V}_k} \mathcal{V}_i$ (not in $\bigcup_{\mathcal{V}_i \in \boldsymbol{V}_k} \mathcal{V}_i^e$) such that $\bar{B}_k \subset \bigcup_{\mathcal{V}_i \in \boldsymbol{V}_k} \mathcal{V}_i^e$ and that $\operatorname{grad} U \neq 0$ on ∂B_k We define

$$\boldsymbol{E}_k = \boldsymbol{D} \backslash \bigcup_{1 \leq k' \leq L_0, k' \neq k} \bar{B}_{k'}, \qquad 1 \leqq k \leqq L_0,$$

$$\mathcal{U}_k = \bigcap_{\mathcal{V}_i, \mathcal{V}_{i'} \in \boldsymbol{V}_k, \mathcal{V}_i \neq \mathcal{V}_{i'}} \mathcal{V}_i^e \cup \mathcal{V}_{i'}^e \cup \bigcup_{M_\alpha \subset \bar{\mathcal{V}}_i \cap \bar{\mathcal{V}}_{i'}} M_\alpha, \qquad 1 \leqq k \leqq L_0,$$

and claim the next theorem.

THEOREM 4.2. *For every* $1 \leqq k_1, k_2 \leqq L_0$, $k_1 \neq k_2$, *we have*

$$\lim_{\varepsilon \downarrow 0} P_x(x^\varepsilon_{\tau^\varepsilon_{E_{k_1}}} \in \partial B_{k_2}) = p_{k_1, k_2} \tag{4.1}$$

uniformly in x *belonging to any compact subset* F *of* $\boldsymbol{E}_{k_1} \cap \mathcal{D}(\mathcal{U}_{k_1})$.

In order to prove the above theorem, we fix $1 \leqq k_1 \leqq L_0$ and $1 \leqq i_2 \leqq L$ so that $\mathcal{V}_{i_2} \notin \boldsymbol{V}_{k_1}$ and $\bar{\mathcal{U}}_{k_1} \cap \bar{\mathcal{V}}_{i_2} \neq \emptyset$. Let $V_1 = \max_{x \in K: U(x) < V_0} U(x)$. Then, one can find $\tilde{U} \in C^\infty(\mathcal{M})$ satisfying the conditions (C_1), (C_2) given by replacing U with $\tilde{U}$ and the following conditions:

$(\tilde{C}_1)$ $\tilde{U}(x) = U(x)$ for $x \notin \bigcup_{M_\alpha \not\subset \bar{\mathcal{U}}_{k_1}} M''_\alpha(\delta_2) \cup \{x \in \mathcal{V}_{i_2};\ U(x) < (V_0 + V_1)/2\}$;

(C_2) $\tilde{U}(x) > U(x)$ for $x \in \bigcup_{M_\alpha \not\subset \bar{\mathcal{U}}_{k_1}} [W_\alpha]_0$;

(C_3) $\{x \in \boldsymbol{K};\ U(x) > (V_0 + V_1)/2\} = \{x \in \tilde{\boldsymbol{K}};\ U(x) > (V_0 + V_1)/2\}$;

(C_4) $\{x \in \mathcal{V}_{i_2} \cap \tilde{\boldsymbol{K}};\ U(x) < (V_0 + 2V_1)/3\}$ consists of exactly one point b satisfying that $\tilde{U}(b) = -1$ and that $H_{\tilde{U}}(b)$ is positive definite.

Here we use the same notations as those in Sections 2 and 3 and denote the set of critical points of $\tilde{U}$ and the Hessian of $\tilde{U}$, respectively, by $\tilde{\boldsymbol{K}}$ and $H_{\tilde{U}} = H_{\tilde{U}}(x)$.

We write by $\tilde{\boldsymbol{D}}$ the connected component of $\{x \in \mathcal{M};\ \tilde{U}(x) < (V_0 + \tilde{V}_{-1})/2\}$ which contains the bottom b, where $\tilde{V}_{-1} = [\min_{x \in \tilde{\boldsymbol{K}}: \tilde{U}(x) > V_0} \tilde{U}(x)] \wedge V_{-1}$. Then, from $(\tilde{C}_2)$, one notices $\tilde{\boldsymbol{D}}\cap \mathcal{V}_i^{\varepsilon}=\emptyset$ if $\bar{\mathcal{U}}_{k_1}\cap\bar{\mathcal{V}}_i=\emptyset$, $\mathcal{V}_i\in \boldsymbol{V}$. We also set

$$\tilde{B}_i = \{x \in \mathcal{V}_i\,;\ \tilde{U}(x) \leqq (V_0+V_1)/2\}, \qquad 1 \leqq i \leqq L,$$
$$\tilde{\boldsymbol{D}}_{k_1} = \tilde{\boldsymbol{D}}\backslash \cup_{\mathcal{V}_i\notin \boldsymbol{V}_{k_1}:\bar{\mathcal{V}}_i\cap\bar{\mathcal{U}}_{k_1}\neq\emptyset}\tilde{B}_i\,.$$

Let $(\tilde{x}_t^{\varepsilon}, P_x)$ be the diffusion process generated by $\tilde{\mathcal{L}}^{\varepsilon}=(\varepsilon^2/2)\Delta-(1/2)\,\mathrm{grad}\tilde{U}$ and let $\tilde{\tau}_{k_1}^{\varepsilon}$ be the first exit time of $\{\tilde{x}_t^{\varepsilon}\}$ from the domain $\tilde{\boldsymbol{D}}_{k_1}$.

LEMMA 4.3. *We have*

$$\lim_{0\downarrow\varepsilon} P_x(\tilde{x}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}_{k_1}} \in\partial\tilde{B}_{i_2}) = \frac{\sum_{\mathcal{V}_i\in \boldsymbol{V}_{k_1}} H^{(\mu_{k_1})}_{\mathcal{V}_i\mathcal{V}_{i_2}}}{\sum_{1\leq k\leq L_0,\, k\neq k_1}\kappa_{k_1,k}} \tag{4.2}$$

uniformly in x belonging to any compact subset F of $\tilde{\boldsymbol{D}}\cap\mathcal{D}(\mathcal{U}_{k_1})$.

PROOF. Consider the principal eigenvalue $\tilde{\lambda}^{\varepsilon}$ and the associated eigenfunction $\tilde{\varphi}^{\varepsilon}$ for the following boundary value problem:

$$\tilde{\mathcal{L}}^{\varepsilon}\varphi+\lambda\varphi = 0 \ \text{ in } \ \tilde{\boldsymbol{D}}_0, \qquad \text{with } \ \varphi = 0 \ \text{ on } \ \partial\tilde{\boldsymbol{D}}_0, \tag{4.3}$$

where we set $\tilde{\boldsymbol{D}}_0=\tilde{\boldsymbol{D}}\backslash\cup_{\mathcal{V}_i\notin \boldsymbol{V}_{k_1}:\mathcal{V}_i\neq\mathcal{V}_{i_2},\bar{\mathcal{V}}_i\cap\bar{\mathcal{U}}_{k_1}\neq\emptyset}\tilde{B}_i$ and, as Section 3, normalize $\tilde{\varphi}^{\varepsilon}$ in a similar manner to (3.4). Then, from Theorem 3.1, if one writes the RHS of (4.2) by ζ, one has

$$\lim_{\varepsilon\downarrow 0}\sup_{x\in F}|\tilde{\varphi}^{\varepsilon}(x)-\zeta| = 0$$

for every compact subset F in $\cup_{\mathcal{V}_i\in \boldsymbol{V}_{k_1}}\mathcal{D}(\mathcal{V}_i^{\varepsilon})\cap\tilde{\boldsymbol{D}}_0$, where one notes $U(x)=\tilde{U}(x)$ at every point $x\in\mathcal{D}(\mathcal{U}_{k_1})$. By using Itô's formula, (4.3) verifies

$$E_x[\tilde{\varphi}^{\varepsilon}(\tilde{x}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}_{k_1}})\,;\ \tilde{x}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}_{k_1}} \in \partial\tilde{B}_{i_2}]-\tilde{\varphi}^{\varepsilon}(x) = -\tilde{\lambda}^{\varepsilon}E_x\Big[\int_0^{\tilde{\tau}^{\varepsilon}_{k_1}}\tilde{\varphi}^{\varepsilon}(\tilde{x}_t^{\varepsilon})dt\Big].$$

On the other hand, we know rough asymptotics:

$$\lim_{\varepsilon\downarrow 0}\varepsilon^2\log\tilde{\lambda}^{\varepsilon} = -(V_0+1), \qquad \limsup_{\varepsilon\downarrow 0}\varepsilon^2\log\sup_{x\in\tilde{\boldsymbol{D}}_{k_1}} E_x[\tilde{\tau}^{\varepsilon}_{k_1}] \leqq V_0,$$

respectively, from Theorem 2.7 in [**18**] and Theorem 1 in [**17**]. Combining with Proposition 3.6 (i), one can find $r>0$ so that

$$\sup_{x\in F}|P_x(\tilde{x}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}_{k_1}} \in \partial\tilde{B}_{i_2})-\zeta| \leqq 2e^{-r/\varepsilon^2}+\sup_{x\in F}|\tilde{\varphi}(x)-\zeta|,$$

for all $\varepsilon>0$ sufficiently small and compact subsets F. Therefore (4.2) is obtained for every compact subset F in $\cup_{\mathcal{V}_i\in \boldsymbol{V}_{k_1}}\mathcal{D}(\mathcal{V}_i^{\varepsilon})\cap\tilde{\boldsymbol{D}}_0$. Together with the strong Markov property, one can easily obtain (4.2) for every compact subset F of $\tilde{\boldsymbol{D}}\cap\mathcal{D}(\mathcal{U}_{k_1})$. □

PROOF OF THEOREM 4.2. Let $\boldsymbol{D}'$ be a connected domain satisfying that $\tilde{\boldsymbol{D}}\supset\boldsymbol{D}'\supset\tilde{\boldsymbol{D}}\backslash\cup_{M_\alpha\subset\mathcal{U}_{k_1}}M'_\alpha(\delta_2)$ and that $U=\tilde{U}$ on $\boldsymbol{D}'_{k_1}=\boldsymbol{D}'\backslash\cup_{\mathcal{V}_i\notin \boldsymbol{V}_{k_1}:\bar{\mathcal{V}}_i\cap\bar{\mathcal{U}}_{k_1}\neq\emptyset}\tilde{B}_{i'}$. For every compact set F_1 in $\boldsymbol{D}'\cap\mathcal{D}(\mathcal{U}_{k_1})$, one knows

(4.4) $$\limsup_{\varepsilon\downarrow 0} \varepsilon^2 \log \sup_{x\in F_1} P_x(\tilde{x}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}_{D'_{k_1}}} \in \partial \boldsymbol{D}') < 0,$$

from Proposition 2.2 in [**18**] and, together with Lemma 4.3, has

$$\lim_{\varepsilon\downarrow 0} \sup_{x\in F_1} |P_x(\tilde{x}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}_{D'_{k_1}}} \in \partial \tilde{B}_{i_2}) - \zeta| = 0,$$

where $\tilde{\tau}^{\varepsilon}_{D'_{k_1}}$ stands for the first exit time of $\tilde{x}^{\varepsilon}_t$ from $\boldsymbol{D}'_{k_1}$ and ζ denotes the RHS of (4.2). Hence, noting that, for $x\in F_1$,

$$P_x(\tilde{x}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}_{k_1}} \in \partial \tilde{B}_{i_2}) - P_x(\tilde{x}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}_{D'_{k_1}}} \in \partial \tilde{\boldsymbol{D}}') \leqq P_x(\tilde{x}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}_{D'_{k_1}}} \in \partial \tilde{B}_{i_2}) \leqq P_x(\tilde{x}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}_{k_1}} \in \partial \tilde{B}_{i_2})$$

and $P_x(x^{\varepsilon}_{\tau^{\varepsilon}_{D'_{k_1}}} \in \partial \tilde{B}_{i_2}) = P_x(\tilde{x}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}_{D'_{k_1}}} \in \partial \tilde{B}_{i_2})$, we obtain

(4.5) $$\lim_{\varepsilon\downarrow 0} \sup_{x\in F_1} |P_x(x^{\varepsilon}_{\tau^{\varepsilon}_{D'_{k_1}}} \in \partial \tilde{B}_{i_2}) - \zeta| = 0.$$

Set $B'_i = \{x\in \mathcal{V}_i\,;\, U(x) \leqq (V_0+V_1)/2\}$, $1\leqq i\leqq L$, and $\boldsymbol{G} = \boldsymbol{D}\setminus \bigcup_{\mathcal{V}_i \notin \boldsymbol{V}_{k_1}} B'_i$. Then, by combining (4.4) with $P_x(x^{\varepsilon}_{\tau^{\varepsilon}_{D'_{k_1}}} \in \partial \boldsymbol{D}') = P_x(\tilde{x}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}_{D'_{k_1}}} \in \partial \boldsymbol{D}')$, (4.5) implies

$$\lim_{\varepsilon\downarrow 0} \sup_{x\in F_1} |P_x(x^{\varepsilon}_{\tau^{\varepsilon}_G} \in \partial B'_{i_2}) - \zeta| = 0.$$

Hence, one can easily obtain, for $1\leqq k_2 \leqq L_0$, $k_2 \neq k_1$,

(4.6) $$\lim_{\varepsilon\downarrow 0} P_x(x^{\varepsilon}_{\tau^{\varepsilon}_G} \in \bigcup_{\mathcal{V}_{i'} \in \boldsymbol{V}_{k_2}} \partial B'_{i'}) = p_{k_1, k_2}$$

uniformly in x belonging to any compact subset F of $\boldsymbol{G} \cap \mathcal{D}(\mathcal{U}_{k_1})$. On the other hand, suppose $B_{k_2} \subset \mathcal{V}_{i_2} (\in \boldsymbol{V}_{k_2})$ and set $\boldsymbol{G}' = \boldsymbol{D}\setminus (B'_{i_2} \cup \bigcup_{\mathcal{V}_i \notin \boldsymbol{V}_{k_2}} B'_i)$. Then, in a similar manner to (4.6), one can also prove

(4.7) $$\lim_{\varepsilon\downarrow 0} P_x(x^{\varepsilon}_{\tau^{\varepsilon}_{G'}} \in \partial B'_{i_2}) = 1$$

uniformly in x belonging to any compact subset of $\boldsymbol{G}' \cap \mathcal{D}(\mathcal{U}_{k_2})$. Furthermore, from [**18**, Proposition 2.2], we have

(4.8) $$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \sup_{x\in B'_i} P_x(x^{\varepsilon}_{\tau^{\varepsilon}_{E_{k_1}}} \notin \partial B_{k_2}) < 0.$$

Therefore, together with (4.6)-(4.8), the strong Markov property immediately verifies Theorem 4.1. □

PROPOSITION 4.4. *Set* $\boldsymbol{E}_0 = \boldsymbol{D}\setminus \bigcup_{1\leq k\leq L_0} \overline{B_k}$. *Then, for* $1\leqq k\leqq L_0$, *we have*

$$\lim_{\varepsilon\downarrow 0} P_x(x^{\varepsilon}_{\tau^{\varepsilon}_{E_0}} \in \partial B_k) = 1$$

uniformly in x *belonging to every compact subset* F *of* $\boldsymbol{E}_0 \cap \mathcal{D}(\mathcal{U}_k)$.

The proof is quite similar to Lemma 4.3 and we omit it.

We introduce the Wentzell and Freidlin W-graph. Let J be a finite set and

W be a subset of J. A graph consisting of arrows $\alpha\to\beta$ ($\alpha\in J\setminus W$, $\beta\in J$, $\alpha\neq\beta$) is called W-graph on J if it satisfies the following conditions:

(1) every point $\alpha\in J\setminus W$ is the initial point of exactly one arrow;

(2) there are no closed cycles in the graph.

We denote by $\mathfrak{G}^J(W)$ the set of W-graphs on J. For $\alpha\in J\setminus W$ and $\beta\in W$, $\mathfrak{G}^J_{\alpha\beta}(W)$ stands for the set of W-graphs on J containing the sequence of arrows leading from α to β. (See Wentzell and Freidlin [4, pp. 177-182].)

The following lemma is a slight modification of Lemma 3.3 in [4, Chapter 6]. One can prove it in a quite parallel manner.

LEMMA 4.5. *Let us consider a Markov chain on a phase space* $X=\bigcup_{i\in J}X_i$, $X_i\cap X_{i'}=\emptyset$ $(i\neq i')$, *the transition probabilities of which satisfy the inequalities*

$$\underline{p}_{i,i'}\leqq P(x, X_{i'})\leqq \bar{p}_{i,i'},\qquad x\in X_i,\ i\neq i'.$$

For $x\in X$ *and* $B\subset\bigcup_{j\in W}X_j$, *we denote by* $q_W(x, B)$ *the probability that the chain starting form* x *hits* B *at the first exit time from* $\bigcup_{j\in W}X_j$. *Then, we have*

$$a(W)^{-2r}\frac{\sum_{\mathfrak{g}\in\mathfrak{G}^J_{ij}(W)}\pi_-(\mathfrak{g})}{\sum_{\mathfrak{g}\in\mathfrak{G}^J(W)}\pi_-(\mathfrak{g})}\leqq q_W(x, X_j)\leqq a(W)^{2r}\frac{\sum_{\mathfrak{g}\in\mathfrak{G}^J_{ij}(W)}\pi_+(\mathfrak{g})}{\sum_{\mathfrak{g}\in\mathfrak{G}^J(W)}\pi_+(\mathfrak{g})},$$
$$x\in X_i,\ i\in J\setminus W,\ j\in W,$$

if the denominator $\sum_{\mathfrak{g}\in\mathfrak{G}^J(W)}\pi_-(\mathfrak{g})$ *is positive. Here we write* $r=\#[J\setminus W]$,

$$a(W)=\sum_{W\subset W'\subset J}\frac{\sum_{\mathfrak{g}\in\mathfrak{G}^J(W')}\pi_+(\mathfrak{g})}{\sum_{\mathfrak{g}\in\mathfrak{G}^J(W')}\pi_-(\mathfrak{g})},$$

and $\pi_+(\mathfrak{g})=\prod_{(i\to i')\in\mathfrak{g}}\bar{p}_{i,i'}$, $\pi_-(\mathfrak{g})=\prod_{(i\to i')\in\mathfrak{g}}\underline{p}_{i,i'}$ *for* $\mathfrak{g}\in\mathfrak{G}^J(W)$.

PROOF OF THEOREM 2. Let $1\leqq j_0\leqq l$ be fixed. We set $J=\{\partial\boldsymbol{D}, \partial B_1, \cdots, \partial B_{L_0}\}$, $W=W^{(j_0)}=\{\partial\boldsymbol{D}, \partial B_1, \cdots, \partial B_{j_0-1}, \partial B_{j_0+1}, \cdots, \partial B_l\}$ and consider the W-graph on J. Fix a sufficiently small $r>0$. Then, from Theorem 4.2, there is $\varepsilon_0>0$ so that, for all $0<\varepsilon<\varepsilon_0$ and $i=j_0, l+1, \cdots, L_0$, $i'=0, \cdots, L_0$,

$$\underline{p}_{i,i'}\leqq P_x(x^\varepsilon_{\tau^\varepsilon_{\boldsymbol{E}_i}}\in\partial B_{i'})\leqq\bar{p}_{i,i'},\qquad x\in\partial B_i,$$

where $\underline{p}_{i,i'}=(p_{i,i'}-r)\vee 0$, $\bar{p}_{i,i'}=(p_{i,i'}+r)\wedge 1$ and $\partial B_0=\partial\boldsymbol{D}$, $p_{i,0}=0$. We assume $\underline{p}_{i,i'}>0$ if $p_{i,i'}>0$ without loss of generality. By combining with the strong Markov property, Lemma 4.5 verifies the estimates:

$$a^r(W)^{-2L_0-l+1}\frac{\sum_{\mathfrak{g}\in\mathfrak{G}^J_{\partial B_{j_0}\partial B_{j_1}}(W)}\pi^r_-(\mathfrak{g})}{\sum_{\mathfrak{g}\in\mathfrak{G}^J(W)}\pi^r_-(\mathfrak{g})}\leqq P_x(x^\varepsilon_{\tau^\varepsilon_{\boldsymbol{D}_{j_0}}}\in\partial B_{j_1})$$
$$\leqq a^r(W)^{2L_0-l+1}\frac{\sum_{\mathfrak{g}\in\mathfrak{G}^J_{\partial B_{j_0}\partial B_{j_1}}(W)}\pi^r_+(\mathfrak{g})}{\sum_{\mathfrak{g}\in\mathfrak{G}^J(W)}\pi^r_+(\mathfrak{g})},\qquad x\in\overline{B_{j_0}},$$

for $1\leqq j_1\leqq l$, $j_1\neq j_0$, provided that $\sum_{\mathfrak{g}\in\mathfrak{G}^J(W)}\pi_-^r(\mathfrak{g})$ is positive, where

$$\pi_+^r(\mathfrak{g})=\prod_{(\partial B_i\to\partial B_{i'})\in\mathfrak{g}}\bar{p}_{i,i'},\quad \pi_-^r(\mathfrak{g})=\prod_{(\partial B_i\to\partial B_{i'})\in\mathfrak{g}}\underline{p}_{i,i'},\qquad \mathfrak{g}\in\mathfrak{G}^J(W),$$

$$a^r(W)=\prod_{W\subset W'\subset J}\frac{\sum_{\mathfrak{g}\in\mathfrak{G}^J(W')}\pi_+^r(\mathfrak{g})}{\sum_{\mathfrak{g}\in\mathfrak{G}^J(W')}\pi_-^r(\mathfrak{g})}.$$

Noticing

$$\lim_{r\downarrow 0}\pi_+^r(\mathfrak{g})=\lim_{r\downarrow 0}\pi_-^r(\mathfrak{g})=\pi(\mathfrak{g})\equiv\prod_{(\partial B_i\to B\partial_{i'})\in\mathfrak{g}}p_{i,i'},\qquad \mathfrak{g}\in\mathfrak{G}^J(W),$$

and $\lim_{r\downarrow 0}a^r(W)=1$, we have (2.9) uniformly on $\overline{B_{j_0}}$. Here we write

$$q_{j_0,j_1}=\frac{\sum_{\mathfrak{g}\in\mathfrak{G}^J_{\partial B_{j_0}\partial B_{j_1}}(W)}\pi(\mathfrak{g})}{\sum_{\mathfrak{g}\in\mathfrak{G}^J(W)}\pi(\mathfrak{g})},\tag{4.9}$$

if the denominator in (4.9) is positive, which will be proved in the next lemma. Hence, by virtue of the strong Markov property combined with (4.7), we obtain (2.9) uniformly in x belonging to any compact subset of $\boldsymbol{D}\cap\mathcal{D}(\mathcal{V}_{j_0})$. □

LEMMA 4.6. *The denominator in* (4.9) *is positive.*

PROOF. As the proof of the previous theorem, we consider the Markov chain whose transition probabilities are given by $p_{i,i'}$. If the denominator in (4.9) vanishes, there is a closed cycle $\{\partial B_{i_1},\cdots,\partial B_{i_n}\}\subset\{\partial B_{j_0},\partial B_{l+1},\cdots,\partial B_{L_0}\}$ in the chain. For the corresponding $\mathcal{V}_{i_1},\cdots,\mathcal{V}_{i_n}$, one has

$$\min_{1\leq p\leq n}\max_{1\leq p'\leq n,\,p\neq p'}\max_{M_\alpha\subset\mathcal{V}_{i_p}\cap\mathcal{V}_{i_{p'}}}m_\alpha>\max_{1\leq p\leq n,\,k\neq i_1,\cdots,i_n}\max_{M_\alpha\subset\mathcal{V}_{i_p}\cap\mathcal{V}_k}m_\alpha.$$

Then, when m denotes the LHS of the above inequality, we have $C_{m-1}(\mathcal{V}_i)\supset\bigcup_{q=1}^n\boldsymbol{V}_{i_q}$ and $\#[C_{m-1}(\mathcal{V}_i)\cap\boldsymbol{V}_0]\leqq 1$. But this contradicts the definition of the decomposition $\{\boldsymbol{V}_k\}$. □

REMARK 4.7. For $1\leqq j_1$, $j_2\leqq l$, $j_1\neq j_2$, one can easily know that $q_{j_1,j_2}>0$ if and only if $m^{(j_1)}=m_{j_1,j_2}$, where

$$m_{j_1,j_2}=\max_{\mathfrak{r}\in\mathfrak{R}_{\mathcal{V}_{i_1}\mathcal{V}_{i_2}}(V\setminus\bigcup_{1\leq j\leq l,\,j\neq j_1,j_2}V_j)}\min_{(\mathcal{V}_i\to\mathcal{V}_{i'})\in\mathfrak{r}}\max_{M_\alpha\subset\mathcal{V}_i\cap\mathcal{V}_{i'}}m_\alpha.$$

REMARK 4.8. Set $W_0=\{\partial\boldsymbol{D},\partial B_1,\cdots,\partial B_l\}$ and $\boldsymbol{D}_0=\boldsymbol{D}\setminus\bigcup_{1\leq j\leq l}\overline{B_j}$. By considering the W_0-graph on J with Theorem 4.2, we have, for $1\leqq j\leqq l$ and $l+1\leqq k\leqq L_0$,

$$\lim_{\varepsilon\downarrow 0}P_x(x^\varepsilon_{\tau^\varepsilon_{D_0}}\in\partial B_j)=\frac{\sum_{\mathfrak{g}\in\mathfrak{G}^J_{\partial B_k\partial B_j}(W_0)}\pi(\mathfrak{g})}{\sum_{\mathfrak{g}\in\mathfrak{G}^J(W_0)}\pi(\mathfrak{g})},\tag{4.10}$$

uniformly in x belonging to any compact subset of $\boldsymbol{D}\cap\mathcal{D}(\mathcal{V}_k)$ in the above manner.

REMARK 4.9. We have (2.8) uniformly in x belonging to any compact subset of $\boldsymbol{D}\cap\mathcal{D}(\mathcal{U}_j)$. Indeed, it follows immediately if one uses the strong Markov property with Proposition 4.4.

5. Metastable behaviors.

In this section, we shall show Theorem 3. To this end, we fix $1\leqq j_0\leqq l$ throughout this section. The first task is to prove the uniform exponential exit law.

THEOREM 5.1. *Let F be a compact subset of $\boldsymbol{D}\cap\mathcal{D}(\mathcal{U}_{j_0})$. If $m^{(j_0)}-n^{(j_0)}=\mu$, we have*

$$\lim_{\varepsilon\downarrow 0}\sup_{t\geqq 0}\sup_{x\in F}|P_x(\alpha_\varepsilon^{-1}\tau_{D_{j_0}}^\varepsilon>t,\ x_{\tau_{D_{j_0}}^\varepsilon}^\varepsilon\in\bar{B}_{j_1})-q_{j_0,j_1}e^{-c_{j_0}t}|=0 \tag{5.1}$$

for $1\leqq j_1\leqq l$, $j_1\neq j_0$. In case that $m^{(j_0)}-n^{(j_0)}<\mu$, we have

$$\lim_{\varepsilon\downarrow 0}\inf_{x\in F}P_x(\alpha_\varepsilon^{-1}\tau_{D_{j_0}}^\varepsilon>t)=1 \tag{5.2}$$

for $t\geqq 0$.

In order to prove Theorem 5.1, we need some preparations. One can find that the following lemma concerning the pointwise exponential exit law holds from [**18**, Theorem 3.7] combined with Theorem 3.1.

LEMMA 5.2. *We have $\lim_{\varepsilon\downarrow 0}P_x(\varepsilon^{-(m^{(j_0)}-n^{(j_0)})}e^{-V_0/\varepsilon}\tau_{D_{j_0}}^\varepsilon>t)=e^{-t\cdot H_{j_0}/2\nu^{(j_0)}}$ for all $x\in\boldsymbol{D}\cap\mathcal{D}(\mathcal{V}_{j_0}^\varepsilon)$ and $t\geqq 0$.*

For uniform estimates, we consider the harmonic measures. Set $E=\{x\in\mathcal{V}_{j_0};\,U(x)<(V_0+V_1)/2\}$; recall $V_1=\max_{x\in K:U(x)<V_0}U(x)$.

LEMMA 5.3. *Let F_1 be a compact subset of E. There exist $r_1,\varepsilon_1>0$ so that*

$$\sup_{x,y\in F_1}|E_x[f(x_{\tau_E^\varepsilon}^\varepsilon)]-E_y[f(x_{\tau_E^\varepsilon}^\varepsilon)]|\leqq C_f\cdot e^{-r_1/\varepsilon^2} \tag{5.3}$$

for all bounded continuous functions f and $0<\varepsilon<\varepsilon_1$, where $C_f=\sup_{y\in\partial E}|f(y)|$.

REMARK 5.4. By the bounded convergence theorem, (5.3) holds for all bounded measurable functions f.

PROOF. Let us consider the Dirichlet boundary value problem:

$$\mathcal{L}^\varepsilon u=0\quad\text{in } E,\qquad\text{with } u=f\quad\text{on }\partial E\,. \tag{5.4}$$

It is known that there is a unique solution $u^\varepsilon\in C^0(\bar{E})\cap C^\infty(E)$ of (5.4). (See Theorem 6.13 in [7].) By using Itô's formula, one can write $u^\varepsilon(x)=E_x[f(x_{\tau_E^\varepsilon}^\varepsilon)]$. Hence, for a compact subset F_1 of E, it suffices to show the existence of $r_2>0$

so that

(5.5) $$\sup_{x\in F_1}\|\mathrm{grad}\, u^{\varepsilon}(x)\| \leqq C_f e^{-r_2/\varepsilon^2}.$$

In a similar manner to [**1**, Lemma 7], for a compact subset F_1 of E, there is $C_2>0$ so that $\sup_{x\in F_1}\|\mathrm{grad}\, u^{\varepsilon}(x)\|\leqq C_2C_f\varepsilon^{-2}$, $0<\varepsilon<1$. Then, we can use the same technique as in the proof of Lemma 2.1 of [**2**] and obtain the estimate (5.5). □

We state the following two lemmas without proof since they are shown in a quite similar manner to Lemmas 3 and 4 in [**6**], respectively. Here one notices that they require exponential estimates in [**17**], [**18**] and Proposition 4.4 as well as the previous lemma.

LEMMA 5.5. *Let F be a compact subset of $\boldsymbol{D}\cap\mathcal{D}(\mathcal{V}_{j_0})$ and let $b\in \boldsymbol{N}^{(j_0)}$. One can find $r>0$ and $\delta_\varepsilon>0$ so that $\lim_{\varepsilon\downarrow 0}\delta_\varepsilon=0$ and that*

$$\sup_{y\in F} P_y(\tau^{\varepsilon}_{D_{j_0}}>t,\ x^{\varepsilon}_{\tau^{\varepsilon}_{D_{j_0}}}\in\overline{B_{j_1}}) \leqq P_b(\tau^{\varepsilon}_{D_{j_0}}>t-\eta_\varepsilon,\ x^{\varepsilon}_{\tau^{\varepsilon}_{D_{j_0}}}\in\overline{B_{j_1}})+\delta_\varepsilon,$$

$$\inf_{y\in F} P_y(\tau^{\varepsilon}_{D_{j_0}}>t,\ x^{\varepsilon}_{\tau^{\varepsilon}_{D_{j_0}}}\in\overline{B_{j_1}}) \geqq P_b(\tau^{\varepsilon}_{D_{j_0}}>t+\eta_\varepsilon,\ x^{\varepsilon}_{\tau^{\varepsilon}_{D_{j_0}}}\in\overline{B_{j_1}})-\delta_\varepsilon,$$

for all $\varepsilon>0$ sufficiently small and $1\leqq j_1\leqq l$, $j_1\neq j_0$, where $\eta_\varepsilon=e^{(V_0-r)/\varepsilon^2}$.

LEMMA 5.6. *Let $b\in\boldsymbol{N}^{(j_0)}$ and set $f^{\varepsilon}_{j_1}(t)=P_b(\alpha_\varepsilon^{-1}\tau^{\varepsilon}_{D_{j_0}}>t,\ x^{\varepsilon}_{\tau^{\varepsilon}_{D_{j_0}}}\in\overline{B_{j_1}})$ for $1\leqq j_1\leqq l$, $j_1\neq j_0$, and $t\geqq 0$. For every $t_0>0$, there exist positive numbers r, ε_0 so that*

$$P_b(\alpha_\varepsilon^{-1}\tau^{\varepsilon}_{D_{j_0}}>s+\delta_\varepsilon)f^{\varepsilon}_{j_1}(t+\delta_\varepsilon)-\delta_\varepsilon \leqq f^{\varepsilon}_{j_1}(t+s) \leqq P_b(\alpha_\varepsilon^{-1}\tau^{\varepsilon}_{D_{j_0}}>s)f^{\varepsilon}_{j_1}(t-\delta_\varepsilon)+\delta_\varepsilon$$

for all $s>0$, $t\geqq t_0$ and $0<\varepsilon\leqq\varepsilon_0$, where $\delta_\varepsilon=e^{-r/\varepsilon^2}$.

PROOF OF THEOREM 5.1. Since (5.2) is obvious from Lemmas 5.2 and 5.5, it suffices to show the case of $m^{(j_0)}-n^{(j_0)}=\mu$. For an arbitrarily fixed $b\in\boldsymbol{N}^{(j_0)}$, we set

$$f^{\varepsilon}_{j}(t)=P_b(\alpha_\varepsilon^{-1}\tau^{\varepsilon}_{D_{j_0}}>t,\ x^{\varepsilon}_{\tau^{\varepsilon}_{D_{j_0}}}\in\overline{B_j}),\qquad t\geqq 0,\ 1\leqq j\leqq l,\ j\neq j_0,$$

and $f^{\varepsilon}_{j_0}(t)=0$, $t\geqq 0$. From Lemma 5.6, every $\{f_j(t)\}_{\varepsilon>0}$, $1\leqq j\leqq l$, is a uniformly equicontinuous family on each bounded interval. Hence, combining with Lemma 5.2 and the fact that $P_b(x^{\varepsilon}_{\tau^{\varepsilon}_{D_{j_0}}}\in\partial\boldsymbol{D})$ vanishes exponentially fast, we have

$$\lim_{\varepsilon\downarrow 0}\sup_{t\geqq 0}\Big|\sum_{j=1}^{l} f^{\varepsilon}_{j}(t)-e^{-c_{j_0}t}\Big|=0,\qquad \lim_{\varepsilon\downarrow 0}\sup_{t\geqq 0}\Big|\sum_{j=1}^{l}\{f^{\varepsilon}_{j}(0)-f^{\varepsilon}_{j}(t)\}-(1-e^{-c_{j_0}t})\Big|=0.$$

From the above formulae and Theorem 2, in case that $q_{j_0,j_1}>0$ the distribution function $g^{\varepsilon}_{j_1}(t)=1-f^{\varepsilon}_{j_1}(t)/f^{\varepsilon}_{j_1}(0)$ satisfies

$$\liminf_{\varepsilon\downarrow 0} g^{\varepsilon}_{j_1}(T)\geqq 1-e^{-c_{j_0}T}/q_{j_0,j_1},\quad \limsup_{\varepsilon\downarrow 0} g^{\varepsilon}_{j_1}(t)\leqq(1-e^{-c_{j_0}t})/q_{j_0,j_1},\quad T>t>0.$$

Hence, the family of the corresponding probability measures is tight. If $g_{j_1}(t)$ denotes the distribution function of its arbitrary limit point, we have

$$1-g_{j_1}(t+s)=(1-g_{j_1}(t))e^{-c_{j_0}s}, \qquad s,\ t\geqq 0,$$

from Lemma 5.6 and, especially, $1-g_{j_1}(t)=e^{-c_{j_0}t}$, $t\geqq 0$. However, since $\{g_{j_1}^{\varepsilon}(t)\}_{\varepsilon>0}$ is uniformly equicontinuous, we have the uniform convergence:

$$\lim_{\varepsilon\downarrow 0}\sup_{t\geqq 0}|(1-g_{j_1}^{\varepsilon}(t))-e^{-c_{j_0}t}|=0$$

and (5.1) for $F=\{b\}$. In order to obtain (5.1) for an arbitrary compact subset F of $\boldsymbol{D}\cap\mathcal{D}(\mathcal{U}_{j_0})$, one has only to combine with Lemma 5.5. □

LEMMA 5.7. *There is* $\gamma>0$ *so that*

(5.6) $$\lim_{\varepsilon\downarrow 0}\sup_{y\in \boldsymbol{D}_{j_0}} P_y(\tau_{\boldsymbol{D}_{j_0}}^{\varepsilon}>t_\varepsilon,\ x_{t_\varepsilon}^{\varepsilon}\notin\overline{B_{j_0}})=0,$$

where $t_\varepsilon=e^{(V_0-\gamma)/\varepsilon^2}$.

PROOF. Recalling the condition (C_2), we denote by $\boldsymbol{K}_{\mathrm{s}}$ the set of all stable compacta with respect to the flow determined by $-(1/2)\operatorname{grad}U$. Define $V_k^{(1)}$, $C_k^{(1)}$, $V_k^{(2)}$, $k=0, 1, \cdots$, inductively, below: $V_0^{(1)}=V_0$, $C_0^{(1)}=\boldsymbol{D}_{j_0}$ and

$V_k^{(2)}=\max_{x\in K:U(x)<V_k^{(1)}}U(x)$;

$C_k^{(1)}$: the union of all connected components of $\{x\,;U(x)<(V_k^{(1)}+V_k^{(2)})/2\}$ containing some $N_\alpha^{(j)}$, $1\leqq\alpha\leqq l_{j_0}$;

$V_{k+1}^{(1)}=\max_{K_i\in\boldsymbol{K}_{\mathrm{s}}:K_i\subset C_k^{(1)}}\max_{x\in N^{(j_0)},\,y\in K_i:C_{01}^{x,y}(C_k^{(1)})\neq\emptyset}U(x,y)$;

recall (2.1) for the notation $U(x, y)$. Then, write by n the smallest number so that $\boldsymbol{N}^{(j_0)}$ coincides with the union of all stable compacta in $C_n^{(1)}$. Choose $0<\gamma<(1/2)\{\min_{K_i\in\boldsymbol{K}_{\mathrm{s}}:K_i\subset \boldsymbol{D}_{j_0}\setminus F}\min_{x\in K_i}U(x)\wedge\min_{x\in\partial B_{j_0}}U(x)\}$ so that every $V_k^{(1)}-\gamma/2$, $0\leqq k\leqq n$, is a regular value of U, where F is a compact subset of B_{j_0} whose interior contains $\boldsymbol{N}^{(j_0)}$, and put

$$t_k^{\varepsilon}=e^{(V_k^{(1)}-\gamma)/\varepsilon^2}, \qquad 0\leqq k\leqq n, \qquad t_{n+1}^{\varepsilon}=e^{\gamma/\varepsilon^2},$$

$$s_k^{\varepsilon}=e^{(V_k^{(1)}-3\gamma/2)/\varepsilon^2}, \qquad 0\leqq k\leqq n, \qquad s_{n+1}^{\varepsilon}=e^{\gamma/2\varepsilon^2},$$

C_k: the union of all connected components of $\{x\,;U(x)<V_k^{(1)}-\gamma/2\}$ containing some $N_\alpha^{(j_0)}$, $1\leqq\alpha\leqq l_{j_0}$, $0\leqq k\leqq n$,

and $C_{-1}=\boldsymbol{D}_{j_0}$, $C_{n+1}=B_{j_0}$. Here one notices

(5.7a) $$\sup_{y\in C_{k-1}\setminus F}P_y(\tau_{C_{k-1}\setminus F}^{\varepsilon}>s_k^{\varepsilon})\leqq e^{-\gamma/8\varepsilon^2}, \qquad 0\leqq k\leqq n+1.$$

(5.7b) $$\lim_{\varepsilon\downarrow 0}\sup_{y\in F}P_y(\tau_{C_k}^{\varepsilon}<t_k^{\varepsilon})=0, \qquad 0\leqq k\leqq n+1,$$

which follow, respectively, from Theorem 1 in [**17**] with Chebyshev's inequality and from Theorem 2.5 in [**18**]. If A_k^{ε} denotes the event that x_t^{ε} enters in F

during $[0, s_k^\varepsilon)$ and spends the rest of time interval $(0, t_k^\varepsilon]$ in C_k, one has

$$p_k^\varepsilon \equiv \sup_{y\in C_{k-1}} P_y([A_k^\varepsilon]^c, \tau_{C_{k-1}}^\varepsilon > t_k^\varepsilon)$$
$$\leqq \sup_{y\in C_{k-1}} P_y(\tau_{C_{k-1}\setminus F}^\varepsilon > s_k^\varepsilon) + \sup_{y\in F} P_y(\tau_{C_k}^\varepsilon < t_k^\varepsilon)$$

and $\lim_{\varepsilon\downarrow 0} p_k^\varepsilon = 0$, $0\leqq k\leqq n+1$, from (5.7). On the other hand, the Markov property verifies, for $y\in \boldsymbol{D}_{j_0}$,

$$P_y(\tau_{\boldsymbol{D}_{j_0}}^\varepsilon > t_0^\varepsilon,\ x_t^\varepsilon \in C_{k-1} \text{ for all } t\in(t_0^\varepsilon-(t_{k-1}^\varepsilon - s_{k-1}^\varepsilon), t_0^\varepsilon])$$
$$\leqq E_y[P_{x^\varepsilon_{t_0^\varepsilon - t_k^\varepsilon}}(\tau_{\boldsymbol{D}_{j_0}}^\varepsilon > t_k^\varepsilon),\ \tau_{\boldsymbol{D}_{j_0}}^\varepsilon > t_0^\varepsilon - t_k^\varepsilon,\ x^\varepsilon_{t_0^\varepsilon - t_k^\varepsilon} \in C_{k-1}] + p_k^\varepsilon$$
$$\leqq P_y(\tau_{\boldsymbol{D}_{j_0}}^\varepsilon > t_0^\varepsilon,\ x_t^\varepsilon \in C_k \text{ for all } t\in(t_0^\varepsilon-(t_k^\varepsilon - s_k^\varepsilon), t_0^\varepsilon]) + p_k^\varepsilon, \quad 0\leqq k\leqq n+1$$

where $t_{-1}^\varepsilon - s_{-1}^\varepsilon = 0$. From the above estimates, we obtain

$$P_y(\tau_{\boldsymbol{D}_{j_0}}^\varepsilon > t_0^\varepsilon) \leqq P_y(\tau_{\boldsymbol{D}_{j_0}}^\varepsilon > t_0^\varepsilon,\ x_{t_0^\varepsilon}^\varepsilon \in B_{j_0}) + \{p_0^\varepsilon + \cdots p_{n+1}^\varepsilon\}$$

for all $y\in \boldsymbol{D}_{j_0}$. Therefore, (5.6) is immediately derived if one sets $t_\varepsilon = t_0^\varepsilon$. □

We define a sequence $\{T_n^\varepsilon\}$ of stopping times in the following manner: $T_0\equiv 0$ and for $n\geqq 1$

$$T_n^\varepsilon = \inf\{t > T_{n-1}^\varepsilon\,;\, x_t^\varepsilon \notin \boldsymbol{D}_j\}, \qquad \text{if } T_{n-1}^\varepsilon < +\infty \text{ and } x^\varepsilon_{T_{n-1}^\varepsilon} \in \boldsymbol{D} \cap \mathcal{D}(\mathcal{U}_j),$$

and $T_n^\varepsilon = +\infty$ otherwise.

Let $\{Y_n\}_{n=0,1,\dots}$ be a Markov chain on $\boldsymbol{B}\equiv\{\boldsymbol{N}^{(1)}, \cdots, \boldsymbol{N}^{(l)}\}$ starting from $\boldsymbol{N}^{(j_0)}$ with the transition probabilities $q_{j,j'}$. Let $\Delta_0, \Delta_1, \cdots$ be independent and exponentially distributed with parameter 1 and independent of $\{Y_n\}$. Then,

$$X_t = Y_n, \quad \sum_{p=0}^{n-1} \frac{\Delta_p}{c(Y_p)} \leqq t < \sum_{p=0}^{n} \frac{\Delta_p}{c(Y_p)}, \qquad n=0, 1, \cdots,$$

defines a Markov process $\{X_t\}$ in $\boldsymbol{B}$ starting from $\boldsymbol{N}^{(j_0)}$ with the generator $\mathcal{G}$ defined by (2.11), where $c(\boldsymbol{N}^{(j)}) = c_j$, $1\leqq j\leqq l$, and we take $\Delta/0 \equiv +\infty$ and $\sum_{r=0}^{-1} \equiv 0$. (See, e.g., Ethier and Kurtz [**3**].)

PROPOSITION 5.8. *Let F be a compact subset of $\boldsymbol{D}\cap\mathcal{D}(\mathcal{U}_j)$ and let $t>0$. We have*

$$\text{(5.8)} \qquad \lim_{\varepsilon\downarrow 0} \sup_{y\in F} \Big| P_y(x^\varepsilon_{T_1^\varepsilon} \in \overline{B_{j_1}}, \cdots, x^\varepsilon_{T_N^\varepsilon} \in \overline{B_{j_N}},\ T_N^\varepsilon - 1 \leqq t\alpha_\varepsilon < T_N^\varepsilon)$$
$$-\boldsymbol{P}_{N^{(j_0)}}\Big(Y_1 = \boldsymbol{N}^{(j_1)}, \cdots, Y_N = \boldsymbol{N}^{(j_N)},\ \sum_{n=0}^{N-2} \frac{\Delta_n}{c_{j_n}} \leqq t < \sum_{n=0}^{N-1} \frac{\Delta_n}{c_{j_n}}\Big)\Big| = 0$$

for every $j_1, \cdots, j_N \in J_0$ and

$$\limsup_{\varepsilon\downarrow 0}\sup_{y\in F}\Big|P_y(x^\varepsilon_{T^\varepsilon_1}\in\overline{B_{j_1}},\cdots,x^\varepsilon_{T^\varepsilon_N}\in\overline{B_{j_N}},\ T^\varepsilon_N\leqq t\alpha_\varepsilon<T^\varepsilon_{N+1})$$
$$-\boldsymbol{P}_{\boldsymbol{N}^{(j_0)}}\Big(Y_1=\boldsymbol{N}^{(j_1)},\cdots,Y_N=\boldsymbol{N}^{(j_N)},\ \sum_{n=0}^{N-1}\frac{\Delta_n}{c_{j_n}}\leqq t<\sum_{n=0}^{N}\frac{\Delta_n}{c_{j_n}}\Big)\Big|=0$$

for every $j_1,\cdots,j_{N-1}\in J_0$ *and* $1\leqq j_N\leqq l$, *where* $J_0=\{1\leqq j\leqq l;\ m^{(j)}-n^{(j)}=\mu\}$.

PROOF. For (5.8), we use induction on N. The case of $N=1$ is already known in Theorem 5.1. If one writes, for $b=\boldsymbol{N}^{(j_0)}$,

$$P_b(x^\varepsilon_{T^\varepsilon_1}\in\overline{B_{j_1}},\cdots,x^\varepsilon_{T^\varepsilon_N}\in\overline{B_{j_N}},\ T^\varepsilon_{N-1}\leqq t\alpha_\varepsilon<T^\varepsilon_N)$$
$$=\int_{\overline{B_{j_{N-1}}}\times[0,t)}P_y(x^\varepsilon_{\tau^\varepsilon_{D_{j_{N-1}}}}\in\overline{B_{j_N}},\ \alpha_\varepsilon^{-1}\tau^\varepsilon_{D_{j_{N-1}}}>t-s)$$
$$\times P_b(x^\varepsilon_{T^\varepsilon_1}\in\overline{B_{j_1}},\cdots,x^\varepsilon_{T^\varepsilon_{N-2}}\in\overline{B_{j_{N-2}}},\ x^\varepsilon_{T^\varepsilon_{N-1}}\in dy,\ \alpha_\varepsilon^{-1}T^\varepsilon_{N-1}\in ds),$$

Theorem 5.1 and the assumption of induction verify that the RHS of the above formula converges to

$$\int_{[0,t)}\boldsymbol{P}_{\boldsymbol{N}^{(j_{N-1})}}\Big(Y_1=\boldsymbol{N}^{(j_N)},\ \frac{\Delta_0}{c_{j_{N-1}}}>t-s\Big)$$
$$\times\boldsymbol{P}_{\boldsymbol{N}^{(j_0)}}\Big(Y_1=\boldsymbol{N}^{(j_1)},\cdots,Y_{N-2}=\boldsymbol{N}^{(j_{N-2})},\ Y_{N-1}=\boldsymbol{N}^{(j_{N-1})},\ \sum_{n=0}^{N-2}\frac{\Delta_n}{c_{j_n}}\in ds\Big)$$
$$=\boldsymbol{P}_{\boldsymbol{N}^{(j_0)}}\Big(Y_1=\boldsymbol{N}^{(j_1)},\cdots,Y_N=\boldsymbol{N}^{(j_N)},\ \sum_{n=0}^{N-2}\frac{\Delta_n}{c_{j_n}}\leqq t<\sum_{n=0}^{N-1}\frac{\Delta_n}{c_{j_n}}\Big).$$

Hence, (5.8) holds for $F=\{b\}$. In particular, we have

$$\lim_{\varepsilon\downarrow 0}P_b(t\alpha_\varepsilon-e^{(V_0-\gamma)/\varepsilon^2}\leqq T^\varepsilon_n\leqq t\alpha_\varepsilon)=0,\qquad \gamma>0,\ 1\leqq n\leqq N. \tag{5.9}$$

On the other hand, it is known that, for each compact set F in $\boldsymbol{D}\cap\mathcal{D}(\mathcal{V}_{j_0})$, there are $r>0$ and δ_ε such that $\lim_{\varepsilon\downarrow 0}\delta_\varepsilon=0$ and that

$$\sup_{y\in F}P_y(x^\varepsilon_{T^\varepsilon_1}\in\overline{B_{j_1}},\cdots,x^\varepsilon_{T^\varepsilon_N}\in\overline{B_{j_N}},\ T^\varepsilon_{N-1}\leqq t\alpha_\varepsilon<T^\varepsilon_N)$$
$$\leqq P_b(x^\varepsilon_{T^\varepsilon_1}\in\overline{B_{j_1}},\cdots,x^\varepsilon_{T^\varepsilon_N}\in\overline{B_{j_N}},\ T^\varepsilon_{N-1}\leqq t\alpha_\varepsilon,\ t\alpha_\varepsilon-\eta_\varepsilon<T^\varepsilon_N)+\delta_\varepsilon,$$
$$\inf_{y\in F}P_y(x^\varepsilon_{T^\varepsilon_1}\in\overline{B_{j_1}},\cdots,x^\varepsilon_{T^\varepsilon_N}\in\overline{B_{j_N}},\ T^\varepsilon_{N-1}\leqq t\alpha_\varepsilon<T^\varepsilon_N)$$
$$\geqq P_b(x^\varepsilon_{T^\varepsilon_1}\in\overline{B_{j_1}},\cdots,x^\varepsilon_{T^\varepsilon_N}\in\overline{B_{j_N}},\ T^\varepsilon_{N-1}\leqq t\alpha_\varepsilon-\eta_\varepsilon,\ t\alpha_\varepsilon<T^\varepsilon_N)+\delta_\varepsilon,$$

where $\eta_\varepsilon=e^{(V_0-r)/\varepsilon^2}$. In fact, they can immediately be derived in the same manner as Lemma 3 in [**6**]. Therefore, by combining the above estimates with (5.8) for $F=\{b\}$ and (5.9), the assertion is obtained.

The second assertion for $j_N\in J_0$ is obtained from (5.8). For $j_N\notin J_0$, one can get it if one uses the strong Markov property combined with (5.8) and (5.2) in Theorem 5.1. □

The next lemma immediately follows from Proposition 5.8.

LEMMA 5.9. *Let* $j_0 \in J_0$. *We have*

$$\limsup_{n\to\infty} \limsup_{\varepsilon\downarrow 0} \sup_{y\in F} P_y(T_n^\varepsilon \leqq t\alpha_\varepsilon) = 0$$

for every compact subset F *of* $\boldsymbol{D}\cap\mathcal{D}(\mathcal{U}_{j_0})$ *and* $t>0$.

LEMMA 5.10. *Let* F *be a compact subset of* $\boldsymbol{D}\cap\mathcal{D}(\mathcal{U}_{j_0})$. *Then, for every* $1\leqq j\leqq l$ *and* $t>0$, *we have*

$$\lim_{\varepsilon\downarrow 0} \sup_{y\in F} |P_y(y_t^\varepsilon \in B_j) - \boldsymbol{P}_{\boldsymbol{N}^{(j_0)}}(X_t = \boldsymbol{N}^{(j)})| = 0, \tag{5.10}$$

where we recall $y_t^\varepsilon = x_{t\alpha_\varepsilon}^\varepsilon$, $t\geqq 0$.

PROOF. If one sets $t_\varepsilon = e^{(V_0-\gamma)/\varepsilon^2}$, $\gamma>0$, one has

$$P_y(y_t^\varepsilon \notin \overline{B_{j_0}}) \leqq E_y[P_{x^\varepsilon_{t\alpha_\varepsilon - t_\varepsilon}}(x_{t_\varepsilon}^\varepsilon \notin \overline{B_{j_0}}, \tau_{D_{j_0}}^\varepsilon > t_\varepsilon), \tau_{D_{j_0}}^\varepsilon > t\alpha_\varepsilon - t_\varepsilon] + P_y(\tau_{D_{j_0}}^\varepsilon \leqq t\alpha_\varepsilon).$$

However, in case that $m^{(j_0)} - u^{(j_0)} < \mu$, each term of the RHS vanishes as $\varepsilon\downarrow 0$ respectively from Lemma 5.7 and Proposition 5.1. Hence, (5.10) is obtained. We move to the proof in case that $m^{(j_0)} - n^{(j_0)} = \mu$. We write, for $N\geqq 1$,

$$\begin{aligned} &P_y(y_t^\varepsilon \in \overline{B_j}) \\ &= \sum_{n=1}^{N} \sum_{j_1,\cdots,j_{n-1}=1}^{l} P_y(x^\varepsilon_{T_1^\varepsilon} \in \overline{B_{j_1}}, \cdots, x^\varepsilon_{T_{n-1}^\varepsilon} \in \overline{B_{j_{n-1}}}, x^\varepsilon_{t\alpha_\varepsilon} \in \overline{B_j}, T_{n-1}^\varepsilon \leqq t\alpha_\varepsilon < T_n^\varepsilon) \\ &\quad + P_y(y_t^\varepsilon \in \overline{B_j}, T_N^\varepsilon \leqq t\alpha_\varepsilon), \\ &\boldsymbol{P}_{\boldsymbol{N}^{(j_0)}}(X_t = \boldsymbol{N}^{(j)}) \\ &= \sum_{n=1}^{N} \sum_{j_1,\cdots,j_{n-2}=1}^{l} \boldsymbol{P}_{\boldsymbol{N}^{(j_0)}}\Big(Y_1 = \boldsymbol{N}^{(j_1)}, \cdots, Y_{n-2} = \boldsymbol{N}^{(j_{n-2})}, Y_{n-1} = \boldsymbol{N}^{(j)}, \\ &\qquad\qquad \sum_{k=1}^{n-2} \frac{\Delta_k}{c_{j_k}} \leqq t < \sum_{k=0}^{n-1} \frac{\Delta_k}{c_{j_k}}\Big) \\ &\quad + \boldsymbol{P}_{\boldsymbol{N}^{(j_0)}}\Big(Y_N = \boldsymbol{N}^{(j)}, \sum_{n=0}^{N} \frac{\Delta_n}{c_{j_n}} \leqq t\Big). \end{aligned}$$

Then, when we fix $1\leqq j_1, \cdots, j_{n-1}\leqq l$ and set

$$p_j^\varepsilon(y) = P_y(x^\varepsilon_{T_1^\varepsilon} \in \overline{B_{j_1}}, \cdots, x^\varepsilon_{T_{n-1}^\varepsilon} \in \overline{B_{j_{n-1}}}, x^\varepsilon_{t\alpha_\varepsilon} \in \overline{B_j}, T_{n-1}^\varepsilon \leqq t\alpha_\varepsilon < T_n^\varepsilon), \quad 1\leqq j\leqq l,$$

$$p^\varepsilon(y) = P_y(x^\varepsilon_{T_1^\varepsilon} \in \overline{B_{j_1}}, \cdots, x^\varepsilon_{T_{n-1}^\varepsilon} \in \overline{B_{j_{n-1}}}, T_{n-1}^\varepsilon < t\alpha_\varepsilon < T_n^\varepsilon),$$

it is sufficient from Proposition 5.8 and Lemma 5.9 to show

(5.11) $$\limsup_{\varepsilon\downarrow 0}\sup_{y\in F}|p^{\varepsilon}_{j_{n-1}}(y)-p^{\varepsilon}(y)|=0,\quad \limsup_{\varepsilon\downarrow 0}\sup_{y\in F}p^{\varepsilon}_{j}(y)=0,\quad j\neq j_{n-1},$$

for all compact subset F of $\boldsymbol{D}\cap\mathcal{D}(\mathcal{V}_{j_0})$. Note that

$$p^{\varepsilon}_{j}(y)=E_y[P_{x^{\varepsilon}_{t\alpha_\varepsilon-t_\varepsilon}}(x^{\varepsilon}_{t_\varepsilon}\in\overline{B_j},\ \tau^{\varepsilon}_{\boldsymbol{D}_{j_{n-1}}}>t_\varepsilon),$$

$$x^{\varepsilon}_{T^{\varepsilon}_1}\in\overline{B_{j_1}},\ \cdots,\ x^{\varepsilon}_{T^{\varepsilon}_{n-1}}\in\overline{B_{j_{n-1}}},\ T^{\varepsilon}_{n-1}\leqq t\alpha_\varepsilon-t_\varepsilon]$$

$$+P_y(x^{\varepsilon}_{T^{\varepsilon}_1}\in\overline{B_{j_1}},\ \cdots,\ x^{\varepsilon}_{T^{\varepsilon}_{n-1}}\in\overline{B_{j_{n-1}}},\ x^{\varepsilon}_{t\alpha_\varepsilon}\in\overline{B_j},\ t\alpha_\varepsilon-t_\varepsilon<T^{\varepsilon}_{n-1}\leqq t\alpha_\varepsilon),$$

$$p^{\varepsilon}(y)=E_y[P_{x^{\varepsilon}_{t\alpha_\varepsilon-t_\varepsilon}}(\tau^{\varepsilon}_{\boldsymbol{D}_{j_{n-1}}}>t_\varepsilon),\ x^{\varepsilon}_{T^{\varepsilon}_1}\in\overline{B_{j_1}},\ \cdots,\ x^{\varepsilon}_{T^{\varepsilon}_{n-1}}\in\overline{B_{j_{n-1}}},\ T^{\varepsilon}_{n-1}\leqq\alpha_\varepsilon t-t_\varepsilon]$$

$$+P_y(x^{\varepsilon}_{T^{\varepsilon}_1}\in\overline{B_{j_1}},\ \cdots,\ x^{\varepsilon}_{T^{\varepsilon}_{n-1}}\in\overline{B_{j_{n-1}}},\ t\alpha_\varepsilon-t_\varepsilon<T^{\varepsilon}_{n-1}\leqq t\alpha_\varepsilon),$$

for $t_\varepsilon=e^{(V_0-\gamma)/\varepsilon^2}$, $\gamma>0$. Then, since Proposition 5.8 verifies

$$\limsup_{\varepsilon\downarrow 0}\sup_{y\in F}P_y(t\alpha_\varepsilon-t_\varepsilon\leqq T^{\varepsilon}_{n-1}\leqq t\alpha_\varepsilon)=0,$$

one can easily obtain (5.11) from Lemma 5.7. □

By virtue of the Markov property combined with Lemma 5.10, the following proposition is obvious.

PROPOSITION 5.11. *We have*

$$\limsup_{\varepsilon\downarrow 0}\sup_{y\in F}|P_y(y^{\varepsilon}_{t_1}\in\overline{B_{j_1}},\ \cdots,\ y^{\varepsilon}_{t_N}\in\overline{B_{j_N}})$$
$$-\boldsymbol{P}_{N^{(j_0)}}(X_{t_1}=\boldsymbol{N}^{(j_1)},\ \cdots,\ X_{t_N}=\boldsymbol{N}^{(j_N)})|=0$$

for every $0<t_1<t_2<\cdots<t_N$, $1\leqq j_1,\ \cdots,\ j_N\leqq l$ *and compact subset* F *of* $\boldsymbol{D}\cap\mathcal{D}(\mathcal{V}_{j_0})$.

REMARK 5.12. If every α^{ε}_n, $1\leqq n\leqq N$, satisfies $\lim_{\varepsilon\downarrow 0}\alpha^{\varepsilon}_n/\alpha_\varepsilon=1$, then we have

$$\limsup_{\varepsilon\downarrow 0}\sup_{y\in F}|P_y(x^{\varepsilon}_{t_1\alpha^{\varepsilon}_1}\in\overline{B_{j_1}},\ \cdots,\ x^{\varepsilon}_{t_N\alpha^{\varepsilon}_N}\in\overline{B_{j_N}})$$
$$-\boldsymbol{P}_{N^{(j_0)}}(X_{t_1}=\boldsymbol{N}^{(j_1)},\ \cdots,\ X_{t_N}=\boldsymbol{N}^{(j_N)})|=0$$

for every $0<t_1<t_2<\cdots<t_N$, $1\leqq j_1,\ \cdots,\ j_N\leqq l$ and compact subset F of $\boldsymbol{D}\cap\mathcal{D}(\mathcal{V}_{j_0})$.

PROOF OF THEOREM 3. Let us fix $\delta>0$, $1\leqq j_0,\ \cdots,\ j_N\leqq l$, $0<t_1,\ \cdots,\ t_N<\infty$ and $x\in\mathcal{D}(\mathcal{V}_{j_0})$. One can suppose $B_j=\boldsymbol{N}^{(j)}_\delta$, $1\leqq j\leqq l$, and $x\notin\boldsymbol{D}$, since the case that $x\in\boldsymbol{D}$ is already obtained. Note that there exist a compact set F in $\boldsymbol{D}\cap\mathcal{D}(\mathcal{V}_{j_0})$ and T, $\delta_1>0$ such that, if $\phi\in C([0,T],\mathcal{M})$ satisfies $\sup_{0\leq t\leq T}|\phi(t)-\bar{x}_t(x)|<\delta_1$, then $\phi(T)\in F$; recall $\bar{x}_t(x)$ is the solution of the ODE (2.7). Then, Theorem 3.2 in [4, Chapter 5] guarantees the existence of $r>0$ so that

(5.12) $$P_x(\sup_{0\leq t\leq T}|\phi(t)-\bar{x}_t(x)|\geqq\delta_1)\leqq e^{-r/\varepsilon^2}$$

for all sufficiently small $\varepsilon>0$. Hence, when we write

$$\begin{aligned}&P_x(y^{\varepsilon}_{t_1}\in N^{(j_1)}_\delta, \cdots, y^{\varepsilon}_{t_N}\in N^{(j_N)}_\delta)\\ &=E_x[P_{x^{\varepsilon}_T}(x^{\varepsilon}_{t_1\alpha_\varepsilon-T}\in N^{(j_1)}_\delta, \cdots, x^{\varepsilon}_{t_N\alpha_\varepsilon-T}\in N^{(j_N)}_\delta), \sup_{0\leq t\leq T}|\phi(t)-\bar{x}_t(x)|<\delta_1]\\ &\quad+P_x(y^{\varepsilon}_{t_1}\in N^{(j_1)}_\delta, \cdots, y^{\varepsilon}_{t_N}\in N^{(j_N)}_\delta, \sup_{0\leq t\leq T}|\phi(t)-\bar{x}_t(x)|\geqq\delta_1),\end{aligned}$$

(2.12) is immediately obtained from (5.12) and Proposition 5.11 with Remark 5.12. □

REMARK 5.13. Let $x\in\mathcal{D}(\mathcal{V}_k)$, $l+1\leqq k\leqq L_0$. Consider a Markov jump process X_t realized on some probability space $(\boldsymbol{\Omega}, \mathcal{F}, \boldsymbol{P})$ generated by $\mathcal{G}$ satisfying $\boldsymbol{P}(X_0=N^{(j)})=q_{k,j}$, $1\leqq j\leqq l$, where $q_{k,j}$ is defined by the RHS of (4.10) in Remark 4.8. Then, by the above methods, we can show

$$\lim_{\varepsilon\downarrow 0}P_x(y^{\varepsilon}_{t_1}\in N^{(j_1)}_\delta, \cdots, y^{\varepsilon}_{t_N}\in N^{(j_N)}_\delta)=\boldsymbol{P}(X_{t_1}=N^{(j_1)}, \cdots, X_{t_N}=N^{(j_N)})$$

for all $0<t_1<\cdots<t_N$, $N^{(j_1)}, \cdots, N^{(j_N)}\in\boldsymbol{B}$ and sufficiently small $\delta>0$. And if $N^{(j)}$ consists of one point b_j and $H^*(b_j)$ has rank d for every $1\leqq j\leqq l$, then we also have

$$\lim_{\varepsilon\downarrow 0}E_x[f_1(y^{\varepsilon}_{t_1})\cdots f_N(y^{\varepsilon}_{t_N})]=\boldsymbol{E}[f_1(X_{t_1})\cdots f_N(X_{t_N})]$$

for all $x\in\mathcal{D}(\mathcal{V}_k)$, $0<t_1<\cdots<t_N$ and bounded continuous functions $f_1, \cdots, f_N$ on $\mathcal{M}$, where $\boldsymbol{E}$ stands for the expectation with respect to $\boldsymbol{P}$.

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