Asymptotic expansion of an oscillating integral on a hypersurface

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1. Introduction.

Let $f : (\boldsymbol{R}^n, \vec{0}) \to (\boldsymbol{R}, \vec{0})$ be a germ of an analytic function at $\vec{0}$ in $\boldsymbol{R}^n$ and $g : (\boldsymbol{R}^n, \vec{0}) \to (\boldsymbol{R}, \vec{0})$ be a germ of an analytic function at the origin. We consider an oscillating integral of a phase function $f(x)$ with the constraint equation $g(x)=0$ such that

$$I(\tau, \varphi) = \int_{\boldsymbol{R}^n} e^{i\tau f(x)} \delta(g(x)) \varphi(x) dx \tag{1}$$

where τ is a real parameter and $\varphi \in C_0^\infty(\boldsymbol{R}^n)$. $\delta(g(x))$ is the "delta-function" [3, 17] expressing the constraint $g(x)=0$. This type of the oscillating integral sometimes appears in a kind of the path integral in physics as the Faddeev-Popov method for gauge theory ([15]), and also appeared in a mathematical physics (for example, see [16]). It is meaningful to obtain the asymptotic expansion of the integral for a large τ.

The existence of the asymptotic expansion for the oscillating integral (1) is easily proved by a technique of Jeanquatier [11] (see also Arnold, Guzein-Zade and Varchenko [7]) or Malgrange [13]. The purpose of this paper is to calculate the principal term of the asymptotic expansion of the oscillating integral (1) in a neighborhood of a singularity of the phase in terms of Newton's diagram of the phase function and the constraint equation. The asymptotic expansion of an oscillating integral without the constraint $\delta(g(x))$ is already studied by Varchenko [6] by means of the toroidal embedding. Our considerations are not included in their works. It is shown by seeing the next example. Let f_d be the principal term such that $f_d = x^2 y^2 z^2$ in $\boldsymbol{R}^3$. We consider the hyperplane $g = x+y+z=0$. Substituting the equation into f_d, this term has the form $f_d = x^2 y^2 (x+y)^2$. This $f_d(x, y)$ is degenerate type in the meaning of Varchenko. Generally, the non-degeneracy depends on the choice of the coordinate. However in the above case we can prove easily that any coordinate transformation does not change its degeneracy. Hence if one should estimate the integral (1) with substituting directly the constraint equation $g(x)=0$, then the substituted

function f is not necessarily non-degenerate. Thus in the case $n>2$ it is difficult to calculate generally the power index of the asymptotic expansion for our integrals by means of Varchenko [6].

Using a toroidal resolution with a canonical simplicial subdivision of the dual Newton diagram (Oka [1]), we obtain our main result Theorem 14 and Theorem 27 which give a method to estimate the highest weight of the asymptotic expansion

$$\int_{R^n} e^{i\tau f(x)} \delta(g(x))\varphi(x)dx \sim \sum_p \sum_{k=0}^{n-2} a_{p,k}(\varphi)\tau^p(\log \tau)^k$$

for $|\tau|\to\infty$ through the Newton boundary.

In section 2, we will recall the toroidal resolution and non-degenerate complete intersection variety. In section 3, we will prove our main theorem for smooth $g(x)$ function, and in section 4 we extend it to the case of singular function. In Section 5, we will show some examples of the asymptotic expansion.

2. Non-degenerate complete intersection variety and its toroidal resolution.

In this section, we recall briefly the definition and the fundamental facts for a toroidal resolution of a complete intersection variety (see the detail in Khovansky [4, 5] and Oka [2]).

Let $f(x)=\sum_\nu a_\nu x^\nu$ and $g(x)=\sum_\nu b_\nu x^\nu$ be analytic functions of n-variables. We consider the function $f(x)$ and $g(x)$ on R^n as the restriction of the complex analytic function $f(z)$ and $g(z)$ which is defined in a neighborhood U of the origin in C^n. So, we initially states the some definitions and facts for the complex case. Let $h(z)=\sum_\nu c_\nu z^\nu$ be an analytic function of n-variables which is defined in a neighborhood of the origin. The Newton polyhedron $\Gamma_+(h)$ is the convex hull of the union of $\{\nu+R_+^n\}$ for ν such that $c_\nu\neq0$. The Newton boundary $\Gamma(h)$ is the union of the compact faces of the Newton polyhedron $\Gamma_+(h)$. The dual space of R^n can be canonically identified with R^n itself by the Euclidean inner product. Let N be the set of integral dual vectors under this identification and let N^+ be the set of positive integral dual vectors. Let $P={}^t(p_1, \cdots, p_n)$. For each $x\in R^n$, $P(x)$ is defined by $P(x):=\sum_{i=1}^n p_i x_i$. P is called a positive (respectively a strictly positive) dual vector if $p_i\geq0$ (respectively $p_i>0$) for $i=1, \cdots, n$. For a positive integral dual vector $P\in N^+$, we

define $d(P; h)$ as the minimal value of the restriction $P|_{\Gamma_+(h)}$, i.e., $d(P; h)=$ $\min\{P(x); x\in\Gamma_+(h)\}$ and let $\Delta(P; h)=\{x\in\Gamma_+(h); P(x)=d(P; h)\}$. We define $h_P(z)=h_{\Delta(P;h)}(z)$ where $h_\Delta(z)=\sum_{\nu\in\Delta}c_\nu z^\nu$. We call $h_P(z)$ the face function of h with respect to P.

Let $(f(z), g(z))$ be an analytic mapping from a neighborhood U of the origin of C^n to C^2 such that $f(\vec{0})=g(\vec{0})=0$. We say that the variety $V=\{z\in U; f(z)=g(z)=0\}$ is a *non-degenerate complete intersection variety* at $\vec{0}$ (with respect to the Newton boundary) if for any strictly positive dual vector $P={}^t(p_1, \cdots, p_n)$, the 2-form $df_P\wedge dg_P$ does not vanish on $V^*(P)=\{z\in C^{*n}; f_P(z)=g_P(z)=0\}$ (Khovansky [4, 5], Oka [2]). Here f_P and g_P is the face functions of f and g, respectively, with respect to P. Hereafter in this section, we assume that V is a non-degenerate complete intersection variety and that each f and g is convenient. Convenient means that $f^{(i)}$ and $g^{(i)}$ is not identically zero for any $i=1, \cdots, n$. We define an equivalence relation $\sim$ on the space of the positive integral dual vectors N^+ by $P\sim Q$ if and only if $\Delta(P; f)=\Delta(Q; f)$ and $\Delta(P; g)=\Delta(Q; g)$. This defines a polyhedral subdivision $\Gamma^*(f, g)$ of N^+. $\Gamma^*(f, g)$ is called the dual Newton diagram of (f, g). If we define $h=f\cdot g$, the dual Newton diagram $\Gamma^*(h)$ is equal to $\Gamma^*(f, g)$ (see Oka [1, 2]). Let Σ^* be a fixed unimodular simplicial subdivision of $\Gamma^*(f, g)$ and let $\hat{\pi}: X\to C^n$ be the associated toroidal modification map (see Ehler [8], Varchenko [6], Oka [1, 2]). Let $\tilde{V}$ be the proper transform of V and let $\pi: \tilde{V}\to V$ be the restriction of $\hat{\pi}$ to $\tilde{V}$. It is well-known that $\pi: \tilde{V}\to V$ is a good resolution of V. We assume that the set of the vertices of Σ^* which are not strictly positive is equal to $\{R_1, \cdots, R_n\}$ where $R_i={}^t(0, \cdots, \overset{i}{1}, \cdots, 0)$. This implies that $\pi: X-\pi^{-1}(\vec{0})\to C^n-\{\vec{0}\}$ is biholomorphic.

Next, the construction of X is simply described. X is covered by affine space C_σ^n with coordinate $\mathbf{y}_\sigma=(y_{\sigma,1}, \cdots, y_{\sigma,n})$ where σ moves in n-simplicial cone of Σ^*. An n-simplicial cone is always identified with a unimodular $n\times n$ matrix, the corresponding cone in Σ^* is defined by $\{\sum_{i=1}^n t_i P_i; t_1, \cdots, t_n\geq 0\}$. $P_1, \cdots, P_n$ are called vertices of the simplicial cone σ. Let $\sigma=(p_{ij})$ be an n-simplicial cone. Then $\hat{\pi}|_{C_\sigma^n}$ is defined by

$$\hat{\pi}(\mathbf{y}_\sigma) = \mathbf{z} = (z_1, \cdots, z_n)$$

where $z_i=\prod_{j=1}^n y_{\sigma j}^{p_{ij}}$. Let P be a vertex of Σ^*. Then P defines a divisor $\hat{E}(P)$ of X as follows. Let $\sigma=(P_1, \cdots, P_n)$ be an n-simplicial cone of Σ^* such that $P=P_1$. Then $\hat{E}(P)\cap C_\sigma^n$ is defined by the divisor $y_{\sigma 1}=0$. For an n-simplicial cone τ, $\hat{E}(P)\cap C_\tau^n\neq\varnothing$ if and only if P is a vertex of τ. If P is strictly positive, the union of $\{\hat{E}(P)\cap C_\sigma^n; P\in\sigma\}$ for σ is a compact toric variety of dimension $n-1$. For the general properties of the toric varieties, see K-K-M-S [10] and Oda [14]. If P is strictly positive, then $\hat{E}(P)$ is an exceptional divisor,

i. e., $\hat{\pi}(\hat{E}(P))=\{\vec{0}\}$. On the other hand, $\hat{E}(R_i)$ is isomorphic to the hyperplane $\{z_i=0\}$ in the base space $\boldsymbol{C}^n$ by the projection $\hat{\pi}$. In our practical calculation to be continued on the next section, we use the method of the unimodular simplicial subdivision of Σ^* which is developed by Oka [1].

3. Asymptotic expansion for smooth $g(x)$.

In this section, we shall prove Theorem 16 and Corollary 17. Following the definition of the complex case, we can also define a non-degenerate complete intersection variety in the real case. Let $(f(x), g(x))$ be an analytic mapping from a neighborhood U of the origin of $\boldsymbol{R}^n$ to $\boldsymbol{R}^2$ such that $f(\vec{0})=g(\vec{0})=0$ where we hereafter assume $n\geq 3$. We say that the variety $W=\{x\in U\,;\,f(x)=g(x)=0\}$ is a *non-degenerate complete intersection variety* at $\vec{0}$ with respect to the Newton boundary if for any strictly positive dual vector $P={}^t(p_1, \cdots, p_n)$, the 2-form $df_P\wedge dg_P$ does not vanish on $W^*(P)=\{x\in\boldsymbol{R}^{*n}\,;\,f_P(x)=g_P(x)=0\}$. At first we assume that $f(x)$ and $g(x)$ are analytic functions with a singularity at $\vec{0}$, and below Proposition 5 we suppose that $g(x)$ is a smooth analytic function in this section. Moreover $W=\{x\in U\,;\,f(x)=g(x)=0\}$ is assumed as a non-degenerate complete intersection variety in a neighborhood U of the origin, and also each $f(x)$ and $g(x)$ is convenient. Under the above assumptions for $f(x)$ and $g(x)$, we rewrite the oscillating integral $I(\tau, \varphi)$ of equation (1) using the property of the delta function (see detail for Gel'fand-Šilov [3]) as follows:

$$I(\tau, \varphi) = \int_{\boldsymbol{R}^n} e^{i\tau f(x)}\delta(g(x))\varphi(x)dx$$

$$= \int_{g(x)=0} e^{i\tau f(x)}\varphi\omega_1 \tag{2}$$

where τ is a real parameter, $\varphi\in C_0^\infty(\boldsymbol{R}^n)$, and the $(n-1)$-form ω_1 is defined as

$$\omega_1\wedge dg(x) = dx_1\wedge \cdots \wedge dx_n\,. \tag{3}$$

The existence and the uniqueness of the form ω_1 on $g(x)=0$ in $\boldsymbol{R}^n$ is well-known, i. e., we can take the Leray-form for ω_1 ([3]). Our strategy is based on the following formal analytic calculations. At first, the equation (2) is transformed to

$$I(\tau, \varphi) = \int dt\, e^{i\tau t}\int_{f(x)=t}\delta(g(x))\varphi(x)\omega_0$$

$$= \int dt e^{i\tau t}\int_{\substack{f(x)=t \\ g(x)=0}}\varphi(x)\omega_2$$

where the forms ω_2 and ω_0 are defined, respectively, as

$$\omega_0 \wedge df = dx_1 \wedge \cdots \wedge dx_n$$

$$\omega_2 \wedge df \wedge dg = dx_1 \wedge \cdots \wedge dx_n .$$

Remark that we can see that for a sufficiently small t, $f(x)$ is smooth on $\{f(x) = t \ (\neq 0)\}$. Also, by the assumption of the complete intersection for $f(x)$ and $g(x)$, we see that the 2-form $df \wedge dg$ does not vanish on $\{f(x)=t, g(x)=0 \ (t \neq 0)\}$.

We define the kernel function $K(f, g, \varphi, c)$ for $c \in \boldsymbol{R}$ as

$$K(f, g, \varphi, c) = \int_{\substack{f(x)=c \\ g(x)=0}} \varphi(x)\omega_2 .$$

The initial oscillating integral now reads to

$$I(\tau, \varphi) = \int e^{i\tau t} K(f, g, \varphi, t)dt .$$

Thus we expand asymptotically the kernel $K(f, g, \varphi, t)$ for $t \to 0$.

For our purpose, let us consider the integrals:

$$I_+(\tau, \varphi) = \int_{\boldsymbol{R}^n} (f_+(x))^\tau \delta(g(x))\varphi(x)dx_1 \cdots dx_n$$

$$I_-(\tau, \varphi) = \int_{\boldsymbol{R}^n} (f_-(x))^\tau \delta(g(x))\varphi(x)dx_1 \cdots dx_n$$

where

$$f_+(x) = \begin{cases} f(x), & \text{for } f(x) \geq 0 , \\ 0, & \text{for } f(x) < 0 , \end{cases}$$

$$f_-(x) = \begin{cases} 0, & \text{for } f(x) \geq 0 , \\ -f(x), & \text{for } f(x) < 0 , \end{cases}$$

and $\tau \in \boldsymbol{C}$, $\operatorname{Re} \tau > 0$. The integrals I_+ and I_- are analytic functions of the parameter τ (for simplicity, we denote together the both integrals such as $I_\pm$). On the other hand, the integral $I_\pm$ is written as

$$I_\pm(\tau, \varphi) = \int_{\{\pm c > 0\}} dc \ (\pm c)^\tau K(f, g, \varphi, c) .$$

Then the kernel $K(f, g, \varphi, c)$ can be described by $I_\pm(\tau, \varphi)$ with the inverse Mellin transformation, and we will obtain the our final results.

Let $(f, g) : \boldsymbol{R}^n \to \boldsymbol{R}^2$ be an analytic mapping at $\vec{0}$ in $\boldsymbol{R}^n$ with $f(\vec{0})=g(\vec{0})=0$ and 2-form $df \wedge dg$ vanish at $\vec{0}$. Let Y be a non-singular real analytic n-dimensional manifold, U be a neighborhood of $\vec{0} \in \boldsymbol{R}^n$ and $\pi : Y \to U \subset \boldsymbol{R}^n$ a proper analytic mapping such that at each point of the set $S = \pi^{-1}(\vec{0})$, there exist local coordinates $(y_1, \cdots, y_n)$ in the neighborhood of the each point of S, at which (see Atiyah [9])

$(D-1)$
$$\begin{cases} (f \circ \pi)(y_1, \cdots, y_n) = \varepsilon_f y_1^{l_1} \cdots y_n^{l_n}, \\ (g \circ \pi)(y_1, \cdots, y_n) = \varepsilon_g y_1^{n_1} \cdots y_n^{n_n}, \end{cases}$$

where ε_f and ε_g are invertible analytic functions.

$(D-2)$ The Jacobian J_π of the mapping π has the form
$$J_\pi(y_1, \cdots, y_n) = y_1^{m_1} \cdots y_n^{m_n} \bar{J}_\pi(y_1, \cdots, y_n)$$
where $\bar{J}_\pi(0, \cdots, 0) \neq 0$.

$(D-3)$ In a neighborhood U of $\vec{0}$ in $\mathbf{R}^n$, π is an analytic
isomorphism outside a proper analytic subset in $\mathbf{R}^n$.

By means of the method described in the previous section (K-K-M-S [10]), for a given Newton boundary Γ, we can construct a complex manifold $X(\Gamma)$ and its projection $\pi: X(\Gamma) \to \mathbf{C}^n$. The transition functions between the local maps of the manifold $X(\Gamma)$ are real on real parts of the manifold $X(\Gamma)$. We denote it by $Y(\Gamma)$. The restriction of the projection π to $Y(\Gamma)$ is also denoted by π. Then we have that (i) $Y(\Gamma)$ is a non-singular n-dimensional real (algebraic) manifold and (ii) $\pi: Y(\Gamma) \to \mathbf{R}^n$ is a proper mapping onto $\mathbf{R}^n$ that satisfies the condition $(D-1) \sim (D-3)$, for almost all the function (f, g) with a given Newton boundary $\Gamma(f, g) = \Gamma$. We shall show the following Lemma.

LEMMA 1. Let $f(x_1, \cdots, x_n)$ and $g(x_1, \cdots, x_n)$ be germs of real valued real analytic functions at $\vec{0} \in \mathbf{R}^n$. Let Σ^* be a unimodular simplicial subdivision of $\Gamma^*(f, g)$, and $\sigma \in \Sigma^*$ be an n-simplicial cone. For a coordinate neighborhood $\mathbf{R}^n(\sigma)$, the mapping $\pi(\sigma): \mathbf{R}^n(\sigma) \to \mathbf{R}^n$ has the following properties:

(1)
$$\begin{cases} (f \circ \pi(\sigma))[y_1, \cdots, y_n] = y_1^{l_1} \cdots y_n^{l_n} f_\sigma(y_1, \cdots, y_n) \\ (g \circ \pi(\sigma))[y_1, \cdots, y_n] = y_1^{n_1} \cdots y_n^{n_n} g_\sigma(y_1, \cdots, y_n) \end{cases}$$

where $y_1, \cdots, y_n$ is the coordinate in $\mathbf{R}^n(\sigma)$ and $f_\sigma(0, \cdots, 0) \neq 0$, $g_\sigma(0, \cdots, 0) \neq 0$.

(2) The Jacobian of the mapping $\pi(\sigma)$ is equal to $y_1^{m_1} \cdots y_n^{m_n} \cdot C$ where $m_i := |P_i(\sigma)| - 1$, $|P_i(\sigma)| := \sum_{j=1}^n p_{ji}(\sigma)$ for $P_i = {}^t(p_{1i}, \cdots, p_{ni})$ and C is a non-zero constant.

(3) A set of points in $\mathbf{R}^n$ in which π is not an isomorphism is a union of coordinate planes.

PROOF. (1) It is easily verified by the same manner as the Varchenko [6] (Lemma 2.13).

(2) We calculate directly the Jacobian of the mapping $\pi(\sigma)$:

$$J_{\pi(\sigma)} = \frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)}$$

$$= \det \begin{vmatrix} p_{11}(\sigma)y_1^{p_{11}(\sigma)-1} \cdots y_n^{p_{1n}(\sigma)}, & \cdots, & p_{1n}(\sigma)y_1^{p_{11}(\sigma)} \cdots y_n^{p_{1n}(\sigma)-1} \\ \vdots & & \vdots \\ p_{n1}(\sigma)y_1^{p_{n1}(\sigma)-1} \cdots y_n^{p_{nn}(\sigma)}, & \cdots, & p_{nn}(\sigma)y_1^{p_{n1}(\sigma)} \cdots y_n^{p_{nn}(\sigma)-1} \end{vmatrix}$$

$$= y_1^{|P_1(\sigma)|-1} \cdots y_n^{|P_n(\sigma)|-1} \cdot \det(P(\sigma)),$$

where $\det(P(\sigma)) = \pm 1$.

(3) The assertion follows directly from the formula for π. That is, the set in which $\pi(\sigma)$ is not one-to-one mapping is the union of hyperplanes $\{y_i=0\}$ corresponding to the index i of $m_i = |P_i(\sigma)| - 1 \neq 0$. This completes the proof of Lemma 1.

We have the next fundamental results.

PROPOSITION 2. *Let* $(f(x), g(x))$ *be an analytic mapping from a neighborhood* U *of the origin of* $\boldsymbol{R}^n$ *to* $\boldsymbol{R}^2$ *such that* $f(\vec{0})=g(\vec{0})=0$. *If* $W=\{x\in U ; f(x)=g(x) =0\}$ *is a non-degenerate complete intersection variety at* $\vec{0}$, *then the manifold* $Y(\Gamma)$ *and the projection* $\pi : Y(\Gamma)\to\boldsymbol{R}^n$, *together, with the analytic functions defined by* $f(x)$ *and* $g(x)$, *will satisfy the condition* $(D-1)$, $(D-2)$ *and* $(D-3)$.

PROOF. By Lemma 1, the assertion is clear for a neighborhood of $\boldsymbol{y}=\vec{0}$ in $\boldsymbol{R}^n$. According to Lemma 1, $\pi(\sigma)$ gives an isomorphism between $\{y_1 \cdots y_n \neq 0\}$ in $\boldsymbol{R}^n(\sigma)$ and $\{x_1 \cdots x_n \neq 0\}$ in $\boldsymbol{R}^n$. Therefore, in the neighborhood of $\pi^{-1}(\vec{0})$, we may consider the assertion only on the neighborhood of coordinate planes $\{y_j=0\}$ $(j=1, \cdots, n)$ since it is possible that the branch points $f_\sigma(y_1, \cdots, y_n)=0$ or $g_\sigma(y_1, \cdots, y_n)=0$ appear away from the origin. This follows from the next Lemma which is essentially the same one as in the complex case proved by Oka [2], and here we give an elementary proof.

LEMMA 3. *Let* $\sigma\in\Sigma^*$ *be an* n-*simplicial cone,* $I\subset\{1, \cdots, n\}$, *and* $T_I = \{\boldsymbol{y}\in\boldsymbol{R}^n(\sigma) ; y_j=0, j\in I\}$. *Then:*

(1) *For germs of any real valued real analytic functions* $f(x)$ *and* $g(x)$ *with a given Newton boundaries* $\Gamma(f)=\Gamma_1$ *and* $\Gamma(g)=\Gamma_2$ *respectively, if* $\pi_\sigma(T_I)=0$, *then* $f_{\sigma,I} := f_\sigma|_{T_I}$ *and* $g_{\sigma,I} := g_\sigma|_{T_I}$ *will be polynomials.*

(2) *If* $\{x\in U | f(x)=g(x)=0\}$ *is a non-degenerate complete intersection variety and* $\pi(\sigma)(T_I)=0$, *then the set* $\{\boldsymbol{y}\in T_I | f_\sigma(\boldsymbol{y})=g_\sigma(\boldsymbol{y})=0\}$ *will be a non-singular complete intersection variety, i.e.,* 2-*form* $d(f_\sigma|_{T_I})\wedge d(g_\sigma|_{T_I})$ *(or* $df_\sigma\wedge dg_\sigma|_{T_I}$) *will be non-vanishing on them.*

PROOF OF LEMMA 3 (see also [6, 12]).

(1) The condition $\pi_\sigma(T_I)=0$ means that for any i $(i=1, \cdots, n)$, $\sum_{j\in I} p_{ij}(\sigma)$

>0. Then we show that this condition is equivalent to the next condition:

Indefinite equations for α and β

$$\left\langle \sum_{j \in I} P_j(\sigma),\ \alpha \right\rangle = \sum_{j \in I} d(P_j(\sigma),\ f)$$

$$\left\langle \sum_{j \in I} P_j(\sigma),\ \beta \right\rangle = \sum_{j \in I} d(P_j(\sigma),\ g)$$
(4)

have at most finitely many solutions $\alpha \in \Gamma_+(f)$ *and* $\beta \in \Gamma_+(g)$.

The sufficiency is obvious, i.e., under the condition $\sum_{j \in I} p_{ij}(\sigma) > 0$, each component of α and β in equation (4) is bounded. We show the converse. From the definition of $p_{ij}(\sigma)$, we always hold $\sum_{j \in I} p_{ij}(\sigma) \geq 0$. Hence, suppose that there exists i_0 such that $\sum_{j \in I} p_{i_0 j}(\sigma) = 0$. Then, the corresponding components of α and β have the infinite freedom.

For $\alpha \in \Gamma_+(f)$ and $\beta \in \Gamma_+(g)$, by the definition of $d(P_j(\sigma),\ f)$ and $d(P_j(\sigma),\ g)$, the equation of (4) is, respectively, equivalent to

$$\begin{aligned} \langle P_j(\sigma),\ \alpha \rangle &= d(P_j(\sigma),\ f) \\ \langle P_j(\sigma),\ \beta \rangle &= d(P_j(\sigma),\ g). \end{aligned} \qquad \text{for } j \in I \tag{5}$$

The assertion easily follows from the fact that the inequalities $\langle P_j(\sigma),\ \alpha \rangle \geq d(P_j(\sigma),\ f)$ and $\langle P_j(\sigma),\ \beta \rangle \geq d(P_j(\sigma),\ g)$ always hold.

The solutions of equation (5) consist from the points α and β in faces γ and γ' of $\Gamma_+(f)$ and $\Gamma_+(g)$, respectively, which correspond to $\#I$-dimensional faces determined by $\{P_j(\sigma);\ j \in I\}$. Since we have

$$\begin{cases} (f \circ \pi(\sigma))[y_1,\ \cdots,\ y_n] = y_1^{d(P_1 \cdot f)} \cdots y_n^{d(P_n \cdot f)} f_\sigma(y_1,\ \cdots,\ y_n) \\ (g \circ \pi(\sigma))[y_1,\ \cdots,\ y_n] = y_1^{d(P_1 \cdot g)} \cdots y_n^{d(P_n \cdot g)} g_\sigma(y_1,\ \cdots,\ y_n), \end{cases}$$

the non-vanishing terms in $f_\sigma(\boldsymbol{y})$ and $g_\sigma(\boldsymbol{y})$ whenever one puts y_j for $j \in I$ to zero are those only which satisfy the equation (5) in terms y^α and y^β of $(f \circ \pi(\sigma))(y)$ and $(g \circ \pi(\sigma))(y)$, respectively. Therefore $f_{\sigma, I}$ and $g_{\sigma, I}$ have a finitely many terms satisfying equation (5), respectively. This proves the first assertion.

(2) The faces γ and γ' which are determined by the equation (5) are compact since the integral points are finite on them. Hence the faces are also included in the Newton boundary $\Gamma(f)$ and $\Gamma(g)$, respectively. So, we have

$$\begin{cases} (f_\gamma \circ \pi(\sigma))[y_1,\ \cdots,\ y_n] = y_1^{d(P_1 \cdot f)} \cdots y_n^{d(P_n \cdot f)} (f_\sigma(y_1,\ \cdots,\ y_n)|_{T_I}) \\ (g_{\gamma'} \circ \pi(\sigma))[y_1,\ \cdots,\ y_n] = y_1^{d(P_1 \cdot g)} \cdots y_n^{d(P_n \cdot g)} (g_\sigma(y_1,\ \cdots,\ y_n)|_{T_I}). \end{cases} \tag{6}$$

The functions f_γ and $g_{\gamma'}$ are quasi-homogeneous polynomials. Let the hyperplane containing the faces γ and γ' be $\langle m,\ \alpha \rangle = q$ and $\langle m',\ \beta \rangle = q'$, respectively. Then one has the equalities

$$f_\gamma(t^{m_1}x_1, \cdots, t^{m_n}x_n) = t^q f_\gamma(x_1, \cdots, x_n)$$

$$g_{\gamma'}(t^{m'_1}x_1, \cdots, t^{m'_n}x_n) = t^{q'} g_{\gamma'}(x_1, \cdots, x_n).$$

Differentiating the both side of the above equalities with respect to t, and putting $t=1$, one can obtain the so-called "Euler's identities":

$$m_1 x_1 \frac{\partial f_\gamma}{\partial x_1} + m_2 x_2 \frac{\partial f_\gamma}{\partial x_2} + \cdots + m_n x_n \frac{\partial f_\gamma}{\partial x_n} = q f_\gamma$$

$$m'_1 x_1 \frac{\partial g_{\gamma'}}{\partial x_1} + m'_2 x_2 \frac{\partial g_{\gamma'}}{\partial x_2} + \cdots + m'_n x_n \frac{\partial g_{\gamma'}}{\partial x_n} = q' g_{\gamma'}.$$

Now, we can take the hyperplanes containing γ and γ' as $m_i = p_{ij}(\sigma)$, $q = d(P_j, f)$ and $m'_i = p_{ij}(\sigma)$, $q' = d(P_j, g)$ for each $j \in I$. Then, on each $f_\gamma(x) = 0$ and $g_{\gamma'}(x) = 0$ we have, respectively,

$$p_{1j}(\sigma) x_1 \frac{\partial f_\gamma}{\partial x_1} + \cdots + p_{nj}(\sigma) x_n \frac{\partial f_\gamma}{\partial x_n} = 0$$
$$\text{for } j \in I \qquad (7)$$
$$p_{1j}(\sigma) x_1 \frac{\partial g_{\gamma'}}{\partial x_1} + \cdots + p_{nj}(\sigma) x_n \frac{\partial g_{\gamma'}}{\partial x_n} = 0.$$

Let $\tilde{y} = {}^t(\tilde{y}_1, \cdots, \tilde{y}_n)$ and

$$\tilde{y}_j \neq \begin{cases} y_j & (j \notin I) \\ 1 & (j \in I). \end{cases}$$

For brevity's sake, let us $\tilde{y}^{P(\sigma)} = (\tilde{y}_1^{p_{11}(\sigma)} \cdots \tilde{y}_n^{p_{1n}(\sigma)}, \cdots, \tilde{y}_1^{p_{n1}(\sigma)} \cdots \tilde{y}_n^{p_{nn}(\sigma)})$. Using the homogeneity of f_γ and $g_{\gamma'}$, we obtain

$$(f_\gamma \circ \pi(\sigma))[y_1, \cdots, y_n] = f_\gamma(y_1^{p_{11}(\sigma)} \cdots y_n^{p_{1n}(\sigma)}, \cdots, y_1^{p_{n1}(\sigma)} \cdots y_n^{p_{nn}(\sigma)})$$

$$= \prod_{j \in I} y_j^{d(P_j(\sigma), f)} f_\gamma(\tilde{y}^{P(\sigma)})$$

$$(g_{\gamma'} \circ \pi(\sigma))[y_1, \cdots, y_n] = g_{\gamma'}(y_1^{p_{11}(\sigma)} \cdots y_n^{p_{1n}(\sigma)}, \cdots, y_1^{p_{n1}(\sigma)} \cdots y_n^{p_{nn}(\sigma)})$$

$$= \sum_{j \in I} y_j^{d(P_j(\sigma), g)} g_{\gamma'}(\tilde{y}^{P(\sigma)}).$$

Comparing the equation with (6), we have

$$f_\sigma |_{T_I} = \frac{f_\gamma(\tilde{y}^{P(\sigma)})}{\prod_{j \notin I} y_j^{d(P_j(\sigma), f)}}$$
$$\text{on } T_I \qquad (8)$$
$$g_\sigma |_{T_I} = \frac{g_{\gamma'}(\tilde{y}^{P(\sigma)})}{\prod_{j \notin I} y_j^{d(P_j(\sigma), g)}}.$$

The denominators do not vanish for $\{y_j \neq 0 ; j \notin I\}$ on T_I. Hence, for studying $f_\sigma(y)|_{T_I} = 0$ and $g_\sigma(y)|_{T_I} = 0$, it is sufficient to consider the equations, respectively,

$$f_\gamma(\tilde{y}^{P(\sigma)}) = g_{\gamma'}(\tilde{y}^{P(\sigma)}) = 0.$$

Differentiating $f_\gamma(\tilde{\boldsymbol{y}}^{P(\sigma)})$ and $g_\gamma(\tilde{\boldsymbol{y}}^{P(\sigma)})$ with respect to y_j $(j \notin I)$, one obtains that (noting $x_k = \tilde{y}_1^{p_{k1}(\sigma)} \cdots \tilde{y}_n^{p_{kn}(\sigma)} \neq 0$):

$$\frac{\partial}{\partial y_j} f_\gamma(\tilde{\boldsymbol{y}}^{P(\sigma)}) = \sum_{k=1}^{n} \frac{\partial f_\gamma(\boldsymbol{x})}{\partial x_k} \frac{\partial}{\partial y_j}(\tilde{y}_1^{p_{k1}(\sigma)} \cdots \tilde{y}_n^{p_{kn}(\sigma)})$$

$$= \frac{1}{y_j} \sum_{k=1}^{n} p_{kj}(\sigma) x_k \frac{\partial f_\gamma(\boldsymbol{x})}{\partial x_k} \qquad j \notin I, \tag{9}$$

$$\frac{\partial}{\partial y_j} g_{\gamma'}(\tilde{\boldsymbol{y}}^{P(\sigma)}) = \frac{1}{y_j} \sum_{k=1}^{n} p_{kj}(\sigma) x_k \frac{\partial g_{\gamma'}(\boldsymbol{x})}{\partial x_k} \qquad j \notin I.$$

To show the assertion, we suppose that 2-form $d(f_\sigma|_{T_I}) \wedge d(g_\sigma|_{T_I}) = 0$. Then we show the our assertion with the following two steps:

(i) $\mathrm{grad}\,(f_\sigma|_{T_I}) = 0$ or $\mathrm{grad}\,(g_\sigma|_{T_I}) = 0$,

(ii) $\mathrm{grad}\,(f_\sigma|_{T_I}) \neq 0$, $\mathrm{grad}\,(g_\sigma|_{T_I}) \neq 0$ and 2-form $d(f_\sigma|_{T_I}) \wedge d(g_\sigma|_{T_I}) = 0$.

Firstly, we assume the case (i). Then adding the equation (7) to $\partial f_\gamma/\partial y_j|_{T_I} = 0$ $(j \notin I)$ or $\partial g_{\gamma'}/\partial y_j|_{T_I} = 0$ $(j \notin I)$ of the equation (9), we have a simultaneous linear equation with the coefficient $(p_{ij}(\sigma))$ $(i, j = 1, \cdots, n)$. The equation has a solution $(x_1 \partial f_\gamma/\partial x_1, \cdots, x_n \partial f_\gamma/\partial x_n)$ or $(x_1 \partial g_{\gamma'}/\partial x_1, \cdots, x_n \partial g_{\gamma'}/\partial x_n)$ according to whether $\mathrm{grad}\,(f_\sigma|_{T_I}) = 0$ or $\mathrm{grad}\,(g_\sigma|_{T_I}) = 0$, respectively. Note that the point $x_k = \tilde{y}_1^{p_{k1}(\sigma)} \cdots \tilde{y}_n^{p_{kn}(\sigma)}$ satisfies $f_\gamma(\tilde{\boldsymbol{y}}^{P(\sigma)}) = 0$ and $g_{\gamma'}(\tilde{\boldsymbol{y}}^{P(\sigma)}) = 0$. Since $\det(P(\sigma)) = \pm 1$, one obtains $x_k \partial f_\gamma/\partial x_k = 0$ or $x_k \partial g_{\gamma'}/\partial x_k = 0$. Hence, by $x_k \neq 0$, we have $\partial f_\gamma/\partial x_k = 0$ or $\partial g_{\gamma'}/\partial x_k = 0$ for $k = 1, \cdots, n$. This contradicts to the assumption of the non-degenerate complete intersection variety of $\{\boldsymbol{y} \in T_I \mid f_\sigma(\boldsymbol{y}) = g_\sigma(\boldsymbol{y}) = 0\}$.

Next we assume the case (ii). By the assumption, we have the 2-form

$$\sum_{j,j'} \frac{\partial(f_\sigma|_{T_I})}{\partial y_j} \frac{\partial(g_\sigma|_{T_I})}{\partial y_{j'}} dy_j \wedge dy_{j'} = 0 \qquad \text{for } j, j' \notin I \ (j \neq j').$$

Then using equation (8) and (9), one obtains that for each $j, j' \in I$ $(j < j')$,

$$\frac{\partial f_\gamma(\tilde{\boldsymbol{y}}^{P(\sigma)})}{\partial y_j} \frac{\partial g_{\gamma'}(\tilde{\boldsymbol{y}}^{P(\sigma)})}{\partial y_{j'}} - \frac{\partial f_\gamma(\tilde{\boldsymbol{y}}^{P(\sigma)})}{\partial y_{j'}} \frac{\partial g_{\gamma'}(\tilde{\boldsymbol{y}}^{P(\sigma)})}{\partial y_j} = 0.$$

Since y_j and $y_{j'}$ do not vanish for $j, j' \notin I$, we have

$$\sum_{k<k'}^{n} p_{kj}(\sigma) p_{k'j'}(\sigma) x_k x_{k'} \left\{ \frac{\partial f_\gamma}{\partial x_k} \frac{\partial g_{\gamma'}}{\partial x_{k'}} - \frac{\partial f_\gamma}{\partial x_{k'}} \frac{\partial g_{\gamma'}}{\partial x_k} \right\} = 0 \qquad \text{for } j, j' \notin I. \tag{10}$$

On the other hand, the identities (7) mean that

$$\sum_{k<k'}^{n} p_{kj}(\sigma) p_{k'j'}(\sigma) x_k x_{k'} \left\{ \frac{\partial f_\gamma}{\partial x_k} \frac{\partial g_{\gamma'}}{\partial x_{k'}} - \frac{\partial f_\gamma}{\partial x_{k'}} \frac{\partial g_{\gamma'}}{\partial x_k} \right\} = 0 \tag{11}$$

where at least one of j and j' is contained in I. Combining the equation (11) with (10), one has the simultaneous linear equation for $A_{kk'} := \{x_k(\partial f_\gamma/\partial x_k) \cdot x_{k'}(\partial g_{\gamma'}/\partial x_{k'}) - x_{k'}(\partial f_\gamma/\partial x_{k'}) x_k(\partial g_{\gamma'}/\partial x_k)\}$;

$$\sum_{k<k'}^{n} p_{ki_1}(\sigma)p_{k'i_2}(\sigma)A_{kk'} = 0 \,, \tag{12}$$

where i_1 and i_2 $(i_1 < i_2)$ run in $\{1, \cdots, n\}$. We use the next elementary fact from linear algebra.

LEMMA 4 (Sylvester-Spottiswood). *Let* $D = |a_{ik}|$ *be a determinant of an* $n \times n$ *matrix* $A = (a_{ik})$. *Let* $\Delta_{ik} := \Delta\begin{pmatrix} i_1 & i_2 \cdots i_r \\ k_1 & k_2 \cdots k_r \end{pmatrix}$ *be an* $r \times r$ *minor determinant which is constructed by taking the columns* $i_1, i_2, \cdots, i_r$ *and the rows* $k_1, k_2, \cdots, k_r$ *in* D. *Then* $N \times N$ *determinant with components which are* $r \times r$ *minor determinant* Δ_{ik} *of* D *equals to* D^l *where* $N = \binom{n}{r}$ *and* $l = \binom{n-1}{r-1}$.

Taking $r = 2$, we apply Lemma 4 to the equation (12). Then, $N = \binom{n}{2}$ is the number of unknown quantities $A_{kk'}$ in (12), and the determinant of the coefficient equals to $|\det(p_{ij})|^{(n-1)} = (\pm 1)^{(n-1)} \neq 0$. This shows that the linear equation (12) has only a trivial solution $A_{kk'} = 0$ $(k < k')$ in which we already assume that $\partial f_r / \partial x_k$ or $\partial g_{r'} / \partial x_{k'}$ does not vanish. Hence, the 2-form $d f_r(x) \wedge d g_{r'}(x) = \sum_{k<k'} A_{kk'} dx_k \wedge dx_{k'}$ vanishes. This contradicts to our first assumption of non-degenerate complete intersection. This completes the proof of Lemma 3.

We return to the proof of Proposition 2. Note that $\pi(\sigma)^{-1}(0) \subset \{y_1 = 0\} \cup \cdots \cup \{y_n = 0\}$. Hence in the neighborhood of any point except the origin, this set is included in $T_I \cap \{y_j \neq 0, \; j \notin I\}$ for suitable $I \subset \{1, 2, \cdots, n\}$. Then by Lemma 3, our assertion is proved. This completes the proof of Proposition 2.

Now we obtain the pole of $I_\pm(\tau, \varphi)$ and its order. Here we recall the definition of $I_\pm(\tau, \varphi)$ as

$$I_\pm(\tau, \varphi) = \int_{R^n} (f_\pm)^\tau \delta(g(x)) \varphi(x) dx \,,$$

$$= \int_{g=0} (f_\pm)^\tau \varphi(x) \omega_1 \,,$$

where $\varphi(x) \in C_0^\infty(R^n)$. Hereafter we assume that the support of $\varphi(x)$ is sufficiently small so that the support of $\varphi(x)$ lies in an neighborhood U of the origin in R^n, and $g(x)$ is a smooth analytic function. Then we have the following results.

PROPOSITION 5. *The poles of* $I_\pm(\tau, \varphi)$ *belong to the set of next arithmetic progressions*

$$\begin{cases} -1, \ -2, \ \cdots & \text{for } (l, \ m-d(P, \ g))=(1, \ 0) \\ -\dfrac{m-d(P, \ g)+1}{l}, \ -\dfrac{m-d(P, \ g)+2}{l}, \ \cdots & \text{for } (l, \ m-d(P, \ g))\neq(1, \ 0) \end{cases}$$

where $l=d(P, \ f)$, *and* $m=|P|-1$ *for any* P.

PROOF. If necessary, by shrinking the neighborhood U of the origin in R^n, it is possible to assume that the critical value of $f(x)$ with $g(x)=0$ (i. e., the value of t such that $\{f(x)=t, \ g(x)=0\}$ includes a singularity) is only the origin in U. Let $\pi: Y \to U \subset R^n$ be the projection map described in Lemma 1 and $V=\pi^{-1}(U)$. We can choose the partition of unity $\{\varphi_\alpha(y)\}$ on Y satisfying $\sum_\alpha \varphi_\alpha|_V \equiv 1$, where the conditions of Lemma 1(1) and (2) for the local coordinate hold on each $\mathrm{supp}\,\varphi_\alpha$. For any φ_α, there exists an open set V' containing the support of φ_α, and local coordinates on V' such that (D-1) and (D-2) are satisfied on V'. For $\mathrm{Re}\,\tau > -1$ we have

$$I_\pm(\tau, \ \varphi) = \int_Y (f \circ \pi)_\pm^\tau (\varphi \circ \pi)(\omega_1 \circ \pi)$$

$$= \sum_\alpha \int_Y (f \circ \pi)_\pm^\tau \varphi_\alpha (\omega_1 \circ \pi).$$

In each σ, the coordinate neighborhood $R^n(\sigma)$ is contained in V', and we have

$$\begin{cases} (f \circ \pi(\sigma))[y_1, \ \cdots, \ y_n] = y_1^{d(P_1, f)} \cdots y_n^{d(P_n, f)} f_\sigma(y_1, \ \cdots, \ y_n) \\ (g \circ \pi(\sigma))[y_1, \ \cdots, \ y_n] = y_1^{d(P_1, g)} \cdots y_n^{d(P_n, g)} g_\sigma(y_1, \ \cdots, \ y_n), \end{cases}$$

where we take an n-simplicial cone $\sigma = \langle P_1, \ P_2, \ \cdots, \ P_n \rangle$ of Σ^* and let $P_i = {}^t(p_{1i}, \ p_{2i}, \ \cdots, \ p_{ni})$. So we consider the integral on each local chart σ:

$$\int_Y (f \circ \pi)_\pm^\tau (\varphi \circ \pi) \varphi_{\alpha, \sigma}(\omega_1 \circ \pi)$$

$$= \int_{\{\prod_{i=1}^n y_i^{d(P_i, g)} g_\sigma(y)=0\} \cap V',} (\varepsilon(y_1)_{\varepsilon_1}^{d(P_1, f)} \cdots (y_n)_{\varepsilon_n}^{d(P_n, f)} f_\sigma(y)_\pm)^\tau (\varphi \circ \pi) \varphi_{\alpha, \sigma} \pi^*(\omega_1)$$

where $\varepsilon = \pm 1$ and $\varepsilon_j = \pm$ for $j=1, \ \cdots, \ n$. Then we calculate the integral (13) by dividing finitely to each component. In order to estimate (13), we now describe the form $\pi^*(\omega_1)$. We denote the $(n-1)$-form ω_1 by $dx_1 \wedge \cdots \wedge dx_n / dg$. We ask for the local expression of the meromorphic $(n-1)$-form $\pi^*(\omega_1)$ on $g_\sigma(y)=0$. By the condition of Lemma 1(1),

$$\pi^*(dg) = d(\pi^* g) = d\left[\prod_{i=1}^n y_{\sigma, i}^{d(P_i, g)} g_\sigma(y_\sigma) \right]$$

$$= d\left[\prod_{i=1}^n y_{\sigma, i}^{d(P_i, g)} \right] g_\sigma(y_\sigma) + \prod_{i=1}^n y_{\sigma, i}^{d(P_i, g)} dg_\sigma.$$

Here $g_\sigma = 0$ is the defining equation of $\widetilde{W}_1$ in $\boldsymbol{R}_\sigma^n$ which is a proper transform of $W_1 := \{g(\boldsymbol{x}) = 0 \text{ in } \boldsymbol{R}^n\}$, since the proper transform $\widetilde{W}_1$ is defined by the closure in Y of $\pi^{-1}(W_1 - \{0\})$. On the one hand, by Lemma 1(2), one has

$$\pi^*(dx_1 \wedge \cdots \wedge dx_n) = \det(P(\sigma)) \prod_{i=1}^n y_{\sigma,i}^{\beta_i} dy_{\sigma,1} \wedge \cdots \wedge dy_{\sigma,n}$$

where $\beta_i = |P_i| - 1$ and $|P_i| = \sum_{j=1}^n p_{ji}$. We get a meromorphic $(n-1)$-form $\tilde{\omega}_{1\sigma}$ on $\boldsymbol{R}_\sigma^n$:

$$\tilde{\omega}_{1\sigma} = \pi_\sigma^*(dx_1 \wedge \cdots \wedge dx_n)/\pi_\sigma^*(dg_\sigma)$$

$$= \prod_{i=1}^n y_{\sigma,i}^{\alpha(P_i)}(dy_{\sigma,1} \wedge \cdots \wedge dy_{\sigma,n}/dg_\sigma) \quad \text{on } \widetilde{W}_1$$

where $\alpha(P_i) = |P_i| - d(P_i, g) - 1$. Then it is easy to see that the restriction of $\tilde{\omega}_1$ to $\widetilde{W}_1$ is equal to $\pi^*(\omega_1)$. Note that $dy_{\sigma,1} \wedge \cdots \wedge dy_{\sigma,n}/dg_\sigma$ is a nowhere vanishing $(n-1)$-form on $W_1 \cap \boldsymbol{R}_\sigma^n$. We have the following lemma.

LEMMA 6. *For positive vertex P_i, we have $\alpha(P_i) \geqq 0$.*

PROOF OF LEMMA 6. Let $P_i = {}^t(p_1, \cdots, p_n)$. We prove it by dividing two parts.

(i) The case of strictly positive P_i.

Since g is smooth, the Newton boundary $\Gamma(g)$ has a vertex such that $x_{i_0} = {}^t(0, \cdots, \overset{i_0}{1}, \cdots, 0)$ for some i_0. By definition, we have

$$d(P_i, g) = \min\left\{ \prod_{j=1}^n p_j x_j \mid x \in \Gamma(g) \right\}$$

$$\leqq P_i(x_{i_0}) = p_{i_0}.$$

On the other hand,

$$|P_i| = p_1 + \cdots + p_{i_0} + \cdots + p_n \geqq p_{i_0} + (n-1) \geqq d(P_i, g) + (n-1).$$

Hence, we have

$$|P_i| - d(P_i, g) - 1 \geqq (n-1) \geqq 0.$$

(ii) The case of the not strictly positive P_i.

Using the same notations as the above (i) for x_{i_0}, if p_{i_0} equals to zero, then $d(P_i, g) = 0$. Since $|P_i| \neq 0$, we have

$$|P_i| \geqq d(P_i, g).$$

Next, we consider the case $p_{i_0} \neq 0$. As the same as (i), we have $p_{i_0} \geqq d(P_i, g)$. On the other hand,

$$|P_i| \geqq p_{i_0}.$$

Hence $|P_i| \geqq d(P_i, g)$. Here we assume the equality, i.e., $|P_i| = p_{i_0} = d(P_i, g)$

Since $p_1 + \cdots + p_{i_0} + \cdots + p_n = p_{i_0}$, one gets $P_i = {}^t(0, \cdots, p_{i_0}, \cdots, 0)$. Since P_i is primitive vector, $p_{i_0} = 1$ and $|P_i| = 1$. By the assumption that g is convenient, there exists a vertex $x \in \Gamma(g)$ such that i_0-th component is zero. Hence $d(P_i, g) = 0$. So, we complete the proof of Lemma 6.

Then, the $(n-1)$-form $\pi^*(\omega_1)$ on $\widetilde{W}_1$ is extended to the total transform $\pi^{-1}(W_1)$ without the pole at the origin. Hence we consider the next integral $I(\sigma)$ for a fixed σ such that

$$I(\sigma) := \int_{\{\Pi_i\, y_i^{d(P_i,\,g)} = 0\} \cup \{g_\sigma(\nu) = 0\}} (\varepsilon y_1^{d(P_1,\,f)} \cdots y_n^{d(P_n,\,f)})_{\pm}^{\tau}(f_\sigma)_{\pm}^{\tau}(\varphi \circ \pi)\varphi_{\alpha,\sigma}\pi^*(\omega_1)$$

$$= \int_{\{\Pi_i y_i^{d(P_i,\,g)} = 0\} \cup \{g_\sigma(\nu) = 0\}} (\varepsilon y_1^{d(P_1,\,f)} \cdots y_n^{d(P_n,\,f)})_{\pm}^{\tau}(f_\sigma)_{\pm}^{\tau}(\varphi \circ \pi)\varphi_{\alpha,\sigma}$$

$$\cdot \prod_{i=1}^{n} y_{\sigma,i}^{\alpha(P_i,\,g)}\left(\frac{dy_{\sigma,1} \wedge \cdots \wedge dy_{\sigma,n}}{dg_\sigma}\right).$$

Since $\mathrm{Re}\,\tau > -1$ and $\alpha(P_i, g) \geq 0$, the integral over $\{\Pi_i\, y_i^{d(P_i,\,g)} = 0\}$ vanishes. Note that g_σ and f_σ are smooth on the neighborhood of the origin in Y from Lemma 1 and Lemma 3. Therefore we have an integral

$$I(\sigma) = \int_{\{g_\sigma(\nu) = 0\} \cap V'} \prod_{i=1}^{n} \varepsilon(y_{\sigma,i})_{\pm}^{d(P_i,\,f)\tau + \alpha(P_i,\,g)}(f_\sigma)_{\pm}^{\tau}(\varphi \circ \pi)\varphi_{\alpha,\sigma}\left(\frac{dy_{\sigma,1} \wedge \cdots \wedge dy_{\sigma,n}}{dg_\sigma}\right). \quad (14)$$

Now, our assertion follows directly from the next Lemma.

LEMMA 7 (Gel'fand and Šilov [3]). *Let* $\phi(y_1, \cdots, y_n, \mu)$ *be a* C_0^∞-*class function on* $\boldsymbol{y}$ *and a meromorphic function of the parameter* $\mu \in C$. *Then the integral*

$$I(\nu_1, \cdots, \nu_n, \mu) = \int_{R^n} \prod_{i=1}^{n} (\pm y_{\sigma,i})_{\pm}^{\nu_i}\phi(y_1, \cdots, y_n, \mu)dy_1 \cdots dy_n,$$

can be analytically continued at all the values of $\nu_1, \cdots, \nu_n$ *and* μ *as meromorphic function, and all its poles other than those already possessed by* ϕ *can be lie only on* $\nu_i = -1, -2, \cdots$ ($i = 1, 2, \cdots, n$).

For the later discussion we give the proof of the lemma.

PROOF OF LEMMA 7. We use the integration by parts. For simplicity, we take only the $+$ sign in the integral. Then we have

$$\int_0^\infty \cdots \int_0^\infty y_1^{\nu_1} \cdots y_n^{\nu_n}\phi(y_1, \cdots, y_n, \mu)dy_1 \cdots dy_n$$

$$= \frac{(-1)^{s_1 + \cdots + s_n}}{(\nu_1 + 1) \cdots (\nu_1 + s_1) \cdot (\nu_2 + 1) \cdots (\nu_2 + s_2) \cdots (\nu_n + 1) \cdots (\nu_n + s_n)} \quad (15)$$

$$\times \int_0^\infty \cdots \int_0^\infty y_1^{\nu_1 + s_1} \cdots y_n^{\nu_n + s_n}\left(\frac{\partial}{\partial y}\right)^s \phi(y, \mu)dy_1 \cdots dy_n$$

for $\mathrm{Re}\,\nu_i > -1$. The right hand side of the integral is analytically continued to $\mathrm{Re}\,\nu_i > -s_i-1$. Hence the original integral is analytically continued to $\mathrm{Re}\,\nu_i > -s_i-1$ as an analytic function with simple poles of $\nu_i = -1, -2, \cdots, -s_i$. This completes the proof of the Lemma 7.

We consider again the integral (14). Then, in the neighborhood of the origin and $\{y_i=0; i\in\{1, 2, \cdots, n\}\}$ (or T_I), $\{f_\sigma(\boldsymbol{y})=g_\sigma(\boldsymbol{y})=0\}$ is a smooth complete intersection variety. Hence the substituted term of $g_\sigma(\boldsymbol{y})=0$ to $f_\sigma(\boldsymbol{y})=0$ is smooth. Therefore, if the branch points of $f_\sigma(\boldsymbol{y})=0$ exist, then they are contained in the non-singular normal crossing component. Therefore the poles corresponding to f_σ belong to the arithmetic progressions $\{-1, -2, \cdots\}$.

Here we remark that (Oka [1], p. 412) if P_j is strictly positive, then we have

$$\{g_\sigma=0\}\cap\{y_{\sigma,j}=0\} \neq \varnothing \tag{16}$$

if and only if $\dim\Delta_j(g)>0$, where $\Delta_j(g):=\Delta(P_j, g)=\{x\in\Gamma_+(g)\,;\,P_j(x)=d(P_j, g)\}$. Thus, applying Lemma 7 to integral (14), the position of the poles in $I(\sigma)$ belongs to

$$d(P_i, f)\tau+\alpha(P_i, g) = -1, -2, \cdots, \quad \text{(for } \dim\Delta_i(g)>0, \ i=1, \cdots, n\text{)},$$

and hence

$$\tau = -\frac{\alpha(P_i, g)+1}{d(P_i, f)}, \ -\frac{\alpha(P_i, g)+2}{d(P_i, f)}, \ \cdots \quad \text{(for } \dim\Delta_i(g)>0, \ i=1, \cdots, n\text{)}. \tag{17}$$

We notice that one variable of $y_{\sigma,1}, \cdots, y_{\sigma,n}$ in $I(\sigma)$ is redundant by the definition of the integral, i.e., the redundancy in the variables arises from the substitution of $g_\sigma(\boldsymbol{y})=0$ to $\prod_{i=1}^n (y_{\sigma,i})_\pm^{d(P_i,f)\tau+\alpha(P_i,g)}(\varphi\circ\pi)\varphi_{\alpha,\sigma}(f_\sigma)_\pm^\tau(dg_{\sigma,1}\wedge\cdots\wedge dg_{\sigma,n}/dg_\sigma)$. We can consider the any substitution of them. However, by the smoothness of $g_\sigma(\boldsymbol{y})=0$ in the neighborhood of the origin and $g_\sigma(0, \cdots, 0)\neq0$, the variable which is eliminated by the substitution of $g_\sigma(\boldsymbol{y})=0$ does not vanish at the origin. Hence one may consider the position of the poles for the $(n-1)$ variables excluding the eliminated variable by $g_\sigma=0$. This does not depend on the substitution because of the uniqueness of the form $\pi^*(\omega_1)$ on $g_\sigma(\boldsymbol{y})=0$. However the vertex corresponding the eliminated variable in a simplicial cone σ exists also in the coordinate of another simplicial cone τ. Therefore, to obtain the position of the pole in $I(\sigma)$, we may consider the arithmetic progressions for all i ($i=1, 2, \cdots, n$) forgetting the eliminated variable. Noting that $m_i=|P_i|-1$ and $\alpha(P_i, g)=m_i-d(P_i, g)$, we complete the proof of Proposition 5.

DEFINITION 8. For the resolution $\pi: Y\to U\subset R^n$, we denote by M_Y a set of pairs $(d(P_i, f), \alpha(P_i, g))$ such that $d(P_i, f)>0$ and $(d(P_i, f), \alpha(P_i, g))\neq(1, 0)$ for $i=1, \cdots, n$ and in any local coordinate system. The M_Y will be called *a set*

of multiplicitives of the resolution (Y, π). We define the *weight* β_Y of the resolution (Y, π) by

$$\beta_Y = \begin{cases} \max\left\{-\dfrac{\alpha(P_i, g)+1}{d(P_i, f)} \ ; \ (d(P_i, f), \alpha(P_i, g))\in M_Y, \dim \Delta_i(g)>0\right\} & \text{(for } M_Y \neq \varnothing), \\ -\infty & \text{(for } M_Y = \varnothing). \end{cases}$$

Next we prove the following Proposition.

PROPOSITION 9. *Let* $\beta_Y > -1$, *and* $\bar{j} = \max\{j \,|\, j \text{ numbers among } -(\alpha(P_i, g)+1)$ $\cdot/d(P_i, f)$ $(i=1, \cdots, n-1)$ *are equal to* β_Y *such that* $\dim \Delta_i(g)>0\}$ *where the maximum is taken over all local coordinate systems with a vertex such that* $\dim \Delta_k(g)$ >0 $(k=1, \cdots, n)$. *If* φ *has a support that is concentrated in a sufficiently small neighborhood of the origin in* $\boldsymbol{R}^n$, $\varphi(\vec{0})\neq 0$ *and* $\varphi(\boldsymbol{x})\geqq 0$, *then the order of a pole of* $I_\pm(\tau, \varphi)$ *at* $\tau=\beta_Y$ *is not higher than* $\bar{j}$, *and the sum of coefficients of* $1/(\tau-\beta_Y)^{\bar{j}}$ *in the Laurent expansion of the* $I_\pm(\tau, \varphi)$ *at* $\tau=\beta_Y$ *is non-zero.* (*Each* $I_\pm(\tau, \varphi)$ *is non-negative* (*resp. non-positive*), *and at least one of them is positive* (*resp. negative*).)

PROOF OF PROPOSITION 9. Let us consider the integral (14) with any fixed σ. If the number of subscripts i of the variables in (14) such that $-(\alpha(P_i, g)+1)/d(P_i, f)$ $=\beta_Y$, is smaller than the number $\bar{j}$, then it is easy to see that the integral (14) has a pole of order strictly smaller than $\bar{j}$ at the point $\tau=\beta_Y$ for the simplicial cone σ. Hence we can suppose that by changing suitably the indexes

$$-\frac{\alpha(P_1, g)+1}{d(P_1, f)} = \cdots = -\frac{\alpha(P_{\bar{j}}, g)+1}{d(P_{\bar{j}}, f)} = \beta_Y \,.$$

We rewrite the integral (14) by a change of the variables according to the above change of indexes as

$$I(\sigma) = \int_{g_\sigma(\boldsymbol{y})=0} \prod_{k=1}^{n-1}(y_{\sigma, k})_\pm^{d(P_k, f)\tau+\alpha(P_k, g)}(y_{\sigma, n})_\pm^{d(P_n, f)\tau+\alpha(P_n, g)}$$
$$\times (f_\sigma)_\pm^\tau(\varphi \circ \pi)\varphi_{\alpha, \sigma} h_\sigma dy_{\sigma, 1} \cdots dy_{\sigma, n-1} \tag{18}$$

where the meromorphic function h_σ is formally defined by $dy_{\sigma, 1}\wedge \cdots \wedge dy_{\sigma, n}/dg_\sigma$ $=h_\sigma dy_{\sigma, 1} \cdots dy_{\sigma, n-1}$ and then, it does not vanish on $g_\sigma(\boldsymbol{y})=0$. Now we suppose that the variable $y_{\sigma, n}$ is eliminated by the substitution of $g_\sigma(\boldsymbol{y})=0$. According to Lemma 7 and (17), the integral (18) has poles at $\nu_i=-1, -2, \cdots$ $(i=1, \cdots, n-1)$ where we set $\nu_j=d(P_j, f)\tau+\alpha(P_j, g)$ $(j=1, \cdots, n)$. By the integral (15) and the assumption, we have

$$\frac{1}{(\nu_1+1)\cdots(\nu_{\bar{j}}+1)} = \frac{1}{d(P_1, f)\cdots d(P_{\bar{j}}, f)} \times \frac{1}{(\tau-\beta_Y)^{\bar{j}}} \,.$$

Hence the coefficients of $1/(\tau-\beta_Y)^{\bar{j}}$ in the Laurent expansion of the $I(\sigma)$ is obtained by calculating the residue of the $I(\sigma)$ at $\nu_1=-1, \cdots, \nu_{\bar{j}}=-1$. By itera-

tively using the residue formula of $\int_R x_{\pm}^{\tau}\psi(x)dx$ for a smooth function $\psi(x)$ (Gel'fand-Šilov [3]), the residue of the $I(\sigma)$ is equal to

$$\frac{1}{d(P_1,\ f)\cdots d(P_{\bar{j}},\ f)}\int_{V'}(y_{\sigma,\bar{j}+1})_{\varepsilon_{\bar{j}+1}}^{\beta_Y d(P_{\bar{j}+1},\ f)+\alpha(P_{\bar{j}+1},\ g)}\cdots(y_{\sigma,\ n-1})_{\varepsilon_{n-1}}^{\beta_Y d(P_{n-1},\ f)+\alpha(P_{n-1},\ g)}$$

$$\times\left\{(y_{\sigma,\ n})_{\varepsilon_n}^{\beta_Y d(P_n,\ f)+\alpha(P_n,\ g)}(f_{\sigma})_{\pm}^{\beta_Y}(\varphi\circ\pi)\varphi_{\alpha,\sigma}h_{\sigma}\right\}\Big|_{y_1=\cdots=y_{\bar{j}}=0}dy_{\sigma,\bar{j}+1}\cdots dy_{\sigma,\ n-1}. \tag{19}$$

Since the sign of the h_{σ} (i.e., the orientation) is definite, hereafter we may assume the orientation to be the positive definite. By assumption, $\varphi(0)>0$ and $\varphi(x)\geqq0$. Hence the residue of (19) is positive, and the integral of this type appears necessarily in either $I_+(\tau,\ \varphi)$ or $I_-(\tau,\ \varphi)$. Therefore summing over each quadrant and coordinate, the same situation holds with respect to the coefficients of $1/(\tau-\beta_Y)^{\bar{j}}$ in the Laurent expansion of $I_{\pm}(\tau,\ \varphi)$. We complete the proof of Proposition 9.

For the relation between $I_{\pm}(\tau,\ \varphi)$ and $K(f,\ g,\ \varphi,\ c)$, we have the following result.

LEMMA 10 (Gel'fand-Šilov [3], p. 407). *If $I_{\pm}(\tau,\ \varphi)$ has the poles $\tau=-\tau_1$, $-\tau_2$, $\cdots(\tau_1<\tau_2<\cdots)$ of the order $k_1,\ k_2,\ \cdots$, respectively, then*

$$K(f,\ g,\ \varphi,\ c)\sim\sum_{\bar{j}=1}^{\infty}\sum_{k=1}^{k_{\bar{j}}}a_{\bar{j},\ k}^{\pm}(\varphi)c^{\tau_{\bar{j}}-1}(\log c)^{k-1}$$

for $c\rightarrow\pm0$, where

$$a_{\bar{j},\ k}^{\pm}(\varphi)=\frac{(-1)^{k-1}}{(k-1)!}\times\left(coefficients\ of\ \frac{1}{(\tau+\tau_{\bar{j}})^k}\ in\ I_{\pm}(\tau,\ \varphi)\right).$$

We shall use the following two well-known Lemmas (Gel'fand-Šilov [3], Varchenko [6], Kaneko [12]).

LEMMA 11. *Let $\theta(c)$ be a function in $C_0^{\infty}(R)$ such that $\theta(c)$ identically equals to 1 in a neighborhood of zero. Then, for $|\tau|\rightarrow+\infty$, we have the following asymptotic expansions:*

$$\int_0^{\infty}e^{i\tau c}c^p(\log c)^m\theta(c)dc\sim\left(\frac{d}{dp}\right)^m\left\{\frac{\Gamma(p+1)}{(-i\tau)^{p+1}}\right\}$$

$$\int_{-\infty}^0e^{i\tau c}|c|^p(\log|c|)^m\theta(c)dc\sim\left(\frac{d}{dp}\right)^m\left\{\frac{\Gamma(p+1)}{(i\tau)^{p+1}}\right\}$$

where $m\geqq0$ is an integer and $p>-1$ is a real value.

LEMMA 12. *For $|\tau|\rightarrow\infty$, and $p\neq-1,\ -2,\ \cdots$,*

$$\left(\frac{d}{dp}\right)^m \left\{\frac{\Gamma(p+1)}{(\mp i\tau)^{p+1}}\right\} \sim \frac{\Gamma(p+1)}{\tau^{p+1}} e^{\pm(\pi/2)i(p+1)} (\log \tau)^m \ .$$

Here we define the oscillation index.

DEFINITION 13. The *oscillation index* of the function f and g at zero in R^n is defined by $\beta(f, g)$ which is the maximum of the numbers p having the following property: For any neighborhood of the origin in R^n, there exists a $\varphi \in C_0^\infty(R^n)$ for the asymptotic expansion of $I(\tau, \varphi)$ and a k such that $a_{p, k}(\varphi) \neq 0$.

Now, we show the main theorem.

THEOREM 14. *Let* $(f(\boldsymbol{x}), g(\boldsymbol{x}))$ *be an analytic mapping from a neighborhood* U *of the origin of* R^n *to* R^2 *such that* $f(\vec{0}) = g(\vec{0}) = 0$, *and* $\{\boldsymbol{x} \in R^n ; f(\boldsymbol{x}) = g(\boldsymbol{x}) = 0\}$ *is a complete intersection variety with a singularity at* $\vec{0}$ *in* R^n. *We assume that* $g(\boldsymbol{x})$ *is a smooth analytic function. Further suppose that* $\{\boldsymbol{x} \in U ; f(\boldsymbol{x}) = g(\boldsymbol{x}) = 0\}$ *is a non-degenerate complete intersection variety at* $\vec{0}$ *in* R^n *and,* $f(\boldsymbol{x})$ *and* $g(\boldsymbol{x})$ *are convenient. Then we have an asymptotic expansion*

$$\int_{R^n} e^{i\tau f(\boldsymbol{x})} \delta(g(\boldsymbol{x})) \varphi(\boldsymbol{x}) dx \sim \sum_p \sum_{k=0}^{n-2} a_{p, k}(\varphi) \tau^p (\log \tau)^k \tag{20}$$

for $|\tau| \to \infty$, *where* $\varphi(\boldsymbol{x}) \in C_0^\infty(R^n)$ *with a support concentrating in a sufficiently small neighborhood of the origin in* R^n, *such that:*

1. *There exists a method of calculation of the power* p, *on the basis of toroidal resolution, in which the powers belong to finitely many arithmetic progressions constructed from negative rational numbers.*

2. *If* $\beta_Y > -1$, *the oscillation index* $\beta(f, g)$ *is not exceeding* β_Y. *Further if* $\varphi(0) > 0$ *and* $\varphi(\boldsymbol{x}) \geq 0$, *we have* $\beta(f, g) = \beta_Y$.

3. *If* $\beta_Y > -1$ *and* $\varphi(0) > 0$, $\varphi(\boldsymbol{x}) \geq 0$, *then the power index of* $(\log \tau)$ *corresponding to the highest power* $p = \beta(f, g) = \beta_Y$ *is also calculated as* $j - 1$ *by the toroidal resolution.*

PROOF. 1. By (17) and Proposition 9, we apply Lemma 10 to $I_{\pm}(\tau, \varphi)$ and $K(f, g, \varphi, c)$. We consider

$$I(\tau, \varphi) = \int_{-\infty}^{\infty} dc\, e^{i\tau c} K(f, g, \varphi, c)$$

$$= \int_{-\infty}^{0} dc\, e^{i\tau c} K(f, g, \varphi, c) + \int_{0}^{\infty} dc\, e^{i\tau c} K(f, g, \varphi, c) , \tag{21}$$

where $K(f, g, \varphi, c)$ have a compact support as the function of c, and C^∞ except the origin such that

$$K(f, g, \varphi, c) \approx \sum_{j=1}^{\infty} \sum_{k=1}^{k_j} a_{j,k}^{+} c^{\tau_j - 1} (\log c)^{k-1} \quad \text{for } c \to +0$$

$$K(f, g, \varphi, c) \approx \sum_{j=1}^{\infty} \sum_{k=1}^{k_j} a_{j,k}^{-} (-c)^{\tau_j - 1} (\log (-c))^{k-1} \quad \text{for } c \to -0 .$$

(22)

Then we apply Lemma 11 and Lemma 12 to the equations. Inserting (22) to (21) we obtain the following equation

$$I(\tau, \varphi) = \sum_{j=1}^{\infty} \sum_{k=1}^{k_j} \int_{-\infty}^{0} e^{i\tau c} a_{j,k}^{-} (-c)^{\tau_j - 1} (\log (-c))^{k-1} \theta(c) dc$$

$$+ \sum_{j=1}^{\infty} \sum_{k=1}^{k_j} \int_{0}^{\infty} e^{i\tau c} a_{j,k}^{+} c^{\tau_j - 1} (\log c)^{k-1} \theta(c) dc$$

$$\approx \sum_{j=1}^{\infty} \sum_{k=1}^{k_j} a_{j,k}^{-} \frac{\Gamma(\tau_j)}{\tau^{\tau_j}} (\log \tau)^{k-1} e^{-(\pi/2) i\tau_j}$$

$$+ \sum_{j=1}^{\infty} \sum_{k=1}^{k_j} a_{j,k}^{+} \frac{\Gamma(\tau_j)}{\tau^{\tau_j}} (\log \tau)^{k-1} e^{+(\pi/2) i\tau_j} .$$

(23)

After all, we have the asymptotic expansion of (20). From Proposition 5 and (17), the indexes of the powers belong to finitely many arithmetic progression:

$$\begin{cases} -1, -2, \cdots & \text{for } (d(P, f), \alpha(P, g)) = (1, 0) , \\ -\dfrac{\alpha(P, g)+1}{d(P, f)}, -\dfrac{\alpha(P, g)+2}{d(P, f)}, \cdots & \text{for } (d(P, f), \alpha(P, g)) \in M_Y . \end{cases}$$

2. and 3. By definition, the oscillation index $\beta(f, g)$ is not exceeding β_Y. If $\varphi(0) > 0$ and $\varphi(x) \geq 0$, then by Proposition 9, we will obtain more precise results. In equation (23), setting $\tau_1 = -\beta_Y$ and $k_1 = j$, we evaluate the coefficient of the highest power term $\tau^{\beta_Y} (\log \tau)^{j-1}$ for τ. By Lemma 10, we have

$$a_{j,k}^{\pm} = \frac{(-1)^k}{(k-1)!} b_{j,k}^{\pm}$$

where $b_{j,k}^{\pm}$ is the coefficients of $1/(\tau + \tau_j)^k$ in $I_{\pm}(\tau, \varphi)$. Hence the term in $I(\tau, \varphi)$ corresponding to $\tau^{\beta_Y} (\log \tau)^{j-1}$ is written as

$$\frac{(-1)^{j-1}}{(j-1)!} \Gamma(-\beta_Y)(e^{-(\pi/2) i\beta_Y} b_{1,j}^{+} + e^{+(\pi/2) i\beta_Y} b_{1,j}^{-}) .$$

On the one hand, by Proposition 9 we have $b_{1,j}^{\pm} \geq 0$ where either $b_{1,j}^{+}$ or $b_{1,j}^{-}$ does not vanish. Then, for $\beta_Y > -1$, we obtain

$$\mathrm{Re} \, (e^{-(\pi/2) i\beta_Y} b_{1,j}^{+} + e^{+(\pi/2) i\beta_Y} b_{1,j}^{-}) = (b_{1,j}^{+} + b_{1,j}^{-}) 2 \cos \left(\frac{\pi}{2} |\beta_Y| \right) > 0 .$$

Therefore the coefficient of the highest power term $\tau^{\beta_Y} (\log \tau)^{j-1}$ does not vanish. This completes the proof of the theorem.

The weight β_Y is more easily calculated with the help of the Newton boundary.

COROLLARY 15. *Under the same assumption as Theorem* 14, *we have the next method of calculations through the Newton boundary.*

(1) *Let* $d_f(P_i)$ *or* $d_g(P_i)$ *be the rational number which is the intersection points between the diagonal line* $x_1 = x_2 = \cdots = x_n = t$, $t \in \mathbf{R}$ *and* $P_i(x) = d(P_i, f)$ *or* $P_i(x) = d(P_i, g)$, *respectively. Then we have that*

$$\beta_Y = \max_i \left\{ -\frac{1 - d_g(P_i)}{d_f(P_i)} \right\},$$

where the maximum is taken over all i *such that* $\dim \Delta(P_i, g) > 0$.

(2) *In the same notations as* (1), *let*

$$j_0 := \max_\sigma \left\{ \bar{j} \mid -\frac{1 - d_g(P_1)}{d_f(P_1)} = \cdots = -\frac{1 - d_g(P_j)}{d_f(P_j)} = \beta_Y, \right.$$

$$\left. for\ \dim \Delta(P_i, g) > 0 \ (i = 1, \cdots, n) \right\},$$

for each σ, *and define the index* $\bar{j}_0 = \min\{j_0, n-1\}$. *Then the power of* $\log \tau$ *corresponding to the* β_Y *is at most* $\bar{j}_0 - 1$ (*more precisely saying*, $\bar{j}_0 - 1$ *for* $\bar{j}_0 = 1$, $\bar{j}_0 - 2$ *for* $\bar{j}_0 = n$ *and* $\bar{j}_0 - 1$ *or* $\bar{j}_0 - 2$ *for the other case*).

REMARK 16. (a) The precise power of $\log \tau$ at β_Y is determined by looking over concretely the variables in $g_\sigma(\mathbf{y})$ for each σ which are eliminated by the substitution of $g_\sigma(\mathbf{y}) = 0$. Hence this process could not be obtained by only the word of Newton boundary. It needs to calculate concretely the integral $I_\pm(\tau, \varphi)$ by the toroidal embedding. (b) Here each simplicial cone σ corresponds to the n faces (proto-vertices) which are $(n-1)$-dimensional ones intersecting each other in the Newton polyhedron $\Gamma_+(f, g)$. This situation is also allowed for the case existing of the simplicial unimodular subdivisions among the each proto-vertices (see also Remark 22) where by the proto-vertex we mean the vertex before the canonical simplicial subdivision.

PROOF OF COROLLARY 15. Let us consider two hyperplanes $P_i(x) = d(P_i, f)$ and $P_i(x) = d(P_i, g)$ in $\Gamma_+(f)$ and $\Gamma_+(g)$, respectively, for each vertex P_i. Then, by definition, d_f and d_g satisfy the equations

$$|P_i| \cdot d_f(P_i) = d(P_i, f),$$

$$|P_i| \cdot d_g(P_i) = d(P_i, g).$$

Then,

$$-\frac{\alpha(P_i, g)+1}{d(P_i, f)} = -\frac{|P_i|-d(P_i, g)}{d(P_i, f)}$$

$$= -\left(\frac{1}{d_f(P_i)} - \frac{d_g(P_i)}{d_f(P_i)}\right).$$

Hence we obtain

$$\beta_Y = \max_i \left\{ -\frac{1-d_g(P_i)}{d_f(P_i)} ; \left(|P_i| \cdot d_f(P_i), \; |P_i|(1-d_g(P_i))-1\right) \in M_Y \right\}$$

where the maximum is taken over all i such that dim $\Delta(P_i, g) > 0$. Clearly, this quantity is obtained geometrically if one knows the explicit form of each P_i. Using the above equality, we also obtain $\bar{j}_0$. But it is possible in $\bar{j}_0$ to contain the contribution of the eliminated variable with $g_\sigma(\boldsymbol{y}) = 0$, which is redundant in $\bar{j}_0$. So, the highest power of $\log \tau$ corresponding to β_Y is at most $\bar{j}_0 - 1$. This completes the proof of Corollary 15.

REMARK 17. The condition $\beta_Y > -1$ is equivalent to $\max_i \{d_f(P_i) + d_g(P_i)\} > 1$.

Next we supplement the case $\beta_Y \leq -1$ (i.e., $\max_i \{d_f(P_i) + d_g(P_i)\} \leq 1$). In this case, the oscillation index generally can not be obtained. Now, we state the next Proposition corresponding to Proposition 5. The next Proposition 18 is essentially the same type with Varchenko [6] (Proposition 1.4(2)) (see also [12]).

PROPOSITION 18. *Suppose that for any point $\boldsymbol{y} \in S = \pi^{-1}(\bar{0})$ and any local coordinate system $(y_1, \cdots, y_n)$ centered at $\boldsymbol{y}$ satisfying (D-1) and (D-2), there exists at most one pairs equal to $(1, 0)$ among the pair $(l_i, m_i - d(P_i, g))$ $(i=1, \cdots, n)$ in (D-1) and (D-2). Suppose $\beta_Y \leq -1$. Let $1, \cdots, \bar{j}$ be all the natural numbers strictly smaller than $-\beta_Y$. Then $I_\pm(\tau, \varphi)$ have at the points $\tau = -1, -2, \cdots, -\bar{j}$ poles of multiplicity not higher than 1. If a_j^+ (resp. a_j^-) is the residue of $I_+(\tau, \varphi)$ (resp. of $I_-(\tau, \varphi)$) at $\tau = -j$, where $j = 1, \cdots, \bar{j}$, then $a_j^+ = (-1)^{j-1} a_j^-$. For Re $\tau > \beta_Y$, $I_\pm(\tau, \varphi)$ does not have other poles.*

PROOF. We investigate the equations (13) and (14) for a fixed coordinate system. By definition of β_Y, the numbers which are greater than β_Y in the arithmetic progressions of Proposition 5 are only $-1, -2, \cdots, -\bar{j}$. These numbers correspond directly to the poles of $I_\pm(\tau, \varphi)$ if and only if there exist pairs $(l_i, m_i - d(P_i, g)) = (1, 0)$ by the proof of Proposition 5. By our assumption, there exists at most such one pair. If the pair is nothing, the assertion is trivial. Hence we may assume that there exists a pair $(l_1, m_1 - d(P_1, g)) = (d(P_1, f), \alpha(P_1)) = (1, 0)$. By the proof of Proposition 5, the contribution to the pole from the other pairs does not exist. Therefore the order of the pole of the pair is at most one (i.e., simple pole). We calculated the residue of the pole. The contributions to $I_\pm(\tau, \varphi)$ from the pair are expressed as

$$\int_{V' \cap \{g_\sigma = 0\}} (y_1)^{\tau}_{\pm}(y_2)^{d(P_2, f)\tau + \alpha(P_2)}_{\varepsilon_2}$$

$$\times \cdots (y_n)^{d(P_n, f)\tau + \alpha(P_n)}_{\varepsilon_n} (f_\sigma)^{\tau}_{\pm}(\varphi \circ \pi)\varphi_{\alpha, \sigma}(dy_1 \wedge \cdots \wedge dy_n/dg_\sigma)$$

where $\varepsilon_j = \pm$. Then applying the integration (15) to y_1, the residue at $\tau = -s$ $> \beta_Y$ is obtained as follows:

$$\frac{(\pm 1)^{s-1}}{(s-1)!} \int_{V' \cap \{g_\sigma = 0\}} (y_2)^{-d(P_2, f)s + \alpha(P_2)}_{\varepsilon_2} \cdots (y_{n-1})^{-d(P_{n-1}, f)s + \alpha(P_{n-1})}_{\varepsilon_{n-1}}$$

$$\times \left(\frac{\partial}{\partial y_1}\right)^{s-1} \left\{(y_n)^{d(P_n, f)\tau + \alpha(P_n)}_{\varepsilon_n} (\varphi \circ \pi)(f_\sigma)^{\tau}_{\pm}\varphi_{\alpha, \sigma}(dy_1 \wedge \cdots \wedge dy_n/dg_\sigma)\right\}\Big|_{\substack{\tau = -s \\ y_1 = 0}}.$$

Hence we obtain $a^+_s = (-1)^{s-1} a^-_s$. This completes the proof of Proposition 18.

With this Proposition 18, we have the next Proposition.

PROPOSITION 19. *Under the same assumption as Proposition* 18, *suppose that* $\beta_Y \leq -1$. *If* $1, \cdots, j$ *are the natural numbers strictly smaller than* β_Y, *the index of the asymptotic expansion of* $I(\tau, \varphi)$ *is contained in those excepting* $-1, \cdots, -j$ *in the arithmetic progression of Proposition* 5. *The highest power of the asymptotic expansion is less than or equal to* β_Y.

PROOF. We investigate the asymptotic expansion (23).

$$I(\tau, \varphi) = \sum_{j=1}^{\infty} \sum_{k=1}^{k_j} \frac{(\log \tau_j)^{k-1}}{\tau^{\tau_j}} \Gamma(\tau_j)(\varepsilon^{(\pi/2) i\tau_j} a^+_{j, k} + e^{-(\pi/2) i\tau_j} a^-_{j, k}).$$

The coefficient corresponding to the pole $j = -1, \cdots, -j$ which is simple by Proposition 18 is written as

$$\Gamma(\tau_j)(e^{(\pi/2) i\tau_j} a^+_{j, k} + e^{-(\pi/2) i\tau_j} a^-_{j, k})$$

$$= \Gamma(\tau_j)e^{(\pi/2) i\tau_j}((-1)^{j-1} + e^{-\pi i\tau_j}) a^-_j.$$

Since $\tau_j = -j$, the coefficient vanishes. Therefore, the term corresponding to the pole $j = -1, \cdots, -j$ does not appear in the asymptotic expansion of $I(\tau, \varphi)$, completing the proof of Proposition 19.

Finally we obtain the next theorem.

THEOREM 20. *Under the same assumption as Theorem* 14, *if* $\max_i\{d_f(P_i) + d_g(P_i)\} \leq 1$, *then the maximum of the power index of the asymptotic expansion is less than or equal to* $\max_i\{-(1 - d_g(P_i))/d_f(P_i)$ *in* $M_Y\}$.

PROOF. Applying Proposition 18 and Proposition 19 to the asymptotic expansion (23), our assertion is easily obtained. This completes the proof of Theorem 20.

Note that the statement of Theorem 14 (1) for $\beta_Y \leq -1$ is also held.

REMARK 21. For $\beta_Y \leq -1$ case, the oscillation index is not necessarily obtained. Since, in the residue formula (19), one must differentiate the term $y_{\sigma,n}^{\beta_Y \, d(P_n, f)}(f_\sigma)_{\pm}^{\beta_Y}(\varphi \circ \pi)\varphi_\sigma h_\sigma$ by the number of times corresponding to the poles $-1, \cdots, -\bar{j}$, the integral has not necessarily the positive (or negative) definite value even if $\varphi(0) > 0$ and $\varphi(x) \geq 0$.

REMARK 22. The canonical unimodular subdivision of Σ^* is not unique. However we can see that the weight β_Y does not depend on the ambiguity of the subdivision. That is, we first subdivide the $S_2 \Gamma^*(f, g)$ (=the two-skeleton of $\Gamma^*(f, g)$ which is considered as a graph by a plane section, for example, for the case $n=3$). In this stage, there does not exist the ambiguity of the canonical unimodular subdivision. Next, we obtain inductively the subdivision of $S_k \Gamma^*(f, g)$ $(k=2, 3, \cdots, n)$. But, in general, the subdivision depends on the geometrical triangulation. Since we construct the unimodular subdivision of $S_k \Gamma^*(f, g)$ $(k>2)$ after the geometrical triangulation, the obtained subdivision is not unique. However we can easily prove that $\tau(P_i) < \tau(T) < \tau(P_j)$ (or $\tau(P_i) > \tau(T) > \tau(P_j)$) if the vertex T is a canonical primitive sequence of the segment $\overline{P_i P_j}$ (Oka [1]). The same inequality holds for the canonical primitive sequence of $S_k \Gamma^*(f, g)$ $(k>2)$. Moreover, it is easily seen that if $\dim \Delta(P_i, g) > 0$ and $\dim \Delta(P_j, g) = 0$, then we have $\dim \Delta(T, g) = 0$ for the vertex T. Therefore if we consider the simplicial cone σ containing the proto-vertices such that $\dim \Delta_i(g) > 0$, then by definition the weight β_Y is evaluated by their only proto-vertices, and hence it shows that the weight β_Y is independent of the subdivision.

4. The singular $g(x)$ case.

The purpose of this section is to prove Theorem 27 and Corollary 28 for the singular $g(x)$ case. In the singular case, the asymptotic expansion of the equation (1) is held with a weak sense. We use the same notation as in section 3, and put the same assumption for $f(x)$ and $g(x)$ except the smoothness of $g(x)$. Then we have the following Proposition.

PROPOSITION 23. *The poles of $I_\pm(\tau, \varphi)$ belong to the set of next arithmetic progressions*

$$
\begin{cases}
-1, -2, \cdots & \text{for } (d(P, f), |P| - d(P, g) - 1) = (1, 0), \\
-\dfrac{|P| - d(P, g)}{d(P, f)}, -\dfrac{|P| - d(P, g) + 1}{d(P, f)}, \cdots & \text{for } (d(P, f), |P| - d(P, g) - 1) \neq (1, 0).
\end{cases}
$$

PROOF. With the same notations and situation as the proof of Proposition

5, we consider the integral

$$I_{\pm}(\tau, \varphi) = \sum_{\alpha} \int_Y (f \circ \pi)_{\pm}^{\tau} \varphi_{\alpha}(\omega_1 \circ \pi).$$

For $\mathrm{Re}\,\tau > -1$ we have the integral on each local chart σ:

$$\int_Y (f \circ \pi)_{\pm}^{\tau} (\varphi \circ \pi) \varphi_{\alpha, \sigma}(\omega_1 \circ \pi)$$

$$= \int_{\{\prod_{i=1}^n y_i^{d(P_i, g)} g_\sigma(\mathbf{y}) = 0\} \cap V'} (\varepsilon(y_1)_{\varepsilon_1}^{d(P_1, f)} \cdots (y_n)_{\varepsilon_n}^{d(P_n, f)} f_\sigma(\mathbf{y})_{\pm})^{\tau} (\varphi \circ \pi) \varphi_{\alpha, \sigma} \pi^*(\omega_1), \quad (24)$$

where $\varepsilon = \pm 1$ and $\varepsilon_j = \pm$ for $j = 1, \cdots, n$. Then we calculate the integral (24) by dividing finitely to each component. We need the following definition.

DEFINITION 24. We call the function g α-class for f if the function g satisfies that $\alpha(P, g) \geq 0$ for any P in $\Gamma(f, g)$.

Then we classify the our asymptotic integral for two cases, that is, (i) the function g belongs to the α-class for f, and (ii) does not belong to it.

For the case (i), one can easily see that the same results as the smooth case of section 3 hold. That is, for any positive vertex P_i, we have that $\alpha(P_i, g) \geq 0$ from the definition. Hence, from the same argument as the proof of Proposition 5 we have

$$\tau = -\frac{\alpha(P_i, g) + 1}{d(P_i, f)}, \quad -\frac{\alpha(P_i, g) + 2}{d(P_i, f)}, \quad \cdots \quad (\text{for } \dim \Delta_i(g) > 0, \ i = 1, \cdots, n). \quad (25)$$

Next we consider the case (ii). In this case, there exist the vertices P_i such that $\alpha(P_i, g) < 0$. We consider the next integral $I(\sigma)$ for a fixed σ such that

$$I(\sigma) = \int_{\{\prod_i y_i^{d(P_i, g)} = 0\} \cup \{g_\sigma(\mathbf{y}) = 0\}} (\varepsilon y_1^{d(P_1, f)} \cdots y_n^{d(P_n, f)})_{\pm}^{\tau} (f_\sigma)_{\pm}^{\tau} (\varphi \circ \pi) \varphi_{\alpha, \sigma} \pi^*(\omega_1).$$

Then taking the sufficiently large $\mathrm{Re}\,\tau \gg 0$, we have an integral

$$I(\sigma) = \int_{\{g_\sigma(\mathbf{y}) = 0\} \cap V'} \prod_{i=1}^n \varepsilon(y_{\sigma, i})_{\pm}^{d(P_i, f)\tau + \alpha(P_i, g)} (f_\sigma)_{\pm}^{\tau} (\varphi \circ \pi) \varphi_{\alpha, \sigma} \left(\frac{dy_{\sigma, 1} \wedge \cdots \wedge dy_{\sigma, n}}{dg_\sigma} \right), \quad (26)$$

where V' represents an open set which contains the support of φ_α such that (D-1) and (D-2) are satisfied on it. Then, again applying Lemma 7 to the integral (26), we have the position of the pole as the same as (25). We complete the proof of Proposition 23.

Next we prove the following Proposition.

PROPOSITION 25. *Let $\beta_Y > -1$. If φ has a support that is concentrated in a*

sufficiently small neighborhood of the origin in $\boldsymbol{R}^n$, $\varphi(\vec{0}) \neq 0$ *and* $\varphi(\boldsymbol{x}) \geq 0$, *then the order of a pole of* $I_\pm(\tau, \varphi)$ *at* $\tau = \beta_Y$ *is not higher than* j *which is defined in Proposition 9, and the sum of coefficients of* $1/(\tau - \beta_Y)^j$ *in the Laurent expansion of the* $I_\pm(\tau, \varphi)$ *at* $\tau = \beta_Y$ *is non-zero. (Each* $I_\pm(\tau, \varphi)$ *is non-negative (resp. non-positive), and at least one of them is positive (resp. negative).)*

PROOF OF PROPOSITION 25. With the same procedure as Proposition 9, the statement is easily verified.

The relation between $I_\pm(\tau, \varphi)$ and $K(f, g, \varphi, c)$, is easily obtained by Lemma 10. We shall use the following Lemma (Gel'fand-Šilov [3]) which also contains the case for $p \leq -1$.

LEMMA 26. *Let* $\theta(c)$ *be a function in* $C_0^\infty(\boldsymbol{R})$ *such that* $\theta(c)$ *identically equals to 1 in a neighborhood of zero. Then, for* $|\tau| \to +\infty$, *we have the following asymptotic expansions*:

$$\int_0^\infty e^{i\tau c} c^p (\log c)^m \theta(c) dc \sim \left(\frac{d}{dp}\right)^m \left\{\frac{\Gamma(p+1)}{(-i\tau)^{p+1}}\right\},$$

$$\int_{-\infty}^0 e^{i\tau c} |c|^p (\log |c|)^m \theta(c) dc \sim \left(\frac{d}{dp}\right)^m \left\{\frac{\Gamma(p+1)}{(i\tau)^{p+1}}\right\},$$

where $m \geq 0$ *is a non-negative integer and* p *is a real value such that* $p \neq -1, -2, \cdots$. *For* $p = -n$ $(n=1, 2, 3, \cdots)$ *case, in the meaning of an improper integral of the generalized function, we have the asymptotic expansion*

$$\int_0^\infty e^{i\tau c} c^{-n} (\log c)^m \theta(c) dc \sim \tau^{n-1} \sum_{l=-1}^m \left\{A_l^{(n)} \frac{(-1)^{m-l} m!}{(m-l)!} (\log(\tau+i0))^{m-l}\right\},$$

$$\int_{-\infty}^0 e^{i\tau c} |c|^p (\log |c|)^m \theta(c) dc \sim \tau^{n-1} \sum_{l=-1}^m \left\{B_l^{(n)} \frac{(-1)^{m-l} m!}{(m-l)!} (\log(\tau-i0))^{m-l}\right\}.$$

The coefficient $A_l^{(n)}$ *and* $B_l^{(n)}$ *are given by*

$$A_l^{(n)} = \lim_{\lambda \to -\infty} \frac{d^{l+1}}{d\lambda^{l+1}} \{i(\lambda+n) e^{i\lambda(\pi/2)} \Gamma(\lambda+1)\},$$

$$B_l^{(n)} = \overline{A_l^{(n)}}.$$

where the overline represents the complex conjugation.

Now, we show the main theorem.

THEOREM 27. *Let* $(f(\boldsymbol{x}), g(\boldsymbol{x}))$ *be an analytic mapping from a neighborhood* U *of the origin of* $\boldsymbol{R}^n$ *to* $\boldsymbol{R}^2$ *such that* $f(\vec{0}) = g(\vec{0}) = 0$, *and* $\{\boldsymbol{x} \in \boldsymbol{R}^n ; f(\boldsymbol{x}) = g(\boldsymbol{x}) = 0\}$ *is a complete intersection variety with a singularity at* $\vec{0}$ *in* $\boldsymbol{R}^n$. *We assume that* $\{\boldsymbol{x} \in U ; f(\boldsymbol{x}) = g(\boldsymbol{x}) = 0\}$ *is a non-degenerate complete intersection variety at* $\vec{0}$ *in* $\boldsymbol{R}^n$ *and,* $f(\boldsymbol{x})$ *and* $g(\boldsymbol{x})$ *are convenient. Then we have an asymptotic expansion,*

in the sense of improper integral of generalized function,

$$\int_{R^n} e^{i\tau f(x)} \delta(g(x))\varphi(x)dx \sim \sum_p \sum_{k=0}^{n-2} a_{p,k}(\varphi)\tau^p(\log \tau)^k \tag{27}$$

for $|\tau| \to \infty$, where $\varphi(x) \in C_0^\infty(R^n)$ with a support concentrating in a sufficiently small neighborhood of the origin in R^n, such that:

1. *There exists a method of calculation of the power p, on the basis of toroidal resolution, in which the powers belong to finitely many arithmetic progressions constructed from rational numbers.*

2. *If $\beta_Y > -1$, the oscillation index $\beta(f, g)$ is not exceeding β_Y. Further if $\varphi(0) > 0$ and $\varphi(x) \geq 0$, we have $\beta(f, g) = \beta_Y$.*

3. *If $\beta_Y > -1$ and $\varphi(0) > 0$, $\varphi(x) \geq 0$, then the power index of $\log \tau$ corresponding to the highest power $p = \beta(f, g) = \beta_Y$ is also calculated as j or $j-1$ by the toroidal resolution.*

PROOF. 1. By (25) and Proposition 25, we apply Lemma 10 to $I_\pm(\tau, \varphi)$ and $K(f, g, \varphi, c)$. Next applying Lemma 26, Lemma 11 and Lemma 12 to (21) and (22), we obtain the following equation.

$$I(\tau, \varphi) = \sum_{j=1}^\infty \sum_{k=1}^{k_j} \int_{-\infty}^0 e^{i\tau c} a_{j,k}^-(-c)^{\tau_j - 1}(\log(-c))^{k-1}\theta(c)dc$$

$$+ \sum_{j=1}^\infty \sum_{k=1}^{k_j} \int_0^\infty e^{i\tau c} a_{j,k}^+ c^{\tau_j - 1}(\log c)^{k-1}\theta(c)dc .$$

Hence,

$$I(\tau, \varphi) \approx \sum_{j=1}^\infty \sum_{k=1}^{k_j} \begin{cases} a_{j,k}^- \dfrac{\Gamma(\tau_j)}{\tau^{\tau_j}}(\log \tau)^{k-1}e^{-(\pi/2)i\tau_j} + a_{j,k}^+ \dfrac{\Gamma(\tau_j)}{\tau^{\tau_j}}(\log \tau)^{k-1}e^{+(\pi/2)i\tau_j} \\ \qquad\qquad\qquad\qquad \text{for } \tau_j \neq -1, -2, \cdots, \\ a_{j,k}^- \dfrac{1}{\tau^{\tau_j}}\sum_{l=-1}^{k-1}\left\{B_l^{-\tau_j}\dfrac{(-1)^{k-1-l}(k-1)!}{(k-1-l)!}(\log(\tau - i0))^{k-1-l}\right\} \\ + a_{j,k}^+ \dfrac{1}{\tau^{\tau_j}}\sum_{l=-1}^{k-1}\left\{A_l^{-\tau_j}\dfrac{(-1)^{k-1-l}(k-1)!}{(k-1-l)!}(\log(\tau + i0))^{k-1-l}\right\} \\ \qquad\qquad\qquad\qquad \text{for } \tau_j = -1, -2, \cdots . \end{cases} \tag{28}$$

Then, we have our asymptotic expansion of (27). From Proposition 23 and (25), the indexes of the powers of τ belong to finitely many arithmetic progressions:

$$\begin{cases} -1, -2, \cdots & \text{for } (d(P, f), \alpha(P)) = (1, 0) \\ -\dfrac{\alpha(P)+1}{d(P, f)}, -\dfrac{\alpha(P)+2}{d(P, f)}, \cdots & \text{for } (d(P, f), \alpha(P)) \in M_Y . \end{cases}$$

2. and 3. The first assertion of 2 is clear. In (28), setting $\tau_1 = -\beta_Y$ and $k_1 = j$, we evaluate the coefficient of the highest power term τ^{β_Y} for τ. By

Lemma 10, we have

$$a^{\pm}_{j,k} = \frac{(-1)^k}{(k-1)!}b^{\pm}_{j,k}$$

where $b^{\pm}_{j,k}$ is the coefficients of $1/(\tau+\tau_j)^k$ in $I_{\pm}(\tau,\varphi)$. For $\beta_Y \neq 1, 2, 3, \cdots$, the term in $I(\tau,\varphi)$ of (28) corresponding to $\tau^{\beta_Y}(\log \tau)^{j-1}$ is written as

$$\frac{(-1)^{j-1}}{(j-1)!}\Gamma(-\beta_Y)(e^{-(\pi/2)i\beta_Y}b^+_{\mathrm{I},j}+e^{+(\pi/2)i\beta_Y}b^-_{\mathrm{I},j}).$$

By Proposition 25, we have $b^{\pm}_{\mathrm{I},j} \geq 0$ where either $b^+_{\mathrm{I},j}$ or $b^-_{\mathrm{I},j}$ does not vanish. Then, for $\beta_Y > -1$, we obtain

$$\mathrm{Re}\,(e^{-(\pi/2)i\beta_Y}b^+_{\mathrm{I},j}+e^{+(\pi/2)i\beta_Y}b^-_{\mathrm{I},j}) = (b^+_{\mathrm{I},j}+b^-_{\mathrm{I},j})2\cos\left(\frac{\pi}{2}|\beta_Y|\right) > 0.$$

Hence the coefficient of the highest power term $\tau^{\beta_Y}(\log\tau)^{j-1}$ does not vanish for $\beta_Y \neq 1, 2, 3, \cdots$. Next, we consider the case for $\beta_Y = 1, 2, 3, \cdots$. Using (28), the highest power term and the next leading term of $\log\tau$ corresponding to τ^{β_Y} are $\tau^{\beta_Y}(\log\tau)^j$ and $\tau^{\beta_Y}(\log\tau)^{j-1}$, respectively. Their terms are written by

$$\tau^{\beta_Y}(\log\tau)^j\frac{1}{j!}(b^-_{\mathrm{I},j}B^{(\beta_Y)}_{-1}+b^+_{\mathrm{I},j}A^{(\beta_Y)}_{-1})$$

$$+\tau^{\beta_Y}(\log\tau)^{j-1}\frac{1}{j!}(b^-_{\mathrm{I},j}B^{(\beta_Y)}_0+b^+_{\mathrm{I},j}A^{(\beta_Y)}_0),$$

(29)

where we have (Gel'fand-Šilov [3])

$$A^{(\beta_Y)}_{-1} = \frac{i^{\beta_Y-1}}{(\beta_Y-1)!},$$

$$B^{(\beta_Y)}_{-1} = \frac{(-i)^{\beta_Y-1}}{(\beta_Y-1)!} = (-1)^{\beta_Y-1}A^{(\beta_Y)}_{-1},$$

$$A^{(\beta_Y)}_0 = \frac{i^{\beta_Y-1}}{(\beta_Y-1)!}\left(1+\frac{1}{2}+\cdots+\frac{1}{\beta_Y-1}+\Gamma'(1)+i\frac{\pi}{2}\right),$$

(30)

$$B^{(\beta_Y)}_0 = \overline{A^{(\beta_Y)}_0}.$$

Then, the first term in (29) is written as

$$(b^-_{\mathrm{I},j}B^{(\beta_Y)}_{-1}+b^+_{\mathrm{I},j}A^{(\beta_Y)}_{-1}) = A^{(\beta_Y)}_{-1}(b^+_{\mathrm{I},j}+(-1)^{\beta_Y-1}b^-_{\mathrm{I},j}).$$

Using $b^{\pm}_{\mathrm{I},j} \geq 0$ where either $b^+_{\mathrm{I},j}$ or $b^-_{\mathrm{I},j}$ does not vanish, the term $(b^+_{\mathrm{I},j}+(-1)^{\beta_Y-1}b^-_{\mathrm{I},j})$ is not equal to zero if β_Y is positive odd integer. But when β_Y is positive even integer, there is a possibility that their terms cancel out. Next, we see that the second term in (29) does not vanish. That is,

$$\mathrm{Re}\,(b_{\bar{1},j}^- B_0^{\langle \beta_Y \rangle} + b_{\bar{1},j}^+ A_0^{\langle \beta_Y \rangle}) = \mathrm{Re}\,(b_{\bar{1},j}^- \overline{A_0^{\langle \beta_Y \rangle}} + b_{\bar{1},j}^+ A_0^{\langle \beta_Y \rangle})$$

$$= (b_{\bar{1},j}^- + b_{\bar{1},j}^+)\,\mathrm{Re}\,A_0^{\langle \beta_Y \rangle} \neq 0\,,$$

in which $\mathrm{Re}\,A_0^{\langle \beta_Y \rangle}$ is not equal to zero by (30). Hence we obtain the result that the highest power term in $I(\tau, \varphi)$ corresponding to τ^{β_Y} is

$$\begin{cases} \tau^{\beta_Y}(\log \tau)^j & \text{for } \beta_Y = 1,\, 3,\, 5,\, \cdots \\[2mm] \tau^{\beta_Y}(\log \tau)^j \text{ or } \tau^{\beta_Y}(\log \tau)^{j-1} & \text{for } \beta_Y = 2,\, 4,\, \cdots. \end{cases}$$

This completes the proof of the theorem.

The next Corollary shows that the weight β_Y is evaluated by the Newton boundary.

COROLLARY 28. *Under the same assumption as Theorem 27, we have the method of calculations through the Newton boundary which is described below.*

(1) *Let $d_f(P_i)$ or $d_g(P_i)$ be the rational number which is the intersection points between the diagonal line $x_1 = x_2 = \cdots = x_n = t$, $t \in R$ and $P_i(x) = d(P_i, f)$ or $P_i(x) = d(P_i, g)$, respectively. Then we have that*

$$\beta_Y = \max_i \left\{ -\frac{1 - d_g(P_i)}{d_f(P_i)} \right\},$$

where the maximum is taken over all i such that $\dim \Delta(P_i, g) > 0$.

(2) *In the same notations as (1), let*

$$j_0 := \max_\sigma \left\{ \bar{j}\,\Big|\, -\frac{1 - d_g(P_1)}{d_f(P_1)} = \cdots = -\frac{1 - d_g(P_j)}{d_f(P_j)} = \beta_Y, \right.$$

$$\left. \text{for } \dim \Delta(P_i, g) > 0 \ (i = 1, \cdots, n) \right\},$$

for each σ, and define the index $\bar{j}_0 = \min\{j_0, n-1\}$. Then, for $\beta_Y \neq 1, 2, 3, \cdots$, the power of $\log \tau$ corresponding to the β_Y is at most $\bar{j}_0 - 1$ (precisely saying, $\bar{j}_0 - 1$ or $\bar{j}_0 - 2$). For $\beta_Y = 1, 3, 5, \cdots$, the power is $\bar{j}_0$ or $\bar{j}_0 - 1$, and for $\beta_Y = 2, 4, \cdots$, it is $\bar{j}_0$ or $\bar{j}_0 - 1$ or $\bar{j}_0 - 2$.

PROOF OF COROLLARY 28. With the same procedure as the proof of Corollary 15, we obtain

$$\beta_Y = \max_i \left\{ -\frac{1 - d_g(P_i)}{d_f(P_i)}\,;\, (|P_i| \cdot d_f(P_i),\, |P_i|(1 - d_g(P_i)) - 1) \in M_Y \right\},$$

where the maximum is taken over all i such that $\dim \Delta(P_i, g) > 0$. Clearly, this quantity is obtained from the geometry of the Newton boundary if one knows the explicit form of each P_i. So, the first part is proved. Next we show the second assertions. Using the above equality, we also obtain j_0. Considering

the redundant contribution of $\bar{j}_0$ as the eliminated variable in $g_\sigma(\boldsymbol{y})=0$, the highest power of $\log \tau$ corresponding to the power $\beta_Y \neq 1, 2, 3, \cdots$ is at most $\bar{j}_0 - 1$. The other cases for β_Y is easily obtained by Theorem 27. This completes the proof of Corollary 28.

It is also held that the condition $\beta_Y > -1$ is equivalent to $\max_i \{ d_f(P_i) + d_g(P_i) \} > 1$. Finally we remark the case $\beta_Y \leqq -1$ (i. e., $\max_i \{ d_f(P_i) + d_g(P_i) \} \leqq 1$). Then, we can easily obtain the statements which are the quite same forms as Proposition 18, Proposition 19 and Theorem 20. Note that for $\beta_Y \leqq -1$ the statement of Theorem 27 (1) is also held. However the oscillation index is not necessarily obtained.

5. Examples.

First we show some examples of the smooth case which is described in the previous section 3. Let $f(x, y, z) = x^4 + y^4 + z^4 + xyz$ and $g(x, y, z) = x + y + z$ where we can choose suitably the coefficient of the each terms to have the meaning as the real case, so we omit to specify them. Then it is easy to see that $\{ (x, y, z) \in \boldsymbol{R}^3 ; f(x, y, z) = g(x, y, z) = 0 \}$ is a complete intersection variety with a singularity at the origin. By the toric embedding, the dual Newton diagram $\Gamma^*(f, g)$ is as follows.
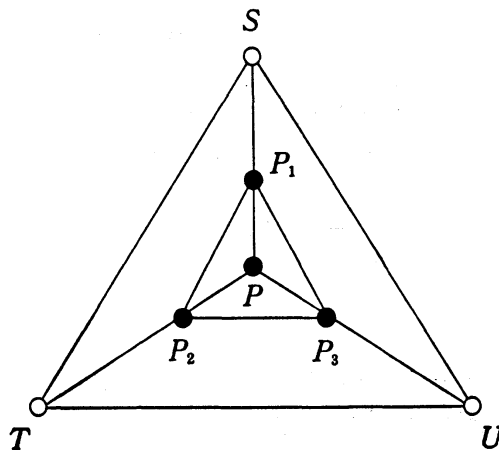
where $S = {}^t(1, 0, 0)$, $T = {}^t(0, 1, 0)$ and $U = {}^t(0, 0, 1)$. The other vertices P and P_i ($i = 1, 2, 3$) are strictly positive vertices such that $P = {}^t(1, 1, 1)$, $P_1 = {}^t(2, 1, 1)$, $P_2 = {}^t(1, 2, 1)$ and $P_3 = {}^t(1, 1, 2)$. Then we can easily prove that $\{ (x, y, z) \in \boldsymbol{R}^3 ; f(x, y, z) = g(x, y, z) = 0 \}$ is a non-degenerate complete intersection variety at $\vec{0}$ in $\boldsymbol{R}^3$. By the unimodular subdivision, we get nine 3-simplicial cones as shown in the figure. However since the contribution of the vertices S, T and

U to the weight β_Y is vanishing, we may consider only the strictly positive vertices P and P_i ($i=1, 2, 3$). By the symmetry, it is enough to consider the 3-simplicial cone $\sigma_1=\langle P_1, P_2, P_3\rangle$. By definition, we obtain

$$\begin{cases} d(P, f) = 3 \quad \text{and} \quad d(P_i, f) = 4 \\ d(P, g) = 1 \quad \text{and} \quad d(P_i, g) = 1. \end{cases} \quad \text{for } i=1, 2, 3$$

Similarly the power $\alpha(P_k, g)=|P_k|-d(P_k, g)-1$ ($k=$index of all vertices) is calculated such that $\alpha(P)=1$ and $\alpha(P_i)=2$ ($i=1, 2, 3$). For $R^3_{\sigma_1}$, we take the local coordinate $y_{\sigma_1}=(y_1, y_2, y_3)$. Then we have

$$\begin{cases} (\pi(\sigma_1)\circ f)(\boldsymbol{y}) = y_1^3 y_2^4 y_3^4 (y_1+y_1 y_2^4+y_1 y_3^4+1) \\ (\pi(\sigma_1)\circ g)(\boldsymbol{y}) = y_1 y_2 y_3 (1+y_2+y_3) \end{cases}$$

where $g_{\sigma_1}=1+y_2+y_3$. Then by the symmetry the position of poles $\tau(P_k)= -(\alpha(P_k)+1)/d(P_k, f)$ are calculated as follows:

$$\begin{cases} \tau(P) = -\dfrac{2}{3} \\ \tau(P_i) = -\dfrac{3}{4} \quad (i=1, 2, 3). \end{cases} \tag{31}$$

Hence, by definition, we have $\beta_Y=-2/3$ (>-1) and $j=1$. So, the oscillation index is $\beta(f, g)=-2/3$. Next we show how to calculate it by the Newton boundary. We consider the hyperplanes $P_k(x)=d(P_k, f)$ and $P_k(x)=d(P_k, g)$ in $\Gamma_+(f)$ and $\Gamma_+(g)$, respectively, where k runs over all strictly positive vertices. For each Newton boundary, let d_f and d_g be the intersection point of the diagonal line $x=y=z=t$ ($t\in R$) with their hyperplanes, respectively, as stated in Corollary 15. In this case, we obtain $d_f=1$ for any P_k and

$$d_g=\begin{cases} \dfrac{1}{3} & \text{for } P \\ \dfrac{1}{4} & \text{for } P_1, P_2 \text{ and } P_3. \end{cases}$$

Therefore the value of $\tau(P_k)=-(1-d_g)/d_f$ is the same as (31).

The second example is given by

$$\begin{cases} f(x, y, z) = x^8+y^8+z^8+x^2 y^2 z^2, \\ g(x, y, z) = x+y+z. \end{cases}$$

The variety $\{(x, y, z)\in R^3; f(\boldsymbol{x})=g(\boldsymbol{x})=0\}$ is also a non-degenerate complete intersection variety at $\vec{0}$ in R^3. The dual Newton diagram and each vertices S, T, U, P and P_i ($i=1, 2, 3$) are the same as the above case. Then we have

$$\begin{cases} d(P, f) = 6 \quad \text{and} \quad d(P_i, f) = 8 \\ d(P, g) = 1 \quad \text{and} \quad d(P_i, g) = 1 \end{cases}$$

for $i=1, 2, 3$. Thus we have

$$\begin{cases} \tau(P) = -\dfrac{1}{3} \\ \tau(P_i) = -\dfrac{3}{8} \quad (i=1, 2, 3). \end{cases}$$

Hence, we obtain the weight $\beta_Y = -1/3$ and the oscillation index $\beta(f, g) = -1/3$. Note that this example is not calculated by the method of Varchenko with the reason mentioned in Introduction.

REMARK 29. Generally, we can consider $f(x, y, z) = x^p + y^p + z^p + x^{s_1} y^{s_2} z^{s_3}$ and $g(x, y, z) = x + y + z$ which include the above examples as a special case. We assume that $p > s_1 + s_2 + s_3$. Then there exist four original vertices, P, P_i ($i=1, \cdots, 3$). Let r_1 be the greatest common divisor of $p - s_2 - s_3$ and s_1. The natural numbers r_2 and r_3 are also defined as the same manner. By an easy calculation, we obtain $P = {}^t(1, 1, 1)$, $P_1 = {}^t(a_1, b_1, b_1)$, $P_2 = {}^t(b_2, a_2, b_2)$ and $P_3 = {}^t(b_3, b_3, a_3)$ where $a_1 = (p - s_2 - s_3)/r_1$, $a_2 = (p - s_1 - s_3)/r_2$, $a_3 = (p - s_1 - s_2)/r_3$ and $b_i = s_i/r_i$ ($i=1, 2, 3$). To obtain the weight β_Y, we may take into account only the contribution of the original four vertices by Remark 22. By definition, we calculate the quantities:

$$\begin{cases} d(P, f) = s_1 + s_2 + s_3, \\ d(P, g) = 1, \end{cases}$$

and

$$\begin{cases} d(P_i, f) = b_i p, \\ d(P_i, g) = b_i. \end{cases} \quad (i=1, 2, 3)$$

Hence, $\tau(P) = -2/(s_1 + s_2 + s_3)$ and $\tau(P_i) = -(a_i + b_i)/b_i p$ for $i=1, \cdots, 3$. Finally we obtain

$$\beta_Y = \max\left(-\frac{2}{s_1 + s_2 + s_3}, -\frac{a_i + b_i}{b_i p} \quad (i=1, 2, 3) \right).$$

Next, we give two examples of the singular case that the function $g(x)$ belongs to the α-class for f, and not is included into it. Let $f(x, y, z) = x^n + y^n + z^n$ ($n \geq 2$) and $g(x, y, z) = x^2 + y^3 + z^3$. Then it is easy to see that $\{(x, y, z) \in \mathbf{R}^3 ; f(x, y, z) = g(x, y, z) = 0\}$ is a non-degenerate complete intersection variety with a singularity at the origin. By the torus embedding, the dual Newton diagram $\Gamma^*(f, g)$ is as follows.
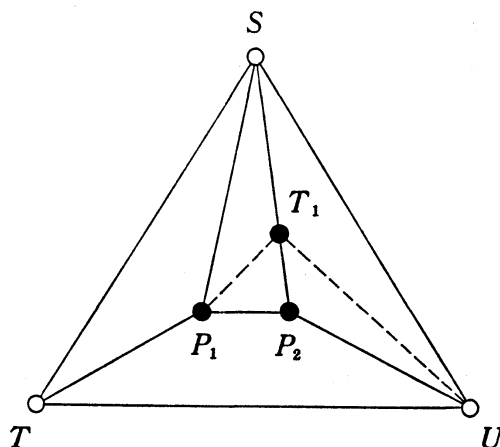
where $S={}^t(1, 0, 0)$, $T={}^t(0, 1, 0)$ and $U={}^t(0, 0, 1)$. The vertices P_1 and P_2 are strictly positive vertices such that $P_1={}^t(1, 1, 1)$, $P_2={}^t(3, 2, 2)$. By the unimodular subdivision, we get a vertex $T_1=(S+P_2)/2={}^t(2, 1, 1)$. Since the contribution of the vertices S, T and U to the weight β_Y is vanishing, we may consider mainly the strictly positive vertices P_1, P_2 and T_1. Thus by the definition $\tau(P_i)=-(1-d_g)/d_f$, we get the quantities such that

$$\tau(P_1) = -\frac{1}{n}$$

$$\tau(P_2) = -\frac{1}{2n}$$

$$\tau(T_1) = -\frac{1}{n}.$$

Hence, we have $\beta_Y=-1/2n(>-1)$ and $\bar{j}=1$. So, the oscillation index is $\beta(f, g)$ $=-1/2n$. Note that each $\alpha(P_i, g)$ is given by

$$\alpha(P_1, g) = \alpha(P_2, g) = \alpha(T_1, g) = 0.$$

This shows that the function g belongs to the α-class for f.

Finally, let $f(x, y, z)=x^4+y^4+z^4+xyz$ and $g(x, y, z)=x^n+y^n+z^n$ $(n>3)$. We can easily verify that the variety $\{(x, y, z)\in R^3 ; f(x)=g(x)=0\}$ is also a non-degenerate complete intersection variety at $\bar{0}$ in R^3. The dual Newton diagram $\Gamma^*(f, g)$ and each vertices S, T, U, P and P_i $(i=1, 2, 3)$ are the same as the first smooth case. Then we obtain

$$\tau(P) = \frac{n-3}{3}, \qquad \tau(P_i) = \frac{n-4}{4} \quad (i=1, 2, 3).$$

Hence we obtain $\beta_Y=(n-3)/3$, and the oscillation index $\beta(f, g)=(n-3)/3$. Since $\alpha(P)=2-n$ and $\alpha(P_i)=3-n$ $(i=1, 2, 3)$, the function g does not belong

to the α-*class* for f.　Moreover, if $n \neq 3m$ $(\geq 2,\ m \in \mathbf{Z})$, then $\log \tau$ term does not appear in $\tau^{\beta \gamma}$ term.　For $n = 3m$ $(m \geq 2,\ m \in \mathbf{Z})$, the $\log \tau$ term appears at most as $\log \tau$ in $\tau^{\beta \gamma}$ term.

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