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L^p-mapping properties of functions of Schrödinger operators and their applications to scattering theory

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§ 1. Introduction.

In the first part of this paper, we study operators $f(H)$ and $e^{-itH}f(H)$ in $L^p(\boldsymbol{R}^d)$, where $H=-\Delta+V(x)$ is a Schrödinger operator defined primarily as a self-adjoint operator in $L^2(\boldsymbol{R}^d)$. For $H_0=-\Delta$, mapping properties of $f(H_0)$ between L^p-spaces and norm estimates for $e^{-itH_0}f(H_0)$ follow from the theory of Fourier multipliers. One of our goals is to extend these results to a fairly large class of Schrödinger operators $H=H_0+V(x)$. To attain this goal we use several tools, including properties of the Schrödinger semigroup: e^{-tH}, the spaces $l^p(L^q)$ which are sometimes called amalgams of l^p and L^q, commutator estimates, and a result (Theorem 2.4) which can be viewed as a version of the Beurling-Carlson theorem on Fourier multipliers (see [**BTW**]).

Throughout this paper we suppose the potential $V(x)$ satisfies the following condition:

ASSUMPTION (A). V is real-valued function on $\boldsymbol{R}^d$, and it is decomposed as $V(x)=V_+(x)-V_-(x)$ such that $V_\pm\geqq 0$, $V_+\in K_d^{\mathrm{loc}}$ and $V_-\in K_d$, where K_d is the Kato class of potentials.

For the sake of completeness, we recall the definitions of K_d and K_d^{loc} (cf. Simon [**S**: Section A2] for the detail):

DEFINITION. $V\in K_d$, if:

For $d\geqq 3$, $$\lim_{r\to 0}\sup_{x\in\boldsymbol{R}^d}\int_{|x-y|\leq r}\frac{|V(y)|}{|x-y|^{d-2}}dy=0\,;$$

For $d=2$, $$\lim_{r\to 0}\sup_{x\in\boldsymbol{R}^d}\int_{|x-y|\leq r}\log\{|x-y|^{-1}\}|V(y)|dy=0\,;$$

For $d=1$, $$\sup_{x\in\boldsymbol{R}^d}\int_{|x-y|\leq 1}|V(y)|dy<\infty\,.$$

$V \in K_d^{\mathrm{loc}}$ if $\chi_{\{|x|<R\}}(x)V(x) \in K_d$ for any $R>0$, where χ_Ω denotes the characteristic function of Ω.

Then it is known that H defines a closed quadratic form with the form domain $Q(H)=Q(H_0)\cap Q(V_+)$, where $H_0=-\Delta$ and $Q(H_0)=H^1(\boldsymbol{R}^d)$, the usual Sobolev space. Hence H has a self-adjoint realization (the Friedrichs extension) which is semi-bounded (cf. [**S**: Section A2]). For a bounded function $f(\lambda)$, $f(H)$ and $e^{-itH}f(H)$ are defined in $L^2(\boldsymbol{R}^d)$ using the spectral decomposition for H. We consider continuous extensions of $f(H)$ and $e^{-itH}f(H)$ in $L^p(\boldsymbol{R}^d)$.

We define a class of symbols $S(\beta)$, $\beta \in \boldsymbol{R}$, as follows:

DEFINITION. $f \in S(\beta)$ if $f \in C^\infty(\boldsymbol{R})$ and $f(\lambda)$ has an asymptotic expansion in λ^{-1} as $\lambda \to \infty$ in the following sense: for any $N>0$,

$$f(\lambda)=\sum_{k=0}^{N} a_k \lambda^{-\beta-k}+r_N(\lambda), \qquad \lambda \geqq 1,$$

where the remainder term $r_N(\lambda)$ satisfies

$$\left|\left(\frac{d}{d\lambda}\right)^k r_N(\lambda)\right| \leqq C_{Nk}(1+|\lambda|)^{-\beta-N-1}, \qquad \lambda \geqq 1,\ k=0, 1, 2, \cdots.$$

We write $S(\infty)=\bigcap_{m=0}^{\infty} S(m)$, and note $S(\infty)\supset \mathcal{S}(\boldsymbol{R}^d)$, the Schwartz space. Since we are interested in $f(H)$ and H is semi-bounded, we may always assume $\operatorname{supp} f \subset [-M, \infty)$ for some $M>0$ without loss of generality.

THEOREM 1.1. *Let $f \in S(0)$. Then $f(H)$ is extended to a bounded operator in $L^p(\boldsymbol{R}^d)$ for $1 \leqq p \leqq \infty$.*

COROLLARY 1.2. *Let $1 \leqq p \leqq q \leqq \infty$, and let $\beta>(d/2)(1/p-1/q)$. If $f \in S(\beta)$ then $f(H)$ is extended to a bounded operator from $L^p(\boldsymbol{R}^d)$ to $L^q(\boldsymbol{R}^d)$.*

PROOF OF COROLLARY. We decompose $f(\lambda)$ as $f(\lambda)=(\lambda+M)^{-\beta}g(\lambda)$ where $g \in S(0)$ and M is a sufficiently large number. By [**S**: Theorem B.2.1], $(H+M)^{-\beta}$ is bounded from $L^p(\boldsymbol{R}^d)$ to $L^q(\boldsymbol{R}^d)$, and combining this with the boundedness of $g(H)$ in $L^p(\boldsymbol{R}^d)$, we learn that $f(H)=(H+M)^{-\beta}g(H)$ is bounded from $L^p(\boldsymbol{R}^d)$ to $L^q(\boldsymbol{R}^d)$. $\square$

We remark that Corollary 1.2 gives an answer to an open question in Simon [**S**: Section B2].

Now we would like to consider the time evolution e^{-itH} in $L^p(\boldsymbol{R}^d)$. It is known, however, that e^{-itH_0} is not bounded in $L^p(\boldsymbol{R}^d)$ if $p \neq 2$ (see, e.g., [**BTW**] p. 27). Instead, we consider $e^{-itH}f(H)$ where $f(\lambda)$ decays rapidly as $\lambda \to \infty$.

THEOREM 1.3. *Let $1 \leqq p \leqq \infty$ and let $f \in S(\infty)$. Then $e^{-itH}f(H)$ is bounded in $L^p(\boldsymbol{R}^d)$ for $t \in \boldsymbol{R}$. Moreover, for any $\beta>d|1/p-1/2|$,*

$$\|e^{-itH}f(H)\|_{B(L^p(\boldsymbol{R}^d))} \leqq C(1+|t|)^{\beta}, \qquad t\in\boldsymbol{R}. \tag{1.1}$$

The estimate (1.1) is almost, but not exactly, optimal. In fact, it is known that for the free case,

$$c(1+|t|)^{d|1/p-1/2|} \leqq \|e^{-itH_0}f(H_0)\|_{B(L^p(\boldsymbol{R}^d))} \leqq C(1+|t|)^{d|1/p-1/2|}$$

holds for $t\in\boldsymbol{R}$, with c, $C>0$ (cf. [**BTW**] p. 134). In many cases, we can prove the optimal upper bound.

THEOREM 1.4. *Suppose $d\leqq 3$ and let $1\leqq p\leqq\infty$. If $f\in S(\beta)$ for some $\beta>2+d/4$, then*

$$\|e^{-itH}f(H)\|_{B(L^p(\boldsymbol{R}^d))} \leqq C(1+|t|)^{d|1/p-1/2|}, \qquad t\in\boldsymbol{R}. \tag{1.2}$$

In Section 5, we shall discuss several generalizations of Theorem 1.4. For example, if V is sufficiently smooth and $f\in S(\infty)$, then (1.2) holds for any dimension (Theorem 5.2).

In the second part of this paper, we study the mapping properties of the wave operators for short-range scattering between L^p-spaces.

For $s\in\boldsymbol{R}_+=(0,\infty)$, we write $[s]=\min\{l\in\boldsymbol{N}\,|\,l>s\}$. Note that $[s]$ is different from the usual integer part. For the potential $V(x)$, we suppose:

ASSUMPTION (B). $V\in C^{[d/2]}(\boldsymbol{R}^d)$, real-valued and for some $\rho>d$ and for all multi-indices α with $|\alpha|\leqq[d/2]$ it satisfies

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}V(x)\right| \leqq C(1+|x|)^{-\rho}, \qquad x\in\boldsymbol{R}^d.$$

If V satisfies Assumption (B), H is self-adjoint with $D(H)=D(H_0)$, and the wave operators: $W_{\pm}=s\text{-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$ exist and are asymptotically complete (see, e.g., [**RS**: Vol. III] and references therein).

The main result is the following mapping property of the wave operators:

THEOREM 1.5. *Assume $d\geqq 3$ and V satisfies Assumption (B). Let $f\in C_0^\infty(\boldsymbol{R}_+)$, and let p, q: $1\leqq p<q\leqq\infty$ satisfy:*

(i) *If* $1\leqq p<2$: $\dfrac{1}{q}<\dfrac{d-2}{d}\cdot\dfrac{1}{p}+\dfrac{1}{d}$;

(ii) *If* $2<p<\infty$: $\dfrac{1}{q}<\dfrac{d}{d-2}\cdot\dfrac{1}{p}-\dfrac{1}{d-2}$.

Then $W_{\pm}f(H_0)$ and the L^2-adjoints: $(W_{\pm}f(H_0))^*$ define bounded operators from $L^p(\boldsymbol{R}^d)$ to $L^q(\boldsymbol{R}^d)$.

A series of more precise results is given in Section 7, with the assumptions on the potential depending on the values of p and q.

L^p-properties of Schrödinger operators have been studied by several authors, mainly using the Schrödinger semigroup: $\exp(-tH)$. We refer to Aizenman-Simon [**AS**], Simon [**S**], Pang [**P**], Davies [**D**] and references therein (see also [**CFKS**: Ch. 2] for an overview). In particular, Pang proved an estimate of the form (1.1) with $f(\lambda)=(\lambda+M)^{-\alpha}$, $\alpha>d+1$ and $\beta=d+2$; Davies studied the integral kernel of $f(-\Delta)$ on Riemannian manifolds. In the recent paper [**BD**], an estimate of the type (1.1) is obtained with $\beta=2d|1/p-1/2|$ for potentials $V=V_+-V_-$ such that $V_+\in K_d$ and $V_-\in L^\infty(\boldsymbol{R}^d)$.

We also mention the work by Hempel and Voigt [**HV**], in which they proved that the spectrum of H in $L^p(\boldsymbol{R}^d)$ is independent of p. As far as we know, Theorem 1.5 is the first result giving mapping properties of the wave operators between the L^p-spaces. For mapping properties between the weighted L^2-spaces, see [**JN**] and references therein.

The idea of the proof of Theorems 1.1-1.4 is the following: we reduce the problems in $L^p(\boldsymbol{R}^d)$ $(1\leqq p<2)$ to those in $l^p(L^2)$, which is defined by

$$l^p(L^q)=\left\{\varphi\in L^q_{\mathrm{loc}}(\boldsymbol{R}^d)\,\middle|\,\sum_{n\in\boldsymbol{Z}^d}\|\varphi\|^p_{L^q(C(n))}<\infty\right\},$$

where $C(n)$ is the unit cube at $n\in\boldsymbol{Z}^d$:

$$C(n)=\left\{x\in\boldsymbol{R}^d\,\middle|\,\max_{i=1,\cdots,d}|x_i-n_i|\leqq\frac{1}{2}\right\}.$$

The norm of $l^p(L^q)$ is defined by

$$\|\varphi\|_{l^p(L^q)}=\Big(\sum_{n\in\boldsymbol{Z}^d}\|\varphi\|^p_{L^q(C(n))}\Big)^{1/p},\qquad \varphi\in l^p(L^q),$$

and $l^p(L^q)$ is a Banach space. More on $l^p(L^q)$-spaces can be found in [**FS**] and the references therein. We first show that some power of the resolvent for H is bounded from $L^p(\boldsymbol{R}^d)$ to $l^p(L^2)$. Then we study the boundedness of $f(H)$ or $e^{-itH}f(H)$ in $l^p(L^2)$, which is continuously embedded in $L^p(\boldsymbol{R}^d)$. We mainly consider the case $p=1$. The general case follows by duality and an interpolation argument.

In the proof of Theorem 1.5, the following estimate plays an essential role: let $s<-d/2$, and let $\chi\in C^\infty(\boldsymbol{R})$ be bounded and supported away from 0, then

$$\|e^{-itH}\chi(H)\|_{B(L^1,L^{2,s})}\leqq C|t|^{-d/2},\qquad t\neq 0,\tag{1.3}$$

where $L^{2,s}$ is the weighted L^2-space of order s (see Section 6 for the definition). The estimate looks similar to a result in [**JSS**], but the proof given in Section 6 is quite different from theirs.

This paper is organized as follows: In Section 2 we prepare several basic estimates. In Sections 3 and 4, we prove Theorems 1.1 and 1.3 respectively, in slightly more general forms. Section 5 is devoted to the discussion of the estimate (1.2). In Section 6 we prepare several estimates, including (1.3), for the proof of Theorem 1.5, which is proved in Section 7 in a somewhat generalized setting.

We shall use the following notation: For $x\in\boldsymbol{R}^d$ or $\boldsymbol{R}$, we write $\langle x\rangle=(1+|x|^2)^{1/2}$. For Banach spaces X and Y, $B(X, Y)$ denotes the space of bounded operators from X to Y and $B(X)=B(X, X)$. The operator norm is denoted by $\|\cdot\|_{B(X,Y)}$ or $\|\cdot\|_{X\to Y}$. We sometimes write L^p instead of $L^p(\boldsymbol{R}^d)$. $\|\cdot\|$ denotes the $L^2(\boldsymbol{R}^d)$-norm or the operator norm in $L^2(\boldsymbol{R}^d)$ unless otherwise specified.

At last we want to mention recent progress on this subject made after this work was completed. Yajima proved strong L^p-mapping properties of wave operators for a class of Schrödinger operators [**Y**]. His method is completely different from ours. We have made some improvements and generalizations for the mapping properties of $f(H)$ and it will be published in [**JN2**].

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§ 2. Preliminary estimates in $l^p(L^2)$-spaces.

A. Boundedness of $(H+M)^{-\beta}$ from $L^p(\boldsymbol{R}^d)$ to $l^p(L^q)$.

The goal of this subsection is the next theorem:

THEOREM 2.1. *Let $1\leqq p<q\leqq\infty$ and let $\beta>(d/2)(1/p-1/q)$. Then there exists $M_0>0$, depending only on H, such that for $M>M_0$, $(H+M)^{-\beta}$ is extended to a bounded operator from $L^p(\boldsymbol{R}^d)$ to $l^p(L^q)$.*

LEMMA 2.2 (Young's inequality). *Let $1\leqq p, q, r, s, t, u\leqq\infty$ such that $1/p+1/q-1=1/r$ and $1/s+1/t-1=1/u$. If $f\in l^p(L^s)$ and $g\in l^q(L^t)$, then $f*g\in l^r(L^u)$ and*

$$\|f*g\|_{l^r(L^u)}\leqq 3^d\|f\|_{l^p(L^s)}\|g\|_{l^q(L^t)}\,. \tag{2.1}$$

PROOF. This result is well-known, see [**FS**, § 2]. We include the proof for the sake of completeness. Let $n\in\boldsymbol{Z}^d$. Then by Young's inequality on $\boldsymbol{R}^d$, we have

$$\begin{aligned}\|f*g\|_{L^u(C(n))}&=\Big\|\sum_m\int_{C(m)}f(y)g(\cdot-y)dy\Big\|_{L^u(C(n))}\\&\leqq\sum_m\|f\|_{L^s(C(m))}\|g\|_{L^t(C(n)-C(m))}\\&\leqq\sum_e\Big(\sum_m\|f\|_{L^s(C(m))}\|g(\cdot+e)\|_{L^t(C(n-m))}\Big),\end{aligned}$$

where e runs over $\{e\in\boldsymbol{Z}^d\,|\,e_j=\pm1,\text{ or }0,\ j=1,\cdots,d\}$. We now use Young's inequality for l^p-spaces to obtain

$$\|f*g\|_{l^r(L^u)} \leqq \sum_e \|f\|_{l^p(L^s)}\|g(\cdot+e)\|_{l^q(L^t)}$$

$$\leqq 3^d\|f\|_{l^p(L^s)}\|g\|_{l^q(L^t)}. \qquad \square$$

LEMMA 2.3. *Let $1\leqq p<q\leqq\infty$. Then e^{-tH} is bounded from $L^p(\boldsymbol{R}^d)$ to $l^p(L^q)$ for $t>0$, and there exists C, $L>0$ such that*

$$\|e^{-tH}\|_{B(L^p,l^p(L^q))} \leqq Ce^{Lt}(t^{-d(1/p-1/q)/2}+1), \qquad t>0. \tag{2.2}$$

PROOF. Let $k(t; x, y)$ be the integral kernel of e^{-tH}. Then it is known that for some $L\in\boldsymbol{R}$ and any $\varepsilon>0$,

$$|k(t; x, y)| \leqq C_\varepsilon t^{-d/2}e^{Lt}\exp(-|x-y|^2/4(1+\varepsilon)t) \tag{2.3}$$

(see, e.g., [**S**: Theorem B.6.7]). We set $\varepsilon=1$ and let $k_0(t; x-y)$ be the right hand side of (2.3). We first prove:

$$\|k_0(t; \cdot)\|_{l^1(L^p)} \leqq Ce^{Lt}(t^{-d(1-1/p)/2}+1), \qquad t>0. \tag{2.4}$$

We compute $\|k_0(t; \cdot)\|_{L^p(C(n))}$ for the case $n=0$, and $n\neq 0$, respectively:

$$\begin{aligned}\|k_0(t; \cdot)\|_{L^p(C(0))} &= \Big(\int_{|x|<1/2} k_0(t; x)^p dx\Big)^{1/p}\\ &\leqq Ct^{-d/2}e^{Lt}\Big(\int e^{-p|x|^2/8t}dx\Big)^{1/p}\\ &= Ct^{-d/2}e^{Lt}\Big(\int e^{-p|x|^2/8}t^{d/2}dx\Big)^{1/p}\\ &= Ct^{-d(1-1/p)/2}e^{Lt};\end{aligned} \tag{2.5}$$

$$\begin{aligned}\sum_{n\neq 0}\|k_0(t; \cdot)\|_{L^p(C(n))} &\leqq Ct^{-d/2}e^{Lt}\sum_{n\neq 0}\sup_{x\in C(n)} e^{-|x|^2/8t}\\ &\leqq Ct^{-d/2}e^{Lt}\sum_{n\neq 0}e^{-\alpha|n|^2/t}, \qquad \exists\alpha>0,\\ &\leqq Ce^{Lt}.\end{aligned} \tag{2.6}$$

Estimates (2.5) and (2.6) prove (2.4). We apply Lemma 2.2 with $f=k_0(t; \cdot)\in l^1(L^r)$ with $1/p+1/r-1=1/q$, and $g=\varphi\in l^p(L^p)=L^p(\boldsymbol{R}^d)$. Then we have

$$\begin{aligned}\|e^{-tH}\varphi\|_{l^p(L^q)} &\leqq \|k_0(t; \cdot)*|\varphi|\|_{l^p(L^q)}\\ &\leqq 3^d\|k_0(t; \cdot)\|_{l^1(L^r)}\|\varphi\|_{l^p(L^p)}\\ &\leqq Ce^{Lt}(t^{-d(1-1/r)/2}+1)\|\varphi\|_{L^p(\boldsymbol{R}^d)}\\ &\leqq Ce^{Lt}(t^{-d(1/p-1/q)/2}+1)\|\varphi\|_{L^p(\boldsymbol{R}^d)}.\end{aligned} \tag{2.7}$$

This proves the assertion. $\square$

PROOF OF THEOREM 2.1. Let us fix $M_0=\max\{L, -\inf \sigma(H)\}$ with $L>0$ in Lemma 2.3. We use the formula:

$$(H+M)^{-\beta}=\frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}e^{-Mt}e^{-tH}dt\,.$$

By Lemma 2.3, we have

$$\begin{aligned}\|(H+M)^{-\beta}\varphi\|_{l^p(L^q)} &\leqq \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}e^{-Mt}\|e^{-tH}\varphi\|_{l^p(L^q)}dt\\ &\leqq C\int_0^\infty (t^{\beta-d(1/p-1/q)/2-1}+1)e^{-(M-L)t}dt\|\varphi\|_{L^p(R^d)}\,.\end{aligned}$$

Since $\beta>d(1/p-1/q)/2$, the integral is finite, and $(H+M)^{-\beta}$ is bounded from $L^p(\boldsymbol{R}^d)$ to $l^p(L^q)$. $\square$

REMARK. In fact, we can take $M_0=-\inf \sigma(H)$, since in the above proof we may take any L such that $L>-\inf \sigma(H)$ (see [**S**: Section B.5]).

B. Bounded operators in $l^1(L^2)$.

Here we give a sufficient condition for an L^2-bounded operator A to be bounded in $l^1(L^2)$, and also give an estimate on the operator norm. Let us note that if an L^2-bounded operator A is local (i.e., $\mathrm{supp}(Af)\subseteqq \mathrm{supp}(f)$ for all $f\in L^2$), then it is trivially bounded on $l^1(L^2)$.

For $\beta>0$ we define a class of operators $\mathcal{A}_\beta$ as follows:

DEFINITION. $A\in\mathcal{A}_\beta$ if $A\in B(L^2(\boldsymbol{R}^d))$ and there is $C>0$ such that

$$\sup_{n\in\boldsymbol{Z}^d}\|\langle\cdot-n\rangle^\beta A\chi_{C(n)}\varphi\|\leqq C\|\varphi\|$$

for $\varphi\in L^2(\boldsymbol{R}^d)$. For $A\in\mathcal{A}_\beta$, we write

$$|||A|||_\beta=\|A\|+\sup_{n\in\boldsymbol{Z}^d}\|\langle\cdot-n\rangle^\beta A\chi_{C(n)}\|\,.$$

THEOREM 2.4. *If $A\in\mathcal{A}_\beta$ for some $\beta>d/2$, then A is bounded in $l^1(L^2)$ and*

$$\|A\|_{B(l^1(L^2))}\leqq C|||A|||_\beta^{d/2\beta}\|A\|^{1-d/2\beta}\,, \tag{2.8}$$

where C depends only on d and β.

PROOF. We write $\chi_n=\chi_{C(n)}$ for simplicity. We first note that if $A\in\mathcal{A}_\beta$,

$$\Big(\sum_{m\in\boldsymbol{Z}^d}\langle m-n\rangle^{2\beta}\|\chi_m A\chi_n\varphi\|^2\Big)^{1/2}\leqq C|||A|||_\beta\|\chi_n\varphi\|$$

for $\varphi\in L^2(\boldsymbol{R}^d)$ and $n\in\boldsymbol{Z}^d$. For any $\omega>1$, we have

$$\begin{aligned}\sum_{m\in Z^d}\|\chi_m A\chi_n\varphi\| &= \sum_{|m-n|>\omega}|m-n|^{-\beta}|m-n|^{\beta}\|\chi_m A\chi_n\varphi\|+\sum_{|m-n|\leq\omega}\|\chi_m A\chi_n\varphi\| \\ &\leqq\Big(\sum_{|m-n|>\omega}|m-n|^{-2\beta}\Big)^{1/2}\Big(\sum_{|m-n|>\omega}|m-n|^{2\beta}\|\chi_m A\chi_n\varphi\|^2\Big)^{1/2} \\ &\quad+\Big(\sum_{|m-n|\leq\omega}1\Big)^{1/2}\Big(\sum_{|m-n|\leq\omega}\|\chi_m A\chi_n\varphi\|^2\Big)^{1/2} \\ &\leqq C\{\omega^{-(\beta-d/2)}|||A|||_\beta+\omega^{d/2}\|A\|\}\,\|\chi_n\varphi\|\,.\end{aligned}$$

In the first inequality we have used the Schwarz inequality. Setting $\omega=(|||A|||_\beta/\|A\|)^{1/\beta}$, we obtain

$$\sum_{m\in Z^d}\|\chi_m A\chi_n\varphi\|\leqq C|||A|||_\beta^{d/2\beta}\|A\|^{1-d/2n}\|\chi_n\varphi\|\,,$$

and this implies

$$\|A\varphi\|_{l^1(L^2)}\leqq C|||A|||_\beta^{d/2\beta}\|A\|^{1-d/2\beta}\|\varphi\|_{l^1(L^2)}\,. \qquad \square$$

§ 3. Mapping properties of $f(H)$ in $L^p(\boldsymbol{R}^d)$.

At first we prepare an algebraic lemma which is useful for proving $A\in\mathcal{A}_\beta$.

Lemma 3.1. *Let X and Y be topological vector spaces. Let A and B be continuous linear operators in X and Y, respectively. For a continuous linear operator L from X to Y, $\mathrm{Ad}^k(L)\colon X\to Y$, $h=0, 1, \cdots$, is defined inductively by*

$$\mathrm{Ad}^0(L)=L\,,\quad \mathrm{Ad}^k(L)=\mathrm{Ad}^{k-1}(BL-LA)\,,\qquad k\geqq 1\,.$$

Then there exists a set of constants: $\{\Gamma(n, m)\,|\,n\geqq 1, 0\leqq m\leqq n\}$ *such that*

$$B^nL=\sum_{m=0}^{n}\Gamma(n, m)\,\mathrm{Ad}^m(L)A^{n-m}\,. \tag{3.1}$$

Proof. For $n\geqq 1$, we set $\Gamma(n, 0)=1$ and define $\Gamma(n, m)$ inductively by

$$\Gamma(n, m+1)=\sum_{k=m}^{n-1}\Gamma(k, m)\,,\qquad 1\leqq m\leqq n-1\,.$$

For example, $\Gamma(n, 1)=n$, $\Gamma(n, 2)=n(n-1)/2$, etc.. We prove (3.1) by induction. For $n=1$,

$$BL=LA+(BL-LA)=\sum_{m=0}^{1}\Gamma(1, m)\,\mathrm{Ad}^m(L)A^{1-m}\,.$$

Let us suppose (3.1) holds for $n=1, 2, \cdots, l$. Then

$$
\begin{aligned}
B^{l+1}L &= \sum_{k=0}^{l} B^k(BL-LA)A^{l-k}+LA^{l+1} \\
&= \sum_{k=0}^{l}\sum_{m=0}^{k}\Gamma(k,m)\,\mathrm{Ad}^m(BL-LA)A^{k-m}A^{l-k}+LA^{l+1} \\
&= \sum_{m=0}^{l}\left\{\sum_{k=m}^{l}\Gamma(k,m)\right\}\mathrm{Ad}^{m+1}(L)A^{l-m}+LA^{l+1} \\
&= \sum_{m=0}^{l+1}\Gamma(l+1,m)\,\mathrm{Ad}^m(L)A^{l+1-m}. \qquad \square
\end{aligned}
$$

In what follows we fix $M>0$ as in Theorem 2.1 and we let $R=(H+M)^{-1}$.

Commutator estimates of the following type have been used for proving weighted L^2-estimates (see, e.g., [**J1**], [**J2**], [**JN**] and references therein). In our context, it is crucial that the estimates are uniform with respect to the translations.

LEMMA 3.2. *For any $\beta>0$, there is $C>0$ such that*

$$
\|\langle\cdot-n\rangle^{\beta}e^{-itR}\langle\cdot-n\rangle^{-\beta}\| \leqq C\langle t\rangle^{\beta}, \qquad n\in \boldsymbol{Z}^d,\ t\in\boldsymbol{R}. \tag{3.2}
$$

PROOF. We use Lemma 3.1 with $X=\mathcal{D}=C_0^\infty(\boldsymbol{R}^d)$, $Y=\mathcal{D}'$, $A=B=(x_i-n_i)\cdot$ with fixed $n\in\boldsymbol{Z}^d$ and $i\in\{1,\cdots,d\}$. H is a continuous linear operator from $\mathcal{D}$ to $\mathcal{D}'$ and it is easy to see

$$
\begin{aligned}
&\mathrm{Ad}^1(H)=[x_i,H]=2\partial_i; \qquad \mathrm{Ad}^2(H)=[x_i,2\partial_i]=-2;\\
&\mathrm{Ad}^k(H)=0 \qquad \text{for } k\geqq 3,
\end{aligned} \tag{3.3}
$$

where $\partial_i=(\partial/\partial x_i)$. From (3.3) we learn that

$$
\mathrm{Ad}^k(R)=P_k(R,\partial_i R), \qquad k=1,2,\cdots,
$$

where P_k is an (ordered) polynomial of order $k+1$. Since $Q(H)\subset H^1(\boldsymbol{R}^d)$, $\partial_i R$ is bounded in $L^2(\boldsymbol{R}^d)$. These imply that $\mathrm{Ad}^k(R)$ is bounded in $L^2(\boldsymbol{R}^d)$ for any $k\geqq 0$. Using this fact and the formula

$$
\mathrm{Ad}^1(e^{-itR})=-i\int_0^t e^{-isR}\,\mathrm{Ad}^1(R)e^{-i(t-s)R}ds
$$

repeatedly, we obtain

$$
\|\mathrm{Ad}^k(e^{-itR})\|\leqq C_k\langle t\rangle^k, \qquad t\in\boldsymbol{R} \tag{3.4}
$$

for $k\geqq 1$. Combining (3.4) with Lemma 3.1, we have

$$
\begin{aligned}
&\|(x_i-n_i)^l e^{-itR}\langle x-n\rangle^{-2N}\| \\
&\leqq \sum_{m=0}^{l}\Gamma(l,m)\|\mathrm{Ad}^m(e^{-itR})\|\|(x_i-n_i)^{l-m}\langle x-n\rangle^{-2N}\| \\
&\leqq C\langle t\rangle^l, \qquad t\in\boldsymbol{R}
\end{aligned}
$$

if $l \leqq 2N$. Since the estimate is independent of $n \in \boldsymbol{Z}^d$ and $i \in \{1, \cdots, d\}$, this implies

$$\|\langle x-n\rangle^{2N} e^{-itR} \langle x-n\rangle^{-2N}\| \leqq C_N \langle t\rangle^{2N}, \qquad t \in \boldsymbol{R},\ n \in \boldsymbol{Z}^d,$$

for any integer $N \geqq 0$. Now (3.2) follows by the Calderón-Lions interpolation theorem ([**RS**: Theorem IX.20]). □

LEMMA 3.3. *Let $\beta > 0$ and let $f \in C_0^m(\boldsymbol{R})$ with $m > \beta + 1/2$. Then $f(R) \in \mathcal{A}_\beta$ and*

$$|||f(R)|||_\beta \leqq C \int |\hat{f}(t)| \langle t\rangle^\beta dt, \tag{3.5}$$

where $\hat{f}$ is the Fourier transform of f, and C depends only on d and β.

PROOF. By Lemma 3.2 and the representation

$$f(R) = (2\pi)^{-1/2} \int e^{itR} \hat{f}(t) dt,$$

we learn

$$\begin{aligned} \|\langle \cdot - n\rangle^\beta f(R) \langle \cdot - n\rangle^{-\beta}\| &\leqq (2\pi)^{-1/2} \int \|\langle \cdot - n\rangle^\beta e^{itR} \langle \cdot - n\rangle^{-\beta}\| \, |\hat{f}(t)| dt \\ &\leqq C \int \langle t\rangle^\beta |\hat{f}(t)| dt. \end{aligned} \tag{3.6}$$

By the assumption on f, $\langle t\rangle^m \hat{f}(t) \in L^2(\boldsymbol{R})$ and the right hand side is finite by $m > \beta + 1/2$ and the Schwarz inequality. Since

$$\|\langle \cdot - n\rangle^\beta A \chi_n\| \leqq (1+d)^{\beta/2} \|\langle \cdot - n\rangle^\beta A \langle \cdot - n\rangle^{-\beta}\| \tag{3.7}$$

for any $A \in B(L^2(\boldsymbol{R}^d))$, (3.5) follows from (3.6). □

LEMMA 3.4. *Let $m = [(d+1)/2]$. If $f \in C_0^m(\boldsymbol{R})$, then $f(R)$ is bounded on $l^1(L^2)$.*

PROOF. Since $m > (d+1)/2$, we can determine β with $d/2 < \beta < m - 1/2$ and apply Lemma 3.3 and Theorem 2.4. □

THEOREM 3.5. *Let $m = [(d+1)/2]$. If $f \in C^m(\boldsymbol{R})$ and f has an asymptotic expansion*

$$f(\lambda) = \sum_{k=0}^{2m} a_k \lambda^{-k} + r_{2m}(\lambda), \qquad \lambda > 1, \tag{3.8}$$

where $r_{2m}(\lambda)$ satisfies $|(d/d\lambda)^k r_{2m}(\lambda)| \leqq C \langle \lambda\rangle^{-2m-1}$, $\lambda \geqq 1$, for $k = 0, 1, \cdots, m$, then $f(H)$ extends to a bounded operator on $l^1(L^2)$.

PROOF. Using the Neumann series expansion ($\lambda \geqq 1$)

$$\lambda^{-1}=((\lambda+M)-M)^{-1}=\sum_{k=0}^{\infty}M^k(\lambda+M)^{-k-1},$$

we can rewrite (3.8) as

$$f(\lambda)=\sum_{k=0}^{2m}b_k(\lambda+M)^{-k}+\tilde{r}_{2m}(\lambda), \qquad \lambda\geqq 1, \tag{3.9}$$

where $\tilde{r}_{2m}$ satisfies the same condition as r_{2m}. Thus the condition (3.8) implies that there is $g\in C_0^m(\boldsymbol{R})$ such that

$$g((\lambda+M)^{-1})=f(\lambda) \qquad \text{for } \lambda\in\sigma(H).$$

Hence $f(H)=g(R)$ and the claim follows from Lemma 3.4. □

THEOREM 3.6. *Let m be an integer with $m>[(d+1)/2]+(d/4)$. If $f\in C^m(\boldsymbol{R})$ and f has an asymptotic expansion* (3.8), *then $f(H)$ is bounded on $L^1(\boldsymbol{R}^d)$.*

PROOF. Choose $g(\lambda)\in C^m(\boldsymbol{R})$ so that $\operatorname{supp} g\subset(-M, \infty)$ and

$$g(\lambda)=f(\lambda)-\sum_{k=0}^{2m}b_k(\lambda+M)^{-k} \qquad \text{for } \lambda\in\sigma(H),$$

with $\{b_k\}$ in (3.9). Let $\beta=m-[(d+1)/2]$ and $h(\lambda)\equiv(\lambda+M)^{\beta}g(\lambda)$. Then h satisfies the assumption of Theorem 3.5, and $h(H)$ is bounded on $l^1(L^2)$. Combining this with Theorem 2.1, we learn that $g(H)=h(H)(H+M)^{-\beta}$ is bounded from $L^1(\boldsymbol{R}^d)$ to $l^1(L^2)$, and hence bounded on $L^1(\boldsymbol{R}^d)$.

On the other hand, $(H+M)^{-k}$ is bounded on $L^1(\boldsymbol{R}^d)$ by [**S**: Theorem B.2.1] for $k\geqq 0$. Thus $f(H)=\sum_{k=0}^{m}b_k(H+M)^{-k}+g(H)$ is bounded on $L^1(\boldsymbol{R}^d)$. □

COROLLARY 3.7. *If f satisfies the conditions of Theorem* 3.6, *then $f(H)$ is bounded on $L^p(\boldsymbol{R}^d)$ for $1\leqq p\leqq\infty$.*

PROOF. $f(H)$ is bounded on $L^\infty(\boldsymbol{R}^d)$ by duality, and the claim follows by the Riesz-Thorin interpolation theorem ([**RS**: Theorem IX.17]). □

Theorem 1.1 now follows immediately from Corollary 3.7.

§4. Estimates for $e^{-itH}f(H)$ in $L^p(\boldsymbol{R}^d)$.

For a bounded function $g(\mu)$, we set

$$g_t(\mu)=e^{it(M-\mu^{-1})}g(\mu), \qquad \mu\in\boldsymbol{R},\ t\in\boldsymbol{R}.$$

By the functional calculus, it is easy to see $g_t(R)=e^{-itH}g(R)$.

LEMMA 4.1. *Let $m>0$ be an integer, and let $\kappa>2m-1/2$ be a real number. Assume $g\in C_0^m(\boldsymbol{R})$ such that $|(d/d\mu)^j g(\mu)|\leqq C|\mu|^\kappa$ for $j=0, 1, \cdots, m$. Assume $\beta<m-1/2$. Then*

$$\int |\hat{g}(s)|\langle s\rangle^{\beta} ds \leqq C\langle t\rangle^{m}, \qquad t\in\boldsymbol{R}. \tag{4.1}$$

PROOF. We note that for $h\in C_0^m(\boldsymbol{R})$, $\beta<m-1/2$, we have

$$\begin{aligned}\int |\hat{h}(s)|\langle s\rangle^{\beta} ds &= \int |\langle s\rangle^{m}\hat{h}(s)|\langle s\rangle^{\beta-m} ds \\ &\leqq \|\langle\cdot\rangle^{m}\hat{h}(\cdot)\|_{L^2}\|\langle\cdot\rangle^{\beta-m}\|_{L^2} \\ &\leqq C(\|h\|_{L^2}+\|h^{(m)}\|_{L^2}),\end{aligned}$$

where we used the Schwarz inequality, well-known properties of the Fourier transform, and $\beta-m<-1/2$. Thus we only need to estimate $(d/d\mu)^m g_t(\mu)$. Straightforward differentiation and the assumptions yield

$$|(d/d\mu)^m g_t(\mu)| \leqq C\langle t\rangle^{m}\mu^{-2m}\sum_{j=0}^{m}|g^{(j)}(\mu)| \leqq C\langle t\rangle^{m}\mu^{-2m+\kappa}.$$

Since $-2m+\kappa>-1/2$, the derivative is square integrable, and the result follows. □

THEOREM 4.2. *Let $m>(d+1)/2$ be an integer. Let $\kappa>4m-1/2$ be a real number. Suppose $f\in C^m(\boldsymbol{R})$ such that $|(d/d\lambda)^j f(\lambda)|\leqq C\langle\lambda\rangle^{-\kappa}$, $\lambda\in\boldsymbol{R}$, for $j=0, 1, \cdots, m$. Then $e^{-itH}f(H)$ is bounded on $l^1(L^2)$ for $t\in\boldsymbol{R}$, and for any $\gamma>\frac{m}{m-1/2}\cdot\frac{d}{2}$,*

$$\|e^{-itH}f(H)\|_{B(l^1(L^2))} \leqq C_\gamma\langle t\rangle^{\gamma}, \qquad t\in\boldsymbol{R}. \tag{4.2}$$

PROOF. The condition on m implies $m<\frac{m}{m-1/2}\cdot\frac{d}{2}$, and we may suppose $m>\gamma>\frac{m}{m-1/2}\cdot\frac{d}{2}$ without loss of generality. Let $\beta=md/2\gamma$, then $d/2<\beta<m-1/2$. By the condition on f, we can find $g\in C_0^m(\boldsymbol{R})$, such that $g((\lambda+M)^{-1})=f(\lambda)$, $\lambda\in\sigma(H)$, and such that g satisfies the conditions in Lemma 4.1. Now we can apply this lemma to obtain (4.1) with the above m and β. Combining this with Lemma 3.3, we have

$$\begin{aligned}|||e^{-itH}f(H)|||_{\beta} &= |||e^{-itH}g(R)|||_{\beta} = |||g_t(R)|||_{\beta} \\ &\leqq C\int |\hat{g}_t(s)|\langle s\rangle^{\beta} ds \leqq C\langle t\rangle^{m}\end{aligned}$$

for $t\in\boldsymbol{R}$. By Theorem 2.4, this implies

$$\begin{aligned}\|e^{-itH}f(H)\|_{B(l^1(L^2))} &\leqq C|||e^{-itH}f(H)|||_{\beta}^{d/2\beta}\|e^{-itH}f(H)\|^{1-d/2\beta} \\ &\leqq C\langle t\rangle^{dm/2\beta} = C\langle t\rangle^{\gamma}, \qquad t\in\boldsymbol{R}. \qquad \square\end{aligned}$$

THEOREM 4.3. *Let $m>(d+1)/2$ be an integer. Suppose $f\in C^m(\boldsymbol{R})$ such that* *for some real number* $\kappa > 4m + (d-2)/4$ *and* $j = 0, 1, \cdots, m$, $|(d/d\lambda)^j f(\lambda)| \leqq C\langle\lambda\rangle^{-\kappa}$, $\lambda \in \boldsymbol{R}$. *Then* $e^{-itH} f(H)$ *is bounded on* $L^1(\boldsymbol{R}^d)$ *and for any* $\gamma > \frac{m}{m-1/2} \cdot \frac{d}{2}$

$$\|e^{-itH} f(H)\|_{B(L^1)} \leqq C_\gamma \langle t\rangle^\gamma, \qquad t \in \boldsymbol{R}. \tag{4.3}$$

In particular, if $f \in S(\infty)$ *then* (4.3) *holds for any* $\gamma > d/2$.

PROOF. The claim follows from Theorem 2.1 and Theorem 4.2. □

COROLLARY 4.4. *Let* m *and* f *satisfy the conditions in Theorem* 4.3. *Then* $e^{-itH} f(H)$ *is bounded on* $L^p(\boldsymbol{R}^d)$ *for* $t \in \boldsymbol{R}$, $1 \leqq p \leqq \infty$, *and for any* $\gamma > \frac{m}{m-1/2} \times d|1/p - 1/2|$,

$$\|e^{-itH} f(H)\|_{B(L^p)} \leqq C_\gamma \langle t\rangle^\gamma, \qquad t \in \boldsymbol{R}. \tag{4.4}$$

In particular, if $f \in S(\infty)$ *then* (4.4) *holds for any* $\gamma < d|1/p - 1/2|$.

PROOF. For $1 < p < 2$, using interpolation between (4.3) and the trivial estimate: $\|e^{-itH} f(H)\|_{B(L^2)} < C$ we obtain (4.4). The case: $2 < p \leqq \infty$ follows by duality. □

§5. Sharp estimates for $e^{-itH} f(H)$ on $L^p(\boldsymbol{R}^d)$.

Here we prove Theorem 1.4 and discuss its generalizations. In order that we employ more direct method than in the last section. Namely, we study $l^1(L^2)$-mapping properties of $e^{-itH} R^\kappa$ using the commutator method directly, instead of looking at e^{-itR}. At first we prepare a lemma:

LEMMA 5.1. *There is* $C > 0$ *such that*

$$\|\langle\cdot - n\rangle^2 e^{-itH} R^2 \langle\cdot - n\rangle^{-2}\| \leqq C\langle t\rangle^2, \qquad t \in \boldsymbol{R},\ n \in \boldsymbol{Z}^d. \tag{5.1}$$

PROOF. As in the proof of Lemma 3.2, it suffices to show

$$\|[(x_i - n_i), e^{-itH} R^2]\| \leqq C\langle t\rangle; \tag{5.2}$$

$$\|[(x_i - n_i), [(x_i - n_i), e^{-itH} R^2]]\| \leqq C\langle t\rangle^2 \tag{5.3}$$

for $t \in \boldsymbol{R}$, $n \in \boldsymbol{Z}^d$ and $i \in \{1, \cdots, d\}$. We first compute:

$$\begin{aligned}[(x_i - n_i), e^{-itH} R^2] &= [x_i, Re^{-itH} R] \\ &= -i\int_0^t e^{-isH} R[x_i, H] Re^{-i(t-s)H} ds + [x_i, R] e^{-itH} R + Re^{-itH}[x_i, R].\end{aligned}$$

Using (3.3) and noting that $\partial_i R$ is bounded, we obtain (5.2). The double commutator is

$$
\begin{aligned}
&[(x_i-n_i), [(x_i-n_i), e^{-itH}R^2]] \\
&\quad = [x_i, [x_i, R]e^{-itH}R + R[x_i, e^{-itH}]R + Re^{-itH}[x_i, R]] \\
&\quad = 2(-i)^2\int_0^t\int_0^s e^{-iuH}(R[x_i, H])e^{-i(s-u)H}([x_i, H]R)e^{-i(t-s)H}duds \\
&\quad\quad +(-i)\int_0^t e^{-isH}R[x_i, [x_i, H]]Re^{-i(t-s)H}ds \\
&\quad\quad +2\{[x_i, R]e^{-itH}[x_i, R]+[x_i, R][x_i, e^{-itH}]R+R[x_i, e^{-itH}][x_i, R]\} \\
&\quad\quad +[x_i, [x_i, R]]e^{-itH}R+Re^{-itH}[x_i, [x_i, R]].
\end{aligned}
$$

Using (3.3) again, we learn that the first term of the right hand side is $O(\langle t\rangle^2)$. The other terms can be estimated as in the proof of (5.2) using (3.3). □

PROOF OF THEOREM 1.4. By Lemma 5.1 and (3.7), we immediately have

$$|||e^{-itH}R^2|||_2 \leqq C\langle t\rangle^2, \qquad t\in \boldsymbol{R}.$$

Since $d\leqq 3$, Theorem 2.4 applies with $\beta=2$ and

$$
\begin{aligned}
\|e^{-itH}R^2\|_{B(l^1(L^2))} &\leqq C|||e^{-itH}R^2|||_2^{d/4}\|e^{-it}R^2\|^{1-d/4} \\
&\leqq C\langle t\rangle^{d/2}, \qquad t\in \boldsymbol{R}.
\end{aligned}
$$

Combining this with Theorem 2.1 and Theorem 1.1, we obtain

$$
\begin{aligned}
\|e^{-itH}f(H)\|_{B(L^1)} &\leqq C\|e^{-itH}R^2\|_{B(l^1(L^2))}\|R^{\beta-2}\|_{B(L^1, l^1(L^2))}\|R^{-\beta}f(H)\|_{B(L^1)} \\
&\leqq C\langle t\rangle^{d/2}, \qquad t\in \boldsymbol{R} \qquad (5.4)
\end{aligned}
$$

if $f\in S(\beta)$ with $\beta>2+d/4$. As in the proof of Corollary 4.4, (5.4) implies (1.2) for $1\leqq p\leqq\infty$. □

For the case $d\geqq 4$, we need an additional condition to prove (1.2). We let $H^l(\boldsymbol{R}^d)$ denote the usual Sobolev space of order l.

THEOREM 5.2. *Let $d\geqq 4$ and suppose $D(|H|^{l/2})=H^l(\boldsymbol{R}^d)$ for $0\leqq l\leqq [d/4]$. If $f\in S(\beta)$ for some $\beta>2[d/4]+d/4$, then*

$$\|e^{-itH}f(H)\|_{B(L^1(\boldsymbol{R}^d))} \leqq C\langle t\rangle^{d/2}, \qquad t\in \boldsymbol{R}. \qquad (5.5)$$

SKETCH OF PROOF. Let $k=[d/4]$. The main step of the proof is to show

$$\|\mathrm{Ad}^l(e^{-itH}R^{2k})\| \leqq C\langle t\rangle^l, \qquad t\in \boldsymbol{R}, \qquad (5.6)$$

for $0\leqq l\leqq 2k$, where $\mathrm{Ad}^l(\cdot)$ is defined as in the proof of Lemma 3.2. In order to prove (5.6), we compute as follows:

$$\begin{aligned}\mathrm{Ad}^l(e^{-itH}R^{2k}) &= \mathrm{Ad}^l(R^k e^{-itH}R^k)\\ &= \sum_{a+b+c=l}\frac{l!}{a!b!c!}\mathrm{Ad}^a(R^k)\,\mathrm{Ad}^b(e^{-itH})\,\mathrm{Ad}^c(R^k).\end{aligned}$$

We expand the term $\mathrm{Ad}(e^{-itH})$ as in the proof of (5.3). Using the assumption: $D(|H|^{l/2})=H^l(\boldsymbol{R}^d)$, we can show

(i) $\mathrm{Ad}^a(R^k)$ is bounded from $L^2(\boldsymbol{R}^d)$ to $H^k(\boldsymbol{R}^d)$, and from $H^{-k}(\boldsymbol{R}^d)$ to $L^2(\boldsymbol{R}^d)$;

(ii) $\mathrm{Ad}^b(e^{-itH})$ is bounded from $H^{[b/2]}(\boldsymbol{R}^d)$ to $H^{-[b/2]}(\boldsymbol{R}^d)$ and

$$\|\mathrm{Ad}^b(e^{-itH})\|_{B(H^{[b/2]},H^{-[b/2]})}\leqq C\langle t\rangle^b, \qquad t\in\boldsymbol{R}.$$

These imply (5.6). Now it follows from (5.6): $|||e^{-itH}R^{2k}|||_{2k}\leqq C\langle t\rangle^{2k}$. Since $2k>d/2$, this implies (5.5) as in the proof of Theorem 1.4. □

COROLLARY 5.3. *Let $1\leqq p\leqq\infty$ and suppose H and f satisfy the assumptions of Theorem* 5.2. *Then*

$$\|e^{-itH}f(H)\|_{B(L^p)}\leqq C\langle t\rangle^{d|1/p-1/2|}, \qquad t\in\boldsymbol{R}. \tag{5.7}$$

In particular, if $d\leqq 7$ and $D(H)=H^2(\boldsymbol{R}^d)$, then (5.7) *holds for $f\in S(\beta)$ with $\beta>4+d/4$.*

If we suppose stronger assumption on H we can relax a condition on f:

THEOREM 5.4. *Let $1\leqq p\leqq\infty$ and suppose $D(|H|^{l/2})=H^l(\boldsymbol{R}^d)$ for $0\leqq l\leqq[d/2]$. If $f\in S(\beta)$ for some $\beta>[d/2]+d/4$, then* (5.7) *holds. In particular, if $d\leqq 3$, $D(H)=H^2(\boldsymbol{R}^d)$ and $f\in S(\beta)$ for some $\beta>2+d/4$, then* (5.7) *holds.*

The proof is similar, but simpler than that of Theorem 5.2.

§6. Mapping properties of e^{-itH_0} and e^{-itH} between L^p-spaces.

In this section we establish some mapping properties of e^{-itH_0} and e^{-itH} between L^p-spaces or weighted L^2-spaces:

$$L^{2,s}(\boldsymbol{R}^d)=\{f\in L^2_{\mathrm{loc}}(\boldsymbol{R}^d)\,|\,\langle x\rangle^s f(x)\in L^2(\boldsymbol{R}^d)\}.$$

We first recall a couple of well-known results:

LEMMA 6.1. (i) *Assume $1\leqq p<2$ and $s>d(1/p-1/2)$. Then $L^{2,s}(\boldsymbol{R}^d)$ is continuously embedded in $L^p(\boldsymbol{R}^d)$.*

(ii) *Assume $2<p\leqq\infty$ and $s<d(1/p-1/2)$. Then $L^p(\boldsymbol{R}^d)$ is continuously embedded in $L^{2,s}(\boldsymbol{R}^d)$.*

PROOF. The result (i) is a direct consequence of Hölder's inequality, and (ii) follows from (i) by duality. □

LEMMA 6.2. *Assume $1 \leqq p \leqq 2$ and let q be the conjugate exponent*: $1/p+1/q=1$. *Then for $t \neq 0$, e^{-itH_0} is bounded from $L^p(\boldsymbol{R}^d)$ to $L^q(\boldsymbol{R}^d)$ and*

$$\|e^{-itH_0}\|_{B(L^p, L^q)} \leqq (4\pi|t|)^{-d(1/p-1/2)}, \qquad t \neq 0.$$

PROOF. If we note the integral kernel of e^{-itH_0} is

$$k_0(t; x, y) = (4\pi it)^{-d/2} \exp(-|x-y|^2/4it),$$

the result follows immediately for $p=1$. If $p=2$ the result is trivial, and the other cases follow by the Riesz-Thorin interpolation theorem. □

REMARK. In stating Lemma 6.2 we have abused notation. *A priori*, e^{-itH_0} is defined on $L^2(\boldsymbol{R}^d)$, hence on $L^2 \cap L^p$ which is dense in $L^p(\boldsymbol{R}^d)$. Thus $e^{-itH_0} \in B(L^p, L^q)$ means that this densely defined operator on L^p maps into L^q and extends to a bounded operator from L^p to L^q. We will continue this abuse of notation without comment.

LEMMA 6.3. *Let $2 \leqq p \leqq \infty$ and let $s > d/p$. Then the operator $\langle D \rangle^{-s} e^{-itH_0}$ is bounded from $L^1(\boldsymbol{R}^d)$ to $L^p(\boldsymbol{R}^d)$ for $t \neq 0$ and*

$$\|\langle D \rangle^{-s} e^{-itH_0}\|_{B(L^1, L^p)} \leqq C|t|^{-d(1/2-1/p)}, \qquad t \neq 0. \tag{6.1}$$

PROOF. Let q denote the exponent conjugate to p. Then $s > d(1-1/q)$, and $\langle D \rangle^{-s} = (H_0+1)^{-s/2}$ is bounded from L^1 to L^q by the Sobolev embedding theorem. Using Lemma 6.2, we see that $e^{-itH_0}\langle D \rangle^{-s}$ is bounded from L^1 to L^p, and (6.1) follows from the estimate given in Lemma 6.2. □

DEFINITION. Let $\chi \in C^\infty(\boldsymbol{R})$ be a real-valued function such that for some $\lambda_0 > 0$, it satisfies: $\chi(\lambda)=1$ for $\lambda \geqq \lambda_0$; $\chi(\lambda)=0$ for $\lambda \leqq \lambda_0/2$. Then χ is called a low energy cut-off function.

We will need the following resolvent estimate, essentially due to Murata [**M**]:

PROPOSITION 6.4. *Suppose that V is real-valued, and for some $\rho > 2$ it satisfies $|V(x)| \leqq C\langle x \rangle^{-\rho}$, $x \in \boldsymbol{R}^d$. Let χ be a low energy cut-off function, and let $M > -\inf \sigma(H)$. Let $0 \leqq s < \rho - 1$ and $0 \leqq s_1 \leqq s$. Then*

$$\|(H+M)^{s_1/2} e^{-itH} \chi(H)\|_{B(L^{2,s}, L^{2,-s})} \leqq C\langle t \rangle^{-s+s_1}|t|^{-s_1}, \qquad t \neq 0. \tag{6.2}$$

PROOF. The result follows from [**M**: Theorem 3.3] and complex interpolation. □

REMARK. For $s_1=0$ the result is well-known and holds for a much larger class of potentials. The result (6.2) can be proved for a class of generalized Schrödinger operators with smooth long-range potentials by combining the estimates in [**J1, J3**]. In this case s in the right hand side of (6.2) should be replaced by s': $s'<s$.

We now suppose V satisfies the following two hypotheses:

(H1) V is real-valued and there exists $\rho>d$ such that $|V(x)|\leqq C\langle x\rangle^{-\rho}$, $x\in\boldsymbol{R}^d$;

(H2) There exist $s_0>d/2-1$, $s_1>d/2$ and $2\leqq p\leqq\infty$ such that $1/2-1/d<1/p<s_0/d$, and that for $M>-\inf\sigma(H)$ the operator $\langle x\rangle^{s_1}(H+M)^{-s_0/2}\cdot V\langle D\rangle^{s_0}$ extends to a bounded operator from $L^p(\boldsymbol{R}^d)$ to $L^2(\boldsymbol{R}^d)$.

The main result of this section is the following estimate for e^{-itH}:

THEOREM 6.5. *Assume $d\geqq 3$ and V satisfies* (H1) *and* (H2). *Let χ be a low energy cut-off function and let $s>d/2$. Then $e^{-itH}\chi(H)$ is bounded from $L^1(\boldsymbol{R}^d)$ to $L^{2,-s}(\boldsymbol{R}^d)$ for $t\neq 0$ and*

$$\|e^{-itH}\chi(H)\|_{B(L^1,L^{2,-s})}\leqq C|t|^{-d/2},\qquad t\neq 0.$$

PROOF. Let ρ, s_0, s_1 and p as in (H1) and (H2). We may assume

$$\frac{d}{2}-1<s_0\leqq\frac{d}{2}<s=s_1=\frac{\rho}{2}$$

without loss of generality. We start by writing:

$$e^{-itH}\chi(H)=\chi(H)e^{-itH_0}-i\int_0^t\chi(H)e^{-i(t-\tau)H}Ve^{-i\tau H_0}d\tau. \tag{6.3}$$

The first term is estimated by Lemmas 6.1, 6.2, and Theorem 1.1:

$$\begin{aligned}\|\chi(H)e^{-itH_0}\|_{B(L^1,L^{2,-s})}&\leqq C\|\chi(H)\|_{B(L^\infty)}\|e^{-itH_0}\|_{B(L^1,L^\infty)}\\&\leqq C|t|^{-d/2},\qquad t\neq 0.\end{aligned}$$

We write $F(t)=e^{-itH}\chi(H)$ and $U(t)=e^{-itH_0}$ for simplicity. By Lemma 6.3, Proposition 6.4 and (H2), we obtain

$$\begin{aligned}&\|F(t-\tau)VU(\tau)\|_{B(L^1,L^{2,-s})}\\&\quad\leqq\|\langle x\rangle^{-s}F(t-\tau)(H+M)^{s_0/2}\langle x\rangle^{-s}\|_{B(L^2)}\\&\qquad\times\|\langle x\rangle^{s}(H+M)^{-s_0/2}V\langle D\rangle^{s_0}\|_{B(L^p,L^2)}\|\langle D\rangle^{-s_0}U(\tau)\|_{B(L^1,L^p)}\\&\quad\leqq C\langle t-\tau\rangle^{-s+s_0}|t-\tau|^{-s_0}|\tau|^{-d(1/2-1/p)}\\&\quad\leqq C\min\{|t-\tau|^{-s_0},\ |t-\tau|^{-s}\}|\tau|^{-d(1/2-1/p)},\end{aligned}\tag{6.4}$$

where $M>-\inf\sigma(H)$. On the other hand, using Proposition 6.4, Lemma 6.2 and (H1) we also have

$$\begin{aligned}
&\|F(t-\tau)VU(\tau)\|_{B(L^1, L^{2,-s})} \\
&\quad \leqq \|F(t-\tau)\|_{B(L^{2,s}, L^{2,-s})}\|V\|_{B(L^\infty, L^{2,s})}\|U(\tau)\|_{B(L^1, L^\infty)} \\
&\quad \leqq C\langle t-\tau\rangle^{-s}|\tau|^{-d/2}. \qquad (6.5)
\end{aligned}$$

We now estimate the second term in the right hand side of (6.3). If $0<t\leqq 1$, it is estimated by

$$\begin{aligned}
&\int_0^t \|F(t-\tau)VU(\tau)\|_{B(L^1, L^{2-s})}d\tau \\
&\quad \leqq \int_0^{t/2} C|t-\tau|^{-s_0}|\tau|^{-d(1/2-1/p)}d\tau + \int_{t/2}^t C\langle t-\tau\rangle^{-s}|\tau|^{-d/2}d\tau \\
&\quad \leqq C|t|^{-s_0}+C|t|^{-d/2} \leqq C|t|^{-d/2},
\end{aligned}$$

where we have applied (6.4) to the first integral and (6.5) to the second. Note that $d(1/2-1/p)<1$ by (H2). If $t>1$ we divide the integral into three pieces:

$$\int_0^t \|F(t-\tau)VU(\tau)\|_{B(L^1, L^{2,-s})}d\tau = \int_0^{1/2} + \int_{1/2}^{t/2} + \int_{t/2}^t = I+II+III.$$

For I we apply (6.4):

$$\begin{aligned}
I &= \int_0^{1/2} \|F(t-\tau)VU(\tau)\|_{B(L^1, L^{2,-s})}d\tau \\
&\leqq C\int_0^{1/2} |t-\tau|^{-s}|\tau|^{-d(1/2-1/p)}d\tau \leqq Ct^{-s}\int_0^{1/2}|\tau|^{-d(1/2-1/p)}d\tau \\
&\leqq Ct^{-s} \leqq Ct^{-d/2}.
\end{aligned}$$

For II and III we apply (6.5) (note that we need $d\geqq 3$ in the last step for II):

$$\begin{aligned}
II &= \int_{1/2}^{t/2} \|F(t-\tau)VU(\tau)\|_{B(L^1, L^{2,-s})}d\tau \\
&\leqq C\int_{1/2}^{t/2} \langle t-\tau\rangle^{-s}|\tau|^{-d/2}d\tau \leqq Ct^{-s}\int_{1/2}^{\infty}|\tau|^{-d/2}d\tau \leqq Ct^{-d/2}; \\
III &= \int_{t/2}^t \|F(t-\tau)VU(\tau)\|_{B(L^1, L^{2,-s})}d\tau \leqq C\int_{t/2}^t \langle t-\tau\rangle^{-s}|\tau|^{-d/2}d\tau \\
&\leqq Ct^{-d/2}\int_{t/2}^t \langle t-\tau\rangle^{-s}d\tau = Ct^{-d/2}\int_0^{t/2}\langle \tau\rangle^{-s}d\tau \leqq Ct^{-d/2}.
\end{aligned}$$

These computations complete the proof. □

REMARK. Estimates of the form:

$$\|e^{-itH}\chi(H)\|_{B(L^1, L^\infty)} \leqq C|t|^{-d/2}, \qquad t\neq 0$$

were obtained in Journé-Sogge-Soffer [**JSS**] for a different class of potentials.

This result is stronger since $L^\infty(\boldsymbol{R}^d)$ is continuously embedded in $L^{2,-s}(\boldsymbol{R}^d)$, $s>d/2$. The proof given above is quite different from the one in [**JSS**].

§7. Mapping properties of the wave operators.

In this section we obtain several results on the mapping properties of the wave operators between L^p-spaces. These results are combined to give a proof of Theorem 1.5.

The first result is an immediate consequence of a result in Simon [**S**] and the intertwining property of the wave operators:

PROPOSITION 7.1. *Let V satisfies Assumption (A) and assume that the wave operators $W_\pm$ exist. Let $f\in C_0^\infty(\boldsymbol{R})$, and let $1\leqq p\leqq 2\leqq q\leqq\infty$. Then $W_\pm f(H_0)$ and $(W_\pm f(H_0))^*$ are bounded from $L^p(\boldsymbol{R}^d)$ to $L^q(\boldsymbol{R}^d)$.*

PROOF. Let $l>d/4$ and $M>-\inf\sigma(H)$. Then $(H_0+M)^{-l}$ and $(H+M)^{-l}$ are both bounded from $L^p(\boldsymbol{R}^d)$ to $L^2(\boldsymbol{R}^d)$, and from $L^2(\boldsymbol{R}^d)$ to $L^q(\boldsymbol{R}^d)$ ([**S**: Theorem B.2.1]). We write $g(\lambda)=(\lambda+M)^{2l}f(\lambda)\in C_0^\infty(\boldsymbol{R})$. Then by the interwining property, we have

$$W_\pm f(H_0)=(H+M)^{-l}(W_\pm g(H_0))(H_0+M)^{-l}$$

and hence it is bounded from $L^p(\boldsymbol{R}^d)$ to $L^q(\boldsymbol{R}^d)$. The statement for $(W_\pm f(H_0))^*$ is proved analogously. □

The next proposition is a consequence of Theorem 1.3 and Lemma 6.2:

PROPOSITION 7.2. *Assume $d\geqq 3$ and $V\in B(L^\infty, L^q)$ for some $q: 1\leqq q\leqq 2$, $1/q<1-1/d$. Assume moreover that $W_\pm$ exist. Then $W_\pm f(H_0)$ is bounded from $L^1(\boldsymbol{R}^d)$ to $L^q(\boldsymbol{R}^d)$ for any $f\in C_0^\infty(\boldsymbol{R})$.*

PROOF. We consider the $+$-case only. We choose $f_1\in C_0^\infty(\boldsymbol{R})$ so that $f_1(\lambda)f(\lambda)=f(\lambda)$. By the standard Cook's method we have

$$\begin{aligned}W_+f(H_0)&=f(H)W_+f_1(H_0)\\&=f(H)e^{iH}e^{-iH_0}f_1(H_0)+i\int_1^\infty f(H)e^{itH}Ve^{-itH_0}f_1(H_0)dt.\end{aligned}$$

The first term in the right hand side is bounded from $L^1(\boldsymbol{R}^d)$ to $L^1\cap L^\infty$ by Corollary 1.2. The integrand in the second term is estimated by Theorem 1.3 and Lemma 6.2: Let $d(1/q-1/2)<\beta<d/2-1$, then

$$\begin{aligned}&\|f(H)e^{itH}Ve^{-itH_0}f_1(H_0)\|_{B(L^1,L^q)}\\&\leqq\|e^{itH}f(H)\|_{B(L^q)}\|V\|_{B(L^\infty,L^q)}\|e^{-itH_0}f_1(H_0)\|_{B(L^1,L^\infty)}\\&\leqq C\langle t\rangle^\beta|t|^{-d/2}\leqq C|t|^{-d/2+\beta}.\end{aligned}$$

By the choice of β, $d/2-\beta>1$, and the integral is absolutely convergent in the $B(L^1, L^q)$ norm. The result follows from this. $\square$

PROPOSITION 7.3. *Assume* $d\geqq 3$, *and let* q *such that* $1\leqq q<2$ *and* $1/q<1-1/d$. *Assume* V *satisfies* (H1) *and* (H2), *and let* $f\in C_0^\infty(\boldsymbol{R}_+)$. *Then* $(W_\pm f(H_0))^*$ *is bounded from* $L^1(\boldsymbol{R}^d)$ *to* $L^q(\boldsymbol{R}^d)$.

PROOF. We first note that (H1) implies the existence of $W_\pm$. Without loss of generality, we may suppose f is real-valued. As in the proof of the previous proposition, we have

$$(W_+ f(H_0))^* = f(H_0)e^{iH_0}e^{-iH}f_1(H) - i\int_1^\infty f(H_0)e^{itH_0}Ve^{-itH}f_1(H)dt.$$

The first term is bounded from $L^1(\boldsymbol{R}^d)$ to $L^q(\boldsymbol{R}^d)$ as before. We estimate the integrand in the second term using Corollary 5.3 and Theorems 6.5: We set $s=\rho/2$ in (H1), then

$$\begin{aligned}
&\|f(H_0)e^{itH_0}Ve^{-itH}f_1(H)\|_{B(L^1, L^q)} \\
&\leqq \|e^{itH_0}f(H_0)\|_{B(L^q)}\|V\|_{B(L^{2,-s}, L^q)}\|e^{-itH}f_1(H)\|_{B(L^1, L^{2,-s})} \\
&\leqq C\langle t\rangle^{d(1/q-1/2)}\sup_x |\langle x\rangle^\rho V(x)|\cdot|t|^{-d/2} \leqq C|t|^{-d(1-1/q)}, \quad t\neq 0.
\end{aligned}$$

By the assumption on q, $d(1-1/q)>1$ and the integral converges absolutely in $B(L^1, L^q)$-norm. The assertion again follows from this. $\square$

REMARK. If we use results in [**JSS**], imposing their conditions on V and H, we can replace the assumption $f\in C_0^\infty(\boldsymbol{R}_+)$ by $f\in C_0^\infty(\boldsymbol{R})$.

PROOF OF THEOREM 1.5. We first note that Assumption (B) implies the existence of $W_\pm$. Hence if $1\leqq p\leqq 2\leqq q\leqq\infty$, it follows from Proposition 7.1 that $W_\pm f(H_0)$ and $(W_\pm f(H_0))^*$ are bounded from $L^p(\boldsymbol{R}^d)$ to $L^q(\boldsymbol{R}^d)$. Since Assumption (B) implies $V\in B(L^\infty, L^1\cap L^\infty)$, we can apply Proposition 7.2 to obtain $W_\pm f(H_0)\in B(L^1, L^q)$ if $1/2<1/q<1-1/d$.

We now apply Proposition 7.3 to prove $(W_\pm f(H_0))^*\in B(L^1, L^q)$. The condition (H1) follows immediately from Assumption (B). In verifying (H2), we take $p=2$, $s_1=\rho/2$, $s_0=[d/2]$ and use the differentiability of V to commute with differentiation of order less than s_0. Then we note that $(H+M)^{-s_0/2}\langle D\rangle^{s_0}$ extends to a bounded operator on $L^2(\boldsymbol{R}^d)$. Thus (H2) is satisfied, and the Proposition 7.3 implies the boundedness of $(W_\pm f(H_0))^*$ from $L^1(\boldsymbol{R}^d)$ to $L^q(\boldsymbol{R}^d)$ if $1/2<1/q<1-1/d$. Combining these results with a duality argument and the Riesz-Thorin interpolation theorem, we conclude the proof. $\square$

References

[AS] M. Aizenman and B. Simon, Brownian motion and Harnack's inequality for Schrödinger operators, Comm. Pure Appl. Math., **35** (1982), 209-273.

[BD] K. Boyadzhiev and R. Delaubenfels, Boundary values of holomorphic semigroups, Proc. Amer. Math. Soc., **118** (1993), 113-118.

[BTW] P. Brenner, V. Thomée and L. B. Wahlbin, Besov Spaces and Applications to Difference Methods for Initial Value Problems, Lecture Notes in Math., **434**, Springer, 1975.

[CFKS] H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry, 1987, Springer-Verlag.

[D] E. B. Davies, Heat Kernels and Spectral Theory, 1989, Cambridge University Press.

[FS] J. F. Fournier and J. Stewart, Amalgams of L^p and l^q, Bull. Amer. Math. Soc., **13** (1985), 1-21.

[GK] D. Gurarie and M. A. Kon, Radial bounds for perturbations of elliptic operators, J. Funct. Anal., **56** (1984), 99-123.

[HV] R. Hempel and J. Voigt, The spectrum of a Schrödinger operator in $L^p(\boldsymbol{R}^\nu)$ is p-independent, Comm. Math. Phys., **104** (1986), 243-250.

[J1] A. Jensen, Propagation estimates for Schrödinger type operators, Trans. Amer. Math. Soc., **291** (1985), 129-144.

[J2] A. Jensen, Commutator methods and a smoothing property of the Schrödinger evolution group, Math. Z., **191** (1986), 53-59.

[J3] A. Jensen, High energy resolvent estimates for generalized many-body Schrödinger operators, Publ. Res. Inst. Math. Sci., Kyoto Univ., **25** (1989), 155-167,

[JN] A. Jensen and S. Nakamura, Mapping properties of wave and scattering operators for two-body Schrödinger operators, Lett. Math. Phys., **24** (1992), 295-305.

[JN2] A. Jensen and S. Nakamura, Mapping properties of Schrödinger operators between L^p-spaces and Besov spaces, in Spectral and Scattering Theory and Applications, (ed. K. Yajima), Adv. Stud. Pure Math., **23** (1994), 187-209.

[JSS] J. L. Journé, A. Soffer and C. D. Sogge, Decay estimates for Schrödinger operators, Comm. Pure Appl. Math., **44** (1991), 573-604.

[M] M. Murata, High energy resolvent estimates, II, higher order elliptic operators, J. Math. Soc. Japan, **36** (1984), 1-10.

[P] M. M. H. Pang, Resolvent estimates for Schrödinger operators in $L^p(\boldsymbol{R}^N)$ and the theory of exponentially bounded C-semigroups, Semigroup Forum, **41** (1990), 97-114.

[RS] M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol. I-IV, 1972-1979, Academic Press, New York.

[S] B. Simon, Schrödinger semigroup, Bull. Amer. Math. Soc. (N.S.), **7** (1982), 447-526.

[Y] K. Yajima, The $W^{k,p}$-continuity of wave operators for Schrödinger operators, to appear in J. Math. Soc. Japan.

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