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Collapsing of quotient spaces of $SO(n)\backslash SL(n, \boldsymbol{R})$ at infinity

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Introduction.

Let Γ be a principal congruence subgroup of $SL(n, \boldsymbol{Z})$ of level $m\geqq 3$. Then $M=SO(n)\backslash SL(n, \boldsymbol{R})/\Gamma$ is a locally symmetric space with finite volume. The purpose of this paper is to study the end of M from the point of view of Riemannian geometry. We investigate the distribution of flat, totally geodesic, isometrically embedded submanifolds of M and determine the tangent cone of M at infinity. (The tangent cone at infinity of a metric space (Y, d) is the limit space $\lim_{t\to\infty}((Y, d/t), q)$ when it exists, where q is an arbitrary point of Y and the limit is taken in the sense of Gromov's Hausdorff distance ([**14**], [**15**]).)

Tits buildings are known to be very valuable for studying the geometry at infinity of manifolds of nonpositive curvature (see [**1**], [**6**]). In [**16**], we studied the case $n=3$ and used, instead of the Tits building itself, its quotient by Γ. The same method works in studying the general case.

Let $|T_\Gamma|$ be the quotient space of the geometric realization of the Tits building of $SL(n, \boldsymbol{Q})$ by Γ. This space is a union of a finite number of simplices corresponding to the Γ-conjugacy classes of proper parabolic $\boldsymbol{Q}$-subgroups of $G=SL(n, \boldsymbol{R})$. And we can label the simplices with subsets Θ's of a fundamental system Υ of the roots of G relative to A, where A is a maximal torus of G.

By means of the fundamental open set for Γ, we decompose M into a finite number of pieces $M_1, \cdots, M_\lambda$. These correspond bijectively to the maximal simplices $\triangle^1, \cdots, \triangle^\lambda$ of $|T_\Gamma|$. We can find a totally geodesic, isometrically embedded Euclidean sector S_i in M_i for each i, which is isometric to the closure of Weyl chamber in the Lie algebra $\mathfrak{a}_\mathfrak{p}$ of A. The manifold M is contained in a δ-neighborhood of the union of S_i for sufficiently large $\delta>0$. So $\bigcup_{i=1}^{\lambda} S_i$ is, as it were, a skeleton of M. The more we rescale and shrink the metric g of M, the more M becomes thin and resembles the union of S_i, which is almost the cone $C|T_\Gamma|$ of $|T_\Gamma|$.

On the other hand, there is a natural compactification $\overline{M}$ of M due to Borel-Serre ([4]). Its boundary $\partial\overline{M}$ is a disjoint union of faces $e'(Q)$ corresponding to the Γ-conjugacy classes of proper parabolic $\boldsymbol{Q}$-subgroups of G (or simplices in $|T_\Gamma|$). We can construct $\partial\overline{M}$ as follows. Let P be the group of upper triangular matrices in G. For each subset Θ of Υ, there exists a unique parabolic $\boldsymbol{Q}$-subgroup P_Θ containing P. Let $P_\Theta=M_\Theta A_\Theta N_\Theta$ be the standard Levi decomposition of P_Θ and $V'_\Theta=(K\cap M_\Theta N_\Theta)\backslash M_\Theta N_\Theta/(\Gamma\cap M_\Theta N_\Theta)$. This manifold V'_Θ has a fiber bundle structure, whose fiber is the nilmanifold $N_\Theta/(\Gamma\cap N_\Theta)$ and whose base space is the locally symmetric space $(K\cap M_\Theta)\backslash M_\Theta/(\Gamma\cap M_\Theta)$. We cut off the ends of V'_Θ and denote by V_Θ the resulting compact manifold. If a parabolic $\boldsymbol{Q}$-subgroup Q is conjugate to P_Θ, then the face $e'(Q)$ is the interior of $V_{\Upsilon-\Theta}$. We put $V_{\Upsilon-\Theta}$'s on the simplices of $|T_\Gamma|$ labelled with Θ, and paste them according to the face relation of $|T_\Gamma|$ to get the boundary $\partial\overline{M}$.

We extract a family $\{\gamma_y\}_{y\in|T_\Gamma|}$ of geodesic rays from $\bigcup_{i=1}^{\lambda}S_i$ which is in one-to-one correspondence with the set of the points of $|T_\Gamma|$. If a parabolic $\boldsymbol{Q}$-subgroup Q corresponds to a point $y\in|T_\Gamma|$ and Q is conjugate to P_Θ, then $\gamma_y([s,\infty))$ has a neighborhood U_y which is diffeomorphic to the product $V_{\Upsilon-\Theta}\times[s,\infty)$ for sufficiently large $s>0$. Let $U_{y,t}$ be the transversal section $V_{\Upsilon-\Theta}\times\{t\}$ of γ_y. We take a divergent sequence $\{p_l\}$ of points along γ_y and study the limit space $\lim_{l\to\infty}(M, p_l)$. Then we can see that the fiber $N_\Theta/(\Gamma\cap N_\Theta)$ of $V_{\Upsilon-\Theta}$ in $U_{y,t}$ shrinks as t goes to infinity (Proposition C of §4). The portion around γ_y collapses in different ways when y runs over $|T_\Gamma|$.

Our main results are as follows.

THEOREM A. *There exists a family $\{\gamma_y\}_{y\in|T_\Gamma|}$ of geodesic rays in M which corresponds bijectively to the points of $|T_\Gamma|$.*

REMARK. (1) When y ranges over the interior of a maximal simplex Δ^i of $|T_\Gamma|$, the interior of S_i is filled with $\gamma_y((0,\infty))$'s.

(2) Let us say that two geodesic rays $\gamma_1, \gamma_2:[0,\infty)\to M$ are equivalent if and only if there exists $C>0$ such that $d_M(\gamma_1(t),\gamma_2(t))\leqq C$ for all $t\geqq 0$. Then among the rays in $\{\gamma_y\}_{y\in|T_\Gamma|}$, γ_y and $\gamma_{y'}$ are not equivalent for $y\neq y'$. We state this and related matters in §7.

THEOREM B. *There exists a metric $d_{C|T_\Gamma|}$ on the cone $C|T_\Gamma|$ of $|T_\Gamma|$ and*

$$\lim_{t\to\infty}\left(\left(M, \frac{1}{t}g\right), q_0\right)=((C|T_\Gamma|, d_{C|T_\Gamma|}), O),$$

where O is the vertex of the cone and q_0 is the coset of the identity element $e\in G$.

REMARK. The cone $C|T_\Gamma|$ is also constructed by pasting a finite number of copies $B_1, \cdots, B_\lambda$ of the closure of a Weyl chamber in $\mathfrak{a}_\mathfrak{p}$. The restriction of the above metric $d_{C|T_\Gamma|}$ to each B_i coincides with the original Euclidean metric.

For $R>0$, let

$$B_R(q_0, M)=\{p\in M \mid d_M(p, q_0)\leqq R\},$$

$$\partial B_R(q_0, M)=\{p\in M \mid d_M(p, q_0)=R\},$$

and let V be a space obtained from $\partial\overline{X}$ as follows. For each Θ, we replace the collar neighborhood $\partial V_{Y-\Theta}\times[0, 1]$ of $V_{Y-\Theta}$ by $\partial V_{Y-\Theta}\times[-a, 1]$ with $a\gg 0$. And then collapse all the fibers $N_{Y-\Theta}/(\Gamma\cap N_{Y-\Theta})$'s over $\partial V_{Y-\Theta}\times[-a, -b]$ $(0<b<a)$ for each Θ.

From Theorem A, B and the study of the limit space $\lim_{t\to\infty}(M, \gamma_y(l))$, $\partial B_R(p_0, M)$ resembles V in the sense of Hausdorff distance d_H for sufficiently large R. Thus we can visualize the "collapsing" phenomenon of M by using the space $|T_\Gamma|$.

In this paper we suppose $n\geqq 4$. (Though our argument is valid for $n=2, 3$, the case $n=2$ is already well known (see [8]) and the case $n=3$ is studied in [**16**].)

The organization of this paper is as follows. In §1, we construct $|T_\Gamma|$ and associated family $\{\gamma_y\}_{y\in|T_\Gamma|}$ of geodesics in M. We also construct the cone $C|T_\Gamma|$ and give it a polyhedral metric $d_{C|T_\Gamma|}$. In §2, we calculate the Busemann function with respect to each γ_y, and prove Theorem A. In §3, we show the existence of isometrically embedded Euclidean spaces in M. In §4, we investigate the limit spaces along the ray γ_y. In §5, we decompose M into the pieces and study how they are pasted together. In §6, we construct an ε-pointed Hausdorff approximation from $((M, g/t), q_0)$ to $((C|T_\Gamma|, d_{C|T_\Gamma|}), O)$ and prove Theorem B. In §7, we study a certain equivalence relation between rays in M and related matters.

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Notation.

For a metric space (X, d), a point p of X, and a positive number D, we denote by $B_D(p, X)$ the set $\{q\in X \mid d(p, q)\leqq D\}$.

§1. Preliminaries.

We recall that the principal congruence subgroup Γ of level m is given by

$$\Gamma=\{g=(g_{ij})\in SL(n, \boldsymbol{Z}) \mid g_{ij}\equiv\delta_{ij} \bmod m\},$$

where δ_{ij} is the Kronecker's delta. Let $n \geqq 4$ and $|T_\Gamma|$ be the quotient space of the geometric realization of the Tits building of $SL(n, \boldsymbol{Q})$ by Γ. Then there exists a family $\{\gamma_y\}_{y \in |T_\Gamma|}$ of geodesics in $SO(n) \backslash SL(n, \boldsymbol{R})/\Gamma$ naturally associated with $|T_\Gamma|$. In this section, we construct $|T_\Gamma|$, $\{\gamma_y\}_{y \in |T_\Gamma|}$, the cone $C|T_\Gamma|$ of $|T_\Gamma|$, and the metric $d_{|T_\Gamma|}$ on $C|T_\Gamma|$.

1-1. Construction of $|T_\Gamma|$.

Let $G=SL(n, \boldsymbol{R})$, $K=SO(n)$. Let P be the group of upper triangular matrices in G, A the group of diagonal matrices in G with positive entries. For a subgroup H of G, we denote by $H_{\boldsymbol{Q}}$ (resp. $H_{\boldsymbol{Z}}$), the group $H \cap SL(n, \boldsymbol{Q})$ (resp. $H \cap SL(n, \boldsymbol{Z})$).

For each $i=1, \cdots, n-1$, we define the map $\theta_i: A \to \boldsymbol{R}^+$ as follows.

$$\theta_i(a) = \frac{a_i}{a_{i+1}} \quad \text{for } a=diag(a_1, \cdots, a_n) \in A . \tag{1-1-1}$$

We put $\Upsilon = \{\theta_1, \cdots, \theta_{n-1}\}$. For a subset $\Theta \subset \Upsilon$, we put

$$P_\Theta = \{g=(g_{ij}) \in G \mid g_{ij}=0 \text{ if } i>j \text{ and } \{\theta_j, \theta_{j+1}, \cdots, \theta_{i-1}\} \not\subset \Theta\}. \tag{1-1-2}$$

Notice that $P_\Upsilon = G$ and $P_\phi = P$. The P_Θ's are called the standard parabolic subgroups of G, and each proper parabolic $\boldsymbol{Q}$-subgroup Q of G is conjugate by some element of $SL(n, \boldsymbol{Q})$ to one of the P_Θ's with $\Theta \neq \Upsilon$.

Let $\mathcal{V}$ be the set of all (proper) maximal parabolic $\boldsymbol{Q}$-subgroups of G. And let Σ be the collection of finite subsets of $\mathcal{V}$ such that $\mathcal{S} \subset \mathcal{V}$ is an element of Σ if and only if $Q_{\mathcal{S}} := \bigcap_{Q \in \mathcal{S}} Q$ is a parabolic $\boldsymbol{Q}$-subgroup of G. We include the empty set in Σ. The pair $(\mathcal{V}, \Sigma)$ gives a simplicial complex T. Then T is a building and we call this the Tits building of $SL(n, \boldsymbol{Q})$ (see §5 of [18]). We denote by $|T|$ the geometric realization of T which is constructed as follows. Let $\boldsymbol{R}^{\mathcal{V}}$ be the set of all maps from $\mathcal{V}$ to $\boldsymbol{R}$. We identify $Q \in \mathcal{V}$ with the map $\varphi \in \boldsymbol{R}^{\mathcal{V}}$ defined by $\varphi(Q)=1$ and $\varphi(P)=0$ for $P \neq Q$. For each $\mathcal{S} = \{Q_1, \cdots, Q_l\} \in \Sigma$, we put $|\mathcal{S}| = \{\sum_{i=1}^l t_i Q_i \in \boldsymbol{R}^{\mathcal{V}} \mid 0 \leqq t_i, \sum_{i=1}^l t_i = 1\}$. Let $|T| = \bigcup_{\mathcal{S} \in \Sigma} |\mathcal{S}|$. We give $|\mathcal{S}|$ the topology as a subset of the finite dimensional vector space spanned by $Q_1, \cdots, Q_l$, and give $|T|$ the weak topology.

The group Γ acts on $\mathcal{V}$ by $Q \cdot g = g^{-1} Q g$ for $Q \in \mathcal{V}$ and $g \in \Gamma$. We also write Q^g instead of $g^{-1} Q g$. This action induces the action of Γ on T (and hence on $|T|$) in the obvious way.

Let us reconstruct the quotient space $|T_\Gamma|$ in a combinatorial way. We remark that each simplex of $|T|$ is either a maximal one or a boundary simplex of some maximal one. So it suffices to consider how Γ-equivalence classes of maximal simplices of $|T|$ are pasted together in $|T_\Gamma|$.

Notice that a maximal simplex of T is the set of maximal parabolic $\boldsymbol{Q}$-subgroups which contain a fixed minimal parabolic $\boldsymbol{Q}$-subgroup.

Let $z_1, \cdots, z_\lambda$ be a complete representative system of $P_{\boldsymbol{Q}}\backslash G_{\boldsymbol{Q}}/\Gamma$ (these double coset classes are known to be finite by Borel [2]). We can take $z_1, \cdots, z_\lambda$ in $G_{\boldsymbol{Z}}$ because $G=SL(n, \boldsymbol{R})$. We put $z_1=e$.

Then the Γ-conjugacy classes of minimal parabolic $\boldsymbol{Q}$-subgroups of G are represented by $P^{z_1}, \cdots, P^{z_\lambda}$. Hence $P^{z_1}, \cdots, P^{z_\lambda}$ correspond bijectively to the maximal simplices in $|T_\Gamma|$.

Let $\triangle=|v_1v_2\cdots v_{n-1}|$ be a simplex whose vertices are $v_1, v_2, \cdots, v_{n-1}$. For a non-empty subset $\Theta=\{\theta_{i_1}, \cdots, \theta_{i_q}\}\subset\Upsilon$; $1\leqq i_1<i_2<\cdots<i_q\leqq n-1$ we put $\triangle(\Theta)=|v_{i_1}\cdots v_{i_q}|$. This is one of the boundary simplices of $\triangle$.

Let us prepare λ copies of $\triangle$, and number them from 1 to λ; i.e., $\triangle^1, \cdots, \triangle^\lambda$. For $\rho=1, \cdots, \lambda$, let the vertices of $\triangle^\rho$ be $v_1^\rho, v_2^\rho, \cdots, v_{n-1}^\rho$; i.e., $\triangle^\rho=|v_1^\rho v_2^\rho\cdots v_{n-1}^\rho|$. We define the simplicial map φ_ρ: $\{v_1, \cdots, v_{n-1}\}\to\{v_1^\rho, \cdots, v_{n-1}^\rho\}$ by $\varphi_\rho(v_j)=v_j^\rho$ for $j=1, \cdots, n-1$, and denote by Φ_ρ the homeomorphism from $\triangle$ to $\triangle^\rho$ induced by φ_ρ. We also denote by $\triangle^\rho(\Theta)$ the boundary simplex $\Phi_\rho(\triangle(\Theta))$ of $\triangle^\rho$.

The simplex $\triangle^\rho(\Theta)$ corresponds to the simplex $\{Q\in\mathcal{V}\,|\,Q\supset(P_{\Upsilon-\Theta})^{z_\rho}\}$ in T. But the two simplices $\{Q\in\mathcal{V}\,|\,Q\supset(P_{\Upsilon-\Theta})^{z_\rho}\}$ and $\{Q\in\mathcal{V}\,|\,Q\supset(P_{\Upsilon-\Theta})^{z_\mu}\}$ might be Γ-equivalent. This is the case where there exists an element γ of Γ such that $(P_{\Upsilon-\Theta})^{z_\rho\gamma}=(P_{\Upsilon-\Theta})^{z_\mu}$. This condition is equivalent to the following: $(P_{\Upsilon-\Theta})_{\boldsymbol{Q}}z_\rho\Gamma=(P_{\Upsilon-\Theta})_{\boldsymbol{Q}}z_\mu\Gamma$.

Therefore we paste $\triangle^1, \cdots, \triangle^\lambda$ together in accordance with the condition below to get a space $|T_\Gamma|$.

(1-1-3) We paste $\triangle^\rho$ and $\triangle^\mu$ along $\triangle^\rho(\Theta)$ and $\triangle^\mu(\Theta)$ by the homeomorphism $\Phi_\mu\circ\Phi_\rho^{-1}|_{\triangle^\rho(\Theta)}$ if and only if $(P_{\Upsilon-\Theta})_{\boldsymbol{Q}}z_\rho\Gamma=(P_{\Upsilon-\Theta})_{\boldsymbol{Q}}z_\mu\Gamma$.

1-2. A family of geodesics.

Let $\tilde{M}=K\backslash G$ and $P(n, \boldsymbol{R})$ be the set of all positive definite, symmetric matrices contained in $SL(n, \boldsymbol{R})$. We identify $\tilde{M}$ with $P(n, \boldsymbol{R})$ in the usual way; i.e., $G=SL(n, \boldsymbol{R})$ operates transitively on $P(n, \boldsymbol{R})$ by conjugation $(x\cdot g={}^tgxg$ for $x\in P(n, \boldsymbol{R})$, $g\in G)$, and the isotropy group of $x_0=I_n=diag(1, \cdots, 1)\in P(n, \boldsymbol{R})$ is $K=SO(n)$. We give $\tilde{M}$ the canonical metric $\tilde{g}$ associated with the Killing form of the Lie algebra $\mathfrak{g}$ of G. (The Killing form of $\mathfrak{g}$ is given by $2n\cdot\operatorname{trace}(XY)$ for $X, Y\in\mathfrak{g}$.)

We remark that $x_0\cdot A$ is a totally geodesic, flat submanifold of $\tilde{M}$ which is isometric to the Euclidean space $\boldsymbol{R}^{n-1}$. We define unit speed geodesics in $x_0\cdot A$ which issue from x_0 as follows.

Let $\mathfrak{a}_\mathfrak{p}$ be the Lie algebra of A, i.e., $\mathfrak{a}_\mathfrak{p}=\{diag(\alpha_1, \cdots, \alpha_n)\,|\,\alpha_1, \cdots, \alpha_n\in\boldsymbol{R};\ \alpha_1+\cdots+\alpha_n=0\}$. For $\alpha=diag(\alpha_1, \cdots, \alpha_n)\in\mathfrak{a}_\mathfrak{p}$, we put

$$D=D(\alpha)=\|\alpha\|=2\sqrt{n\sum_{1\leq i\leq j\leq n-1}\alpha_i\alpha_j}\,.$$

If $\alpha\neq0$, the unit vector in the direction of α is

$$X = X(\alpha) = diag(\alpha_1/D, \cdots, \alpha_n/D)$$

and the unit speed geodesic $\tilde{\gamma}=\tilde{\gamma}(\alpha)\colon [0, \infty)\to\tilde{M}$ in the direction of α is defined by

$$\tilde{\gamma}(t) = x_0\cdot\exp tX = x_0\cdot diag(e^{\alpha_1 t/D}, \cdots, e^{\alpha_n t/D})$$
$$= diag(e^{2\alpha_1 t/D}, \cdots, e^{2\alpha_n t/D}) \qquad \text{for } t\geqq 0,$$

where $\exp\colon \mathfrak{a}_\mathfrak{p}\to A$ is the exponential map.

Let $d\theta_i\colon \mathfrak{a}_\mathfrak{p}\to \boldsymbol{R}$ be the differential of θ_i and put $\Lambda=\{d\theta_1, \cdots, d\theta_{n-1}\}$. Then Λ is a fundamental system of positive roots for the pair $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$. We put

$$\mathfrak{a}_\mathfrak{p}^+ = \{\alpha=diag(\alpha_1, \cdots, \alpha_n)\in\mathfrak{a}_\mathfrak{p} \mid \alpha\neq 0,\ \alpha_1\leqq\cdots\leqq\alpha_n\}.$$

Then $\mathfrak{a}_\mathfrak{p}^+\cup\{0\}$ is the closure of a Weyl chamber of $\alpha_\mathfrak{p}$.

Let $M=\tilde{M}/\Gamma$ and g be the metric of M induced by $\tilde{g}$. Let $\pi\colon \tilde{M}\to M$ be the projection. We put $\mathfrak{R}_\rho=\{\pi\circ(\tilde{\gamma}(\alpha)\cdot z_\rho) \mid \alpha\in\mathfrak{a}_\mathfrak{p}^+\}$.

Let us assign the point $\dfrac{\alpha_2-\alpha_1}{\alpha_n-\alpha_1}v_1^\rho+\dfrac{\alpha_3-\alpha_2}{\alpha_n-\alpha_1}v_2^\rho+\cdots+\dfrac{\alpha_n-\alpha_{n-1}}{\alpha_n-\alpha_1}v_{n-1}^\rho$ of $\triangle^\rho$ to the geodesic $\pi\circ(\tilde{\gamma}(\alpha)\cdot z_\rho)$ in $\mathfrak{R}_\rho$, where $\alpha=diag(\alpha, \cdots, \alpha_n)$. Then for each ρ we get a one to one correspondence of $\mathfrak{R}_\rho$ and the set of all points of $\triangle^\rho$. Since $|T_\Gamma|$ is obtained by pasting the faces of $\triangle^1, \cdots, \triangle^\lambda$ together a finite number of times, we obtain the required family $\{\gamma_y\}_{y\in|T_\Gamma|}$ of geodesics by deleting some geodesics from $\bigcup_{\rho=1}^\lambda \mathfrak{R}_\rho$.

1-3. Polyhedral metric on $C|T_\Gamma|$.

We identify the Lie algebra $\mathfrak{a}_\mathfrak{p}$ of A to the Euclidean space $\boldsymbol{R}^{n-1}$ by using the Killing form as the inner product. Let $\varphi_1\colon \boldsymbol{R}^{n-1}\cong\mathfrak{a}_\mathfrak{p}$ be the identification map. We define a diffeomorphism $\varphi_2\colon A\to x_0\cdot A(\subset\tilde{M})$ by $\varphi_2(a)=x_0\cdot a$ for $a\in A$. Define a map $\Psi\colon x_0\cdot A\to\boldsymbol{R}^{n-1}$ by $\Psi(x)=\varphi_1^{-1}\circ\log\circ\varphi_2^{-1}(x)$ for $x\in x_0\cdot A$, where log is the inverse map of exp.

We define unit speed geodesics $\tilde{\gamma}_i\colon [0, \infty)\to\tilde{M}$ $(i=1, \cdots, n-1)$ by

$$\tilde{\gamma}_i(t) = diag(e^{2(i-n)t/C_i}, \cdots, e^{2(i-n)t/C_i}, e^{2it/C_i}, \cdots, e^{2it/C_i}) \qquad \text{for } t\geqq 0,$$

where the $i-i$ (resp. $(i+1)-(i+1)$) entry of the matrix on the right side is equal to $e^{2(i-n)t/C_i}$ (resp. e^{2it/C_i}), and $C_i=\sqrt{2n}\sqrt{i(n-i)}$.

Let $A_1=\{a\in A \mid \theta_i(a)\leqq 1 \text{ for } i=1, \cdots, n-1\}$. Then A_1 is the image under the exponential map of the closure $\mathfrak{a}_\mathfrak{p}^+\cup\{0\}$ of a Weyl chamber. And the $\tilde{\gamma}_i$ are the geodesics corresponding to the edges of the Weyl chamber.

If we put $E_i=\Psi(\tilde{\gamma}_i([0, \infty)))$, then $E_1, \cdots, E_{n-1}$ are half-lines through the origin. Let B be the convex cone spanned by $E_1, \cdots, E_{n-1}$. Then $\Psi(x_0\cdot A_1)=\{(\pi\circ\tilde{\gamma}(\alpha))([0, \infty)) \mid \alpha\in\mathfrak{a}_\mathfrak{p}^+\}$ coincides with the cone B.

For a subset $\Theta\subset\Upsilon$, we define $B(\Theta)$ to be a face spanned by $\{E_k \mid \theta_k\in\Theta\}$.

This face $B(\Theta)$ is the cone over the simplex $\triangle(\Theta)$ defined in §1-1.

Let us prepare λ copies of B, and number them from 1 to λ; i.e., $B_1, \cdots, B_\lambda$. We denote by $B_\rho(\Theta)$ the face of B_ρ which corresponds to $B(\Theta)$, and by $\Psi_\rho: x_0\cdot A_1 \to B_\rho$ the diffeomorphism which corresponds to Ψ.

Now we get $C|T_\Gamma|$ by pasting $B_1, \cdots, B_\lambda$ together in accordance with the condition below.

(1-3-1) We paste B_ρ and B_μ together along $B_\rho(\Theta)$ and $B_\mu(\Theta)$ by the isometry $\Psi_\mu \circ \Psi_\rho^{-1}|_{B_\rho(\Theta)}$ if and only if

$$(P_{\Upsilon-\Theta})_Q z_\rho \Gamma = (P_{\Upsilon-\Theta})_Q z_\mu \Gamma .$$

Remark that $B_\rho(\Theta)$ is the cone over $\triangle^\rho(\Theta)$ for each non-empty subset Θ of Υ and $\rho=1, \cdots, \lambda$. We give B_ρ the induced metric d_ρ from $\boldsymbol{R}^{n-1}$ for each ρ, and let $d_{C|T_\Gamma|}$ be the metric whose restriction to B_ρ coincides with d_ρ. We call this metric $d_{C|T_\Gamma|}$ the polyhedral metric.

§ 2. Busemann functions and the proof of Theorem A.

In this section we use the following notations (cf. [9], [10]). We parametrize a geodesic by arc length unless otherwise mentioned. Two geodesics $\tilde{\gamma}$, $\tilde{\sigma}$ of $\tilde{M}$ are said to be equivalent if the function $d(\tilde{\gamma}(t), \tilde{\sigma}(t))$ is uniformly bounded on $[0, \infty)$. We denote by $\tilde{M}(\infty)$ the set of all equivalence classes of geodesics of $\tilde{M}$. The equivalence class represented by a geodesic $\tilde{\gamma}$ is denoted by $\tilde{\gamma}(\infty)$. For $p \in \tilde{M}$, $x \in \tilde{M}(\infty)$, we denote by $\tilde{\gamma}_{px}$ the unique geodesic such that $\tilde{\gamma}_{px}(0)=p$, $\tilde{\gamma}_{px}(\infty)=x$.

DEFINITION 2-1. For $x \in \tilde{M}(\infty)$, $p \in \tilde{M}$, we define the Busemann function $h_{x,p}$ to be $h_{x,p}(q)=\lim_{t\to\infty} \{d(q, \tilde{\gamma}_{px}(t))-t\}$ for all $q \in \tilde{M}$.

LEMMA 2-2 (Lemma 2.1 of [16]). *Let $\tilde{\gamma}: [0, \infty] \to \tilde{M}$ be any unit speed geodesic. Then $\pi\circ\tilde{\gamma}: [0, \infty) \to M$ is a ray if and only if $h_{\tilde{\gamma}(\infty),\tilde{\gamma}(0)}(\tilde{\gamma}(0)\cdot g) \geqq 0$ for all $g \in \Gamma$.*

Recall that a ray is a geodesic which realizes the distance between any two points on it.

For simplicity we denote $h_{\tilde{\gamma}(\alpha)(\infty),\tilde{\gamma}(\alpha)(0)}$ by $h(\alpha)$ for $\alpha \in \mathfrak{a}_\mathfrak{p}-\{0\}$. Let $N=\left\{\begin{pmatrix} 1 & & n_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in G\right\}$.

LEMMA 2-3. *Let $\alpha \in \mathfrak{a}_\mathfrak{p}^+$. Then the Busemann function $h(\alpha)$ is invariant under the action of N on $\tilde{M}$.*

PROOF. First we show that $h(\alpha)(x)$ is a continuous function of $\alpha \in \mathfrak{a}_\mathfrak{p}^+$ for a fixed $x\in\tilde{M}$. Let $h_s(\alpha)(x)=d_{\tilde{M}}(x, \tilde{\gamma}(\alpha)(s))-s$ for $s>0$. For each positive integer n, we put $\theta_n=\angle_{\tilde{\gamma}(\alpha)(n)}(x_0, x)$ and $l_n=d_{\tilde{M}}(x, \tilde{\gamma}(\alpha)(n))$. Then, from the Rauch comparison theorem ([7]), we have

$$\begin{aligned} 0 &\leqq h_n(\alpha)(x)-h_{n+s}(\alpha)(x) = d_{\tilde{M}}(x, \tilde{\gamma}(\alpha)(n))-d_{\tilde{M}}(x, \tilde{\gamma}(\alpha)(n+s))+s \\ &\leqq l_n-\sqrt{l_n^2+s^2+2l_n s\cos\theta_n}+s\,. \end{aligned}$$

Hence, $0\leqq h_n(\alpha)(x)-h(\alpha)(x)\leqq l_n(1-\cos\theta_n)$. Again by the comparison theorem, we have $\cos\theta_n\geqq n^2+l_n^2-l_0^2$, where $l_0=d_{\tilde{M}}(x_0, x)$. So we obtain

$$\begin{aligned} 0 &\leqq h_n(\alpha)(x)-h(\alpha)(x) \\ &\leqq l_n\left(1-\frac{n^2+l_n^2-l_0^2}{2nl_n}\right)=\frac{l_0^2-(l_n-n)^2}{2n} \\ &\leqq \frac{l_0^2}{2n}=\frac{1}{2n}\{d_{\tilde{M}}(x_0, x)\}^2. \end{aligned}$$

Thus the convergence $h_n(\alpha)(x)\to h(\alpha)(x)$ $(n\to\infty)$ is uniform on $\mathfrak{a}_{\mathfrak{p}}^+$ and the function $\alpha\mapsto h(\alpha)(x)$ is continuous on $\mathfrak{a}_{\mathfrak{p}}^+$. Therefore it suffices to show that $h(\alpha)(x\cdot g^{-1})=h(\alpha)(x)$ for all $g\in N$, $x\in\tilde{M}$ in the case where α lies in the interior $Int\,\mathfrak{a}_{\mathfrak{p}}^+$ of the Weyl chamber.

Suppose that $\alpha=diag(\alpha_1, \cdots, \alpha_n)$, i.e., $\alpha_1<\alpha_2<\cdots<\alpha_n$. Let $a_t=diag(e^{\alpha_1 t/D}, \cdots, e^{\alpha_n t/D})$. Then $\tilde{\gamma}(\alpha)(t)=x_0\cdot a_t$. If we put $g=\begin{pmatrix}1 & & g_{ij}\\ & \ddots & \\ 0 & & 1\end{pmatrix}\in N$, then

$$a_t g a_t^{-1}=\begin{pmatrix} 1 & e^{(\alpha_1-\alpha_2)t/D}g_{12} & e^{(\alpha_1-\alpha_3)t/D}g_{13} & \cdots & e^{(\alpha_1-\alpha_n)t/D}g_{1n} \\ & 1 & e^{(\alpha_2-\alpha_3)t/D}g_{23} & \cdots & e^{(\alpha_2-\alpha_n)t/D}g_{2n} \\ & & 1 & \cdots & e^{(\alpha_3-\alpha_n)t/D}g_{3n} \\ & & & \ddots & \vdots \\ & & & & 1 \end{pmatrix}.$$

Notice that

$$\begin{aligned} d_{\tilde{M}}(\tilde{\gamma}(\alpha)(t)\cdot g, \tilde{\gamma}(\alpha)(t)) &= d_{\tilde{M}}(x_0\cdot a_t g, x_0\cdot a_t) \\ &= d_{\tilde{M}}(x_0\cdot a_t g a_t^{-1}, x_0)\,. \end{aligned}$$

Since $\lim_{t\to\infty} a_t g a_t^{-1}=I_n$, we have

$$\lim_{t\to\infty} d_{\tilde{M}}(\tilde{\gamma}(\alpha)(t)\cdot g, \tilde{\gamma}(\alpha)(t))=0\,.$$

Therefore,

$$\begin{aligned} &|h(\alpha)(x\cdot g^{-1})-h(\alpha)(x)| \\ &= |\lim_{t\to\infty}\{d_{\tilde{M}}(\tilde{\gamma}(\alpha)(t)\cdot g, x)-t\}-\lim_{t\to\infty}\{d_{\tilde{M}}(\tilde{\gamma}(\alpha)(t), x)-t\}| \end{aligned}$$

$$= |\lim_{t\to\infty}\{d_{\tilde{M}}(\tilde{\gamma}(\alpha)(t)\cdot g,\ x)-d_{\tilde{M}}(\tilde{\gamma}(\alpha)(t),\ x)\}|$$
$$\leqq \lim_{t\to\infty}|d_{\tilde{M}}(\tilde{\gamma}(\alpha)(t)\cdot g,\ x)-d_{\tilde{M}}(\tilde{\gamma}(\alpha)(t),\ x)|$$
$$\leqq \lim_{t\to\infty} d_{\tilde{M}}(\tilde{\gamma}(\alpha)(t)\cdot g,\ \tilde{\gamma}(\alpha)(t))=0\,,$$

and $h(\alpha)(x\cdot g^{-1})=h(\alpha)(x)$ for all $x\in\tilde{M}$. □

For $x=(x_{ij})\in P(n,\ \boldsymbol{R})$, we denote by $\Delta_k(x)$, the $(k\times k)$-minor determinant $\det\begin{pmatrix} x_{11} & \cdots & x_{1k} \\ x_{21} & \cdots & x_{2k} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk}\end{pmatrix}$ in the top left corner.

LEMMA 2-4. *Let $\alpha=diag(\alpha_1,\ \cdots,\ \alpha_n)\in\mathfrak{a}_{\mathfrak{p}}^+$, and $x=(x_{ij})\in P(n,\ \boldsymbol{R})$. Then we have*

$$h(\alpha)(x)=C\log\left(\prod_{k=1}^{n-1}\Delta_k(x)^{\alpha_{k+1}-\alpha_k}\right)$$
$$=n\log\left(\prod_{k=1}^{n-1}(\Delta_k(x))^{-d\theta_k(\alpha/\|\alpha\|)}\right),$$

where

$$C=C(\alpha_1,\ \cdots,\ \alpha_n)=\frac{\sqrt{n}}{2}\frac{1}{\sqrt{\sum_{1\leq i\leq j\leq n-1}\alpha_i\alpha_j}}=\frac{n}{D}\,.$$

PROOF. First observe that we can calculate directly the Busemann functions on the Euclidean space $\boldsymbol{R}^{n-1}$.

Let $\tilde{\sigma}:[0,\ \infty)\to\boldsymbol{R}^{n-1}$ be a unit speed geodesic and $y=(y_1,\ \cdots,\ y_{n-1})\in\boldsymbol{R}^{n-1}$ be a point. We extend $\tilde{\sigma}$ in the opposite direction and get the geodesic $\tilde{\sigma}:(-\infty,\ \infty)\to\boldsymbol{R}^{n-1}$. Let $t_y\in\boldsymbol{R}$ be the (unique) number such that the line through y and $\tilde{\sigma}(t_y)$ is perpendicular to $\tilde{\sigma}(\boldsymbol{R})$. Then the Busemann function $h_{\tilde{\sigma}(\infty),\tilde{\sigma}(0)}$ is given by $h_{\tilde{\sigma}(\infty),\tilde{\sigma}(0)}(y)=-t_y$.

Since the totally geodesic submanifold $x_0\cdot A$ is isometric to $\boldsymbol{R}^{n-1}$ and $\tilde{\gamma}(\alpha)$ lies in $x_0\cdot A$, we compute $h(\alpha)$ on $x_0\cdot A$.

Notice that any element $x=diag(x_1,\ \cdots,\ x_n)\in x_0\cdot A$ can be (uniquely) written as $x=x_0\cdot(\exp\beta)$ with $\beta=((\log x_1)/2,\ \cdots,\ (\log x_n)/2)\in\mathfrak{a}_{\mathfrak{p}}$. Let $\tilde{\tau}:(-\infty,\ \infty)\to\mathfrak{a}_{\mathfrak{p}}$ be the geodesic given by $\tilde{\tau}(t)=t\cdot\alpha/\|\alpha\|$, and t_β be the unique number such that the line through β and $\tilde{\tau}(t_\beta)$ is perpendicular to $\tilde{\tau}(\boldsymbol{R})$. Then $(t_\beta\cdot\alpha/\|\alpha\|-\beta,\ \alpha)=0$, where $(\ ,\ )$ is the inner product given by the Killing form of $\mathfrak{a}_{\mathfrak{p}}$.

Recall that the inner product is given by $(X,\ Y)=2n\cdot\mathrm{trace}(XY)$ for $X,\ Y\in\mathfrak{a}_{\mathfrak{p}}$. Since $x_1\cdot x_2\cdot\ \cdots\ \cdot x_n=1$, we have

$$t_\beta = \frac{(\alpha, \beta)}{\|\alpha\|} = \frac{1}{D} \cdot 2n\Big(\sum_{k=1}^{n} \alpha_k \cdot \frac{\log x_k}{2}\Big)$$

$$= C\Big\{\sum_{k=1}^{n-1} \alpha_k \log x_k + \alpha_n\Big(-\sum_{i=1}^{n-1} \log x_i\Big)\Big\}$$

$$= C\Big\{\sum_{k=1}^{n-1} (\alpha_k - \alpha_n) \log x_k\Big\}.$$

Therefore we have

(2-4-1) $\quad h(\alpha)(x) = -C\Big\{\sum_{k=1}^{n-1} (\alpha_k - \alpha_n) \log x_k\Big\}$ for $x = diag(x_1, \cdots, x_n) \in x_0 \cdot A$.

Recall that $\tilde{M} = x_0 \cdot AN$.

So from Lemma 2-3, $h(\alpha)$ is the unique function which is N-invariant and satisfies the equation (2-4-1) on $x_0 \cdot A$.

Let

$$g = \begin{pmatrix} 1 & & g_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in N, \quad \begin{pmatrix} x_1' & & \\ & \ddots & \\ & & x_n' \end{pmatrix} \in x_0 \cdot A,$$

$x = (x_{ij}) \in P(n, \boldsymbol{R})$, and

$$x = diag(x_1', \cdots, x_n') \cdot g = {}^t g(diag(x_1', \cdots, x_n'))g.$$

Since

$$\begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots \\ x_{1k} & \cdots & x_{kk} \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ g_{12} & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ g_{1k} & g_{2k} & \cdots & g_{k-1,k} & 1 \end{pmatrix} \begin{pmatrix} x_1' & & \\ & \ddots & \\ & & x_k' \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & g_{12} & g_{13} & \cdots & g_{1k} \\ & 1 & g_{23} & \cdots & g_{2k} \\ & & 1 & & \vdots \\ & & & \ddots & g_{k-1,k} \\ & & & & 1 \end{pmatrix},$$

we have $\Delta_k(x) = \det \begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} \end{pmatrix} = x_1' \cdot \cdots \cdot x_k'$ and $x_{k+1}' = \Delta_{k+1}(x)/\Delta_k(x)$ for $k = 1, \cdots, n-1$. Therefore

$$h(\alpha)(x) = h(\alpha)(diag(x_1', \cdots, x_n'))$$

$$= -C\Big\{\sum_{k=1}^{n-1} (\alpha_k - \alpha_n) \log x_k'\Big\}$$

$$= -C \log \left\{ \Delta_1(x)^{\alpha_1 - \alpha_n} \times \prod_{k=2}^{n-1} \left(\frac{\Delta_k(x)}{\Delta_{k-1}(x)} \right)^{\alpha_k - \alpha_n} \right\}$$

$$= -C \log \left(\prod_{k=1}^{n-1} \Delta_k(x)^{\alpha_k - \alpha_{k+1}} \right)$$

$$= C \log \left(\prod_{k=1}^{n-1} \Delta_k(x)^{\alpha_{k+1} - \alpha_k} \right). \qquad \square$$

For each $n \times n$ matrix $x = (x_{ij})$ and each permutation σ of n letters, we denote by $x \cdot \sigma$, the matrix

$$\begin{pmatrix} x_{\sigma(1)\sigma(1)} & x_{\sigma(1)\sigma(2)} & \cdots & x_{\sigma(1)\sigma(n)} \\ x_{\sigma(2)\sigma(1)} & x_{\sigma(2)\sigma(2)} & \cdots & x_{\sigma(2)\sigma(n)} \\ \vdots & \vdots & & \vdots \\ x_{\sigma(n)\sigma(1)} & x_{\sigma(n)\sigma(2)} & \cdots & x_{\sigma(n)\sigma(n)} \end{pmatrix}.$$

LEMMA 2-5. *Let $\alpha = diag(\alpha_1, \cdots, \alpha_n) \in \mathfrak{a}_{\mathfrak{p}} - \{0\}$ and $x \in P(n, \boldsymbol{R})$. Take a permutation σ of n letters such that $\alpha_{\sigma(1)} \leqq \alpha_{\sigma(2)} \leqq \cdots \leqq \alpha_{\sigma(n)}$. Then the Busemann function $h(\alpha)$ is given by the following.*

$$h(\alpha)(x) = C \log \left\{ \prod_{k=1}^{n-1} (\Delta_k(x \cdot \sigma))^{\alpha_{\sigma(k+1)} - \alpha_{\sigma(k)}} \right\}$$

$$= n \log \left\{ \prod_{k=1}^{n-1} (\Delta_k(x \cdot \sigma))^{-d\theta_k((\alpha/\|\alpha\|) \cdot \sigma)} \right\}.$$

PROOF. Since $\tilde{\gamma}(\alpha) = \tilde{\gamma}(\alpha \cdot \sigma) \cdot \sigma^{-1}$, we have $h(\alpha)(x) = h(\alpha \cdot \sigma)(x \cdot \sigma)$. So, from Lemma 2-4,

$$h(\alpha)(x) = h(\alpha \cdot \sigma)(x \cdot \sigma)$$

$$= C \log \left\{ \prod_{k=1}^{n-1} (\Delta_k(x \cdot \sigma))^{\alpha_{\sigma(k+1)} - \alpha_{\sigma(k)}} \right\}.$$

$\square$

LEMMA 2-6. *$h(\alpha)(x_0 \cdot g) \geqq 0$ for all $\alpha \in \mathfrak{a}_{\mathfrak{p}} - \{0\}$ and $g \in SL(n, \boldsymbol{Z})$.*

PROOF. Let σ be a permutation such that $\alpha_{\sigma(1)} \leqq \cdots \leqq \alpha_{\sigma(n)}$. Recall that $x_0 \cdot g = {}^t g g$ is a positive definite symmetric matrix. So $(x_0 \cdot g) \cdot \sigma$ is also positive definite, symmetric and hence the minor determinants $\Delta_k((x_0 \cdot g) \cdot \sigma)$; $k = 1, \cdots, n-1$ are all positive. Notice that each $\Delta_k((x_0 \cdot g) \cdot \sigma)$ is also an integer. Therefore $\Delta_k((x_0 \cdot g) \cdot \sigma) \geqq 1$ for $k = 1, \cdots, n-1$, and the assertion follows immediately from Lemma 2-5. $\square$

COROLLARY 2-7. *$\pi \circ (\tilde{\gamma}(\alpha) \cdot g)$ is a ray in M for all $\alpha \in \mathfrak{a}_{\mathfrak{p}} - \{0\}$ and $g \in SL(n, \boldsymbol{Z})$.*

PROOF. Immediate from $\Gamma \subset SL(n, \boldsymbol{Z})$ and Lemma 2-2. $\square$

PROOF OF THEOREM A. Take the family $\{\gamma_y\}_{y\in|T_\Gamma|}$ of geodesics in M constructed in §1-2. It corresponds bijectively to the points of $|T_\Gamma|$. From Corollary 2-7, γ_y is a geodesic ray for each $y\in|T_\Gamma|$. We have thus proved the theorem. Q.E.D.

REMARK. As stated in [1], we can construct (another) geometric realization of T in the ideal boundary $\tilde{M}(\infty)$. By using this, we can state the relation between γ_y and $y\in|T_\Gamma|$ in Theorem A more clearly. Let

$$(x_0\cdot A_1 g)(\infty)=\{c(\infty)\,|\,c(t)=x_0\cdot(\exp t\alpha)g \quad \text{for } t\geqq 0,\ \alpha\in\mathfrak{a}_\mathfrak{p}^+\},$$

where $g\in G$. We put $|T|'=\bigcup_{g\in SL(n,\boldsymbol{Z})}(x_0\cdot A_1 g)(\infty)$ and give it the topology induced from the restriction of the Tits metric of $\tilde{M}(\infty)$ ([1]) to $|T|'$. We define a homeomorphism $\Phi\colon |T|'\to|T|$ as follows. Let $\alpha=diag(\alpha_1,\cdots,\alpha_n)\in\mathfrak{a}_\mathfrak{p}^+$, $g\in SL(n,\boldsymbol{Z})$, and $c(t)=x_0\cdot(\exp t\alpha)g$ for all $t\geqq 0$. Then $\Phi(c(\infty))=\dfrac{\alpha_2-\alpha_1}{\alpha_n-\alpha_1}(P_{\Upsilon-\{\theta_1\}})^g+\dfrac{\alpha_3-\alpha_2}{\alpha_n-\alpha_1}(P_{\Upsilon-\{\theta_2\}})^g+\cdots+\dfrac{\alpha_n-\alpha_{n-1}}{\alpha_n-\alpha_1}(P_{\Upsilon-\{\theta_{n-1}\}})^g$. We denote by Π the natural projection from $|T|$ to $|T_\Gamma|$.

The relation between γ_y and $y\in|T_\Gamma|$ is as follows. Let $\tilde{\gamma}_y$ be an arbitrary lifting of γ_y in $\tilde{M}$. Then we have $y=\Pi(\Phi(\tilde{\gamma}_y(\infty)))$.

§3. Isometrically embedded Euclidean spaces.

We show in this section that $\pi(x_0\cdot Az_\rho)$; $\rho=1,\cdots,\lambda$ are isometrically embedded Euclidean spaces $\boldsymbol{R}^{n-1}$ (Lemma 3-3). We use this result in §5, 6.

LEMMA 3-1. *The restriction of* $\pi\colon\tilde{M}\to M$ *to* $x_0\cdot Az$ *is injective for each* $z\in SL(n,\boldsymbol{Z})$.

PROOF. Assume that there exist $a, b\in A$ such that $\pi(x_0\cdot az)=\pi(x_0\cdot bz)$. Then $x_0\cdot azh=x_0\cdot bz$ for some $h\in\Gamma$. Let $g=zhz^{-1}\in\Gamma$, because Γ is a normal subgroup of $SL(n,\boldsymbol{Z})$. Then $(x_0\cdot a)\cdot g=x_0\cdot b$. So we have ${}^t g a^2 g=b^2$ and hence $(a^2gb^{-2})^t g=e$.

If we put $g=(g_{ij})$, $a^2=diag(p_1,\cdots,p_n)$, and $b^2=diag(q_1,\cdots,q_n)$, then $a^2gb^{-2}=((p_i/q_j)g_{ij})$. Therefore $(p_k/q_k)g_{kk}^2\leqq\sum_{j=1}^n(p_k/q_j)g_{kj}^2=1$ for $k=1,\cdots,n$. Because $g_{kk}\equiv 1 \pmod m$, we have $g_{kk}^2\geqq 1$ and $p_k/q_k\leqq 1/g_{kk}^2\leqq 1$. But $\prod_{k=1}^n p_k/q_k=1$, so $p_k/q_k=1$ for $k=1,\cdots,n$. Hence $a^2=b^2$. Since the entries in a and b are positive, it follows that $a=b$. □

LEMMA 3-2. $\pi\circ(\tilde{\gamma}(\alpha)\cdot az)$ *is a ray for any* $\alpha\in\mathfrak{a}_\mathfrak{p}-\{0\}$, $a\in A$ *and* $z\in SL(n,\boldsymbol{Z})$.

PROOF. We put $a=diag(p_1,\cdots,p_n)\in A$ and $\tilde{\gamma}'(t)=\tilde{\gamma}(\alpha)(t)\cdot az$. Since $d_{\tilde{M}}(\tilde{\gamma}'(t),x)=d_{\tilde{M}}(\tilde{\gamma}(\alpha)(t),x\cdot z^{-1}a^{-1})$, we have $h(x)=h(\alpha)(x\cdot z^{-1}a^{-1})$ for all $x\in\tilde{M}$, where we denote $h_{\tilde{\gamma}'(\infty), \tilde{\gamma}'(0)}$ by h. We study values of $h(x_0 \cdot azk)=h(\alpha)(x_0 \cdot azkz^{-1}a^{-1})$ for $k \in \Gamma$ (note $\tilde{\gamma}'(0)=x_0 \cdot az$).

Let $g=(g_{ij})=zkz^{-1} \in \Gamma$, because Γ is a normal subgroup of $SL(n, \boldsymbol{Z})$. We put $x=x_0 \cdot (aga^{-1})={}^t(aga^{-1})(aga^{-1})$. We have $aga^{-1}=((p_i/p_j)g_{ij})$.

Let σ be a permutation of n letters such that $\alpha_{\sigma(1)} \leqq \alpha_{\sigma(2)} \leqq \cdots \leqq \alpha_{\sigma(n)}$, and v_i be the i-th column vector of aga^{-1} for $i=1, \cdots, n$. Then

$$\Delta_k(x \cdot \sigma) = \det \begin{pmatrix} (v_{\sigma(1)}, v_{\sigma(1)}) & \cdots & (v_{\sigma(1)}, v_{\sigma(k)}) \\ \vdots & & \vdots \\ (v_{\sigma(k)}, v_{\sigma(1)}) & \cdots & (v_{\sigma(k)}, v_{\sigma(k)}) \end{pmatrix},$$

where (,) denotes the usual inner product of the Euclidean space $\boldsymbol{R}^n$. So we have

$$\Delta_k(x \cdot \sigma) = \sum_{1 \leqq \beta_1 < \cdots < \beta_k \leqq n} \left\{ \det \begin{pmatrix} \frac{p_{\beta_1}}{p_{\sigma(1)}} g_{\beta_1 \sigma(1)} & \cdots & \frac{p_{\beta_1}}{p_{\sigma(k)}} g_{\beta_1 \sigma(k)} \\ \vdots & & \vdots \\ \frac{p_{\beta_k}}{p_{\sigma(1)}} g_{\beta_k \sigma(1)} & \cdots & \frac{p_{\beta_k}}{p_{\sigma(k)}} g_{\beta_k \sigma(k)} \end{pmatrix} \right\}^2$$

$$\geqq \left\{ \det \left(\frac{p_{\sigma(i)}}{p_{\sigma(j)}} g_{\sigma(i)\sigma(j)} \right)_{1 \leqq i, j \leqq k} \right\}^2$$

$$= \left[\det \left\{ \begin{pmatrix} p_{\sigma(1)} & & \\ & \ddots & \\ & & p_{\sigma(k)} \end{pmatrix} \begin{pmatrix} g_{\sigma(1)\sigma(1)} & \cdots & g_{\sigma(1)\sigma(k)} \\ \vdots & & \vdots \\ g_{\sigma(k)\sigma(1)} & \cdots & g_{\sigma(k)\sigma(k)} \end{pmatrix} \right.\right.$$

$$\left.\left. \times \begin{pmatrix} p_{\sigma(1)} & & \\ & \ddots & \\ & & p_{\sigma(k)} \end{pmatrix}^{-1} \right\} \right]^2$$

$$= \left\{ \det \begin{pmatrix} g_{\sigma(1)\sigma(1)} & \cdots & g_{\sigma(1)\sigma(k)} \\ \vdots & & \vdots \\ g_{\sigma(k)\sigma(1)} & \cdots & g_{\sigma(k)\sigma(k)} \end{pmatrix} \right\}^2 .$$

Since $g \in \Gamma$, we have

$$\begin{pmatrix} g_{\sigma(1)\sigma(1)} & \cdots & g_{\sigma(1)\sigma(k)} \\ \vdots & & \vdots \\ g_{\sigma(k)\sigma(1)} & \cdots & g_{\sigma(k)\sigma(k)} \end{pmatrix} \equiv \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \mod m ,$$

and

$$\left\{ \det \begin{pmatrix} g_{\sigma(1)\sigma(1)} & \cdots & g_{\sigma(1)\sigma(k)} \\ \vdots & & \vdots \\ g_{\sigma(k)\sigma(1)} & \cdots & g_{\sigma(k)\sigma(k)} \end{pmatrix} \right\}^2 \equiv 1 \mod m .$$

Therefore $\Delta_k(x \cdot \sigma) \geqq 1$, and from Lemma 2-5, $h(x_0 \cdot azk)=h(\alpha)(x) \geqq 0$. Then]

Lemma 2-2 implies $\pi\circ(\tilde{\gamma}(\alpha)\cdot az)$ is a ray. □

LEMMA 3-3. *Let π_z be the restriction of $\pi:\tilde{M}\to M$ to $x_0\cdot Az$ for $z\in SL(n, Z)$. Then $\pi_z: x_0\cdot Az\to M$ is a globally isometric embedding.*

PROOF. Immediate from Lemmas 3-1 and 3-2. □

§4. Limit spaces along rays.

For each geodesic ray γ_y in Theorem A, we study the limit space $\lim_{l\to\infty}(M, \gamma_y(l))$. Our argument is similar to ones used in [**11**], [**12**]. Let $\alpha=diag(\alpha_1, \cdots, \alpha_n)\in\mathfrak{a}_{\mathfrak{p}}^{+}$, $\rho\in\{1, \cdots, \lambda\}$, and $\gamma_y=\pi\circ(\tilde{\gamma}(\alpha)\cdot z_\rho)$.

For $R>0$, we put

$$L(R)=\{g\in SL(n, \boldsymbol{R})\,|\,d_{\tilde{M}}(x_0, x_0\cdot g)<R\},$$

$$H_l(R)=\{e^{l\alpha}z_\rho k z_\rho^{-1}e^{-l\alpha}\,|\,k\in\Gamma,\ d_{\tilde{M}}(x_0\cdot e^{l\alpha}z_\rho k, x_0\cdot e^{l\alpha}z_\rho)<R\}.$$

Since Γ is a normal subgroup of $SL(n, \boldsymbol{Z})$, we can also write

$$H_l(R)=\{e^{l\alpha}k'e^{-l\alpha}\in L(R)\,|\,k'\in\Gamma\}.$$

We define a metric d on $L(R)$ by

$$d(g, g')=\sup\{d_{\tilde{M}}(x\cdot g, x\cdot g')\,|\,x\in B_R(x_0, \tilde{M})\}\qquad\text{for } g, g'\in L(R).$$

Then $(L(R), d)$ is a compact metric space and $H_l(R)$ is a closed subset of $L(R)$ for each positive integer l. We may assume, by taking a subsequence if necessary, that $H_l(R)$ converges to a subset $H(R)$ with respect to the Hausdorff distance in $L(R)$. For $R<R'$, there is a natural inclusion $I_{R,R'}^{l}: H_l(R)\to H_l(R')$ such that $I_{R,R'}^{l}(g)=g$ on $B_R(x_0, \tilde{M})$, and these maps induce an inclusion $I: H(R)\to H(R')$. We put $H=\bigcup_{R>0}H(R)$ and give it a compact open topology. It is easy to see that H is a closed subgroup of $SL(n, \boldsymbol{R})$. Therefore H is a Lie group.

Hence, we have immediately

$$\lim_{l\to\infty} d_{p.e.H}((\tilde{M}, H, x_0), (\tilde{M}, \Gamma, \tilde{\gamma}(\alpha)(l)\cdot z_\rho))=0\,,$$

where $d_{p.e.H}$ is the equivariant pointed Hausdorff distance (see §3 of [**13**]). So from Lemma 1-11 of [**11**], we have

$$\lim_{l\to\infty}(M, \pi(\tilde{\gamma}(\alpha)(l)\cdot z_\rho))=(\tilde{M}/H, \bar{x}_0)\,,$$

where $\bar{x}_0$ is the equivalence class of x_0.

It remains only to determine H. We need some preparation.

For a proper subset $\Theta\subset\Upsilon$ with $\Upsilon-\Theta=\{\theta_{i_1}, \theta_{i_2}, \cdots, \theta_{i_l}\}$ and $i_1<\cdots<i_l$, we define a sequence (of numbers) $\chi(\Theta)$ as follows: Let us line up the numbers

$i_1, \cdots, i_l$, and insert a vertical line between i_k and i_{k+1} if and only if $i_{k+1}\neq i_k+1$. We line up the lengths of the parts which are placed among the vertical lines and denote this sequence by $\chi(\Theta)$. For example, when $\Upsilon-\Theta=\{\theta_2, \theta_4, \theta_5, \theta_6, \theta_8, \theta_{10}\}$, we get $\chi(\Theta)=(1, 3, 1, 1)$ from the sequence $2|4\ 5\ 6|8|10$.

If $\Theta\neq\Upsilon$ and $\chi(\Theta)=(\omega_1, \cdots, \omega_u)$, we define a subgroup $M_{\Upsilon-\Theta}$ of $P_{\Upsilon-\Theta}$ by the following equation (4-1), and put

$$A_{\Upsilon-\Theta}=\{a\in A\,|\,\beta(a)=1 \text{ for all } \beta\in\Upsilon-\Theta\},$$

$$N_{\Upsilon-\Theta}=\{g=(g_{ij})\in N\,|\,g_{ij}=0 \text{ if } i<j \text{ and } \{\theta_i, \theta_{i+1}, \cdots, \theta_{j-1}\}\subset\Upsilon-\Theta\},$$

where N is the nilpotent group defined after Lemma 2-2 (see § 2).

$$\text{(4-1)}\quad M_{\Upsilon-\Theta}=\{diag\,(\varepsilon_1, \cdots, \varepsilon_{i_1-1}, \mathcal{F}_1, \varepsilon_{i_1+\omega_1+1}, \cdots, \varepsilon_{i_2-1}, \mathcal{F}_2, \varepsilon_{i_2+\omega_2+1}, \cdots, \varepsilon_n)$$
$$\in G\,|\,\mathcal{F}_j\in SL^{\pm}(\omega_j+1, \boldsymbol{R}) \text{ for } j=1, \cdots, u \text{ and } \varepsilon_i=\pm1\}.$$

In the case $\Theta=\Upsilon$, we put $A_\phi=A$, $N_\phi=N$, and $M_\phi={}^0M=\{diag\,(\varepsilon_1, \cdots, \varepsilon_n)\,|\,\varepsilon_j=\pm1 \text{ for } j=1, \cdots, n\}$.

The standard Levi decomposition of $P_{\Upsilon-\Theta}$ is given by $P_{\Upsilon-\Theta}=M_{\Upsilon-\Theta}A_{\Upsilon-\Theta}N_{\Upsilon-\Theta}$ for each subset $\Theta\subset\Upsilon$ (see [**3**]).

LEMMA 4-2. *Let $\alpha=diag\,(\alpha_1, \cdots, \alpha_n)\in\mathfrak{a}_{\mathfrak{p}}^+$, $\Upsilon-\Theta=\{\theta_k\,|\,\alpha_k=\alpha_{k+1}\}$, and H as above. Then $H=N_{\Upsilon-\Theta}(M_{\Upsilon-\Theta}\cap\Gamma)$.*

PROOF. First we notice that for each positive number R there exists a positive number $C(R)$ such that if $g=(g_{ij})\in L(R)$, then $|g_{ij}|\leqq C(R)$ for all i, j.

Step 1. $H\subset(M_{\Upsilon-\Theta}\cap\Gamma)N_{\Upsilon-\Theta}$.

Let $g\in H$ be given. Then there exists a positive number $R>0$ such that $g\in H(R)$. For simplicity, we can assume the following: there exists a sequence $\{k_l\}_{l=1}^\infty$ of elements of Γ such that $g_l=e^{l\alpha}k_le^{-l\alpha}\in H_l(R)\subset L(R)$ and $\lim_{l\to\infty}g_l=g$.

We denote by $g_{l,ij}$ (resp. $k_{l,ij}$) the $i-j$ entry of g_l (resp. k_l). Then we have $g_{l,ij}=e^{l(\alpha_i-\alpha_j)}k_{l,ij}$. We remark that each $k_{l,ij}$ is an integer and that $|g_{l,ij}|\leqq C(R)$. If $i>j$ and $\alpha_i>\alpha_j$ (in other words, if $i>j$ and $\{\theta_j, \theta_{j+1}, \cdots, \theta_{i-1}\}\not\subset\Upsilon-\Theta$), we have $k_{l,ij}=0$ and hence $g_{l,ij}=0$ for sufficiently large l. So hereafter we assume $g_l\in P_{\Upsilon-\Theta}$.

We decompose g_l as

$$g_l=h_ln_l\,;\ h_l\in M_{\Upsilon-\Theta}A_{\Upsilon-\Theta},\ n_l\in N_{\Upsilon-\Theta}\,.$$

We denote by $h_{l,ij}$ (resp. $n_{l,ij}$) the $i-j$ entry of h_l (resp. n_l). We remark that if $\alpha_i=\alpha_j$, then one of the following three conditions is satisfied.

$$\text{(4-2-a)}\qquad i=j\,,$$

$$\text{(4-2-b)}\qquad i>j \text{ and } \{\theta_j, \theta_{j+1}, \cdots, \theta_{i-1}\}\subset\Upsilon-\Theta\,,$$

(4-2-c) $$i<j \text{ and } \{\theta_i, \theta_{i+1}, \cdots, \theta_{j-1}\} \subset \Upsilon-\Theta .$$

Let us investigate the entries of h_l. If the pair (i, j) does not satisfy any of the conditions (4-2-a)$\sim$(4-2-c), we have $h_{l,ij}=0$. If the pair (i, j) satisfies one of the above three conditions, we have $h_{l,ij}=g_{l,ij}=k_{l,ij}$. Therefore $h_l \in \Gamma$, because Γ is the principal congruence subgroup. Moreover we have $h_l \in M_{\Upsilon-\Theta}$ (that is, the $A_{\Upsilon-\Theta}$-factor of h_l is the identity matrix). We obtain $h_l \in M_{\Upsilon-\Theta} \cap \Gamma$, $g_l \in (M_{\Upsilon-\Theta} \cap \Gamma) N_{\Upsilon-\Theta}$ and hence $g \in (M_{\Upsilon-\Theta} \cap \Gamma) N_{\Upsilon-\Theta}$.

Step 2. $(M_{\Upsilon-\Theta} \cap \Gamma) N_{\Upsilon-\Theta} \subset H$.

Let $g=(g_{ij})=hn$; $h \in M_{\Upsilon-\Theta} \cap \Gamma$, $n \in N_{\Upsilon-\Theta}$ be given. We denote by h_{ij} (resp. n_{ij}) the $i-j$ entry of h (resp. n).

Let $g_l'=(g_{l,ij}')=e^{-l\alpha} g e^{l\alpha} \in P_{\Upsilon-\Theta}$. Then we have $g_{l,ij}'=e^{l(\alpha_j-\alpha_i)} g_{ij}$. We can decompose g_l' as $g_l'=hn_l'$; $n_l'=(n_{l,ij}') \in N_{\Upsilon-\Theta}$, $n_{l,ij}'=e^{l(\alpha_j-\alpha_i)} n_{ij}$. We take an element $n_l=(n_{l,ij})$ of Γ such that $|n_{l,ij}-n_{l,ij}'| \leqq m$ for $i, j=1, \cdots, n$. We put $k_l=(k_{l,ij})=hn_l \in \Gamma$ and $g_l=(g_{l,ij})=e^{l\alpha} k_l e^{-l\alpha} \in P_{\Upsilon-\Theta}$.

If $\alpha_i>\alpha_j$ (resp. $\alpha_i=\alpha_j$), then we have $g_{ij}=g_{l,ij}=0$ (resp. $g_{l,ij}=g_{ij}=h_{ij}$). If $\alpha_i<\alpha_j$, then we have

$$\begin{aligned} |g_{ij}-g_{l,ij}| &= e^{l(\alpha_i-\alpha_j)} \left| \sum_{s=1}^{n} h_{is} n_{l,sj}' - \sum_{s=1}^{n} h_{is} n_{l,sj} \right| \\ &\leqq e^{l(\alpha_i-\alpha_j)} \left(\sum_{s=1}^{n} |h_{is}| \, |n_{l,sj}'-n_{l,sj}| \right) \\ &\leqq e^{l(\alpha_i-\alpha_j)} C'mn , \end{aligned}$$

where $C'=\max_{i,j=1,\cdots,n} |h_{ij}|$. Hence $\lim_{l\to\infty} g_l=g$.

Let $\mathcal{S}=\{g'=(g_{ij}') \in G \mid |g_{ij}'-g_{ij}| \leqq C'mn \text{ for } i, j=1, \cdots, n\}$. Notice that $g_l= e^{l\alpha} k_l e^{-l\alpha} \in \mathcal{S}$. Since $\mathcal{S}$ is compact, there exists a positive number R such that $\mathcal{S} \subset L(R)$. Hence $g_l \in H_l(R) \subset L(R)$ and $g \in H(R)$. Therefore $g \in H$.

Step 3. The Levi subgroup $M_{\Upsilon-\Theta} A_{\Upsilon-\Theta}$ normalizes the unipotent radical $N_{\Upsilon-\Theta}$ of $P_{\Upsilon-\Theta}$ ([**3**]). Therefore, from Step 1 and Step 2, we have

$$H=(M_{\Upsilon-\Theta} \cap \Gamma) N_{\Upsilon-\Theta} = N_{\Upsilon-\Theta} (M_{\Upsilon-\Theta} \cap \Gamma) .$$

□

Since $\tilde{M}=x_0 \cdot M_{\Upsilon-\Theta} A_{\Upsilon-\Theta} N_{\Upsilon-\Theta}$, the limit space $\lim_{l\to\infty}(M, \pi(\tilde{\gamma}(\alpha)(l) \cdot z_\rho))$ is diffeomorphic to $((K \cap M_{\Upsilon-\Theta}) \backslash M_{\Upsilon-\Theta} / (\Gamma \cap M_{\Upsilon-\Theta})) \times A_{\Upsilon-\Theta}$, we have thus proved the following.

PROPOSITION C. *Let* $\alpha=diag(\alpha_1, \cdots, \alpha_n) \in \mathfrak{a}_{\mathfrak{p}}^+$, $\rho \in \{1, \cdots, \lambda\}$, *and* $\gamma_y=\pi \circ (\tilde{\gamma}(\alpha) \cdot z_\rho)$. *Suppose that y is an interior point of* $\triangle^\rho(\Theta)$, *that is* $\Upsilon-\Theta=\{\theta_k \mid \alpha_k= \alpha_{k+1}\}$.

(1) *If* $\Theta=\Upsilon$ (*y is an interior point of* $\triangle^\rho$), *then the limit space* $\lim_{l\to\infty}(M, \gamma_y(l))$ *is diffeomorphic to the Euclidean space* $\boldsymbol{R}^{n-1}$.

(2) *If* $\Theta \neq \Upsilon$ *(*y *is an interior point of the boundary simplex* $\triangle^{\rho}(\Theta)$ *of* $\triangle^{\rho}$*), and* $\chi(\Theta)=(\omega_1, \cdots, \omega_u)$*, then the limit space* $\lim_{l\to\infty}(M, \gamma_y(l))$ *is diffeomorphic to* $\mathcal{M}_{\omega_1+1}\times \cdots \times \mathcal{M}_{\omega_u+1}\times \boldsymbol{R}^{\#\Theta}$*, where* $\#\Theta=n-1-(\omega_1+\cdots+\omega_u)$ *is the cardinality of* Θ*,* $\mathcal{M}_{\omega_j+1}=SO(\omega_j+1)\backslash SL(\omega_j+1, \boldsymbol{R})/\Gamma(\omega_j+1; m)$ *for* $j=1, \cdots, u$*, and* $\Gamma(\omega_j+1; m) \subset SL(\omega_j+1, \boldsymbol{Z})$ *is the principal congruence subgroup of level* m*.*

§ 5. Decomposition of M.

5-1. Fundamental open set.

Let ${}^0M=\{diag(\varepsilon_1, \cdots, \varepsilon_n)\in G \mid \varepsilon_j=\pm 1 \text{ for } j=1, \cdots, n\}$, $N=\left\{\begin{pmatrix}1 & & n_{ij}\\ & \ddots & \\ 0 & & 1\end{pmatrix}\in G\right\}$ as in § 4, and put $P^0={}^0MN$.

DEFINITION 5-1-1. For a map $\underline{t}:\Upsilon\to\boldsymbol{R}^+$, we denote by $A_{\underline{t}}$ the set $\{a\in A \mid \beta(a)\leqq \underline{t}(\beta)$ for all $\beta\in\Upsilon\}$, and for a relatively compact open subset $\eta\subset P^0$ and $\underline{t}$, we call the set $S_{\underline{t}\eta}=K\cdot A_{\underline{t}}\cdot\eta$ a Siegel-domain.

The next fact is a special case of Borel's theorem ([**2**]).

THEOREM 5-1-2 (Borel). (1) *If* $\{g_1, \cdots, g_\lambda\}$ *is a complete representative system in* $G_{\boldsymbol{Q}}$ *for the double coset classes* $P_{\boldsymbol{Q}}\backslash G_{\boldsymbol{Q}}/\Gamma$*, then there exists a relatively compact open subset* $\eta_1\subset P^0$ *and a map* $\underline{t}_1:\Upsilon\to\boldsymbol{R}^+$ *which satisfy the following condition: If* $\eta\subset P^0$ *contains* η_1 *and* $\underline{t}:\Upsilon\to\boldsymbol{R}^+$ *is a map such that* $\underline{t}(\beta)\geqq\underline{t}_1(\beta)$ *for all* $\beta\in\Upsilon$ *then* $\bigcup_{i=1}^{\lambda} S_{\underline{t}\eta}g_i\Gamma=G$*.*

(2) *For any relatively compact subset* η *in* P^0*, any map* $\underline{t}:\Upsilon\to\boldsymbol{R}^+$ *and any pair* $g, g'\in G_{\boldsymbol{Q}}$*, the set* $\{\gamma\in\Gamma \mid S_{\underline{t}\eta}g\gamma\cap S_{\underline{t}\eta}g'\neq\phi\}$ *is finite.*

As mentioned in § 1-1, we take $\{z_1, \cdots, z_\lambda\}\subset G_{\boldsymbol{Z}}$ as a complete representative system for $P_{\boldsymbol{Q}}\backslash G_{\boldsymbol{Q}}/\Gamma$, and put $z_1=e$.

DEFINITION 5-1-3. For any $t>0$, we put

$$A_t=\{a\in A \mid \beta(a)\leqq t \text{ for all } \beta\in\Upsilon\}, \quad \text{and}$$
$$\mathring{A}_t=\{a\in A \mid \beta(a)< t \text{ for all } \beta\in\Upsilon\}.$$

From Theorem 5-1-2, we can take a number $t_0>1$ and a relatively compact open subset ω of N suitably, such that

$$M=\bigcup_{i=1}^{\lambda}\pi(x_0\cdot\mathring{A}_{t_0}\omega z_i),$$

where $\pi:\tilde{M}\to M$ is the projection and $x_0=I_n=diag(1, \cdots, 1)$ as in § 1-2.

Furthermore we can choose ω sufficiently large so that it contains a fundamental domain of $N\cap\Gamma$ in N.

DEFINITION 5-1-4. For each subset $\Theta\subset\Upsilon$, we put

$$A_1(\Theta) = \{a \in A_1 \mid \beta(a)=1 \text{ for all } \beta \in \Upsilon - \Theta\}.$$

REMARK. Recall that the inverse image under the isometry $\Psi: x_0 \cdot A \to \boldsymbol{R}^{n-1}$ of the cone $B \subset \boldsymbol{R}^{n-1}$ is $x_0 \cdot A_1$. We notice that $\Psi^{-1}(B(\Theta)) = x_0 \cdot A_1(\Theta)$.

Let $M_i = \pi(x_0 \cdot \mathring{A}_{t_0} \omega z_i)$, $S_i = \pi(x_0 \cdot A_1 z_i)$, and $S_i(\Theta) = \pi(x_0 \cdot A_1(\Theta) z_i)$ for each i, Θ. We decompose M into the pieces $M_1, \cdots, M_\lambda$. Lemma 3-3 implies that S_i is a totally geodesic, isometrically embedded submanifold (of M) contained in M_i. And $S_i(\Theta)$ is a part of the boundary of the submanifold S_i.

In the next two (sub)sections, we examine the correspondence of the way of pasting the pieces $M_1, \cdots, M_\lambda$ together to the one of pasting $B_1, \cdots, B_\lambda$ together.

5-2. In this (sub)section we see the following: if B_i and B_j are pasted together in $C|T_\Gamma|$ along $B_i(\Theta)$ and $B_j(\Theta)$, then $S_i(\Theta)$ ($\subset M_i$) is near to $S_j(\Theta)$ ($\subset M_j$).

PROPOSITION 5-2-1. *For a subset $\Theta \subset \Upsilon$, the following two conditions are equivalent.*

(5-2-1-a) $(P_{\Upsilon-\Theta})_Q z_i \Gamma = (P_{\Upsilon-\Theta})_Q z_j \Gamma$.

(5-2-1-b) *There exists $h \in \Gamma$ such that $\tilde{\gamma}_k(\infty) \cdot z_i = \tilde{\gamma}_k(\infty) \cdot z_j h$ for all k with $\theta_k \in \Theta$.*

PROOF. For $x \in \tilde{M}(\infty)$, we put $G_x = \{g \in G \mid x \cdot g = x\}$. Since $P_{\Upsilon-\Theta} = \bigcap_{\theta_k \in \Theta} P_{\Upsilon-\{\theta_k\}}$, we have only to verify that

$$G_{\tilde{\gamma}_k(\infty)} = P_{\Upsilon-\{\theta_k\}} \qquad \text{for } k=1, \cdots, n-1.$$

These are immediate consequences of the following proposition.

PROPOSITION 5-2-2 (Eberlein [**9**]). *Let $\mathfrak{p}$ be the orthogonal complement of the Lie algebra of K in $\mathfrak{g}$ with respect to the Killing form. (The complement $\mathfrak{p}$ consists of symmetric matrices in $\mathfrak{g}$.) Let $X \in \mathfrak{p}$, and $\tilde{\gamma}(t) = x_0 \cdot e^{tX}$ be a unit speed geodesic in $\tilde{M}$. Then an element $g \in G$ lies in $G_{\tilde{\gamma}(\infty)}$ if and only if $\lim_{t\to\infty} e^{tX} g e^{-tX}$ exists in G.*

We define $h_{\Theta,i,j}$ for each triple (Θ, i, j) as follows, where Θ is a nonempty subset of Υ and $i \neq j$. If there exists $h \in \Gamma$ which satisfies

(5-2-3) $$\tilde{\gamma}_k(\infty) \cdot z_i = \tilde{\gamma}_k(\infty) \cdot z_j h \qquad \text{for all } k \text{ with } \theta_k \in \Theta,$$

we choose such an h arbitrarily and denote it by $h_{\Theta,i,j}$. If there is no element $h \in \Gamma$ which satisfies the relation (5-2-3), we put $h_{\Theta,i,j} = e$.

Let L_3 be as follows.

(5-2-4) $$L_3 = \max\{d_{\tilde{M}}(x_0 \cdot z_i, x_0 \cdot z_j h_{\Theta,i,j}) \mid \phi \neq \Theta \subset \Upsilon;\ i, j=1, \cdots, \lambda;\ i \neq j\}.$$

Then the Hausdorff distance between $S_i(\Theta)$ and $S_j(\Theta)$ in M is not greater than L_3 when $B_i(\Theta)$ and $B_j(\Theta)$ are pasted together in $C|T_\Gamma|$. More precisely, we have the following.

LEMMA 5-2-5. *Let Θ be a nonempty subset of Υ, $(P_{\Upsilon-\Theta})_Q z_i\Gamma=(P_{\Upsilon-\Theta})_Q z_j\Gamma$; $i\neq j$, and $a\in A_1$.*

If $a\in A_1(\Theta)=\{b\in A_1|\beta(a)=1$ for all $\beta\in\Upsilon-\Theta\}$, then we have $d_M(\pi(x_0\cdot az_i), \pi(x_0\cdot az_j))\leqq L_3$.

PROOF. From Proposition 5-2-1, we have $\tilde{\gamma}_k(\infty)\cdot z_i=\tilde{\gamma}_k(\infty)\cdot z_j h_{\Theta,i,j}$ for all k with $\theta_k\in\Theta$.

So,

$$z_i h_{\Theta,i,j}^{-1} z_j^{-1}\in\bigcap_{\theta_k\in\Theta} G_{\tilde{\gamma}_k(\infty)}=\bigcap_{\theta_k\in\Theta} P_{\Upsilon-\{\theta_k\}}=P_{\Upsilon-\Theta}\,.$$

We take the unit vector $X\in\mathfrak{a}_\mathfrak{p}$ and $l>0$ such that $a=e^{lX}$, and define a unit speed geodesic $\tilde{\gamma}:[0, \infty)\to\tilde{M}$ by $\tilde{\gamma}(t)=x_0\cdot e^{tX}$ for all $t\geqq 0$. Since $G_{\tilde{\gamma}(\infty)}\supset P_{\Upsilon-\Theta}$, we have $\tilde{\gamma}(\infty)\cdot z_i h_{\Theta,i,j}^{-1} z_j^{-1}=\tilde{\gamma}(\infty)$ and the function

$$t\longmapsto d_{\tilde{M}}(\tilde{\gamma}(t), \tilde{\gamma}(t)\cdot z_i h_{\Theta,i,j}^{-1} z_j^{-1})=d_{\tilde{M}}(\tilde{\gamma}(t)\cdot z_i, \tilde{\gamma}(t)\cdot z_j h_{\Theta,i,j})$$

is monotone decreasing on $[0, \infty)$. Therefore, we have

$$\begin{aligned} &d_M(\pi(x_0\cdot az_i), \pi(x_0\cdot az_j))\\ &\leqq d_{\tilde{M}}(x_0\cdot az_i, x_0\cdot az_j h_{\Theta,i,j})=d_{\tilde{M}}(\tilde{\gamma}(l)\cdot z_i, \tilde{\gamma}(l)\cdot z_j h_{\Theta,i,j})\\ &\leqq d_{\tilde{M}}(\tilde{\gamma}(0)\cdot z_i, \tilde{\gamma}(0)\cdot z_j h_{\Theta,i,j})=d_{\tilde{M}}(x_0\cdot z_i, x_0\cdot z_j h_{\Theta,i,j})\leqq L_3\,. \end{aligned}$$

□

5-3. We study how the $M_i=\pi(x_0\cdot\mathring{A}_{t_0}\omega z_i)$; $i=1, \cdots, \lambda$ are pasted together.

We define six (positive) constants L_1, L_2, L_4, L_5, t_1, L.

Let $L_1'>0$ be as follows.

$$\text{(5-3-1)}\qquad L_1':=\sup\{d_{\tilde{M}}(x_0\cdot an, x_0\cdot a)|a\in A_{t_0}, n\in\omega\}.$$

The right side of (5-3-1) is finite, because the set $\{ana^{-1}|a\in A_{t_0}, n\in\omega\}$ is relatively compact (see for example [2]). We put $L_1=2L_1'$ and $L_2:=\max_{i,j=1,\ldots,\lambda} d_{\tilde{M}}(x_0\cdot z_i, x_0\cdot z_j)$.

To choose the remainder of the constants, we need the following (due to Borel-Raghunathan).

LEMMA 5-3-2 (Lemma 2.1 of [**17**]). *Let $\eta\subset P^0$ be any relatively compact open subset, $\underline{t}:\Upsilon\to\boldsymbol{R}^+$ any map.*

For $\beta\in\Upsilon$, there exists $s_\beta>0$ such that the following holds; we define the map $\underline{t}':\Upsilon\to\boldsymbol{R}^+$ by $\underline{t}'(\beta)=s_\beta$, $\underline{t}'(\alpha)=\underline{t}(\alpha)$ for $\alpha\neq\beta$ and let $g\in G_{\boldsymbol{Z}}$ be any element, then $S_{\underline{t}'\eta}g\cap S_{\underline{t}\eta}\neq\phi$ only if $g\in P_{\Upsilon-\{\beta\}}$.

Apply this lemma to the case $\eta={}^0M\cdot\omega$, $\underline{t}(\beta)=t_0$, for all $\beta\in\Upsilon$, and take a positive number t_1, such that

$$t_1<\min_{\beta\in\Upsilon}\{s_\beta\},\qquad t_1<1\,. \tag{5-3-3}$$

For each subset $\Theta\subset\Upsilon$, we define a subset $D(\Theta)$ of A as follows.

$$D(\Theta)=\{a\in A\,|\,\theta(a)\leqq t_1 \text{ for all } \theta\in\Theta,\quad t_1<\beta(a)\leqq t_0 \text{ for all } \beta\in\Upsilon-\Theta\}. \tag{5-3-4}$$

We remark that $A_{t_0}-A_{t_1}=\bigcup_{\Theta\neq\Upsilon}D(\Theta)$ (see Fig.).

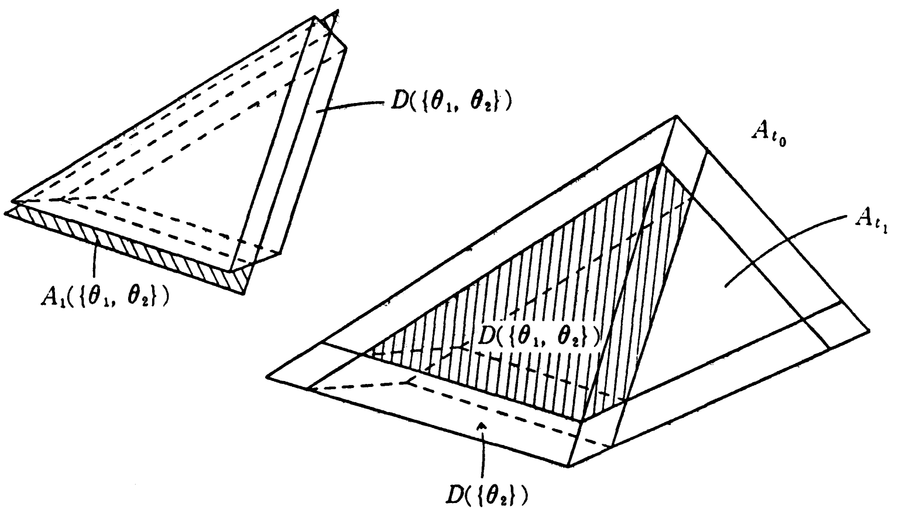

Fig. (n=4). This is a picture of A_{t_0} cut off along the hyperplane which is perpendicular to a line $\exp tv$ for a suitable $v\in\mathfrak{a}_{\mathfrak{p}}^+$.

LEMMA 5-3-5. *For each* $\Theta\subset\Upsilon$, $l_\Theta=\sup\{d_{\tilde{M}}(x_0\cdot b,\ x_0\cdot A_1(\Theta))\,|\,b\in D(\Theta)\}$ *is finite.*

PROOF. The number l_ϕ is finite, because $\{e\}=A_1(\phi)\subset D(\phi)$ and $D(\phi)$ is compact. Since $A_1(\Upsilon)\supset D(\Upsilon)$, we have $l_\Upsilon=0$.

Therefore we assume that $\Theta=\{\theta_{i_1},\cdots,\theta_{i_k}\}\neq\phi,\Upsilon$. By permuting $\theta_1,\cdots,\theta_{n-1}$, we obtain $\theta'_1,\cdots,\theta'_{n-1}$ such that $\Upsilon=\{\theta'_1,\cdots,\theta'_{n-1}\}$ and that $\theta'_j=\theta_{i_j}$ for $j=1,\cdots,k$. Let $b\in D(\Theta)$ and $\theta'_i(b)=s_i$; $i=1,\cdots,n-1$. We take the element b' of $A_1(\Theta)$ with $\theta'_i(b')=s_i$ for $i=1,\cdots,k$ and $\theta'_i(b')=1$ for $i\geqq k+1$.

We show that the number $d_{\tilde{M}}(x_0\cdot b,\ x_0\cdot b')$ is bounded from above by a constant independent of b. For each i, we can take $\mathrm{v}_i\in\mathfrak{a}_{\mathfrak{p}}$ such that $d\theta'_i(X)=(X,\mathrm{v}_i)$ for all $X=\mathfrak{a}_{\mathfrak{p}}$, where $(\,,\,)$ is the inner product induced from the Killing form. Let β (resp. β') be the element of $\mathfrak{a}_{\mathfrak{p}}$ with $b=\exp\beta$ (resp. $b'=\exp\beta'$). Since $\{\mathrm{v}_1,\cdots,\mathrm{v}_{n-1}\}$ is a basis of $\mathfrak{a}_{\mathfrak{p}}$, we can determine the $2(n-1)$ numbers β_i,

β_i'; $i=1, \cdots, n-1$ which satisfy the following equations.

$$\beta=\sum_{i=1}^{n-1}\beta_i \mathrm{v}_i\,, \qquad \beta'=\sum_{i=1}^{n-1}\beta_i' \mathrm{v}_i\,. \tag{5-3-5-a}$$

Then we have

$$d\theta_i'(\beta)=\sum_{j=1}^{n-1}\beta_j(\mathrm{v}_i, \mathrm{v}_j)=\log s_i \qquad \text{for } i=1, \cdots, n-1\,, \tag{5-3-5-b}$$

and

$$d\theta_i'(\beta')=\sum_{j=1}^{n-1}\beta_j'(\mathrm{v}_i, \mathrm{v}_j)=\begin{cases}\log s_i & \text{if } i=1, \cdots, k\,,\\ 0 & \text{if } i\geqq k+1\,.\end{cases} \tag{5-3-5-c}$$

Let H be the $(n-1)\times(n-1)$ matrix whose $i-j$ entry is $(\mathrm{v}_i, \mathrm{v}_j)$. We denote by g_{ij} the $i-j$ entry of H^{-1}. Then we have

$$\beta_i-\beta_i'=\sum_{j=k+1}^{n-1}g_{ij}\log s_j \qquad \text{for } i=1, \cdots, n-1\,.$$

So we have

$$\begin{aligned}\|\beta-\beta'\|^2&=\sum_{i,j=1}^{n-1}\sum_{l,t=k+1}^{n-1}g_{il}(\log s_l)g_{jt}(\log s_t)(\mathrm{v}_i, \mathrm{v}_j)\\ &\leqq (n-1)^2(n-k-1)^2\Big\{\max_{i,j}|g_{ij}|\Big\}^2\Big\{\max_{i,j}|(\mathrm{v}_i, \mathrm{v}_j)|\Big\}\\ &\quad\times[\max\{|\log t_0|, |\log t_1|\}]^2\,.\end{aligned} \tag{5-3-5-d}$$

The right side of the above inequality (5-3-5-d) is independent of β_i, β_i' and hence b. Therefore, $d_{\tilde{M}}(x_0\cdot b, x_0\cdot b')=\|\beta-\beta'\|$ is bounded by the constant independent of b. □

We put

$$L_4=2\max\{l_\Theta\,|\,\Theta\subset\Upsilon\}\,. \tag{5-3-6}$$

PROPOSITION 5-3-7. *If a positive number L satisfies the inequality $L>L_4$, then the following holds.*

If $b\in D(\Theta)$ satisfies the condition $d_{\tilde{M}}(x_0, x_0\cdot b)\geqq L$, then

(1) there exists at least one map $\beta\in\Upsilon$ such that $\beta(b)\leqq t_1$, and

(2) $d_{\tilde{M}}(x_0\cdot b, x_0\cdot A_1(\Theta))\leqq L_4$.

PROOF. Immediate from the above construction. □

We put $\Omega=\bigcup_{i=1}^{\lambda}x_0\cdot A_{t_0}\omega z_i$. By Borel's theorem 5-1-2 (2), the set $\{h\in\Gamma\,|\,\Omega\cdot h\cap\Omega\neq\phi\}$ is finite, so we denote this by $\{h_1, \cdots, h_s\}$. We denote by $g_{kl}^{\mu ij}$ the $k-l$ entry of $z_ih_\mu^{-1}z_j^{-1}$: i.e., $z_ih_\mu^{-1}z_j^{-1}=(g_{kl}^{\mu ij})_{1\leq k, l\leq n}$. We define the compact subset $\mathcal{S}$ of $SL(n, \boldsymbol{R})$ by

$$\mathcal{S} = \left\{(t_{kl} g_{kl}^{\mu ij})_{1 \le k, l \le n} \in SL(n, \boldsymbol{R}) \,|\, 0 \leqq t_{kl} \leqq \left(\max\left(t_0, \frac{1}{t_1}\right)\right)^n ; \right.$$
$$\left. i, j=1, \cdots, \lambda ; \mu=1, \cdots, s \right\}.$$

We put

(5-3-8) $$L_5 = \max\{d_{\tilde{M}}(x_0, x_0 \cdot g) \,|\, g \in \mathcal{S}\}.$$

We take a positive number L such that $L > L_4$.

LEMMA 5-3-9. *Let* $a, b \in A_{t_0}$; $n, m \in \omega$; $i, j \in \{1, \cdots, \lambda\}$ *such that*

$$\pi(x_0 \cdot anz_i) = \pi(x_0 \cdot bmz_j)$$

and suppose $d_{\tilde{M}}(x_0, x_0 \cdot a) \geqq 2L$.

(1) *If* $i=j$, *then* $d_{\tilde{M}}(x_0 \cdot a, x_0 \cdot b) \leqq L_1$.

(2) *If* $i \neq j$, *there exists at least one map* $\beta \in \Upsilon$ *such that* $\beta(a) > t_1$.

(3) *If* $i \neq j$, *let* Θ *be the subset of* Υ *which satisfies the following condition*;

$$\beta(a) \leqq t_1 \textit{ for all } \beta \in \Theta \textit{ and } t_1 < \beta(a) \leqq t_0 \textit{ for all } \beta \in \Upsilon - \Theta.$$

Then,

$$d_{\tilde{M}}(x_0 \cdot A_1(\Theta), x_0 \cdot a) \leqq L_4 \quad \textit{and} \quad d_{\tilde{M}}(x_0 \cdot a, x_0 \cdot b) \leqq L_1 + L_5 .$$

Moreover, $B_i(\Theta)$ *and* $B_j(\Theta)$ *are pasted together in* $C|T_\Gamma|$.

PROOF. Since $\pi(x_0 \cdot anz_i) = \pi(x_0 \cdot bmz_j)$, there exists an element h of Γ such that $x_0 \cdot anz_i = x_0 \cdot bmz_j h$. We can put $k = anz_i h^{-1} z_j^{-1} m^{-1} b^{-1} \in K$.

First, suppose $i=j$.

Then we have $x_0 \cdot an = x_0 \cdot bmz_i h z_i^{-1}$. Since Γ is a normal subgroup of $SL(n, \boldsymbol{Z})$, we can put $h' = z_i h z_i^{-1} \in \Gamma$. So $x_0 \cdot an = x_0 \cdot bmh'$ and $\pi(x_0 \cdot an) = \pi(x_0 \cdot bm)$. From this and Lemma 3-3, we have

$$\begin{aligned} d_{\tilde{M}}(x_0 \cdot a, x_0 \cdot b) &= d_M(\pi(x_0 \cdot a), \pi(x_0 \cdot b)) \\ &\leqq d_M(\pi(x_0 \cdot a), \pi(x_0 \cdot an)) + d_M(\pi(x_0 \cdot bm), \pi(x_0 \cdot b)) \\ &\leqq 2L_1' = L_1 . \end{aligned}$$

Next, suppose $i \neq j$.

If we suppose (2) to be false, we have $anz_i h^{-1} z_j^{-1} = kbm$ and $\beta(a) \leqq t_1$ for all $\beta \in \Upsilon$. So from Lemma 5-3-2, we have

$$p = z_i h^{-1} z_j^{-1} \in \bigcap_{\beta \in \Upsilon} P_{\Upsilon - \{\beta\}} = P .$$

Therefore, $z_i = pz_j h$ and $P_{\boldsymbol{Q}} z_i \Gamma = P_{\boldsymbol{Q}} z_j \Gamma$. This contradicts the hypothesis.

Finally we prove (3).

Let $i\neq j$ and Θ be as in the statement of (3). As can be seen from the conclusion of (2) above, we have $\Theta\neq\Upsilon$. And from Proposition 5-3-7 (1), we have $\Theta\neq\phi$. Since $anz_ih^{-1}z_j^{-1}=kbm$, from Lemma 5-3-2, we have

$$p=z_ih^{-1}z_j^{-1}\in\bigcap_{\beta\in\Theta}P_{\Upsilon-\{\beta\}}=P_{\Upsilon-\Theta}\,.$$

Therefore, $z_i=pz_jh$ and

$$(P_{\Upsilon-\Theta})_Q z_i\Gamma=(P_{\Upsilon-\Theta})_Q z_j\Gamma\,.$$

From this and (1-3-1), $B_i(\Theta)$ and $B_j(\Theta)$ are pasted together in $C|T_\Gamma|$. Let us show that $apa^{-1}\in\mathcal{S}$. Since $x_0\cdot anz_ih=x_0\cdot bmz_j$, we have $\Omega\cdot h\cap\Omega\neq\phi$. Let p_{ij} be the $i-j$ entry of $p=z_ih^{-1}z_j^{-1}$ and a_i the $i-i$ entry of a. Then the $i-j$ entry of apa^{-1} is $(a_i/a_j)p_{ij}$. If $i>j$ and $\{\theta_j, \theta_{j+1}, \cdots, \theta_{i-1}\}\subset\Upsilon-\Theta$, then $a_j/a_i=\theta_j(a)\theta_{j+1}(a)\cdot\cdots\cdot\theta_{i-1}(a)>(t_1)^n$ and $0<a_i/a_j<(1/t_1)^n$. If $i>j$ and $\{\theta_j, \theta_{j+1}, \cdots, \theta_{i-1}\}\not\subset\Upsilon-\Theta$, then $p_{ij}=0$ because $p\in P_{\Upsilon-\Theta}$. If $i<j$, then $a_i/a_j=\theta_i(a)\theta_{i+1}(a)\cdot\cdots\cdot\theta_{j-1}(a)\leqq(t_0)^n$. From the above and that $\Omega\cdot h\cap\Omega\neq\phi$, $h\in\Gamma$, we have $apa^{-1}\in\mathcal{S}$. So $d_{\tilde{M}}(x_0, x_0\cdot apa^{-1})\leqq L_5$.

Therefore we obtain

$$\begin{aligned}
&d_{\tilde{M}}(x_0\cdot a, x_0\cdot b)\\
&\quad\leqq d_{\tilde{M}}(x_0\cdot a, x_0\cdot ap)+d_{\tilde{M}}(x_0\cdot ap, x_0\cdot anp)+d_{\tilde{M}}(x_0\cdot anp, x_0\cdot b)\\
&\quad= d_{\tilde{M}}(x_0, x_0\cdot apa^{-1})+d_{\tilde{M}}(x_0\cdot a, x_0\cdot an)+d_{\tilde{M}}(x_0\cdot bm, x_0\cdot b)\\
&\quad\leqq L_5+2L_1'=L_1+L_5\,.
\end{aligned}$$

The inequality $d_{\tilde{M}}(x_0\cdot A_1(\Theta), x_0\cdot a)\leqq L_4$ is obvious from Proposition 5-3-7, because $a\in D(\Theta)$. □

§ 6. Tangent cone at infinity.

6-1. Preliminary map f.

For each $j\in\{1, \cdots, n-1\}$, we denote by s_j the reflection in $\mathfrak{a}_\mathfrak{p}$ with respect to the hyperplane $H_j=\{\alpha\in\mathfrak{a}_\mathfrak{p}|\, d\theta_j(\alpha)=0\}$. The group W generated by $\{s_1, \cdots, s_{n-1}\}$ is the Weyl group.

Let U' (resp. U) be the normalizer (resp. centralizer) of $\mathfrak{a}_\mathfrak{p}$ in K. (We have $U={}^0M=M_\phi$). Then W is naturally identified with U'/U. Under this identification, we take and fix a representative $g_w\in K_Z$ for each $w\in W$. That is, $Ad(g_w^{-1})\alpha=\alpha\cdot w$ for all $\alpha\in\mathfrak{a}_\mathfrak{p}$. In particular, we put $g_e=e$. We remark that $g_w g_{w'} g_{ww'}^{-1}\in{}^0M$ for all $w, w'\in W$.

Since the reflection s_j fixes every points of the hyperplane H_j, g_{s_j} fixes pointwise the set $x_0\cdot A_1(\Upsilon-\{\theta_j\})$. Hence $g_{s_j}\in P_{\{\theta_j\}}$.

For $w\in W$, $i\in\{1, \cdots, \lambda\}$ we choose $z_{(w,i)}\in\{z_1, \cdots, z_\lambda\}$ such that $P_Q g_w z_i\Gamma$

$=P_Q z_{(w,i)}\Gamma$. The symbol (w, i) stands for one of the numbers $1, \cdots, \lambda$. Moreover we choose $p_{[w,i]}\in P_Q$ and $\gamma_{[w,i]}\in\Gamma$ such that $g_w z_i = p_{[w,i]} z_{(w,i)} \gamma_{[w,i]}$. We decompose $p_{[w,i]}$ as follows.

$$p_{[w,i]} = m_{[w,i]} a_{[w,i]} n_{[w,i]};$$

$$m_{[w,i]} \in {}^0M,\ a_{[w,i]} \in A,\ n_{[w,i]} \in N.$$

By increasing ω if necessary, we may assume that ω contains $n_{[w,i]}$ for all $w\in W$ and $i\in\{1, \cdots, \lambda\}$. We can take a positive number $t_2(>t_0)$ such that the following holds; If $a\in A_{t_0}$, then $a a_{[w,i]}\in A_{t_2}$ for all $w\in W$ and $i\in\{1, \cdots, \lambda\}$.

We replace L_1 by $2\sup\{d_{\tilde{M}}(x_0\cdot an, x_0\cdot a) | a\in A_{t_2}, n\in\omega\}$, and put

$$L_6 = \max\{d_{\tilde{M}}(x_0, x_0\cdot a_{[w,i]}) | w\in W;\ i=1, \cdots, \lambda\}. \tag{6-1-1}$$

By increasing L if necessary, we may assume that

$$L > \lambda(L_1+L_2+L_3+2L_4+L_5+L_6). \tag{6-1-2}$$

We define a map $f: M\rightarrow C|T_\Gamma|$ as follows.

For an arbitrary point v of M, we consider representations $v=\pi(x_0\cdot anz_i)$ with $a\in \mathring{A}_{t_0}$, $n\in\omega$, $i\in\{1, \cdots, \lambda\}$. This representation is not unique and there are two possibilities.

$\langle a\rangle$ There exists a representation $v=\pi(x_0\cdot anz_i)$ such that $d_{\tilde{M}}(x_0, x_0\cdot a) <2L$.

$\langle b\rangle$ For any representation $v=\pi(x_0\cdot anz_i)$, $d_{\tilde{M}}(x_0, x_0\cdot a)\geqq 2L$ is satisfied. In this case, we fix one representation for each point.

In the case of $\langle a\rangle$, we put $f(v)=O$, where O denotes the vertex of the cone $C|T_\Gamma|$.

In the case of $\langle b\rangle$, we take the fixed representation $v=\pi(x_0\cdot anz_i)$ and consider two cases.

$\langle b\rangle$-$\langle 1\rangle$ If $a\in A_1$, we put $f(v)=\Psi_i(x_0\cdot a)\in B_i$.

$\langle b\rangle$-$\langle 2\rangle$ If $a\notin A_1$, we rewrite $x_0\cdot a$ as $x_0\cdot a' g_w$ for suitable $a'\in A_1$ and $w\in W$, and put $f(v)=\Psi_{(w,i)}(x_0\cdot a')\in B_{(w,i)}$.

Concerning the case $\langle b\rangle$, we need some more discussion.

LEMMA 6-1-3. *Let* $x_0\cdot bg_w = x_0\cdot b' g_{w'}$, *where* $w, w'\in W$, $b'\in A_1$ *and* b *is an interior point of* $A_1(\Theta)$.

Then $b=b'$ *and* $B_{(w,i)}$ *and* $B_{(w',i)}$ *are pasted together along* $B_{(w,i)}(\Theta)$ *and* $B_{(w',i)}(\Theta)$ *in* $C|T_\Gamma|$.

PROOF. Take $\beta, \beta'\in\mathfrak{a}_\mathfrak{p}$ such that $b=\exp\beta$, $b'=\exp\beta'$. Then we have $\beta\cdot w=\beta'\cdot w'$ and $\beta=\beta'\cdot w'w^{-1}$. From Theorem 5F in Ch. I of [**5**], $\beta=\beta'$ and $w'w^{-1}$ is a product of elements of the set $\{s_j | \theta_j\in\Upsilon-\Theta\}$. Hence $b=b'$ and $g_{w'}\cdot g_w^{-1}\in(P_{\Upsilon-\Theta})_{\mathbf{Z}}$. Since we can take $q\in(P_{\Upsilon-\Theta})_{\mathbf{Z}}$ such that $g_{w'}=qg_w$, we have

$(P_{\Upsilon-\Theta})_Q z_{(w',i)}\Gamma=(P_{\Upsilon-\Theta})_Q g_{w'} z_i\Gamma=(P_{\Upsilon-\Theta})_Q g_w z_i\Gamma=(P_{\Upsilon-\Theta})_Q z_{(w,i)}\Gamma$ as required. □

If $a\notin A_1$ and a is in the image under the exponential map of some Weyl chamber, then a' and w in $\langle b\rangle$-$\langle 2\rangle$ of the definition of f are uniquely determined. Suppose that $a\notin A_1$ and that a is in the image of some wall of a Weyl chamber, say $x_0\cdot a\in x_0\cdot A_1(\Theta)w$ with $\Theta\neq\Upsilon$. We might be able to choose b, b' $\in A_1$; w, $w'\in W$ such that $x_0\cdot a=x_0\cdot bg_w=x_0\cdot b'g_{w'}$. But from the above lemma, we have $b=b'$ and $\Psi_{(w,i)}(x_0\cdot b)=\Psi_{(w',i)}(x_0\cdot b')$ in $C|T_\Gamma|$. So there is no ambiguity in the case $\langle b\rangle$ of the definition of f.

The submanifold $\pi(x_0\cdot Az_i)$ is divided into the union of $\pi(x_0\cdot A_1 g_w z_i)$; $w\in W$ and each $\pi(x_0\cdot A_1 g_w z_i)$ is isometric to $B_{(w,i)}$. It is important (in the successive discussion) to verify that $B_{(w,i)}$ and $B_{(w',i)}$ intersect in $C|T_\Gamma|$ if $\pi(x_0\cdot A_1 g_w z_i)$ and $\pi(x_0\cdot A_1 g_{w'} z_i)$ intersect in M. So we reformulate the above lemma as follows.

LEMMA 6-1-4. *If $\pi(x_0\cdot A_1 g_w z_i)\cap\pi(x_0\cdot A_1 g_{w'} z_i)=\pi(x_0\cdot A_1(\Theta)g_w z_i)$ with $\Theta\neq\Upsilon$, then $B_{(w,i)}$ and $B_{(w',i)}$ are pasted together along $B_{(w,i)}(\Theta)$ and $B_{(w',i)}(\Theta)$ in $C|T_\Gamma|$.*

6-2. Properties of f.

Our aim is to prove that

$$|d_M(v, v')-d_{C|T_\Gamma|}(f(v), f(v'))|<6\lambda L \qquad \text{for any } v,\ v'\in M.$$

The argument in this (sub)section is almost as same as the one in §4 of [**16**]. So we omit the proof of Proposition 6-2-5.

LEMMA 6-2-1.

$$d_{C|T_\Gamma|}(f(v), f(v'))<d_M(v, v')+(5\lambda+2)L \qquad \textit{for } v,\ v'\in M.$$

PROOF. Let $v=\pi(x_0\cdot anz_i)$, $v'=\pi(x_0\cdot bmz_j)$; a, $b\in\mathring{A}_{t_0}$; n, $m\in\omega$ be arbitrary representations.

Step 1. We join v to v' by a minimizing geodesic $\tau'\colon[0, l']\to M$. Let $0=t_1<t_2<\cdots<t_{\nu'}=l'$ be a partition of the interval $[0, l']$ such that the following condition is satisfied.

(6-2-1-1) There exist curves $\tilde{\tau}'_k\colon[t_k, t_{k+1}]\to x_0\cdot\mathring{A}_{t_0}\omega z_{i_k}\subset\tilde{M}$ such that $\tau'(t)=\pi\circ\tilde{\tau}'_k(t)$ for $t\in[t_k, t_{k+1}]$, where $k=1, \cdots, \nu'-1$.

We deform the curve τ' in the following way. We start from z_{i_1}. Take the largest k such that $i_1=i_k$, and join $\tau'(t_1)$ to $\tau'(t_{k+1})$ as follows: let $\tau'(t_1)=\pi(x_0\cdot a_{t_1}n_{t_1}z_{i_1})$, $\tau'(t_{k+1})=\pi(x_0\cdot a_{t_{k+1}}n_{t_{k+1}}z_{i_1})$, and join $\tau'(t_1)$ to $\pi(x_0\cdot a_{t_1}z_{i_1})$ (resp. $\pi(x_0\cdot a_{t_{k+1}}z_{i_1})$ to $\tau'(t_{k+1})$) by a minimizing geodesic, join $\pi(x_0\cdot a_{t_1}z_{i_1})$ to $\pi(x_0\cdot a_{t_{k+1}}z_{i_1})$ by a minimizing geodesic. We remark that for the last geodesic we can take a geodesic in $\pi(x_0\cdot\mathring{A}_{t_0}z_{i_1})$ by Lemma 3-3, and so we do. Next we consider $z_{i_{k+1}}$. Take the largest k' such that $i_{k+1}=i_{k'}$ and proceed in the same way. Repeating this operation, we get a curve $\tau: [0, l]\to M$.

By construction, there exists a partition of the interval $[0, l]$,

$$0=t_1<t_2<\cdots<t_{\nu-1}<t_\nu=l\,,$$

where $\nu\leqq\lambda+1$, such that the following conditions are satisfied.

(6-2-1-2) $\tau(t_k)=\pi(x_0\cdot a_k n_k z_{i_k})$, $\tau(t_{k+1})=\pi(x_0\cdot b_k m_k z_{i_k})$;

$a_k, b_k\in\mathring{A}_{t_0}$; $n_k, m_k\in\omega$ for each $k\in\{1, \cdots, \nu-1\}$, $i_1=i$, $i_{\nu-1}=j$, and $i_1, \cdots, i_{\nu-1}$ are different from each other.

(6-2-1-3) There exists a subdivision,

$$\begin{aligned}0=t_1\leqq s_1\leqq\eta_1\leqq t_2\leqq s_2\leqq\eta_2\leqq t_3\leqq\cdots\\ \cdots\leqq t_k\leqq s_k\leqq\eta_k\leqq t_{k+1}\leqq\cdots\\ \cdots\leqq t_\nu=l\,,\end{aligned}$$

such that the following hold.

(6-2-1-3-a) There exists a curve $\tilde{\tau}_k: [s_k, \eta_k]\to\tilde{M}$ which can be written as $\tilde{\tau}_k(t)=x_0\cdot a_k(t)z_{i_k}$; $a_k(t)\in\mathring{A}_{t_0}$, such that $\tau(t)=\pi\circ\tilde{\tau}_k(t)$ for $t\in[s_k, \eta_k]$, and $a_k(s_k)=a_k$, $a_k(\eta_k)=b_k$.

(6-2-1-3-b) $\tau|_{[s_k,\eta_k]}$, $\tau|_{[t_k,s_k]}$, $\tau|_{[\eta_k,t_{k+1}]}$ are minimizing geodesics.

We have

(6-2-1-4)
$$\sum_{k=1}^{\nu-1}\text{length }[\tilde{\tau}_k]\leqq d_M(v, v')+\lambda L_1\,.$$

Step 2. We define points P_k, Q_k of $C|T_\Gamma|$ as follows. For $\tilde{\tau}_k(s_k)=x_0\cdot a_k z_{i_k}$, $\tilde{\tau}_k(\eta_k)=x_0\cdot b_k z_{i_k}$, let

$$P_k=\begin{cases}\Psi_{i_k}(x_0\cdot a_k) & \text{if } a_k\in A_1\\ \Psi_{(w,i_k)}(x_0\cdot a_k') & \text{if } a_k\notin A_1 \text{ and } x_0\cdot a_k=x_0\cdot a_k'g_w\,;\\ & \quad a_k'\in A_1;\ w\in W\end{cases}$$

$$Q_k=\begin{cases}\Psi_{i_k}(x_0\cdot b_k) & \text{if } b_k\in A_1\\ \Psi_{(w',i_k)}(x_0\cdot b_k') & \text{if } b_k\notin A_1 \text{ and } x_0\cdot b_k=x_0\cdot b_k'g_{w'}\,;\\ & \quad b_k'\in A_1;\ w'\in W\,.\end{cases}$$

Step 3. We construct a curve in $C|T_{\Gamma}|$ by joining $f(v)$, P_1, Q_1, P_2, Q_2, $\cdots$, $P_{\nu-1}$, $Q_{\nu-1}$, $f(v')$.

(A) To begin with, from Lemma 6-1-4, we can join P_k and Q_k by a (possibly broken) line segment P_kQ_k such that

$$\text{length}[\tilde{\tau}_k]=\overline{P_kQ_k}\,.$$

(B) We join Q_k and P_{k+1} in the following way.

(Case 1) If $\overline{OQ_k}=d_{\tilde{M}}(x_0,\ x_0\cdot b_k)<2L$, we join Q_k to O, and O to P_{k+1} by the line segments in that order. Notice that

$$\begin{aligned}
&|d_{\tilde{M}}(x_0,\ x_0\cdot a_{k+1})-d_{\tilde{M}}(x_0,\ x_0\cdot b_k)|\\
&=|d_{\tilde{M}}(x_0\cdot z_{i_{k+1}},\ x_0\cdot a_{k+1}z_{i_{k+1}})-d_{\tilde{M}}(x_0\cdot z_{i_k},\ x_0\cdot b_kz_{i_k})|\\
&=|d_M(\pi(x_0\cdot z_{i_{k+1}}),\ \pi(x_0\cdot a_{k+1}z_{i_{k+1}}))-d_M(\pi(x_0\cdot z_{i_k}),\ \pi(x_0\cdot b_kz_{i_k}))|\\
&\leqq d_M(\pi(x_0\cdot z_{i_{k+1}}),\ \pi(x_0\cdot z_{i_k}))+d_M(\pi(x_0\cdot a_{k+1}z_{i_{k+1}}),\ \tau(t_{k+1}))\\
&\quad+d_M(\tau(t_{k+1}),\ \pi(x_0\cdot b_kz_{i_k}))\\
&\leqq L_2+2L_1'=L_1+L_2\,.
\end{aligned}$$

So, $\overline{OP_{k+1}}=d_{\tilde{M}}(x,\ x_0\cdot a_{k+1})<2L+L_1+L_2$, and $\overline{Q_kO}+\overline{OP_{k+1}}<4L+L_1+L_2$.

(Case 2) If $\overline{OQ_k}=d_{\tilde{M}}(x_0,\ x_0\cdot b_k)\geqq 2L$, from Lemma 5-3-9, we have $d_{\tilde{M}}(x_0\cdot a_{k+1},\ x_0\cdot b_k)\leqq L_1+L_5$.
Let Θ be the subset of Υ as in (3) of Lemma 5-3-9 and take an element c of $A_1(\Theta)$ such that $d_{\tilde{M}}(x_0\cdot b_k,\ x_0\cdot c)\leqq L_4$. We put $R_k=\Psi_{i_k}(x_0\cdot c)\in B_{i_k}(\Theta)$. Note that $B_{i_k}(\Theta)$ and $B_{i_{k+1}}(\Theta)$ are pasted together. So R_k is also on $B_{i_{k+1}}(\Theta)$ and expressed as $\Psi_{i_{k+1}}(x_0\cdot c)$.

Recall that $\tilde{\tau}_k(\eta_k)=x_0\cdot b_kz_{i_k}$ and

$$Q_k=\begin{cases}\Psi_{i_k}(x_0\cdot b_k) & \text{if } b_k\in A_1\\ \Psi_{(w',i_k)}(x_0\cdot b_k') & \text{if } b_k\notin A_1 \text{ and } x_0\cdot b_k=x_0\cdot b_k'g_{w'}\,;\\ & \quad b_k'\in A_1\,;\ w'\in W\,.\end{cases}$$

We define a point S_{k+1} to be $\Psi_{i_{k+1}}(x_0\cdot b_k)$ if $b_k\in A_1$, $\Psi_{(w',i_{k+1})}(x_0\cdot b_k')$ if $b_k\notin A_1$.

From Lemma 6-1-4, we can join S_{k+1} and P_{k+1} by a (possibly broken) line segment such that $\overline{S_{k+1}P_{k+1}}=d_{\tilde{M}}(x_0\cdot b_k,\ x_0\cdot a_{k+1})$. We can also join Q_k to R_k, R_k to S_{k+1} by (possibly broken) line segments such that $\overline{Q_kR_k}=d_{\tilde{M}}(x_0\cdot b_k,\ x_0\cdot c)$ and $\overline{R_kS_{k+1}}=d_{\tilde{M}}(x_0\cdot c,\ x_0\cdot b_k)$. So we join Q_k to R_k, R_k to S_{k+1}, and S_{k+1} to P_{k+1} by the above segments. Then the sum of the lengths of the added segments is

$$\overline{Q_kR_k}+\overline{R_kS_{k+1}}+\overline{S_{k+1}P_{k+1}} \leqq L_4+L_4+(L_1+L_5)$$
$$= L_1+2L_4+L_5 .$$

(C) Finally we join $f(v)$ with P_1, and $Q_{\nu-1}$ with $f(v')$.

If $f(v)=P_1$, nothing to be done.

If $f(v)\neq P_1$, there are three possibilities.

⟨1⟩ $O=f(v)\neq P_1$

⟨2⟩ The representation of v which we fixed in the definition of f is $v=\pi(x_0\cdot a''n''z_i)$; $a''\in \mathring{A}_{t_0}$, $n''\in\omega$, and $d_{\tilde{M}}(x_0, x_0\cdot a'')\geqq 2L$.
In this case, from Lemma 5-3-9, we have $d_{\tilde{M}}(x_0\cdot a, x_0\cdot a'')\leqq L_1$.

⟨3⟩ The representation of v which we fixed in the definition of f is $v=\pi(x_0\cdot a''n''z_\mu)$; $a''\in \mathring{A}_{t_0}$, $n''\in\omega$, $i\neq\mu$, and $d_{\tilde{M}}(x_0, x_0\cdot a'')\geqq 2L$.

In the case ⟨1⟩ (resp. ⟨2⟩), we join $f(v)$ and P_1 by the line segment. Its length is not greater than $3L$ (resp. L_1).
In the case ⟨3⟩, we join $f(v)$ and P_1 in the same way as one used in (B)-(case 2). The length of the curve joining $f(v)$ and P_1 is not greater than $L_1+2L_4+L_5$.

We join $Q_{\nu-1}$ and $f(v')$ in a similar way.

Step 4. We compare the distance $d_M(v, v')$ with the length ρ of the curve constructed above. We have,

$$\rho \leqq \sum_{k=1}^{\nu-1} \text{length}\,[\tilde{\gamma}_k]+(\lambda-1)\max\{4L+L_1+L_2,\ L_1+2L_4+L_5\}$$
$$+2\max\{3L,\ L_1,\ L_1+2L_4+L_5\}.$$

From (6-2-1-4) and (6-1-2), we obtain

$$d_{C|T_\Gamma|}(f(v),\ f(v')) \leqq \rho < d_M(v,\ v')+L+(\lambda-1)\cdot 5L+2\cdot 3L$$
$$= d_M(v,\ v')+(5\lambda+2)L .$$

□

In particular, as a byproduct of Step 3 of the above proof, we have the following lemma which we need in §7.

LEMMA 6-2-2. *$d_{C|T_\Gamma|}(f(\pi(x_0\cdot az_i)), \Psi_i(x_0\cdot a))<3L$ for all $a\in A_1$ and $i\in\{1, \cdots, \lambda\}$.*

LEMMA 6-2-3.

$$d_M(v,\ v') < d_{C|T_\Gamma|}(f(v),\ f(v'))+9L .$$

PROOF. Let $v=\pi(x_0\cdot anz_i)$, $v'=\pi(x_0\cdot bmz_j)$; $a, b\in A_{t_0}$; $n, m\in\omega$ be arbitrary representations.

We define points P, Q in $C|T_\Gamma|$ as follows.

$$P=\begin{cases} \Psi_i(x_0\cdot a) & \text{if } a\in A_1 \\ \Psi_{(w,i)}(x_0\cdot a') & \text{if } a\notin A_1 \text{ and } x_0\cdot a=x_0\cdot a'g_w\,; \\ & \quad a'\in A_1\,;\ w\in W \end{cases}$$

$$Q=\begin{cases} \Psi_j(x_0\cdot b) & \text{if } b\in A_1 \\ \Psi_{(w',j)}(x_0\cdot b') & \text{if } b\notin A_1 \text{ and } x_0\cdot b=x_0\cdot b'g_{w'}\,; \\ & \quad b'\in A_1\,;\ w'\in W\,. \end{cases}$$

For an arbitrary positive number ε, we join P with Q by a curve $\tau\colon [0, l] \to C|T_\Gamma|$ such that

(6-2-3-a) $\quad \tau(0)=P,\quad \tau(l)=Q,\quad l=\text{length}[\tau]<d_{C|T_\Gamma|}(P, Q)+\varepsilon$

(6-2-3-b) There exists a partition of $[0, l]$,

$0=t_1<t_2<\cdots<t_{\nu-1}<t_\nu=l$ such that

$\tau([t_k, t_{k+1}])\subset B_{i_k}$ for $k=1, \cdots, \nu-1$ and $\nu\leqq\lambda+1$.

Then we have

$$\begin{aligned} l &< d_{C|T_\Gamma|}(f(v), f(v'))+\varepsilon+2\max\{3L, L_1, L_1+2L_4+L_5\} \\ &< d_{C|T_\Gamma|}(f(v), f(v'))+6L+\varepsilon\,. \end{aligned}$$

We define a curve $c_k\colon [t_k, t_{k+1}]\to M$ by $c_k(t)=\pi(\Psi_{i_k}^{-1}(\tau(t))\cdot z_{i_k})\in M$ for each k. We put $v_k=c_k(t_k)$ and $w_k=c_k(t_{k+1})$.

From Lemma 5-2-5, we can join w_k and v_{k+1} by a geodesic whose length is not greater than L_3.

Next we join v with v_1.
If $a\in A_1$, we have $v_1=\pi(x_0\cdot az_i)$ and $d_M(v, v_1)\leqq L_1$.
If $a\notin A_1$, we have $v_1=\pi(\Psi_{(w,i)}^{-1}(\tau(t_1))\cdot z_{(w,i)})=\pi(x_0\cdot a'z_{(w,i)})$. Since $\pi(x_0\cdot az_i)=\pi(x_0\cdot a'g_wz_i)=\pi(x_0\cdot a'p_{[w,i]}z_{[w,i]})=\pi(x_0\cdot a'a_{[w,i]}n_{[w,i]}z_{[w,i]})$, we have $d_M(v, v_1)\leqq L_1+L_6$.
Therefore we can join v and v_1 by a geodesic whose length is not greater than L_1+L_6.

We join $w_{\nu-1}$ and v' in a similar way.

So we have

$$\begin{aligned} d_M(v, v') &\leqq l+(\lambda-1)L_3+2(L_1+L_6) \\ &< d_{C|T_\Gamma|}(f(v), f(v'))+9L-L_2+\varepsilon\,. \end{aligned}$$

Because ε is an arbitrary positive number, we obtain $d_M(v, v')\leqq d_{C|T_\Gamma|}(f(v), f(v'))+9L-L_2$. □

From Lemmas 6-2-1, 6-2-3, and that $\lambda \geqq 2$, we have

(6-2-4) $$|d_M(v, v') - d_{C|T_\Gamma|}(f(v), f(v'))| < 6\lambda L .$$

PROPOSITION 6-2-5. *For sufficiently large $r>0$, $B_r(O, C|T_\Gamma|)$ is contained in the $3L$-neighborhood of $f(B_r(\pi(x_0), M))$.*

6-3. Proof of Theorem B.

Let $\varepsilon>0$ satisfy $10/\varepsilon>L$, and $t>600\lambda L/\varepsilon$. Then $6\lambda L/t<\varepsilon/100$. We define a map $f_t: M \to C|T_\Gamma|$ by $f_t(v)=f(v)/t$ for $v \in M$.

By (6-2-4) and Proposition 6-2-5, we have:

(6-3-1) $f(B_{10/\varepsilon}(\pi(x_0), (M, g/t)))$ is contained in $B_{10/\varepsilon+\varepsilon/100}(O, C|T_\Gamma|)$.

(6-3-2) $B_{10/\varepsilon}(O, C|T_\Gamma|)$ is contained in the $3L/t$-neighborhood of $f_t(B_{10/\varepsilon}(\pi(x_0), (M, g/t)))$.

So, deforming f_t slightly, we get an $\varepsilon/10$-pointed Hausdorff approximation $g_t: ((M, g/t), \pi(x_0)) \to (C|T_\Gamma|, O)$. Consequently, there exists an ε-pointed Hausdorff approximation $\psi_t: (C|T_\Gamma|, O) \to ((M, g/t), \pi(x_0))$. And from the construction there also exists an ε-pointed Hausdorff approximation $\varphi_t: ((M, g/t), \pi(x_0)) \to (C|T_\Gamma|, O)$.

We have $d_{p.H}(((M, g/t), \pi(x_0)), ((C|T_\Gamma|, d_{C|T_\Gamma|}), O)) < \varepsilon$ for $t>600\lambda L/\varepsilon$, where $d_{p.H}$ is the pointed Hausdorff distance. Therefore $\lim_{t\to\infty}((M, g/t), \pi(x_0)) = ((C|T_\Gamma|, d_{C|T_\Gamma|}), O)$. The proof of Theorem B is now complete.

§ 7. Final remarks.

We say (in this paper) that two geodesic rays $\gamma_1, \gamma_2: [0, \infty) \to M$ are equivalent and write $\gamma_1 \sim \gamma_2$ if and only if there exists $C>0$ such that $d_M(\gamma_1(t), \gamma_2(t)) \leqq C$ for all $t \geqq 0$. Since γ_1 and γ_2 are geodesic rays, this condition is equivalent to the following. The Hausdorff distance $Hd(\gamma_1([0, \infty)), \gamma_2([0, \infty)))$ between $\gamma_1([0, \infty))$ and $\gamma_2([0, \infty))$ in M is finite. In fact, if $Hd(\gamma_1([0, \infty)), \gamma_2([0, \infty)))<C$ and $d_M(\gamma_1(0), \gamma_2(0))<C$ then we have $d_M(\gamma_1(t), \gamma_2(t))<3C$ for all $t \geqq 0$.

Let us show that γ_y's in Theorem A are not equivalent one another.

PROPOSITION 7-1. *Let $\{\gamma_y\}_{y\in|T_\Gamma|}$ be the family of geodesic rays in Theorem A. If $y \neq y'$, then γ_y and $\gamma_{y'}$ are not equivalent.*

PROOF. Suppose that γ_y and $\gamma_{y'}$ are equivalent, i.e., $d_M(\gamma_y(t), \gamma_{y'}(t)) \leqq C$ for all $t \geqq 0$.

We take the lines $l_y, l_{y'}: [0, \infty) \to C|T_\Gamma|$ which correspond to y, y' respectively. More precisely, let $\alpha = diag(\alpha_1, \cdots, \alpha_n) \in \mathfrak{a}_\mathfrak{p}^+$, $\beta = diag(\beta_1, \cdots, \beta_n) \in \mathfrak{a}_\mathfrak{p}^+$,

$\gamma_y=\pi\circ(\tilde{\gamma}(\alpha)\cdot z_\rho)$, $\gamma_{y'}=\pi\circ(\tilde{\gamma}(\beta)\cdot z_\mu)$, $y=\frac{\alpha_2-\alpha_1}{\alpha_n-\alpha_1}v_1^\rho+\frac{\alpha_3-\alpha_2}{\alpha_n-\alpha_1}v_2^\rho+\cdots+\frac{\alpha_n-\alpha_{n-1}}{\alpha_n-\alpha_1}v_{n-1}^\rho\in\triangle^\rho$, and $y'=\frac{\beta_2-\beta_1}{\beta_n-\beta_1}v_1^\mu+\frac{\beta_3-\beta_2}{\beta_n-\beta_1}v_2^\mu+\cdots+\frac{\beta_n-\beta_{n-1}}{\beta_n-\beta_1}v_{n-1}^\mu\in\triangle^\mu$. We define $l_y, l_{y'}:[0,\infty)\to C|T_\Gamma|$ by $l_y(s)=\Psi_\rho(x_0\cdot(\exp(s\alpha/\|\alpha\|)))=\Psi_\rho(\tilde{\gamma}(\alpha)(s))$, $l_{y'}(s)=\Psi_\mu(x_0\cdot(\exp(s\beta/\|\beta\|)))=\Psi_\mu(\tilde{\gamma}(\beta)(s))$ for all $s\geqq 0$.

We fix a positive number s. Let n be an arbitrary positive integer. Then, from Lemma 6-2-2, we have $d_{C|T_\Gamma|}(f(\gamma_y(ns)), l_y(ns))<3L$ and $d_{C|T_\Gamma|}(f(\gamma_{y'}(ns)), l_{y'}(ns))<3L$. From Lemma 6-2-1, we also have $d_{C|T_\Gamma|}(f(\gamma_y(ns)), f(\gamma_{y'}(ns)))<d_M(\gamma_y(ns), \gamma_{y'}(ns))+5\lambda L\leqq C+5\lambda L$. So we have $d_{C|T_\Gamma|}(l_y(ns), l_{y'}(ns))<C+(5\lambda+6)L$ and $d_{C|T_\Gamma|}(l_y(s), l_{y'}(s))<(C+(5\lambda+6)L)/n$. Since n is an arbitrary positive integer, we obtain $d_{C|T_\Gamma|}(l_y(s), l_{y'}(s))=0$. Hence $l_y(s)=l_{y'}(s)$ for all $s\geqq 0$, $l_y=l_{y'}$, and $y=y'$. This is a contradiction. □

In Lemma 3-2, we found many geodesic rays in M. We show that each of them is equivalent to some γ_y in Theorem A.

PROPOSITION 7-2. *For any $\alpha\in\mathfrak{a}_\mathfrak{p}-\{0\}$, $a\in A$, and $g\in SL(n, \boldsymbol{Z})$, there exists a point y in $|T_\Gamma|$ such that $\pi\circ(\tilde{\gamma}(\alpha)\cdot ag)\sim\gamma_y$.*

PROOF. *Step* 1. Since $d_M(\pi(\tilde{\gamma}(\alpha)(t)\cdot ag), \pi(\tilde{\gamma}(\alpha)(t)\cdot g))\leqq d_{\tilde{M}}(x_0\cdot a(\exp(t\alpha/\|\alpha\|))g, x_0\cdot(\exp(t\alpha/\|\alpha\|))g)=d_{\tilde{M}}(x_0\cdot a, x_0)$ for all $t\geqq 0$, we have $\pi\circ(\tilde{\gamma}(\alpha)\cdot ag)\sim\pi\circ(\tilde{\gamma}(\alpha)\cdot g)$.

Step 2. We show that there exists $\beta\in\mathfrak{a}_\mathfrak{p}^+$ and $\rho\in\{1, \cdots, \lambda\}$ such that $\pi\circ(\tilde{\gamma}(\alpha)\cdot g)\sim\pi\circ(\tilde{\gamma}(\beta)\cdot z_\rho)$.

We can choose an element w of the Weyl group W and $\beta\in\mathfrak{a}_\mathfrak{p}^+$ such that $\alpha/\|\alpha\|=(\beta/\|\beta\|)\cdot w$. So $x_0\cdot(\exp(t\alpha/\|\alpha\|))=x_0\cdot(\exp(t\beta/\|\beta\|))g_w$ for all $t\geqq 0$. Choose $\rho\in\{1, \cdots, \lambda\}$ with $P_Q g_w g\Gamma=P_Q z_\rho\Gamma$. Since we can write $g_w g=pz_\rho\gamma$; $p\in P_Q$, $\gamma\in\Gamma$, we have $\tilde{\gamma}(\alpha)(t)\cdot g=\tilde{\gamma}(\beta)(t)\cdot g_w g=\tilde{\gamma}(\beta)(t)\cdot pz_\rho\gamma$. Notice that the set $\{apa^{-1}\mid a\in A_1\}$ is compact. Hence, there exists a positive constant C and

$$\begin{aligned}&d_M(\pi(\tilde{\gamma}(\alpha)(t)\cdot g), \pi(\tilde{\gamma}(\beta)(t)\cdot z_\rho))\\&\leqq d_{\tilde{M}}(\tilde{\gamma}(\alpha)(t)\cdot g, \tilde{\gamma}(\beta)(t)\cdot z_\rho\gamma)=d_{\tilde{M}}(\tilde{\gamma}(\beta)(t)\cdot pz_\rho\gamma, \tilde{\gamma}(\beta)(t)\cdot z_\rho\gamma)\\&=d_{\tilde{M}}\Big(x_0\cdot\Big(\exp t\frac{\beta}{\|\beta\|}\Big)p\Big(\exp t\frac{\beta}{\|\beta\|}\Big)^{-1}, x_0\Big)\leqq C\end{aligned}$$

for all $t\geqq 0$.

Step 3. We show that there exists a point $y\in|T_\Gamma|$ such that $\pi\circ(\tilde{\gamma}(\beta)\cdot z_\rho)\sim\gamma_y$.

If $\beta=diag(\beta_1, \cdots, \beta_n)\in Int\ \mathfrak{a}_\mathfrak{p}^+$, then we put $y=\frac{\beta_2-\beta_1}{\beta_n-\beta_1}v_1^\rho+\cdots+\frac{\beta_n-\beta_{n-1}}{\beta_n-\beta_1}v_{n-1}^\rho\in\triangle^\rho$. From the construction of $\{\gamma_y\}_{y\in|T_\Gamma|}$, we have $\pi\circ(\tilde{\gamma}(\beta)\cdot z_\rho)=\gamma_y$.

If $\beta=diag(\beta_1, \cdots, \beta_n)\in\partial\mathfrak{a}_\mathfrak{p}^+$, then we put $y'=\frac{\beta_2-\beta_1}{\beta_n-\beta_1}v_1^\rho+\cdots+\frac{\beta_n-\beta_{n-1}}{\beta_n-\beta_1}v_{n-1}^\rho$

$\in \triangle^{\rho}$. We can find $\mu \in \{1, \cdots, \lambda\}$ such that $y=\frac{\beta_2-\beta_1}{\beta_n-\beta_1}v_1^{\mu}+\cdots+\frac{\beta_n-\beta_{n-1}}{\beta_n-\beta_1}v_{n-1}^{\mu} \in \triangle^{\mu}$ is pasted together with y' in $|T_{\Gamma}|$ and that $\gamma_y=\pi\circ(\tilde{\gamma}(\beta)\cdot z_{\mu})$. Suppose that $\Upsilon-\Theta=\{\theta_i \in \Upsilon \mid \theta_i(\exp\beta)=1\}$. Then y' is on $\triangle^{\rho}(\Theta)$ and y is on $\triangle^{\mu}(\Theta)$. Therefore $(P_{\Upsilon-\Theta})_{\mathbf{Q}}z_{\rho}\Gamma=(P_{\Upsilon-\Theta})_{\mathbf{Q}}z_{\mu}\Gamma$. Notice that $\Theta\neq\phi$. From Lemma 5-2-5, we have $d_M(\pi(x_0\cdot(\exp(t\beta/\|\beta\|))z_{\rho}), \pi(x_0\cdot(\exp(t\beta/\|\beta\|))z_{\mu}))\leqq L_3$ for all $t\geqq 0$. So, $\pi\circ(\tilde{\gamma}(\beta)\cdot z_{\rho}) \sim\pi\circ(\tilde{\gamma}(\beta)\cdot z_{\mu})=\gamma_y$. □

OPEN PROBLEM. *Is the family* $\{\gamma_y\}_{y\in|T_{\Gamma}|}$ *a complete representative system for the set of all equivalence classes of geodesic rays in* M?

As a byproduct of Step 2 of the proof of Proposition 7-2, we obtain the following.

COROLLARY 7-3. *For any* $g\in SL(n, \mathbf{Z})$ *and* $w\in W$, *there exists* $\rho\in\{1, \cdots, \lambda\}$ *such that the Hausdorff distance* $Hd(\pi(x_0\cdot A_1 g_w g), \pi(x_0\cdot A_1 z_{\rho}))$ *between* $\pi(x_0\cdot A_1 g_w g)$ *and* $\pi(x_0\cdot A_1 z_{\rho})$ *in* M *is finite.*

We can show the following in a similar way to Proposition 7-1.

PROPOSITION 7-4. *If* $\rho, \mu\in\{1, \cdots, \lambda\}$ *and* $\rho\neq\mu$, *then* $Hd(\pi(x_0\cdot A_1 z_{\rho}), \pi(x_0\cdot A_1 z_{\mu}))=\infty$.

Let us call the image of a totally geodesic, isometric embedding $B\hookrightarrow M$ a "closed Weyl chamber in M". We say that two closed Weyl chambers in M are equivalent if the Hausdorff distance between them in M is finite.

QUESTION. *Is the family* $\{S_{\rho}=\pi(x_0\cdot A_1 z_{\rho}) \mid \rho=1, \cdots, \lambda\}$ *a complete representative system for the set of all equivalence classes of closed Weyl chambers in* M?

In simply connected case, the set of all equivalence classes of closed Weyl chambers in $\tilde{M}$ forms the Tits building associated with the parabolic ($\mathbf{R}$-)subgroups of $SL(n, \mathbf{R})$ (see Appendix 5 of [**1**]). The above question is a counterpart of this fact.

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Added in Proof. The problem in § 7 was solved by L. Ji and R. MacPherson in more general form as follows (Geometry of compactifications of locally symmetric spaces, preprint, 1993); Let G be a semisimple algebraic group defined over $\boldsymbol{Q}$ with $\boldsymbol{Q}$-rank$\geqq 1$, and X be the symmetric space of maximal compact subgroups of $G_{\boldsymbol{R}}$. Let $\Gamma \subset G_{\boldsymbol{Q}}$ be a neat arithmetic subgroup and $M=X/\Gamma$. Then the set of all equivalence classes of geodesic rays in M corresponds bijectively to the quotient $|T_{\Gamma}|$ of the rational spherical Tits building for G by Γ.