J. Math. Soc. Japan
Vol. 47, No. 1, 1995

Spin^q structures

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(Received Oct. 6, 1992)
(Revised May 31, 1993)

Introduction.

In this paper the notion of Spin^q-structure is introduced and some of the basic materials related to it will be discussed.

To explain the motivation briefly, let us take an n-dimensional compact oriented Riemannian manifold X. The reduced structure group $SO(n)$ $(n\geqq 3)$ has the universal covering group $\mathrm{Spin}(n)$ called the Spin group, together with the short exact sequence

$$1\longrightarrow \boldsymbol{Z}_2\longrightarrow \mathrm{Spin}(n)\xrightarrow{\xi_0} SO(n)\longrightarrow 1\,.$$

A principal $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}(n)}$ with a $\mathrm{Spin}(n)$-equivariant bundle map ξ_0 from $P_{\mathrm{Spin}(n)}$ to the reduced structure bundle $P_{SO(n)}$ is then called a *Spin-structure* on X ([**3**, §5]). As is well-known, it plays a role of great importance particularly in the study of the interrelations between topology, geometry and analysis. However, to our regret, it turns out apparently not always to be effective for researching into a complex manifold X, $w_2(X)\equiv c_1(X)$ (mod 2), because there exists a Spin-structure on X if and only if the second Stiefel-Whitney class vanishes, $w_2(X)=0$. To avoid this disadvantage, the notion of Spin^c-structure was introduced ([**3**, §5 Remark 4]). That is, using the unitary group $U(1)$ $(=SO(2))$, the Spin group is twisted into the Spin^c group, $\mathrm{Spin}^c(n)\equiv\mathrm{Spin}(n)\times_{\boldsymbol{Z}_2}U(1)$, together with the short exact sequence

$$1\longrightarrow \boldsymbol{Z}_2\longrightarrow \mathrm{Spin}^c(n)\xrightarrow{\xi} SO(n)\times U(1)\longrightarrow 1\,,$$

where $\xi([\varphi, z])=(\xi_0(\varphi), z^2)$. The *$\mathrm{Spin}^c$-structure* is then defined to be a principal $\mathrm{Spin}^c(n)$-bundle $P_{\mathrm{Spin}^c(n)}$ with a $\mathrm{Spin}^c(n)$-equivariant bundle map $\xi: P_{\mathrm{Spin}^c(n)}\to P_{SO(n)}\times P_{U(1)}$, where $P_{U(1)}$ is a certain principal $U(1)$-bundle. Since the existence can be characterized by the condition that $w_2(X)$ is the mod 2 reduction of an integral class, a complex structure certainly induces a Spin^c-structure. The study of complex manifolds using this structure is also too vast to survey here.

Let us consider next the case where X has an almost quaternionic structure. The so-called quaternionic Kähler manifolds are examples. The research in this paper started with the question whether such a manifold has a Spin-structure. The answer is apparently negative in general. However, to our joy, on the model of Spin^c-structure, the notion of Spin^q-structure can be consistently introduced by twisting the Spin group into the Spin^q group using the quaternionic unitary group $Sp(1)=Sp(1, \boldsymbol{H})=\{\lambda=\lambda_0+\lambda_1 i+\lambda_2 j+\lambda_3 k \mid \lambda\bar{\lambda}=1\}$ $(=SU(2))$ $(i^2=j^2=k^2=-1,\ ij=-ji=k)$. That is, we set $\mathrm{Spin}^q(n)\equiv\mathrm{Spin}(n)\times_{\boldsymbol{Z}_2} Sp(1)$ and have the short exact sequence

$$1 \longrightarrow \boldsymbol{Z}_2 \longrightarrow \mathrm{Spin}^q(n) \overset{\xi}{\longrightarrow} SO(n)\times SO(3) \longrightarrow 1\,,$$

where $\xi([\varphi, \lambda])=(\xi_0(\varphi), \mathrm{Ad}(\lambda))$. The *Spin^q-structure* on X is then a principal Spin$^q(n)$-bundle $P_{\mathrm{Spin}^q(n)}$ with a Spin$^q(n)$-equivariant bundle map $\xi: P_{\mathrm{Spin}^q(n)}\to P_{SO(n)}\times P_{SO(3)}$, where $P_{SO(3)}$ is an appropriate principal $SO(3)$-bundle. The existence is guaranteed by the condition that $w_2(X)$ coincides with the second Stiefel-Whitney class of some principal $SO(3)$-bundle. Hence, by the argument in [**11**] or [**14**], quaternionic Kähler manifolds certainly satisfy the condition (see (3.18)), just as in the relation between complex manifolds and Spin^c-structures.

The effectiveness of the idea is becoming clear, but only the basic parts are discussed here. Further discussions and applications will be given elsewhere. Finally I should like to add that the argument here follows the line of [**10**, Appendix D], in which Lawson and Michelsohn offer a clear and lucid explanation for the Spin^c-structure.

In §1 we review the Spin group and discuss the quaternionic representations. Just as real and complex ones for Spin and Spin^c, quaternionic ones are crucial for Spin^q. In §2, after defining Spin^q groups and discussing their real representations induced from the quaternionic ones, the strict definition of Spin^q-structure and its characterization in terms of second Stiefel-Whitney classes will be given. In §3 we will show that an almost quaternionic structure induces canonically a Spin^q-structure. In §4, using $P_{\mathrm{Spin}^q(n)}$ and a real representation $\Delta^q: \mathrm{Spin}^q(n)\to GL_{\boldsymbol{R}}(W)$, we construct the Spin^q-vector bundle $S=P_{\mathrm{Spin}^q(n)}\times_{\Delta^q} W$. The Dirac operator D on it will be also constructed and the Bochner-Weitzenböck type formula for the Dirac Laplacian D^2 will be extracted. Finally in §5, we will calculate particularly the index of the Dirac operator D^+ of a Spin^q-manifold of dimension $n\equiv 0 \pmod 4$.

Acknowledgements. I would like to thank Professor H. B. Lawson for his interest in my work and thank the referee for the careful reading of the manuscript.

§ 1. Spin groups and their fundamental $\boldsymbol{H}$-representations.

Let us denote by $Cl(n)$ the *Clifford algebra* associated to the n-dimensional Euclidean space $\boldsymbol{R}^n$ with the standard quadratic form $q(x)=\sum x_i^2$ ($x={}^t(x_1, \cdots, x_n)$): $x\cdot x=-q(x)$. The automorphism α of $Cl(n)$ which extends the map $\alpha(x)=-x$ on $\boldsymbol{R}^n$ gives the $\boldsymbol{Z}_2$-gradation

$$(1.1)\qquad Cl(n)=Cl^0(n)\oplus Cl^1(n), \qquad Cl^p(n)=\{\varphi\in Cl(n)\,|\,\alpha(\varphi)=(-1)^p\varphi\}.$$

On the other hand we consider the multiplicative group of units in $Cl(n)$ which is defined to be the subset

$$Cl^{\times}(n)=\{\varphi\in Cl(n)\,|\,\varphi^{-1}\varphi=\varphi\varphi^{-1}=1 \text{ for some } \varphi^{-1}\in Cl(n)\}.$$

Its subgroup generated by all $x\in\boldsymbol{R}^n$ with $q(x)=1$ is denoted by $\mathrm{Pin}(n)$, called the *Pin group*. Then the associated *Spin group* of $\boldsymbol{R}^n$ is defined by

$$(1.2)\qquad \mathrm{Spin}(n)=Cl^0(n)\cap\mathrm{Pin}(n).$$

For $\varphi\in\mathrm{Spin}(n)$ and $x\in\boldsymbol{R}^n$, we have $\mathrm{Ad}(\varphi)(x)=\varphi x\varphi^{-1}\in\boldsymbol{R}^n$ and $\xi_0(\varphi)\equiv\mathrm{Ad}(\varphi)\in SO(n)$. Further the map ξ_0 is surjective and its kernel is equal to $\boldsymbol{Z}_2=\{1, -1\}$, that is, we have the short exact sequence

$$(1.3)\qquad 1\longrightarrow\boldsymbol{Z}_2\longrightarrow\mathrm{Spin}(n)\overset{\xi_0}{\longrightarrow}SO(n)\longrightarrow 1.$$

If $n\geqq 3$, then, since $\pi_1(SO(n))=\boldsymbol{Z}_2$, $\mathrm{Spin}(n)$ is nothing but the universal covering group of $SO(n)$.

Now let $\boldsymbol{H}$ be the quaternion field $\{a_0+a_1i+a_2j+a_3k\,|\,a_l\in\boldsymbol{R}\}$ ($i^2=j^2=k^2=-1$, $ij=-ji=k$). Unless otherwise stated, an $\boldsymbol{H}$-vector space W means always a vector space with right multiplication of $\boldsymbol{H}$. In this section we will study an $\boldsymbol{H}$-representation of the real algebra $Cl(n)$,

$$(1.4)\qquad \mu\colon Cl(n)\longrightarrow\mathrm{End}_{\boldsymbol{H}}(W)$$

and collect some information about the $\boldsymbol{H}$-representation of $\mathrm{Spin}(n)$ given by the restriction, called a *fundamental* one if μ is irreducible,

$$(1.5)\qquad \Delta\colon\mathrm{Spin}(n)\longrightarrow GL_{\boldsymbol{H}}(W).$$

Let us first provide some lemmata. Set $K=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$. We denote by $K(N)$ the real algebra consisting of $N\times N$-matrices with entries in K and fix the identifications and embeddings,

$$(1.6)\qquad \boldsymbol{H}^N\overset{\iota}{\cong}\boldsymbol{C}^{2N}\overset{\iota'}{\cong}\boldsymbol{R}^{4N},\quad a_0+ia_1+j(a_2+ia_3)\longleftrightarrow\begin{pmatrix}a_0+ia_1\\a_2+ia_3\end{pmatrix}\longleftrightarrow\begin{pmatrix}a_0\\a_2\\a_1\\a_3\end{pmatrix},$$

(1.7) $$\boldsymbol{H}(N) \overset{\iota}{\hookrightarrow} \boldsymbol{C}(2N) \overset{\iota'}{\hookrightarrow} \boldsymbol{R}(4N),$$

$$S^0+iS^1+j(S^2+iS^3) \equiv Z+jW \overset{\iota}{\longmapsto} \begin{pmatrix} Z & -\overline{W} \\ W & \overline{Z} \end{pmatrix},$$

$$A+iB \overset{\iota'}{\longmapsto} \begin{pmatrix} A & -B \\ B & A \end{pmatrix},$$

which will be frequently used also in the following sections without quoting. Remark that (1.7) is obtained by embedding $\boldsymbol{H}(N)$ (acting on $\boldsymbol{H}^N$ from the left), etc. into $\boldsymbol{C}(2N)$ (acting on $\boldsymbol{C}^{2N}$ from the left), etc., through the identifications $\boldsymbol{H}^N=\boldsymbol{C}^{2N}$, etc. at (1.6), and induces the embeddings of the general linear groups

(1.8) $$GL(N, \boldsymbol{H}) \overset{\iota}{\hookrightarrow} GL(2N, \boldsymbol{C}) \overset{\iota'}{\hookrightarrow} GL(4N, \boldsymbol{R}).$$

Further, denoting by $\mathbf{1}=\mathbf{1}_N$ the unit matrix of $\boldsymbol{H}(N)$ (etc.) and putting $Sp(N)=Sp(N, \boldsymbol{H})=\{A\in\boldsymbol{H}(N) \mid {}^t\overline{A}A=\mathbf{1}\}$ called the *quaternionic unitary group* (or the *symplectic group*), we have

(1.9) $$Sp(N) \overset{\iota}{\cong} SU(2N)\cap Sp(2N, \boldsymbol{C}) \overset{\iota'}{\hookrightarrow} SO(4N).$$

LEMMA 1.1. (1) *$K(N)$ has only one irreducible $\boldsymbol{R}$-representation up to $\boldsymbol{R}$-equivalence—the standard one $\nu: K(N)\to(\mathrm{End}_K(K^N)\hookrightarrow)\,\mathrm{End}_{\boldsymbol{R}}(K^N)$.*

(2) *As for the set of $\boldsymbol{C}$-equivalence classes of irreducible $\boldsymbol{C}$-representations of $K(N)$:*

(i) *If $K=\boldsymbol{C}$, it consists of exactly two elements—the standard one $\nu: \boldsymbol{C}(N)\to\mathrm{End}_{\boldsymbol{C}}(\boldsymbol{C}^N)$, $\nu(A)(x)=Ax$, and its conjugate $\bar{\nu}$ given by $\bar{\nu}(A)(x)=\overline{A}x$. An arbitrarily given irreducible one ρ is equivalent to ν if $\rho(i\mathbf{1})=i\mathbf{1}$ and equivalent to $\bar{\nu}$ if $\rho(i\mathbf{1})=-i\mathbf{1}$.*

(ii) *If $K=\boldsymbol{R}$ or $\boldsymbol{H}$, it consists of only one element,*

$$\nu^{\boldsymbol{C}}: \boldsymbol{R}(N) \longrightarrow (\boldsymbol{C}\otimes\mathrm{End}_{\boldsymbol{R}}(\boldsymbol{R}^N)\hookrightarrow)\,\mathrm{End}_{\boldsymbol{C}}(\boldsymbol{C}^N), \qquad K=\boldsymbol{R},$$

$$\nu^{\boldsymbol{C}}: \boldsymbol{H}(N) \longrightarrow (\mathrm{End}_{\boldsymbol{H}}(\boldsymbol{H}^N)\overset{\iota}{\hookrightarrow})\,\mathrm{End}_{\boldsymbol{C}}(\boldsymbol{C}^{2N}), \qquad K=\boldsymbol{H}.$$

(3) *$K(N)$ has only one irreducible $\boldsymbol{H}$-representation up to $\boldsymbol{H}$-equivalence—the standard one $\nu^{\boldsymbol{H}}: K(N)\to\mathrm{End}_{\boldsymbol{H}}(\boldsymbol{H}^N)$, $\nu^{\boldsymbol{H}}(A)(x)=Ax$.*

PROOF. Refer to [8, Chap. 8] for the proofs of (1) and (2)(i). As for (2)(ii): Take an irreducible $\boldsymbol{C}$-representation $\rho: \boldsymbol{R}(N)\to\mathrm{End}_{\boldsymbol{C}}(W)$. It produces an irreducible one

$$\tilde{\rho}: \boldsymbol{C}(N) = \boldsymbol{C}\otimes\boldsymbol{R}(N) \longrightarrow \mathrm{End}_{\boldsymbol{C}}(W), \quad \tilde{\rho}(z\otimes A)(x) = z\rho(A)(x).$$

Since $\tilde{\rho}(i\mathbf{1})(x)=\tilde{\rho}(i\otimes\mathbf{1})(x)=ix$, it is equivalent to the standard one. Hence $\rho\cong\nu^{\boldsymbol{C}}$ because ρ is the restriction of $\tilde{\rho}$ to $\boldsymbol{R}(N)=\{1\}\otimes\boldsymbol{R}(N)$ $(\subset\boldsymbol{C}(N))$. Next we take an irreducible $\boldsymbol{C}$-one $\rho: \boldsymbol{H}(N) \to \mathrm{End}_C(W)$. By using the real algebra isomorphism

$$(1.10) \qquad \boldsymbol{C} \otimes_{\boldsymbol{R}} \boldsymbol{H} \cong \boldsymbol{C}(2), \quad z \otimes (\xi + j\eta) \longleftrightarrow \begin{pmatrix} z\bar{\xi} & z\bar{\eta} \\ -z\eta & z\xi \end{pmatrix},$$

it produces an irreducible one

$$\tilde{\rho}: \boldsymbol{C}(2N) = \boldsymbol{C}(2) \otimes \boldsymbol{R}(N) = \boldsymbol{C} \otimes_{\boldsymbol{R}} \boldsymbol{H} \otimes \boldsymbol{R}(N) = \boldsymbol{C} \otimes_{\boldsymbol{R}} \boldsymbol{H}(N) \longrightarrow \mathrm{End}_C(W).$$

Remark that the identification $\boldsymbol{C}(2N) = \boldsymbol{C}(2) \otimes \boldsymbol{R}(N)$ is given by the Kronecker product,

$$\begin{pmatrix} Ab_{11} & \cdots & Ab_{1N} \\ \vdots & & \vdots \\ Ab_{N1} & \cdots & Ab_{NN} \end{pmatrix} = A \otimes B.$$

If $\tilde{A} \in \boldsymbol{C}(2N)$ corresponds to $z \otimes A \in \boldsymbol{C} \otimes_{\boldsymbol{R}} \boldsymbol{H}(N)$, then $\tilde{\rho}(\tilde{A})(x)$ is defined to be $z\rho(A)(x)$. Since $\boldsymbol{C}(2N) \ni i\mathbf{1} \leftrightarrow i\mathbf{1} \otimes \mathbf{1} \leftrightarrow i \otimes 1 \otimes \mathbf{1} \leftrightarrow i \otimes \mathbf{1} \in \boldsymbol{C} \otimes_{\boldsymbol{R}} \boldsymbol{H}(N)$, we have $\tilde{\rho}(i\mathbf{1})(x) = ix$. Hence $\tilde{\rho}$ is equivalent to the standard one ν. Its restriction to $\boldsymbol{H}(N) = \{1\} \otimes_{\boldsymbol{R}} \boldsymbol{H}(N)$ ($\subset \boldsymbol{C}(2N)$) is obviously equivalent to ν^C. Therefore $\rho \cong \nu^C$. As for (3): Refer to [8, Chap. 8] for the case $K = \boldsymbol{H}$. Let us examine the case $K = \boldsymbol{C}$ or $\boldsymbol{R}$. Take an irreducible $\boldsymbol{H}$-one $\rho: K(N) \to \mathrm{End}_{\boldsymbol{H}}(W)$. Consider the generalization of (1.10)

$$(1.11) \qquad \boldsymbol{C}(N) \otimes_{\boldsymbol{R}} \boldsymbol{H} \cong \mathrm{End}'_C(\boldsymbol{H}^N) \cong \boldsymbol{C}(2N)$$

$$A \otimes \lambda \longleftrightarrow (x \mapsto Ax\bar{\lambda}) \longleftrightarrow \begin{pmatrix} A\bar{\xi} & A\bar{\eta} \\ -A\eta & A\xi \end{pmatrix} \equiv L(A \otimes \lambda)$$

and

$$(1.12) \qquad \boldsymbol{R}(N) \otimes \boldsymbol{H} \cong \mathrm{End}'_{\boldsymbol{H}}(\boldsymbol{H}^N) \cong \boldsymbol{H}(N)$$

$$A \otimes \lambda \longleftrightarrow (x \mapsto Ax\bar{\lambda}) \longleftrightarrow A\lambda.$$

Here $\mathrm{End}'_K(\boldsymbol{H}^N)$ ($K = \boldsymbol{C}$ or $\boldsymbol{H}$) means the real algebra consisting of endomorphisms of $\boldsymbol{H}^N$ on which K acts from the *left*. Through them the representation ρ produces

$$\tilde{\rho}: \boldsymbol{C}(2N) \text{ or } \boldsymbol{H}(N) = K(N) \otimes_{\boldsymbol{R}} \boldsymbol{H} \longrightarrow \mathrm{End}_{\boldsymbol{R}}(W).$$

These are obviously irreducible and, hence, respectively equivalent to the irreducible $\boldsymbol{R}$-ones

$$\nu: \boldsymbol{C}(2N) \longrightarrow \mathrm{End}_{\boldsymbol{R}}(\boldsymbol{C}^{2N}), \qquad \nu: \boldsymbol{H}(N) \longrightarrow \mathrm{End}_{\boldsymbol{R}}(\boldsymbol{H}^N).$$

That is, for example, if $K = \boldsymbol{C}$ then we have the commutative diagram,

$$\begin{array}{ccccc} W & \xrightarrow{f} & C^{2N} & \xleftarrow{\iota} & H^N \\ \tilde{\rho}(A\otimes\lambda)\Big\downarrow & & L(A\otimes\lambda)\Big\downarrow & & \Big\downarrow A\times,\ \times\bar{\lambda} \\ W & \xrightarrow{f} & C^{2N} & \xleftarrow{\iota} & H^N \end{array}$$

where $A\otimes\lambda\in \boldsymbol{C}(N)\otimes_{\boldsymbol{R}}\boldsymbol{H}$ and the maps f and ι are $\boldsymbol{R}$-linear isomorphisms. Further W is originally a right $\boldsymbol{H}$-vector space and the composition $h=\iota^{-1}\circ f$ is an $\boldsymbol{H}$-linear isomorphism. Indeed we have

$$\begin{aligned} h(x\lambda) &= h(\tilde{\rho}(\mathbf{1}\otimes\bar{\lambda})x) = \iota^{-1}(L(\mathbf{1}\otimes\bar{\lambda})f(x)) \\ &= (\iota^{-1}\circ f)(x)\lambda = h(x)\lambda\,. \end{aligned}$$

Hence the ρ, which is the restriction of $\tilde{\rho}$ to $\boldsymbol{C}(N)=\boldsymbol{C}(N)\otimes_{\boldsymbol{R}}\{1\}$ $(\subset \boldsymbol{C}(2N))$, is $\boldsymbol{H}$-equivalent to the standard one $\nu^{\boldsymbol{H}}$. The case $K=\boldsymbol{R}$ also can be shown similarly. ■

LEMMA 1.2. (1) *Any irreducible K-representation of $Cl^0(n)$ restricts to an irreducible K-representation of Spin(n).* (2) *Two K-representations of $Cl^0(n)$ are K-equivalent to each other if and only if so are the restrictions to Spin(n).*

PROOF. It suffices to show Spin(n) contains an additive basis for $Cl^0(n)$. Fix an orthonormal basis $\{e_1, \cdots, e_n\}$ of $\boldsymbol{R}^n$ and consider the basis $\{e_I=e_{i_1}e_{i_2}\cdots e_{i_p}\,|\,I=(i_1<\cdots<i_p)\}$ $(e_\emptyset=1)$ for $Cl(n)$ regarded as an $\boldsymbol{R}$-vector space. Then the subset $\{e_I\,|\,I=(i_1<\cdots<i_p),\ p\equiv 0 \pmod 2\}$ is certainly a basis of $Cl^0(n)$ and the elements belong to Spin(n). ■

Now, by the above lemmata and the well-known identification of the Clifford algebras with the matrix algebras (for example, [**10**, Chap. I. Theorem 4.3]), and with somewhat complicated but routine work, we can get Table I.

Table I.

n	$Cl(n)$	#	W	fundamental Δ
$8m$	$\boldsymbol{R}(2^{4m})$	1	$\boldsymbol{H}^{2^{4m}}$	a direct sum of two $\boldsymbol{H}$-inequiv. $\boldsymbol{H}$-irr. rep.
$1+8m$	$\boldsymbol{C}(2^{4m})$	1	$\boldsymbol{H}^{2^{4m}}$	$\boldsymbol{H}$-irreducible
$2+8m$	$\boldsymbol{H}(2^{4m})$	1	$\boldsymbol{H}^{2^{4m}}$	$\boldsymbol{H}$-irreducible
$3+8m$	$\boldsymbol{H}(2^{4m})\oplus\boldsymbol{H}(2^{4m})$	2	$\boldsymbol{H}^{2^{4m}}$	$\Delta^{\pm}$; $\boldsymbol{H}$-irreducible, $\boldsymbol{H}$-equivalent
$4+8m$	$\boldsymbol{H}(2^{1+4m})$	1	$\boldsymbol{H}^{2^{1+4m}}$	a direct sum of two $\boldsymbol{H}$-inequiv. $\boldsymbol{H}$-irr. rep.
$5+8m$	$\boldsymbol{C}(2^{2+4m})$	1	$\boldsymbol{H}^{2^{2+4m}}$	a direct sum of two $\boldsymbol{H}$-equiv. $\boldsymbol{H}$-irr. rep.
$6+8m$	$\boldsymbol{R}(2^{3+4m})$	1	$\boldsymbol{H}^{2^{3+4m}}$	a direct sum of two $\boldsymbol{H}$-equiv. $\boldsymbol{H}$-irr. rep.
$7+8m$	$\boldsymbol{R}(2^{3+4m})\oplus\boldsymbol{R}(2^{3+4m})$	2	$\boldsymbol{H}^{2^{3+4m}}$	$\Delta^{\pm}$; $\boldsymbol{H}$-irreducible, $\boldsymbol{H}$-equivalent

At the column #, the numbers of the $\boldsymbol{H}$-equivalence classes of irreducible $\boldsymbol{H}$-representations are placed. The vector spaces W are their representation spaces. If $n=3+8m$ or $7+8m$, then $\mu^{\pm}: K(N)\oplus K(N)\to \mathrm{End}_{\boldsymbol{H}}(\boldsymbol{H}^N)$, $\mu^{\pm}(G_+, G_-)(x)=G_{\pm}x$, represent the two irreducible $\boldsymbol{H}$-ones and restrict to $\Delta^{\pm}$.

Let us observe briefly the case $n=8m$ or $4+8m$, which is essential in §5. Set $N=2^{4m}$, 2^{1+4m} and $K=\boldsymbol{R}$, $\boldsymbol{H}$ respectively. Fix an orthonormal basis $\{e_1, \cdots, e_n\}$ of $\boldsymbol{R}^n$ and consider the volume element

$$\boldsymbol{e} = e_1e_2\cdots e_n \in Cl(n). \tag{1.13}$$

Then we have a real algebra isomorphism

$$Cl(n) \cong K(N), \qquad \boldsymbol{e} \longleftrightarrow \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{1.14}$$

where $\mathbf{1}$ is the unit matrix of $K(N/2)$. The identification and the standard $\boldsymbol{H}$-representation $\nu^{\boldsymbol{H}}$ at Lemma 1.1 (3) give the irreducible one

$$\mu: Cl(n) = K(N) \longrightarrow \mathrm{End}_{\boldsymbol{H}}(\boldsymbol{H}^N). \tag{1.15}$$

The element $\varphi\in Cl(n)$ belongs to $Cl^0(n)$ if and only if $\varphi\boldsymbol{e}=\boldsymbol{e}\varphi$. Hence, through (1.14), we have

$$Cl^0(n) = \left\{\begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}\in K(N) \,\middle|\, A_{\pm}\in K(N/2)\right\} \equiv K'(N). \tag{1.16}$$

Accordingly the restriction μ^0 of μ to $Cl^0(n)$ can be decomposed into

$$\begin{aligned} &\mu^0 = \mu^{0+}\oplus\mu^{0-}, \\ &\mu^{0\pm}: Cl^0(n) = K'(N) \longrightarrow \mathrm{End}_{\boldsymbol{H}}(\boldsymbol{H}^{N/2}), \quad \mu^{0\pm}\begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} = A_{\pm}. \end{aligned} \tag{1.17}$$

Further, obviously, $\mu^{0\pm}$ are $\boldsymbol{H}$-irreducible and $\boldsymbol{H}$-inequivalent to each other. They restrict to

$$\Delta_n = \Delta_n^+\oplus\Delta_n^-. \tag{1.18}$$

Lemma 1.2 now implies that so are $\Delta_n^{\pm}$.

§2. Spin^q groups and Spin^q structures.

Let us define the *Spin^q group* by twisting the Spin group using $Sp(1)$. That is, we set

$$\mathrm{Spin}^q(n) \equiv \mathrm{Spin}(n)\times_{Z_2}Sp(1) = \mathrm{Spin}(n)\times Sp(1)/\{\pm(1, 1)\} \tag{2.1}$$

with the group multiplication given by $[\varphi_1, \lambda_1][\varphi_2, \lambda_2]=[\varphi_1\varphi_2, \lambda_1\lambda_2]$.

By identifying naturally $\mathrm{Im}\,\boldsymbol{H}=\{a_1i+a_2j+a_3k\}$ with $\boldsymbol{R}^3$ (*i.e.*, $a_1i+a_2j+a_3k$

$\leftrightarrow {}^t(a_1, a_2, a_3)$) and $SO(\mathrm{Im}\,\boldsymbol{H})$ with $SO(3)$, we have the group homomorphism $\mathrm{Ad}: Sp(1) \to SO(3)$, $\mathrm{Ad}(\lambda)(x) = \lambda x \lambda^{-1}$ for $x \in \mathrm{Im}\,\boldsymbol{H}$. As is well-known, it is surjective and its kernel is equal to $\{\pm 1\}$. Thus, combined with (1.3), it implies the short exact sequence

$$1 \longrightarrow \boldsymbol{Z}_2 \longrightarrow \mathrm{Spin}^q(n) \overset{\xi}{\longrightarrow} SO(n) \times SO(3) \longrightarrow 1 \tag{2.2}$$

$$\xi([\varphi, \lambda]) = (\xi_0(\varphi), \mathrm{Ad}(\lambda))$$

$$\boldsymbol{Z}_2 = \{1 = [1, 1] = [-1, -1], [1, -1] = [-1, 1]\}.$$

Let us first examine the $\boldsymbol{R}$-representation of $\mathrm{Spin}^q(n)$ induced from (1.5), called a *fundamental* one if (1.4) is irreducible,

$$\Delta_n^q : \mathrm{Spin}^q(n) \longrightarrow GL_{\boldsymbol{R}}(W), \qquad \Delta_n^q([\varphi, \lambda])x = \Delta_n(\varphi)x\bar{\lambda}. \tag{2.3}$$

LEMMA 2.1. *Take two $\boldsymbol{H}$-representations μ, μ' of $Cl(n)$ and construct accordingly Δ_n, Δ_n^q, Δ_n', $\Delta_n'^q$ as above. Then we have*

(1) *Δ_n and Δ_n' are $\boldsymbol{H}$-equivalent to each other if and only if Δ_n^q and $\Delta_n'^q$ are $\boldsymbol{R}$-equivalent to each other,*

(2) *Δ_n is $\boldsymbol{H}$-irreducible if and only if Δ_n^q is $\boldsymbol{R}$-irreducible.*

PROOF. Express the representation spaces of μ, μ' by W, W'. As for (1): Assume that there exists an $\boldsymbol{H}$-linear isomorphism $f: W \to W'$ with $f(\Delta(\varphi)x) = \Delta'(\varphi)f(x)$ for all $\varphi \in \mathrm{Spin}(n)$ and $x \in W$. Then, for $[\varphi, \lambda] \in \mathrm{Spin}^q(n)$, we have $f(\Delta^q([\varphi, \lambda])x) = f(\Delta(\varphi)x\bar{\lambda}) = f(\Delta(\varphi)x)\bar{\lambda} = \Delta'(\varphi)f(x)\bar{\lambda} = \Delta'^q([\varphi, \lambda])f(x)$. This means that Δ^q and Δ'^q are $\boldsymbol{R}$-equivalent. Conversely, assume that there exists an $\boldsymbol{R}$-linear isomorphism $f: W \to W'$ with $f(\Delta^q([\varphi, \lambda])x) = \Delta'^q([\varphi, \lambda])f(x)$ for all $[\varphi, \lambda] \in \mathrm{Spin}^q(n)$ and $x \in W$. Then, by putting $\lambda = 1$ we have $f(\Delta(\varphi)x) = \Delta'(\varphi)f(x)$ and, by putting $\varphi = 1$ we have $f(x\lambda) = f(x)\lambda$. As for (2): Assume Δ is $\boldsymbol{H}$-irreducible. And set $W = W_1 \oplus W_2$, a decomposition into $\boldsymbol{R}$-linear $\mathrm{Spin}^q(n)$-invariant subspaces. Then the definition of $\mathrm{Spin}^q(n)$ asserts that each W_i is a right $\boldsymbol{H}$-linear subspace and is, further, $\mathrm{Spin}(n)$-invariant. Thus, by the assumption on Δ, we have $W_1 = \{0\}$ or $W_2 = \{0\}$. The converse can be shown similarly. ■

Hence Table I implies Table II.

Table II.

n	fundamental Δ^q
$8m$	a direct sum of two $\boldsymbol{R}$-inequiv. $\boldsymbol{R}$-irr. rep., $\Delta^q=\Delta^{q+}\oplus\Delta^{q-}$
$1+8m$	$\boldsymbol{R}$-irreducible
$2+8m$	$\boldsymbol{R}$-irreducible
$3+8m$	$\Delta^{q\pm}$; $\boldsymbol{R}$-irreducible, $\boldsymbol{R}$-equivalent
$4+8m$	a direct sum of two $\boldsymbol{R}$-inequiv. $\boldsymbol{R}$-irr. rep., $\Delta^q=\Delta^{q+}\oplus\Delta^{q-}$
$5+8m$	a direct sum of two $\boldsymbol{R}$-equiv. $\boldsymbol{R}$-irr. rep.
$6+8m$	a direct sum of two $\boldsymbol{R}$-equiv. $\boldsymbol{R}$-irr. rep.
$7+8m$	$\Delta^{q\pm}$; $\boldsymbol{R}$-irreducible, $\boldsymbol{R}$-equivalent

Next let us define and study the Spin^q-structure. We take a principal $SO(n)$-bundle $P_{SO(n)}$ over a manifold X.

DEFINITION 2.2. A *Spin^q-structure* on $P_{SO(n)}$ consists of a principal Spin$^q(n)$-bundle $P_{\mathrm{Spin}^q(n)}$ and a principal $SO(3)$-bundle $P_{SO(3)}$ over X together with a Spin$^q(n)$-equivariant bundle map

$$\xi: P_{\mathrm{Spin}^q(n)} \longrightarrow P_{SO(n)}\times P_{SO(3)},$$

i.e., $\xi(pg)=\xi(p)\xi(g)$ for all $p\in P_{\mathrm{Spin}^q(n)}$ and $g\in \mathrm{Spin}^q(n)$. The bundle $P_{SO(3)}$ is called its *fundamental class.*

Let us discuss the existence condition, etc.. The short exact sequence $1\to \boldsymbol{Z}_2 \to Sp(1)\overset{\mathrm{Ad}}{\to} SO(3)\to 1$ yields the long exact sequence of pointed sets:

$$\cdots \longrightarrow H^1(X;Sp(1)) \overset{\mathrm{Ad}}{\longrightarrow} H^1(X;SO(3)) \overset{\tilde{w}_2}{\longrightarrow} H^2(X;\boldsymbol{Z}_2) \tag{2.4}$$

We recall briefly the definition of the cohomology $H^1(X;G)$ with coefficients in some (possibly non-abelian) Lie group G; in general $H^1(X;G)$ is not a group. Take an open cover $\mathcal{U}=\{U_\alpha\}_{\alpha\in A}$ of X. The family {continuous $g_{\alpha\beta}: U_\alpha\cap U_\beta \to G\,|\, g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}\equiv 1$ in $U_\alpha\cap U_\beta\cap U_\gamma$ for all α, β, γ} is called a 1-cocycle on $\mathcal{U}$. We define two 1-cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ on $\mathcal{U}$ to be equivalent if there exists a "0-cochain" {continuous $g_\alpha: U_\alpha\to G$} such that $g'_{\alpha\beta}=g_\alpha^{-1}g_{\alpha\beta}g_\beta$ in $U_\alpha\cap U_\beta$ for all α, β. The set of equivalence classes of 1-cocycles on $\mathcal{U}$ is denoted by $H^1(\mathcal{U};G)$. $H^1(X;G)$ is then defined to be the inductive limit of $H^1(\mathcal{U};G)$ with respect to the refinement of open covers. Since $Sp(1)$ and $SO(3)$ are not abelian, $H^1(X;Sp(1))$ and $H^1(X;SO(3))$ may be just pointed sets, however with the distinguished elements (*i.e.*, $\{g_{\alpha\beta}\equiv 1\}$). By definition $H^1(X; SO(3))$ represents the equivalence classes of principal $SO(3)$-bundles over X, that is,

$$\text{PRIN}_{SO(3)}(X) \cong H^1(X; SO(3)), \tag{2.5}$$

and $\tilde{w}_2$ at (2.4) is just the second Stiefel-Whitney class. Next consider the long exact sequence induced by (2.2),

$$\cdots \to H^1(X; \text{Spin}^q(n)) \xrightarrow{\xi} H^1(X; SO(n)) \oplus H^1(X; SO(3)) \xrightarrow{w_2+\tilde{w}_2} H^2(X; \boldsymbol{Z}_2). \tag{2.6}$$

Here $w_2(P)$ is also the second Stiefel-Whitney class of the element P of $H^1(X; SO(n)) \cong \text{PRIN}_{SO(n)}(X)$. The sequences (2.4) and (2.6) imply

PROPOSITION 2.3. *$P_{SO(n)}$ has a Spin^q-structure if and only if $w_2(P_{SO(n)})$ belongs to the image of the map $\tilde{w}_2$.*

Further, since $\tilde{w}_2(P_{SO(3)})^2$ is the mod 2 reduction of the first Pontryagin class $p_1(P_{SO(3)})$, we have the following: *cf.* the characterization of the existence of Spin^c-structure in terms of characteristic classes.

COROLLARY 2.4. *If $P_{SO(n)}$ has a Spin^q-structure, then $w_2(P_{SO(n)})^2$ is the mod 2 reduction of an integral class.*

Remark that the converse may not hold in general.

Two Spin^q-structures $\xi^\alpha: P^\alpha_{\text{Spin}^q(n)} \to P_{SO(n)} \times P^\alpha_{SO(3)}$ ($\alpha=1, 2$) are defined to be equivalent if there exist Spin$^q(n)$-, $SO(3)$-, equivariant bundle isomorphisms $s: P^1_{\text{Spin}^q(n)} \to P^2_{\text{Spin}^q(n)}$, $t: P^1_{SO(3)} \to P^2_{SO(3)}$ with $\xi^2 \circ s = (\text{id.} \times t) \circ \xi^1$. Then we have

PROPOSITION 2.5. *If $P_{SO(n)}$ has a Spin^q-structure, then the set of equivalence classes of Spin^q-structures corresponds bijectively to* $\text{Ad}(H^1(X; Sp(1))) \oplus H^1(X; \boldsymbol{Z}_2)$.

PROOF. Take such a structure $\xi: P_{\text{Spin}^q(n)} \to P_{SO(n)} \times P_{SO(3)}$. Since $\tilde{w}_2(P_{SO(3)}) = w_2(P_{SO(n)})$, the fundamental class $P_{SO(3)} \in H^1(X; SO(3))$ can run within the subset $\text{Ker}\, \tilde{w}_2 = \text{Ad}\,(H^1(X; Sp(1)))$. Next we fix the class $P_{SO(3)} = P^0_{SO(3)}$ and examine how many $\xi: P_{\text{Spin}^q(n)} \to P_{SO(n)} \times P^0_{SO(3)}$ there are. The fibration $SO(n) \times SO(3) \xrightarrow{i} P_{SO(n)} \times P^0_{SO(3)} \xrightarrow{\pi} X$ induces the exact sequence

$$0 \longrightarrow H^1(X; \boldsymbol{Z}_2) \xrightarrow{\pi^*} H^1(P_{SO(n)} \times P^0_{SO(3)}; \boldsymbol{Z}_2) \xrightarrow{i^*} H^1(SO(n) \times SO(3); \boldsymbol{Z}_2).$$

This asserts that the set of such $P_{\text{Spin}^q(n)}$ bijectively corresponds to $\{\alpha \in H^1(P_{SO(n)} \times P^0_{SO(3)}; \boldsymbol{Z}_2) \mid i^*\alpha$ is equal to the double covering (2.2) $\in H^1(SO(n) \times SO(3); \boldsymbol{Z}_2)\} \cong H^1(X; \boldsymbol{Z}_2)$. ■

Let E be an oriented n-dimensional Riemannian vector bundle over a manifold. A *Spin^q-structure* on E means a Spin^q-structure on the principal $SO(n)$-

bundle $P_{SO(n)}(E)$ consisting of positively oriented orthonormal frames. Remark that the choice of Spin^q-structure for the fibre metric on E canonically determines a Spin^q-structure for any metric. Here we are using the fact that, for any metric, the inclusion from $P_{SO(n)}(E)$ to the bundle $P_{GL+(n)}(E)$ of all positively oriented frames is a homotopy-equivalence. A *Spin^q-manifold* means an oriented Riemannian manifold with a Spin^q-structure on its tangent bundle.

Since the second Stiefel-Whitney class of a bundle $P_{SO(n)}$ which carries a Spin-structure $\xi_0 : P_{\mathrm{Spin}(n)} \to P_{SO(n)}$ vanishes, it has a canonically determined Spin^q-structure. It is obtained by setting $P_{SO(3)} = X \times SO(3)$, $P_{Sp(1)} = X \times Sp(1)$, trivial bundles, and considering the natural map

$$\xi : P_{\mathrm{Spin}^q(n)} \equiv P_{\mathrm{Spin}(n)} \times_{Z_2} P_{Sp(1)} \longrightarrow P_{SO(n)} \times P_{SO(3)} . \tag{2.7}$$

In the next section we will provide an example which is not trivial as above and for which the Spin^q-structure theory will be essentially effective.

§3. Almost quaternionic structures and Spin^q-structures.

Let $E \to X$ be an n-dimensional real vector bundle over a manifold. In this section we will show that an almost quaternionic structure induces canonically a Spin^q-structure.

The *almost quaternionic structure* V is a 3-dimensional subbundle of $E^* \otimes E$ which has locally a basis $\{I, J, K\}$ satisfying the familiar identities

$$I^2 = J^2 = K^2 = -\mathrm{id.}, \qquad IJ = -JI = K , \tag{3.1}$$

called its canonical local basis. Remark that the endomorphism $\phi \in V$ is assumed to act on E from the right: $(IJ)(e) = J(I(e))$. For details refer to Gray [**7**], Ishihara [**9**] and Salamon [**14, 15, 16**], etc..

Assume E has such a structure V. The fibre E_p can be regarded as a right $\boldsymbol{H}$-vector space by identifying the actions of $i, j, k \in \boldsymbol{H}$ with the actions of I, J, K. Take a basis $\{e_1, e_2, \cdots, e_N\}$ of the right $\boldsymbol{H}$-vector space E_p. Then $\{e_1, \cdots, e_N, Ie_1, \cdots, Ie_N, Je_1, \cdots, Je_N, Ke_1, \cdots, Ke_N\}$ forms a basis of E_p as an $\boldsymbol{R}$-vector space. Hence $n = 4N$. Moreover, since two canonical local bases $\{I, J, K\}$ on U, $\{I', J', K'\}$ on U' have the relation

$$\begin{pmatrix} I' \\ J' \\ K' \end{pmatrix} = S \begin{pmatrix} I \\ J \\ K \end{pmatrix} \qquad \text{for some } S \in C^\infty(U \cap U', SO(3)), \tag{3.2}$$

E can be regarded as being oriented by the above ordered basis. Let us next discuss the reduction of its structure group. We embed $GL(N, \boldsymbol{H})$ into $GL(4N, \boldsymbol{R})$ by (1.8) and also embed $GL(1, \boldsymbol{H})$ acting on $\boldsymbol{H}^N$ from the right into $GL(4N, \boldsymbol{R})$ through (1.6),

(3.3)
$$\iota'' : \lambda = \lambda_0+\lambda_1 i+\lambda_2 j+\lambda_3 k \longmapsto \begin{pmatrix} \lambda_0\mathbf{1} & -\lambda_2\mathbf{1} & -\lambda_1\mathbf{1} & \lambda_3\mathbf{1} \\ \lambda_2\mathbf{1} & \lambda_0\mathbf{1} & -\lambda_3\mathbf{1} & -\lambda_1\mathbf{1} \\ \lambda_1\mathbf{1} & \lambda_3\mathbf{1} & \lambda_0\mathbf{1} & \lambda_2\mathbf{1} \\ -\lambda_3\mathbf{1} & \lambda_1\mathbf{1} & -\lambda_2\mathbf{1} & \lambda_0\mathbf{1} \end{pmatrix}.$$

Thus embedded $GL(N, \boldsymbol{H})$ and $GL(1, \boldsymbol{H})$ generate the subgroup $GL(N, \boldsymbol{H})\cdot GL(1, \boldsymbol{H})\equiv GL(N, \boldsymbol{H})\times_{Z_2} GL(1, \boldsymbol{H})$ of $GL(4N, \boldsymbol{R})$, with the group multiplication given by $[G_1, \lambda_1][G_2, \lambda_2]=[G_1G_2, \lambda_2\lambda_1]$. In particular we consider its subgroup

(3.4)
$$Sp(N)\cdot Sp(1) \equiv Sp(N)\times_{Z_2} Sp(1).$$

Similarly to [**4**, 14.61] for instance, the following can be shown easily.

LEMMA 3.1. *The vector bundle E admits an almost quaternionic structure V if and only if $n=4N$ and the structure group $GL(4N, \boldsymbol{R})$ of E can be reduced to $Sp(N)\cdot Sp(1)$.*

We take and fix a fibre metric g on (E^{4N}, V) satisfying

(3.5)
$$g(ve, e')+g(e, ve') = 0$$

for all $v\in V$, $e, e'\in E$. This certainly exists and is called a *quaternion-Hermitian metric*. By the lemma, (V, g) gives a reduction of the frame bundle of E to a principal $Sp(N)\cdot Sp(1)$-bundle

(3.6)
$$P_{Sp(N)\cdot Sp(1)}(E, V, g) \longrightarrow X .$$

On the other hand, by regarding S at (3.2) as the transition function $\Phi_{U'U}$ from U to U', the family $\{\Phi_{U'U}\}$ defines a principal $SO(3)$-bundle

(3.7)
$$P_{SO(3)}(E, V) \longrightarrow X .$$

Using the principal bundles and the following group homomorphism Ξ which makes the diagram commutative, we will show later that (V, g) induces a canonical Spin^q-structure on E.

(3.8)
$$\begin{array}{ccc} & & \mathrm{Spin}^q(4N) \\ & \overset{\Xi}{\nearrow} & \Big\downarrow \xi \\ Sp(N)\cdot Sp(1) & \xrightarrow{\ inc.\times \mathrm{Ad}'\ } & SO(4N)\times SO(3) \end{array}$$

The map $inc.: Sp(N)\cdot Sp(1)\hookrightarrow SO(4N)$ is the inclusion map given above and $\mathrm{Ad}'([G, \lambda])a=\lambda^{-1}a\lambda$ (if N is odd), $=a$ (if N is even) for $a\in \mathrm{Im}\,\boldsymbol{H}=\boldsymbol{R}^3$. To construct Ξ we will give a lemma.

LEMMA 3.2. *There exists a group homomorphism $\tilde{\Xi}$ from $Sp(N)\times Sp(1)$ to $Spin(4N)$ such that the following diagram commutes*:

$$(3.9)\qquad \begin{array}{ccc} Sp(N)\times Sp(1) & \longrightarrow & \mathrm{Spin}(4N) \\ \downarrow & & \downarrow \xi_0 \\ Sp(N)\cdot Sp(1) & \xrightarrow{\ inc.\ } & SO(4N) \end{array}$$

PROOF. In general, for $S\in Sp(N)$, there exists $P\in Sp(N)$ and $\nu_l=\exp(i\theta_l)$ $(\in U(1)\subset Sp(1))$ $(l=1, \cdots, N)$ such that, referring to (1.9), we have

$$(3.10)\qquad P^{-1}SP=\begin{pmatrix} \nu_1 & & 0 \\ & \ddots & \\ 0 & & \nu_N \end{pmatrix}, \quad \iota(P)^{-1}\iota(S)\iota(P)=\begin{pmatrix} \nu_1 & & & & & \\ & \ddots & & & 0 & \\ & & \nu_N & & & \\ & & & \bar{\nu}_1 & & \\ & 0 & & & \ddots & \\ & & & & & \bar{\nu}_N \end{pmatrix}.$$

Indeed, since $\iota(S)$ is a unitary matrix, there exists a unitary matrix U such that $U^{-1}\iota(S)U$ is diagonal. Further such a matrix U can be carefully chosen so that it belongs to $SU(2N)\cap Sp(2N, \boldsymbol{C})$ and the diagonal elements of $U^{-1}\iota(S)U$ are arranged in the order $\nu_1, \nu_2, \cdots, \nu_N, \bar{\nu}_1, \bar{\nu}_2, \cdots, \bar{\nu}_N$. Hence $P=\iota^{-1}(U)$ satisfies (3.10). Now we set $P=(u_1, \cdots, u_N)$ and $Iu_l=u_l i$, $Ju_l=u_l j$, $Ku_l=u_l k$. Then the element

$$(3.11)\qquad \tilde{\Xi}(S)=\prod_{l=1}^{N}\Big(\cos\frac{\theta_l}{2}+\sin\frac{\theta_l}{2}u_l Iu_l\Big)\Big(\cos\frac{\theta_l}{2}+\sin\frac{\theta_l}{2}Ju_l Ku_l\Big)$$

belongs to Spin$(4N)$. It is easily verified that this element does not change when we replace θ_l by $\theta_l+2\pi$ and, moreover, it does not depend on the choice of P. Further we have

$$(3.12)\qquad \xi_0(\tilde{\Xi}(S))=\iota'\iota(S)\,, \qquad \tilde{\Xi}(S)\tilde{\Xi}(T)=\tilde{\Xi}(ST)\,.$$

The first identity implies the second. Indeed the second identity holds for $T=\mathbf{1}$ and the first identity asserts $\xi_0(\tilde{\Xi}(S)\tilde{\Xi}(T))=\xi_0(\tilde{\Xi}(S))\xi_0(\tilde{\Xi}(T))=\iota'\iota(S)\iota'\iota(T)=\iota'\iota(ST)=\xi_0(\tilde{\Xi}(ST))$. That is, $\xi_0=\mathrm{Ad}$: Spin$(4N)\to SO(4N)$ sends both $\tilde{\Xi}(S)\tilde{\Xi}(T)$ and $\tilde{\Xi}(ST)$ to the same element. Therefore, by taking a curve from $\mathbf{1}$ to T in $Sp(N)$, they turn out to be equal to each other. Let us prove the first identity. It will suffice to show the case $N=1$. Put $P=(u_1)=(u)$, $\theta_1=\theta$. Then the matrix representation of $\iota'\iota(S)$ with respect to the ordered basis $\{u, Ju, Iu, -Ku\}$ is

$$\iota'\iota(P^{-1}SP)=\begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

On the other hand, by straightforward computation, we have

$$\begin{aligned}&\xi_0(\tilde{\Xi}(S))(x_0u+x_1Ju+x_2Iu+x_3(-Ku))\\&=(x_0\cos\theta-x_2\sin\theta)u+(x_1\cos\theta+x_3\sin\theta)Ju\\&\quad+(x_0\sin\theta+x_2\cos\theta)Iu+(-x_1\sin\theta+x_3\cos\theta)(-Ku).\end{aligned}$$

Thus the first identity is proved. Now, the action of $[G,\lambda]\in Sp(N)\cdot Sp(1)$ to $x\in \boldsymbol{H}^N$ is given by $[G,\lambda]x=Gx\lambda=G\cdot\lambda\mathbf{1}\cdot\mathrm{Ad}(\lambda^{-1})(x)$. (3.12) asserts that $\xi_0(\tilde{\Xi}(G))=\iota'\iota(G)$ and $\xi_0(\tilde{\Xi}(\lambda\mathbf{1}))=\iota'\iota(\lambda\mathbf{1})$. Using the standard basis $\{e_1,\cdots,e_N\}$ of $\boldsymbol{H}^N$, next let us define the element $\Xi_A(\lambda)$ of Spin$(4N)$ corresponding to $\lambda=\lambda_0+\lambda_1 i+\lambda_2 j+\lambda_3 k$ $(\in Sp(1))$ by

$$\Xi_A(\lambda)=\prod_{l=1}^{N}(\lambda_0-\lambda_1 Je_lKe_l-\lambda_2 Ke_lIe_l-\lambda_3 Ie_lJe_l), \tag{3.13}$$

which satisfies

$$\xi_0(\Xi_A(\lambda))=\iota'\iota(\lambda^{-1}\mathbf{1})\cdot\iota''(\lambda). \tag{3.14}$$

It will suffice to observe also the case $N=1$; the subscript l is removed. For $x={}^t(x_0,x_1,x_2,x_3)=x_0e+x_1Ie+x_2Je+x_3Ke\in\boldsymbol{R}^4$, we have

$$\begin{aligned}\xi_0(\Xi_A(\lambda))(x)=\ &x_0e+\{(\lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2)x_1+2(\lambda_0\lambda_3+\lambda_1\lambda_2)x_2+2(\lambda_1\lambda_3-\lambda_0\lambda_2)x_3\}Ie\\&+\{2(\lambda_1\lambda_2-\lambda_0\lambda_3)x_1+(\lambda_0^2-\lambda_1^2+\lambda_2^2-\lambda_3^2)x_2+2(\lambda_0\lambda_1+\lambda_2\lambda_3)x_3\}Je\\&+\{2(\lambda_0\lambda_2+\lambda_1\lambda_3)x_1+2(\lambda_2\lambda_3-\lambda_0\lambda_1)x_2+(\lambda_0^2-\lambda_1^2-\lambda_2^2+\lambda_3^2)x_3\}Ke\end{aligned}$$

and $\lambda^{-1}(x_0+x_1i+x_2j+x_3k)\lambda$ which corresponds to $\iota'\iota(\lambda^{-1}\mathbf{1})\cdot\iota''(\lambda)(x)\in\boldsymbol{R}^4$ is easily shown to be equal to the right hand side with e, Ie, Je, Ke replaced by 1, i, j, k, respectively. That is, (3.14) is proved. Then, by setting

$$\Xi_r(\lambda)=\tilde{\Xi}(\lambda\mathbf{1})\Xi_A(\lambda), \tag{3.15}$$

the desired map $\bar{\Xi}$ is defined to be

$$\bar{\Xi}(G,\lambda)=\tilde{\Xi}(G)\Xi_r(\lambda). \tag{3.16}$$

The diagram (3.9) is obviously commutative. Moreover, since we have $\bar{\Xi}(H,\eta)\cdot\bar{\Xi}(G,\lambda)=\tilde{\Xi}(H)\Xi_r(\eta)\tilde{\Xi}(G)\Xi_r(\lambda)$ and $\bar{\Xi}((H,\eta)(G,\lambda))=\tilde{\Xi}(H)\tilde{\Xi}(G)\Xi_r(\eta)\Xi_r(\lambda)$, in order to show that $\bar{\Xi}$ is a group homomorphism, it suffices to prove $\tilde{\Xi}(G)\Xi_r(\lambda)=\Xi_r(\lambda)\tilde{\Xi}(G)$, which can be proved along the same lines as that of the second identity in (3.12). ∎

Now, the homomorphism Ξ can be constructed immediately by

$$\Xi([G,\lambda])=\begin{cases}[\bar{\Xi}(G,\lambda),\lambda^{-1}] & N,\ \text{odd}\\ [\bar{\Xi}(G,\lambda),1] & N,\ \text{even}.\end{cases} \tag{3.17}$$

Note that $\bar{\Xi}(-G, -\lambda)=\mp\bar{\Xi}(G, \lambda)$.

THEOREM 3.3. *The vector bundle E^{4N} with an almost quaternionic structure V and with also a fixed quaternion-Hermitian metric g admits a Spin^q-structure given by*

$$P_{\mathrm{Spin}^q(4N)} \equiv P_{Sp(N)\cdot Sp(1)}(E, V, g)\times_{\bar{\Xi}} \mathrm{Spin}^q(4N),$$

$$P_{SO(3)} \equiv P_{Sp(N)\cdot Sp(1)}(E, V, g)\times_{\mathrm{Ad}'} SO(3) \ (=P_{SO(3)}(E, V) \textit{ if } N \textit{ is odd}),$$

together with the natural Spin^q(4N)-equivariant bundle map

$$\xi: P_{\mathrm{Spin}^q(4N)} \longrightarrow P_{SO(4N)}(E, g)\times P_{SO(3)}\,.$$

PROOF. Since $P_{SO(4N)}(E, g)=P_{Sp(4N)\cdot Sp(1)}(E, V, g)\times_{inc.} SO(4N)$, the map $inc.\times\mathrm{Ad}'$ at (3.8) induces the map ξ. ■

Finally let us make a brief comment on the quaternionic Kähler manifold (X^{4N}, V, g), which is a manifold X^{4N} with an almost quaternionic structure V and a quaternion-Hermitian metric g on its tangent bundle and whose holonomy group $Hol(X, g)$ is contained in $Sp(N)\cdot Sp(1)$. [**14**, Proposition 2.3] asserts, by using the notations in § 2,

$$w_2(P_{SO(4N)}(TX, g)) = N\tilde{w}_2(P_{SO(3)}(TX, V))\,. \tag{3.18}$$

This means that (X, V) satisfies the condition of Proposition 2.3 and turns out, with no concrete construction, to admit a Spin^q-structure. Further it is known, referring to [**18**] or [**14**, § 5], that there exist some examples (some Wolf spaces) with $\tilde{w}_2(P_{SO(3)}(TX, V))\neq 0$ and $N\equiv 1 \pmod 2$, that is, there exist quaternionic Kähler manifolds which admit no Spin-structure. (To prevent confusion X is assumed, according to [**14**], to be a quaternionic Kähler manifold. But the assumption $Hol(X, g)\subset Sp(N)\cdot Sp(1)$ was apparently not used in the proof of [**14**, Proposition 2.3]. That is, (3.18) holds for any almost quaternionic manifold.)

§ 4. Spin^q-vector bundles and the Dirac operator.

Let $P_{SO(n)}$ be a principal $SO(n)$-bundle over a manifold X with a Spin^q-structure $\xi: P_{\mathrm{Spin}^q(n)}\to P_{SO(n)}\times P_{SO(3)}$. We consider a real vector bundle, called a *Spin^q-vector bundle,*

$$S = P_{\mathrm{Spin}^q(n)}\times_{\Delta^q} W \tag{4.1}$$

associated to an $\boldsymbol{R}$-representation (2.3) of Spin$^q(n)$. Take an open cover $\{U_\alpha\}_{\alpha\in A}$ of X so that $U_{\alpha_1}\cap\cdots\cap U_{\alpha_l}$ is contractible for all $\alpha_1, \cdots, \alpha_l$.

In general it is impossible to construct principal bundles $P_{\mathrm{Spin}(n)}$, $P_{Sp(1)}$ such that $P_{\mathrm{Spin}^q(n)}=P_{\mathrm{Spin}(n)}\times_{\boldsymbol{Z}_2} P_{Sp(1)}$. We inquire first into the obstructions to their existence. Let $f=\{f_{\alpha\beta}: U_\alpha\cap U_\beta\to SO(n)\}$ be the transition functions for $P_{SO(n)}$: $f_{\alpha\beta}f_{\beta\gamma}f_{\gamma\alpha}\equiv 1$ on $U_\alpha\cap U_\beta\cap U_\gamma$. The "transition functions"

$$f' = \{f'_{\alpha\beta}: U_\alpha\cap U_\beta \to \mathrm{Spin}(n)\} \qquad \text{with } \xi_0\circ f'_{\alpha\beta}=f_{\alpha\beta} \tag{4.2}$$

construct the Spin-structure $P_{\mathrm{Spin}(n)}$ if and only if the Čech cocycle

$$w' = \{w'_{\alpha\beta\gamma}=f'_{\alpha\beta}f'_{\beta\gamma}f'_{\gamma\alpha}: U_\alpha\cap U_\beta\cap U_\gamma \to \boldsymbol{Z}_2(\subset \mathrm{Spin}(n))\} \tag{4.3}$$

is equal to $\{1\}$. The long exact sequence

$$\cdots \longrightarrow H^1(X; \mathrm{Spin}(n)) \xrightarrow{\xi_0} H^1(X; SO(n)) \xrightarrow{\delta=w_2} H^2(X; \boldsymbol{Z}_2) \tag{4.4}$$

associated to (1.3) then asserts, since $\delta([f])=[w']$, that if $[w']=0$ we can take f' with $w'=\{1\}$. Thus *the class* $[w']\in H^2(X; \boldsymbol{Z}_2)$ *is nothing but the obstruction to the existence of the Spin-structure* $P_{\mathrm{Spin}(n)}$. Similarly, let $h=\{h_{\alpha\beta}: U_\alpha\cap U_\beta \to SO(3)\}$ be the transition functions for $P_{SO(3)}$: $h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha}\equiv 1$ on $U_\alpha\cap U_\beta\cap U_\gamma$. The "transition functions"

$$\tilde{h} = \{\tilde{h}_{\alpha\beta}: U_\alpha\cap U_\beta \to Sp(1)\} \qquad \text{with } \mathrm{Ad}\circ\tilde{h}_{\alpha\beta}=h_{\alpha\beta} \tag{4.5}$$

construct the bundle $P_{Sp(1)}$ if and only if the Čech cocycle

$$\tilde{w} = \{\tilde{w}_{\alpha\beta\gamma}=\tilde{h}_{\alpha\beta}\tilde{h}_{\beta\gamma}\tilde{h}_{\gamma\alpha}: U_\alpha\cap U_\beta\cap U_\gamma \to \boldsymbol{Z}_2(\subset Sp(1))\} \tag{4.6}$$

is equal to $\{1\}$. The long exact sequence (2.4) then asserts, similarly to the above, that *the class* $[\tilde{w}]\in H^2(X; \boldsymbol{Z}_2)$ *is nothing but the obstruction to the existence of the desired bundle* $P_{Sp(1)}$.

It is then clear because of the existence of Spin^q-structure that the obstructions $[w']$ and $[\tilde{w}]$ agree, *i.e.*,

$$[w']+[\tilde{w}] = 0 \qquad (\text{in } H^2(X; \boldsymbol{Z}_2)). \tag{4.7}$$

Hence, adjusting by coboundaries, we can choose $f'=\{f'_{\alpha\beta}\}$ and $\tilde{h}=\{\tilde{h}_{\alpha\beta}\}$ so that $w'_{\alpha\beta\gamma}\equiv\tilde{w}_{\alpha\beta\gamma}$ for all α, β, γ. Thus the transition functions

$$F = \{F_{\alpha\beta}\equiv f'_{\alpha\beta}\times\tilde{h}_{\alpha\beta}: U_\alpha\cap U_\beta \to \mathrm{Spin}(n)\times_{\boldsymbol{Z}_2}Sp(1)=\mathrm{Spin}^q(n)\} \tag{4.8}$$

satisfy $F_{\alpha\beta}F_{\beta\gamma}F_{\gamma\alpha}\equiv 1$ on $U_\alpha\cap U_\beta\cap U_\gamma$, and thus determine the global bundle $P_{\mathrm{Spin}^q(n)}$ and the vector bundle (4.1). That is, while we may not construct globally the vector bundles, $S_*=P_{\mathrm{Spin}(n)}\times_\Delta W$—the so-called spinor vector bundle associated to f' and the representation (1.5)—, and $V_H=P_{Sp(1)}\times_{can.}\boldsymbol{H}$ associated to $\tilde{h}$ and the canonical representation $can.: Sp(1)\to GL_H(\boldsymbol{H})$ by left multiplication, both of them exist locally and the $\boldsymbol{H}$-tensor product of S_* and $\overline{V}_H$ (the conjugate of V_H) exists globally and is, moreover, equal to S, *i.e.*,

$$S = S_*\otimes_H \overline{V}_H\,. \tag{4.9}$$

Now let X be an n-dimensional Spin^q-manifold with a fixed metric g. The bundle $P_{SO(n)}$ consisting of positively oriented orthonormal frames carries a Spin^q-structure $\xi: P_{\mathrm{Spin}^q(n)} \to P_{SO(n)} \times P_{SO(3)}$. Accordingly the Spin^q-vector bundle (4.1) is defined. $P_{SO(n)}$ carries the Levi-Civita connection $\omega^R = \{\omega_\alpha^R \in \Gamma(\mathfrak{so}(n) \otimes T^*X | U_\alpha)\}$ associated to the metric g. We fix here a connection $\omega = \{\omega_\alpha \in \Gamma(\mathfrak{so}(3) \otimes T^*X | U_\alpha)\}$ of $P_{SO(3)}$. Then we get the direct sum connection $\omega^R \oplus \omega$ on $P_{SO(n)} \times P_{SO(3)}$. By lifting it, we have a connection $\tilde{\omega} = \{\tilde{\omega}_\alpha \in \Gamma(\mathfrak{spin}^q(n) \otimes T^*X | U_\alpha)\}$ of $P_{\mathrm{Spin}^q(n)}$, which is well-defined globally. The identifications of Lie algebras

$$\mathfrak{spin}^q(n) = \mathfrak{spin}(n) \oplus \mathfrak{sp}(1) \overset{\xi_*}{\cong} \mathfrak{so}(n) \oplus \mathfrak{so}(3) \tag{4.10}$$

$$\tilde{\omega} = \tilde{\omega}^R \oplus \omega_H \longleftrightarrow \omega^R \oplus \omega$$

are convenient: $\omega_H = \{\omega_{H,\alpha} \in \Gamma(\mathfrak{sp}(1) \otimes T^*X | U_\alpha)\}$. The aims below are to define the so-called Dirac operator D and to get the Bochner-Weitzenböck type formula for the Dirac Laplacian D^2.

The connection $\tilde{\omega}$ induces the covariant derivative on S,

$$\nabla : \Gamma(S) \longrightarrow \Gamma(S \otimes T^*X). \tag{4.11}$$

Consider the Clifford bundle

$$Cl(X) = P_{SO(n)} \times_{cl} Cl(n) \tag{4.12}$$

associated to the canonical representation $cl: SO(n) \to \mathrm{Aut}(Cl(n))$. It carries the covariant derivative

$$\nabla^{cl} : \Gamma(Cl(X)) \longrightarrow \Gamma(Cl(X) \otimes T^*X) \tag{4.13}$$

induced from the connection ω^R. Using the representation $cl \circ p \circ \xi: \mathrm{Spin}^q(n) \to \mathrm{Aut}(Cl(n))$ where $p: SO(n) \times SO(3) \to SO(n)$ is the projection, it follows that

$$Cl(X) = P_{\mathrm{Spin}^q(n)} \times_{cl \circ p \circ \xi} Cl(n). \tag{4.14}$$

Hence S is naturally a bundle of left modules over the bundle of algebras $Cl(X)$. Moreover, one can easily verify that the covariant derivative on S is a module derivation, *i.e.*,

$$\nabla(\varphi\sigma) = \nabla^{cl}(\varphi)\sigma + \varphi\nabla(\sigma) \tag{4.15}$$

for all $\varphi \in \Gamma(Cl(X))$ and all $\sigma \in \Gamma(S)$.

Let us introduce a fibre metric on S. We take an inner product $\langle\, ,\rangle$ on the real vector space W satisfying $\langle ew, ew' \rangle = \langle w, w' \rangle = \langle w\lambda, w'\lambda \rangle$ for all $w, w' \in W$, $e \in \boldsymbol{R}^n$ with $\|e\| = 1$ and $\lambda \in Sp(1)$. This certainly exists and induces a fibre metric $\langle\, ,\rangle = \langle\, ,\rangle_S$ on S which satisfies

$$\langle e\sigma_1, e\sigma_2 \rangle = \langle \sigma_1, \sigma_2 \rangle \tag{4.16}$$

for all $\sigma_1, \sigma_2 \in S_x$ and all $e \in T_xX \subset Cl_x(X)$ with $e^2 = -1$. Take a basis $\{w_1, \cdots, w_N\}$ of W as a right $\boldsymbol{H}$-vector space and consider the ordered basis $\{w_1, \cdots, w_N, w_1 i, \cdots, w_N i, w_1 j, \cdots, w_N j, w_1 k, \cdots, w_N k\}$ of W as an $\boldsymbol{R}$-vector space. Then obviously we have $\Delta^q : \mathrm{Spin}^q(n) \to SO(W, \langle , \rangle)$ and hence $\Delta^q_* : \mathfrak{spin}^q(n) \to \mathfrak{so}(W, \langle , \rangle)$. This means that

$$e\langle \sigma_1, \sigma_2 \rangle = \langle \nabla_e \sigma_1, \sigma_2 \rangle + \langle \sigma_1, \nabla_e \sigma_2 \rangle \tag{4.17}$$

holds for all $e \in TX$ and $\sigma_1, \sigma_2 \in \Gamma(S)$.

Then, under these preparations, we can define a first-order differential operator called the *Dirac operator*

$$D : \Gamma(S) \longrightarrow \Gamma(S), \qquad D\sigma = \sum_{a=1}^{n} e_a \cdot \nabla_{e_a} \sigma , \tag{4.18}$$

where, at $x \in X$, $\{e_1, \cdots, e_n\}$ is an orthonormal basis of T_xX. Note that it does not depend on the choice of the basis. Its principal symbol is given by $\sigma_\xi(D) = \sqrt{-1}\xi$ ($\xi \in T^*X$) and hence it is elliptic. Furthermore, with respect to the inner product of $\Gamma(S)$ given by

$$(\sigma_1, \sigma_2) = \int_X \langle \sigma_1, \sigma_2 \rangle dg \tag{4.19}$$

where dg is the volume element of (X, g), it is formally self-adjoint, *i.e.*, $(D\sigma_1, \sigma_2) = (\sigma_1, D\sigma_2)$ for all compactly supported $\sigma_1, \sigma_2 \in \Gamma(S)$. This follows easily from (4.15)-(4.17).

Let us extract the Bochner-Weitzenböck type formula. To any tangent vector fields V_1 and V_2 on X, we associate an invariant second derivative $\nabla^2_{V_1, V_2} : \Gamma(S) \to \Gamma(S)$ by setting $\nabla^2_{V_1, V_2}\sigma = \nabla_{V_1}\nabla_{V_2}\sigma - \nabla_{\nabla^R_{V_1} V_2}\sigma$, where ∇^R is the Riemannian covariant derivative on (X, g). Recall that this is a tensorial operator in the variables V_1, V_2. The connection Laplacian $\nabla^*\nabla$ is defined by taking the trace, *i.e.*,

$$\nabla^*\nabla : \Gamma(S) \longrightarrow \Gamma(S), \qquad \nabla^*\nabla\sigma = -\mathrm{trace}\,(\nabla^2_{\cdot,\cdot}\sigma). \tag{4.20}$$

In terms of local orthonormal frame fields $(e_1, \cdots, e_n)$ induced from normal coordinates at a point $p \in X$, we have, at the point p,

$$\begin{aligned} D^2 &= \sum_{a,b} e_a \nabla_{e_a} e_b \nabla_{e_b} = \sum e_a e_b \nabla_{e_a} \nabla_{e_b} = \sum e_a e_b \nabla^2_{e_a, e_b} \\ &= -\sum \nabla^2_{e_a, e_b} + \sum_{a<b} e_a e_b (\nabla^2_{e_a, e_b} - \nabla^2_{e_b, e_a}). \end{aligned}$$

Hence, using the curvature tensor and the induced curvature transformation

$$\mathcal{R}_{V_1, V_2}(\sigma) = (\nabla_{V_1}\nabla_{V_2} - \nabla_{V_2}\nabla_{V_1} - \nabla_{[V_1, V_2]})\sigma ,$$

(4.21)
$$\mathcal{R}(\sigma)=\frac{1}{2}\sum_{a,b} e_a e_b \mathcal{R}_{e_a,e_b}(\sigma),$$

where $(e_1, \cdots, e_n)$ is any orthonormal tangent frame at the point in question, we get the general Bochner identity

(4.22)
$$D^2=\nabla^*\nabla+\mathcal{R}.$$

We examine $\mathcal{R}$ closely. The computation is local, so the tensor product expression (4.9) is effective. That is, referring to (4.10), we denote by $\tilde{\nabla}^R$ and $\tilde{\mathcal{R}}^R$ the covariant derivative and its curvature tensor on the (local) S_* with the connection $\tilde{\omega}^R$ and by ∇^H and $\mathcal{R}^H$ those on the (local) V_H with the connection ω_H. Then, for a (local) cross-section $\sigma=\sigma_*\otimes\bar{v}$ of (4.9), we have

(4.23)
$$\begin{aligned}\nabla\sigma &= \tilde{\nabla}^R(\sigma_*)\otimes\bar{v}+\sigma_*\otimes\overline{\nabla^H(v)},\\ \mathcal{R}(\sigma) &= \tilde{\mathcal{R}}^R(\sigma_*)\otimes\bar{v}+\frac{1}{2}\sum_{a,b} e_a e_b \sigma_*\otimes\overline{\mathcal{R}^H_{e_a,e_b}(v)}.\end{aligned}$$

We denote by R the curvature tensor on TX with connection ω^R and consider the scalar curvature

(4.24)
$$\kappa=-\sum_{a,b} g(R_{e_a,e_b}(e_a), e_b): X \longrightarrow \boldsymbol{R}.$$

Then the Lichnerowicz theorem ([**10**, Chap. II. Theorem 8.8]) asserts

(4.25)
$$\tilde{\mathcal{R}}^R=\frac{1}{4}\kappa.$$

Let us examine the remaining curvature term $\mathcal{R}^H$. We take the canonical basis $\{f_1, f_2, f_3\}$ of $\boldsymbol{R}^3$ $(=\mathrm{Im}\,\boldsymbol{H})$ and define the basis $\{f_a\wedge f_b\}_{a<b}$ of $\mathfrak{so}(3)$ by setting $(f_a\wedge f_b)(v)=\langle f_a, v\rangle_R f_b-\langle f_b, v\rangle_R f_a$, where $\langle\ ,\ \rangle_R$ is the canonical inner product of $\boldsymbol{R}^3$. The connection ω can be written in the form

(4.26)
$$\omega=\sum_{a<b}(f_a\wedge f_b)\otimes\omega_{ba}=\frac{1}{2}\sum_{a,b}(f_a\wedge f_b)\otimes\omega_{ba},\quad \omega_{ba}=-\omega_{ab}$$

and its curvature 2-form is given by

(4.27)
$$\begin{aligned}\Omega_R\equiv d\omega+\omega\wedge\omega &= \sum_{a<b}(f_a\wedge f_b)\otimes\Omega_{ba}\\ &= \frac{1}{2}\sum_{a,b}(f_a\wedge f_b)\otimes\Omega_{ba}, \qquad \Omega_{ba}=-\Omega_{ab},\\ \Omega_{ba}=d\omega_{ba}+\omega_{bc}\wedge\omega_{ca}, &\qquad \{a, b, c\}=\{1, 2, 3\}.\end{aligned}$$

Since the isomorphism

(4.28)
$$\mathrm{ad}=\mathrm{Ad}_*: \mathrm{Im}\,\boldsymbol{H}=\mathfrak{sp}(1)\overset{\cong}{\longrightarrow}\mathfrak{so}(3)$$

in (4.10) is given by $(a_1 i+a_2 j+a_3 k)/2 \mapsto a_1 f_2\wedge f_3+a_2 f_3\wedge f_1+a_3 f_1\wedge f_2$, we have

$$\omega_H = -\frac{1}{2}(i\omega_{23}+j\omega_{31}+k\omega_{12}) \tag{4.29}$$

and its curvature 2-form is

$$\Omega_H \equiv d\omega_H+\omega_H\wedge\omega_H = -\frac{1}{2}(i\Omega_{23}+j\Omega_{31}+k\Omega_{12}). \tag{4.30}$$

Hence we have

$$\frac{1}{2}\sum e_a e_b \sigma_*\otimes\overline{\mathcal{R}^{\boldsymbol{H}}_{e_a,e_b}(v)} = \sum_{a<b} e_a e_b \sigma \overline{\Omega_H(e_a,\, e_b)} \equiv \Omega_{\boldsymbol{H}}(\sigma). \tag{4.31}$$

In particular, if $W=\boldsymbol{H}^N$, then, by the identification $W=\boldsymbol{R}^{4N}$ through (1.6), $\Omega_{\overline{\boldsymbol{H}}}$ can be written as

$$\Omega_{\boldsymbol{H}} = \frac{1}{2}(I\otimes\Omega_{23}+J\otimes\Omega_{31}+K\otimes\Omega_{12}), \tag{4.32}$$

where

$$I=\begin{pmatrix}0&0&-\mathbf{1}&0\\0&0&0&-\mathbf{1}\\\mathbf{1}&0&0&0\\0&\mathbf{1}&0&0\end{pmatrix},\qquad J=\begin{pmatrix}0&-\mathbf{1}&0&0\\\mathbf{1}&0&0&0\\0&0&0&\mathbf{1}\\0&0&-\mathbf{1}&0\end{pmatrix},$$

$$K=\begin{pmatrix}0&0&0&\mathbf{1}\\0&0&-\mathbf{1}&0\\0&\mathbf{1}&0&0\\-\mathbf{1}&0&0&0\end{pmatrix},\qquad \mathbf{1}=\mathbf{1}_N\,.$$

THEOREM 4.1. $D^2=\nabla^*\nabla+(1/4)\kappa+\Omega_{\overline{\boldsymbol{H}}}$

Let X be compact. Denote by $|\Omega_{\overline{\boldsymbol{H}}}|$ the pointwise operator norm of $\Omega_{\boldsymbol{H}}$ acting on $\Gamma(S)$. Then, since the theorem implies

$$\|\nabla\varphi\|^2+\int\Big(\frac{1}{4}\kappa-|\Omega_{\overline{\boldsymbol{H}}}|\Big)|\varphi|^2 dg \leqq 0 \tag{4.33}$$

for $\varphi\in\operatorname{Ker} D$, we have

COROLLARY 4.2. $\operatorname{Ker} D=\{0\}$ *if* $\kappa\geqq 4|\Omega_{\overline{\boldsymbol{H}}}|$ *and* $>$ *at some point.*

§ 5. Spin^q-manifolds of dimension $n\equiv 0$ (mod 4).

Let X be a compact Spin^q-manifold of dimension $n\equiv 0$ (mod 4). The fundamental representation Δ^q_n given in Table II induces the fundamental Spin^q-vector bundle

$$S(X) = P_{\mathrm{Spin}^q(n)}\times_{\Delta^q_n} W\,, \tag{5.1}$$

$$W=\boldsymbol{H}^N, \qquad N=\begin{cases} 2^{4m}, & n=8m \\ 2^{1+4m}, & n=4+8m\,. \end{cases}$$

Using the volume element $\boldsymbol{e}=e_1e_2\cdots e_n\in\Gamma(Cl(X))$ (see (1.13)), $S(X)$ can be decomposed into

$$\begin{aligned} &S(X)=S^+(X)\oplus S^-(X)\,, \\ (5.2)\qquad &S^\pm(X)=(1\pm\boldsymbol{e})S(X)=P_{\mathrm{Spin}^q(n)}\times_{\Delta_n^{q\pm}}W^\pm\,, \end{aligned}$$

where the splitting $\Delta_n^q=\Delta_n^{q+}\oplus\Delta_n^{q-}$ with $W=W^+\oplus W^-$ was given at Table II (and also (1.14)-(1.18)). Since $\nabla^{cl}\boldsymbol{e}=0$ and $\boldsymbol{e}\xi=-\xi\boldsymbol{e}$ for all $\xi\in TX$, the decomposition (5.2) is parallel with respect to the covariant derivative ∇ (given at (4.11)) and $\xi\cdot S^\pm(X)\subset S^\mp(X)$ for any $\xi\in TX$. Accordingly the Dirac operator is of the form

$$(5.3)\qquad D=\begin{pmatrix}0 & D^-\\ D^+ & 0\end{pmatrix}, \qquad D^\pm:\Gamma(S^\pm(X))\longrightarrow\Gamma(S^\mp(X))\,.$$

Since D is self-adjoint, D^+ and D^- are adjoints of one another with respect to the inner product $(\,,\,)$ given at (4.19). We will calculate the index of the operator D^+, $\operatorname{ind}D^+=\dim\operatorname{Ker}D^+-\dim\operatorname{Coker}D^+$.

Let $\hat{A}(X)\in H^{4*}(X;\boldsymbol{R})$ be the $\hat{A}$-class of X and let V be the vector bundle associated to $P_{SO(3)}$ and the canonical representation of $SO(3)$, *i.e.*, $V=P_{SO(3)}\times_{can.}\boldsymbol{R}^3$. Note that the first Pontryagin class $p_1(V)=p_1(P_{SO(3)})\in H^4(X;\boldsymbol{R})$ can be represented, using the notation (4.27), by the 4-form

$$(5.4)\qquad p_1=\frac{1}{(2\pi)^2}(\Omega_{12}\wedge\Omega_{12}+\Omega_{23}\wedge\Omega_{23}+\Omega_{31}\wedge\Omega_{31})\,.$$

THEOREM 5.1. *If $n=8m$ ($\delta=2$) or $n=4+8m$ ($\delta=1$), then we have*

$$\operatorname{ind}D^+=(-2)^\delta\Big\{\cosh\Big(\frac{1}{2}p_1(V)^{1/2}\Big)\cdot\hat{A}(X)\Big\}[X]\,,$$

where $[X]\in H_n(X;\boldsymbol{R})$ is the fundamental class of X.

This is due to the famous Atiyah-Singer theorem (see [**10**, Chap. III. Theorem 13.8] for instance), which says

$$(5.5)\qquad \operatorname{ind}D^+=\{\pi_!\operatorname{ch}\boldsymbol{\sigma}(D^+)\cdot\hat{A}(X)^2\}[X]\,,$$

where $\pi:TX\to X$ is the projection map, $\boldsymbol{\sigma}(D^+)$ is an element of the K-group $K(TX)$ determined by the principal symbol $\sigma(D^+)$ (and the identification $T^*X\cong TX$ through the metric g),

$$(5.6)\qquad \boldsymbol{\sigma}(D^+)\equiv[\pi^*(S^+(X)\otimes\boldsymbol{C}),\ \pi^*(S^-(X)\otimes\boldsymbol{C})\,;\,\sigma(D^+)]\,,$$

$\operatorname{ch}\boldsymbol{\sigma}(D^+)$ denotes its Chern character ($\in H^{2*}_{cpt}(TX;\boldsymbol{Q})$) and $\pi_!$ is the Thom isomorphism $H^{2*}_{cpt}(TX;\boldsymbol{Q})\cong H^{2*-n}(X;\boldsymbol{Q})$. Our task is to examine $\pi_!\,\mathrm{ch}\,\boldsymbol{\sigma}(D^+)\in H^{2*}(X;\boldsymbol{Q})$ closely. Denoting by $r: X\to TX$ the inclusion given by the zero-section, we have

$$(5.7)\qquad \begin{aligned} r^*\boldsymbol{\sigma}(D^+) &= [S^+(X)\otimes\boldsymbol{C}]-[S^-(X)\otimes\boldsymbol{C}] \equiv \boldsymbol{s}(X)\in K(X),\\ \chi(TX)\cdot\pi_!\,\mathrm{ch}\,\boldsymbol{\sigma}(D^+) &= r^*r_!\pi_!\,\mathrm{ch}\,\boldsymbol{\sigma}(D^+) = r^*\,\mathrm{ch}\,\boldsymbol{\sigma}(D^+) = \mathrm{ch}\,\boldsymbol{s}(X), \end{aligned}$$

where $\chi(TX)\in H^n(X;\boldsymbol{Q})$ is the Euler class of TX. Thus we have only to examine $\mathrm{ch}\,\boldsymbol{s}(X)\in H^{2*}(X;\boldsymbol{Q})$. To do so, it will be convenient to express $\mathrm{ch}\,\boldsymbol{s}(X)$ in terms of the curvature 2-form of $\tilde{\omega}$. Denoting by $\tilde{\Omega}^{R\pm}$ the curvature 2-forms on the local $S^\pm_*(X)$ with the connections $\tilde{\omega}^{R\pm}$ $(=\tilde{\omega}^R)$, we have

$$(5.8)\qquad \begin{aligned} \nabla^2\sigma &= (\tilde{\nabla}^R)^2(\sigma_*)\otimes\bar{v}+\sigma_*\otimes\overline{(\nabla^H)^2(v)}\\ &= (\tilde{\Omega}^{R\pm}+\Omega_{\bar{H}})(\sigma), \end{aligned}$$

for a local cross-section $\sigma=\sigma_*\otimes\bar{v}$ of $S^\pm(X)=S^\pm_*(X)\otimes_H\bar{V}_H$. Hence, by Chern-Weil theory, we have

$$(5.9)\qquad \mathrm{ch}\,\boldsymbol{s}(X) = \mathrm{Tr}\exp\Big(\frac{\sqrt{-1}}{2\pi}(\tilde{\Omega}^{R+}+\Omega_{\bar{H}})\Big)-\mathrm{Tr}\exp\Big(\frac{\sqrt{-1}}{2\pi}(\tilde{\Omega}^{R-}+\Omega_{\bar{H}})\Big).$$

In order to calculate this we make some preparations. Consider the irreducible $\boldsymbol{C}$-representation $\nu^C: Cl(n)\to \mathrm{End}_C(\boldsymbol{C}^N)$ $(n=8m)$, $\mathrm{End}_C(\boldsymbol{C}^{2N})$ $(n=4+8m)$ given at Lemma 1.1(2) (ii). They restrict to $\Delta^C: \mathrm{Spin}(n)\to GL_C(\boldsymbol{C}^N)$, $GL_C(\boldsymbol{C}^{2N})$. The spinor representations Δ^C are decomposable. Note that, in accordance with custom, the complex volume element $\boldsymbol{e}_C=i^{n/2}\boldsymbol{e}$ is used to decompose them: see (1.13) and (1.14).

$$(5.10)\qquad \begin{aligned} \Delta^C &= \Delta^{C+}\oplus\Delta^{C-},\\ \boldsymbol{C}^{N\pm} &= (1\pm\boldsymbol{e}_C)\boldsymbol{C}^N = (1\pm\boldsymbol{e})\boldsymbol{C}^N = \begin{cases} \boldsymbol{C}^{N/2}\oplus\{0\}\\ \{0\}\oplus\boldsymbol{C}^{N/2}\end{cases},\quad n=8m,\\ \boldsymbol{C}^{2N\pm} &= (1\pm\boldsymbol{e}_C)\boldsymbol{C}^{2N} = (1\mp\boldsymbol{e})\boldsymbol{C}^{2N}\\ &= \begin{cases} \{0\}\oplus\boldsymbol{H}^{N/2} = \{0\}\oplus\boldsymbol{C}^N\\ \boldsymbol{H}^{N/2}\oplus\{0\} = \boldsymbol{C}^N\oplus\{0\}\end{cases},\qquad n=4+8m. \end{aligned}$$

Hence, we have the (local) vector bundle with splitting, $S_C(X)=S_C^+(X)\oplus S_C^-(X)$, associated to $P_{\mathrm{Spin}(n)}$ and the above. The connection $\tilde{\omega}^R$ induces the connection $\tilde{\omega}^C$ on it. Denoting by $\tilde{\Omega}^C=\tilde{\Omega}^{C+}\oplus\tilde{\Omega}^{C-}$ the curvature 2-form with corresponding splitting, we have

$$(5.11)\qquad \mathrm{Tr}\exp\Big(\frac{\sqrt{-1}}{2\pi}\tilde{\Omega}^{C+}\Big)-\mathrm{Tr}\exp\Big(\frac{\sqrt{-1}}{2\pi}\tilde{\Omega}^{C-}\Big) = \chi(TX)\hat{A}(X)^{-1}.$$

This is a part of the calculation of the index of Dirac operator on the "Spin manifold" ([**10**, Chap. III. Proposition 11.24]). Now, by computing the right hand side of (5.9), we will show Theorem 5.1.

Proof of Theorem 5.1 for $n=8m$. The irreducible $\boldsymbol{H}$-representation $\mu: Cl(n)=\boldsymbol{R}(N)\to \mathrm{End}_{\boldsymbol{H}}(\boldsymbol{H}^N)$ induces the $\boldsymbol{R}$-representation $\mu_{\boldsymbol{R}}: Cl(n)=\boldsymbol{R}(N)\to \mathrm{End}_{\boldsymbol{R}}(\boldsymbol{R}^{4N})$, $\mu_{\boldsymbol{R}}(A)=A\otimes \mathbf{1}_4$ (the Kronecker product). It further induces the $\boldsymbol{C}$-representation $\mu_{(C)}: Cl(n)=\boldsymbol{R}(N)\to \mathrm{End}_{\boldsymbol{C}}(\boldsymbol{C}\otimes \boldsymbol{R}^{4N})$, which is decomposed into the irreducible ones, $\mu_{(C)}=\nu^C\oplus\nu^C\oplus\nu^C\oplus\nu^C$. Its restriction to Spin(n) is accordingly decomposed into

$$(5.12)\qquad \begin{aligned}\Delta^{(C)} &= \Delta^C\oplus\Delta^C\oplus\Delta^C\oplus\Delta^C\,,\\ \boldsymbol{C}\otimes\boldsymbol{R}^{4N} &= (\boldsymbol{C}\otimes\boldsymbol{R}^N)\oplus\cdots\oplus(\boldsymbol{C}\otimes\boldsymbol{R}^N)\,.\end{aligned}$$

On the other hand, by using the volume element $\boldsymbol{e}$, it admits another decomposition,

$$(5.13)\qquad \begin{aligned}\Delta^{(C)} &= \Delta^{(C)+}\oplus\Delta^{(C)-}\,,\\ (\boldsymbol{C}\otimes\boldsymbol{R}^{4N})^{\pm} &= \boldsymbol{C}\otimes(1\pm\boldsymbol{e})\boldsymbol{R}^{4N} = \boldsymbol{C}\otimes\boldsymbol{R}^{4N\pm}\\ &= \begin{cases}(\boldsymbol{C}\otimes(\boldsymbol{R}^{N/2}\oplus\{0\}))\oplus\cdots\oplus(\boldsymbol{C}\otimes(\boldsymbol{R}^{N/2}\oplus\{0\}))\\ (\boldsymbol{C}\otimes(\{0\}\oplus\boldsymbol{R}^{N/2}))\oplus\cdots\oplus(\boldsymbol{C}\otimes(\{0\}\oplus\boldsymbol{R}^{N/2}))\,.\end{cases}\end{aligned}$$

Observing (5.10), (5.12) and (5.13), it turns out that $\Delta^{(C)\pm}$ can be decomposed into

$$(5.14)\qquad \begin{aligned}\Delta^{(C)\pm} &= \Delta^{C\pm}\oplus\Delta^{C\pm}\oplus\Delta^{C\pm}\oplus\Delta^{C\pm},\\ (\boldsymbol{C}\otimes\boldsymbol{R}^{4N})^{\pm} &= \boldsymbol{C}^{N\pm}\oplus\boldsymbol{C}^{N\pm}\oplus\boldsymbol{C}^{N\pm}\oplus\boldsymbol{C}^{N\pm}\,.\end{aligned}$$

Since the curvature 2-form of the connection $\tilde{\omega}^R$ on the (local) complex vector bundle associated to the (local) $P_{\mathrm{Spin}(n)}$ and the representation (5.13) is exactly $\tilde{\Omega}^R=\tilde{\Omega}^{R+}\oplus\tilde{\Omega}^{R-}$, (5.14) gives the identity

$$(5.15)\qquad \frac{\sqrt{-1}}{2\pi}\tilde{\Omega}^{R\pm} = \frac{\sqrt{-1}}{2\pi}\tilde{\Omega}^{C\pm}\otimes\mathbf{1}_4\,.$$

On the other hand, (5.14) and (4.32) obviously assert

$$(5.16)\qquad \frac{\sqrt{-1}}{2\pi}\Omega_{\boldsymbol{H}} = \mathbf{1}_{N/2}\otimes\frac{\sqrt{-1}}{4\pi}\begin{pmatrix}0 & \Omega_{13} & \Omega_{32} & \Omega_{12}\\ \Omega_{31} & 0 & \Omega_{21} & \Omega_{32}\\ \Omega_{23} & \Omega_{12} & 0 & \Omega_{31}\\ \Omega_{21} & \Omega_{23} & \Omega_{13} & 0\end{pmatrix} \equiv \mathbf{1}_{N/2}\otimes\frac{\sqrt{-1}}{2\pi}\tilde{\Omega}_{\boldsymbol{H}}$$

and we have the formal factorization

$$(5.17)\qquad \det\left(t\mathbf{1}_4-\frac{\sqrt{-1}}{2\pi}\tilde{\Omega}_{\boldsymbol{H}}\right) = \left(t^2-\frac{1}{4}p_1\right)^2 = (t-x)^2(t+x)^2\,.$$

Therefore, denoting the eigenvalues of $(\sqrt{-1}/2\pi)\tilde{\Omega}^{C\pm}$ by $x_1^{\pm}, \cdots, x_{N/2}^{\pm}$, the matrix $(\sqrt{-1}/2\pi)(\tilde{\Omega}^{R\pm}+\Omega_{\boldsymbol{H}})$ can be diagonalized into the form

$$
\begin{pmatrix} x_1^{\pm} & & 0 \\ & \ddots & \\ 0 & & x_{N/2}^{\pm} \end{pmatrix} \otimes \mathbf{1}_4 + \mathbf{1}_{N/2} \otimes \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -x \end{pmatrix}. \tag{5.18}
$$

Now (5.4), (5.9), (5.11) and (5.18) imply

$$
\begin{aligned}
\operatorname{ch} \boldsymbol{s}(X) &= 2(e^{x}+e^{-x})\Big(\sum_{l=1}^{N/2} e^{x_l^+} - \sum_{l=1}^{N/2} e^{x_l^-}\Big) \\
&= 4\cosh\Big(\frac{1}{2}p_1(V)^{1/2}\Big)\chi(TX)\hat{A}(X)^{-1}.
\end{aligned} \tag{5.19}
$$

Combined with (5.7) and (5.5), this implies Theorem 5.1 for $n=8m$. ∎

Proof of Theorem 5.1 for $n=4+8m$. The irreducible $\boldsymbol{H}$-representation $\mu: Cl(n)=\boldsymbol{H}(N) \to \mathrm{End}_{\boldsymbol{H}}(\boldsymbol{H}^N)$ induces the $\boldsymbol{R}$-representation $\mu_{\boldsymbol{R}}: Cl(n)=\boldsymbol{H}(N) \to \mathrm{End}_{\boldsymbol{R}}(\boldsymbol{R}^{4N})$, $\mu_{\boldsymbol{R}}(Z+jW)=\iota'\iota(Z+jW)$; see (1.7). It further induces the $\boldsymbol{C}$-representation $\mu_{(\boldsymbol{C})}: Cl(n)=\boldsymbol{H}(N) \to \mathrm{End}_{\boldsymbol{C}}(\boldsymbol{C}\otimes\boldsymbol{R}^{4N})$. We consider the crucial $\boldsymbol{C}$-linear isomorphisms

$$
\boldsymbol{C}\otimes\boldsymbol{R}^{4N} \cong \boldsymbol{C}\otimes\boldsymbol{C}^{2N} \cong \boldsymbol{C}^{2N}\oplus\boldsymbol{C}^{2N}, \tag{5.20}
$$

$$
z\otimes\begin{pmatrix} a_0 \\ a_2 \\ a_1 \\ a_3 \end{pmatrix} \longleftrightarrow z\otimes\begin{pmatrix} a_0+ia_1 \\ a_2+ia_3 \end{pmatrix} \equiv z\otimes\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longleftrightarrow z\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \oplus z\begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix},
$$

where the tensor products $\otimes$ are over $\boldsymbol{R}$ and the action of $\lambda\in\boldsymbol{C}$ on $\boldsymbol{C}\otimes\boldsymbol{R}^{4N}$ or $\boldsymbol{C}\otimes\boldsymbol{C}^{2N}$ is defined to be $\lambda(z\otimes a)=\lambda z\otimes a$. Through (5.20) the action of $\mu_{(C)}(Z+jW)$ is changed into that of the matrix

$$
\begin{pmatrix} Z & -\bar{W} \\ W & \bar{Z} \end{pmatrix} \oplus \begin{pmatrix} Z & -\bar{W} \\ W & \bar{Z} \end{pmatrix} \tag{5.21}
$$

on $\boldsymbol{C}^{2N}\oplus\boldsymbol{C}^{2N}$. Also (5.20) gives the decomposition of $\mu_{(C)}$ into the irreducible ones, $\mu_{(C)}=\nu^C\oplus\nu^C$. Accordingly the restriction to $\mathrm{Spin}(n)$ is decomposed into

$$
\Delta^{(C)} = \Delta^C \oplus \Delta^C. \tag{5.22}
$$

On the other hand, by using the volume element $\boldsymbol{e}$, it admits another decomposition,

$$
\begin{aligned}
\Delta^{(C)} &= \Delta^{(C)+}\oplus\Delta^{(C)-}, \\
(\boldsymbol{C}\otimes\boldsymbol{R}^{4N})^{\pm} &= \boldsymbol{C}\otimes(1\pm\boldsymbol{e})\boldsymbol{R}^{4N} = \boldsymbol{C}\otimes\boldsymbol{R}^{4N\pm}.
\end{aligned} \tag{5.23}
$$

Further, since $\boldsymbol{R}^{4N\pm}=(1\pm\boldsymbol{e})\boldsymbol{H}^N=\boldsymbol{C}^{2N\mp}$ by (5.10), (5.20) implies

$$\boldsymbol{C}\otimes\boldsymbol{R}^{4N\pm}=\boldsymbol{C}\otimes\boldsymbol{C}^{2N\mp}=\boldsymbol{C}^{2N\mp}\oplus\boldsymbol{C}^{2N\mp}\,. \tag{5.24}$$

Thus we get the decomposition

$$\Delta^{(C)\pm}=\Delta^{C\mp}\oplus\Delta^{C\mp}\,, \tag{5.25}$$

which, combined with (5.21), gives the expression according to (5.24),

$$\frac{\sqrt{-1}}{2\pi}\tilde{\Omega}^{R\pm}=\frac{\sqrt{-1}}{2\pi}\tilde{\Omega}^{C\mp}\otimes\mathbf{1}_2\,. \tag{5.26}$$

Let us express also $(\sqrt{-1}/2\pi)\Omega_{\boldsymbol{H}}$ in terms of the matrix acting on $\boldsymbol{C}^{2N\mp}\oplus\boldsymbol{C}^{2N\mp}$. First consider the identification $\boldsymbol{C}\otimes\boldsymbol{R}^{2N}=\boldsymbol{C}\otimes\boldsymbol{R}^{4N\pm}=\boldsymbol{C}\otimes\boldsymbol{C}^{2N\mp}=\boldsymbol{C}\otimes\boldsymbol{H}^{N/2}=\boldsymbol{C}\otimes\boldsymbol{C}^N$. The curvature $\Omega_{\boldsymbol{H}}$, which acts on $\boldsymbol{C}\otimes\boldsymbol{R}^{2N}$, is expressed as (4.32) with $\mathbf{1}=\mathbf{1}_{N/2}$. This acts via right multiplication by $(i\Omega_{23}+j\Omega_{31}+k\Omega_{12})/2$ on $\boldsymbol{C}\otimes\boldsymbol{H}^{N/2}$ and, hence, gives a $\boldsymbol{C}$-linear map on $\boldsymbol{C}\otimes\boldsymbol{C}^N$ given by

$$z\otimes\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\longmapsto z\otimes\frac{\sqrt{-1}}{2}\Omega_{23}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}+z\otimes\Big(\frac{1}{2}\Omega_{31}-\frac{\sqrt{-1}}{2}\Omega_{12}\Big)\begin{pmatrix}-\bar\beta\\ \bar\alpha\end{pmatrix}. \tag{5.27}$$

Second, consider the identification $\boldsymbol{C}\otimes\boldsymbol{C}^N=\boldsymbol{C}\otimes\boldsymbol{C}^{2N\mp}=\boldsymbol{C}^{2N\mp}\oplus\boldsymbol{C}^{2N\mp}=\boldsymbol{C}^N\oplus\boldsymbol{C}^N$. Then the map (5.27) becomes a $\boldsymbol{C}$-linear map on $\boldsymbol{C}^N\oplus\boldsymbol{C}^N$ given by

$$\begin{aligned}\gamma\oplus\gamma'\longmapsto&\Big\{\frac{\sqrt{-1}}{2}\Omega_{23}\gamma+\Big(\frac{1}{2}\Omega_{31}-\frac{\sqrt{-1}}{2}\Omega_{12}\Big)\gamma'\Big\}\\ &\oplus\Big\{\Big(-\frac{1}{2}\Omega_{31}-\frac{\sqrt{-1}}{2}\Omega_{12}\Big)\gamma-\frac{\sqrt{-1}}{2}\Omega_{23}\gamma'\Big\}.\end{aligned} \tag{5.28}$$

This is easily shown by the fact that the identification $\boldsymbol{C}\otimes\boldsymbol{C}^N=\boldsymbol{C}^N\oplus\boldsymbol{C}^N$ given at (5.24) induces the correspondences:

$$\frac{1}{2}\Big\{1\otimes\begin{pmatrix}\alpha\\ \beta\end{pmatrix}-\sqrt{-1}\otimes\sqrt{-1}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\Big\}\longleftrightarrow\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\oplus\begin{pmatrix}0\\ 0\end{pmatrix},$$

$$\frac{1}{2}\Big\{1\otimes\begin{pmatrix}-\bar\beta\\ \bar\alpha\end{pmatrix}+\sqrt{-1}\otimes\sqrt{-1}\begin{pmatrix}-\bar\beta\\ \bar\alpha\end{pmatrix}\Big\}\longleftrightarrow\begin{pmatrix}0\\ 0\end{pmatrix}\oplus\begin{pmatrix}\alpha\\ \beta\end{pmatrix}.$$

Thus $(\sqrt{-1}/2\pi)\Omega_{\boldsymbol{H}}$ acting on $\boldsymbol{C}^{2N\mp}\oplus\boldsymbol{C}^{2N\mp}$ can be expressed as the matrix

$$\frac{\sqrt{-1}}{2\pi}\Omega_{\boldsymbol{H}}=\mathbf{1}_N\otimes\frac{1}{4\pi}\begin{pmatrix}-\Omega_{23} & \Omega_{12}+\sqrt{-1}\Omega_{31}\\ \Omega_{12}-\sqrt{-1}\Omega_{31} & \Omega_{23}\end{pmatrix}\equiv\mathbf{1}_N\otimes\frac{\sqrt{-1}}{2\pi}\tilde{\Omega}_{\boldsymbol{H}} \tag{5.29}$$

and we have the formal factorization

$$\det\Big(t\mathbf{1}_2-\frac{\sqrt{-1}}{2\pi}\tilde{\Omega}_{\boldsymbol{H}}\Big)=t^2-\frac{1}{4}p_1=(t-x)(t+x). \tag{5.30}$$

Hence, denoting the eigenvalues of $(\sqrt{-1}/2\pi)\Omega^{C\pm}$ by $x_1^\mp,\cdots,x_N^\mp$, the matrix

$(\sqrt{-1}/2\pi)(\tilde{\Omega}^{R\pm}+\Omega_{\bar{H}})$ can be diagonalized into the form

$$\begin{pmatrix} x_1^{\mp} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & x_N^{\mp} \end{pmatrix} \otimes \mathbf{1}_2 + \mathbf{1}_N \otimes \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}. \tag{5.31}$$

Now (5.4), (5.9), (5.11) and (5.31) imply

$$\begin{aligned} \operatorname{ch} \boldsymbol{s}(X) &= (e^{x}+e^{-x})\Big(\sum_{l=1}^{N} e^{x_l^-} - \sum_{l=1}^{N} e^{x_l^+}\Big) \\ &= -2\cosh\Big(\frac{1}{2}p_1(V)^{1/2}\Big)\chi(TX)\hat{A}(X)^{-1}. \end{aligned} \tag{5.32}$$

Combined with (5.7) and (5.5), this implies Theorem 5.1 for $n=4+8m$. ∎

There is an interesting consequence of Theorem 5.1 and Corollary 4.2.

COROLLARY 5.2. *If a compact Spin^q-manifold X of dimension $n\equiv 0$ (mod 4) admits a metric with positive scalar curvature and a fundamental class $P_{SO(3)}$ with a locally flat connection, then the $\hat{A}$-genus of X vanishes, $\hat{A}(X)=0$.*

PROOF. X has a Spin^q-structure associated to such a metric g and such a fundamental class with a locally flat connection ω. The Dirac operator D associated to g and ω is defined on the fundamental Spin^q-vector bundle. Then the condition at Corollary 4.2 is clearly satisfied and hence $\operatorname{Ker} D=\{0\}$. Consequently $\operatorname{ind} D^+ = \dim \operatorname{Ker} D^+ - \dim \operatorname{Ker} D^- = 0$. Moreover $p_1(P_{SO(3)})=0$ because of the existence of a locally flat connection. Combined with Theorem 5.1, this implies $\hat{A}(X)=\hat{A}(X)[X]=0$. ∎

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