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Spectral analysis for N-particle systems with Stark effect: non-existence of bound states and principle of limiting absorption

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§ 1. Introduction.

The local commutator method has been initiated by Mourre [**11**] and major progress has been made in the spectral and scattering theory for many-particle Schrödinger operators during the last decade. By making use of this method, for example, the principle of limiting absorption has been established by [**11, 13**] and the non-existence of positive eigenvalues has been proved by [**3**]. Furthermore, it has also played a basic role in proving the asymptotic completeness of wave operators ([**4, 9, 16, 17, 21**]). In this work, we use this remarkable method to prove the non-existence of bound states and the principle of limiting absorption for many-particle Stark Hamiltonians with homogeneous electric fields. The results obtained have an important application to the problem on the asymptotic completeness of wave operators. We are going to give a full explanation about the matter in another paper.

We now consider a system of N particles moving in a given constant electric field $\mathcal{E}\in R^3$, $\mathcal{E}\neq 0$. We denote by m_j, e_j and $r_j\in R^3$, $1\leqq j\leqq N$, the mass, charge and position vector of the j-th particle, respectively. We also use the notation $\langle\cdot,\cdot\rangle$ to denote the usual scalar product in the Euclidean space. Then the total energy Hamiltonian for such a system is described as

$$-\sum_{1\leqq j\leqq N}\{\Delta/2m_j+e_j\langle\mathcal{E}, r_j\rangle\}+V,$$

where the interaction potential V is given as the sum of pair potentials

$$V=\sum_{1\leqq j<k\leqq N}V_{jk}(r_j-r_k).$$

For notational brevity, the values of masses are fixed throughout as

$$m_j=1, \qquad 1\leqq j\leqq N,$$

but the values of charges are regarded as real parameters. As usual, the Hamiltonian above is considered in the center-of-mass frame. We introduce the configuration space X as

$$X = \{r = (r_1, \cdots, r_N) \in R^{3\times N} : \sum_{1\leq j\leq N} r_j = 0\}$$

and define $E \in X$ by

$$E = \text{projection onto } X \text{ of } (e_1\mathcal{E}, \cdots, e_N\mathcal{E}).$$

We further write x for a generic point in X. Then the energy Hamiltonian H (Schrödinger operator) with identical masses $m_j=1$ takes the following form in the center-of-mass frame and acts on the space $L^2(X)$:

$$H = -\Delta/2 - \langle E, x\rangle + V \qquad \text{on} \quad L^2(X).$$

We here assume that $|E| \neq 0$. This is equivalent to saying that the charges take different values $e_j \neq e_k$ for some pair (j, k). We further assume that the pair potential $V_{jk}(y)$, $y \in R^3$, satisfies the following assumption:

(V) $V_{jk}(y) \in C^2(R^3)$ is a C^2-smooth real function with decay properties

$$|V_{jk}(y)| + |\nabla V_{jk}(y)| = O(|y|^{-\rho}), \ |\nabla\nabla V_{jk}(y)| = O(1) \quad \text{for some } \rho > 1/2.$$

The decay assumption for the first derivatives is mainly used to prove the non-existence of bound states and the uniform boundedness for the second derivatives is required to prove the principle of limiting absorption. Throughout the whole exposition, we use the constant ρ with the meaning ascribed above and assume, without loss of generality, that $1/2 < \rho < 1$. Under assumption (V), the operator H formally defined above admits a unique self-adjoint realization in $L^2(X)$. We denote it by the same notation H.

We are now in a position to formulate the main results obtained in the present work.

THEOREM 1 (NON-EXISTENCE OF BOUND STATES). *Assume that (V) is fulfilled. Then the operator H has no bound states.*

THEOREM 2 (PRINCIPLE OF LIMITING ABSORPTION). *Assume that (V) is fulfilled. Let $\langle x\rangle = (1+|x|^2)^{1/2}$ and denote by $L^2_\nu(X) = L^2(X; \langle x\rangle^{2\nu}dx)$ the weighted L^2 space over X with weight $\langle x\rangle^\nu$. Let $R(\zeta; H) = (H-\zeta)^{-1}$, $\operatorname{Im}\zeta \neq 0$, be the resolvent of H. Then the resolvents have the boundary values to the real axis*

$$\lim_{\varepsilon\downarrow 0} R(\lambda \pm i\varepsilon; H) = R(\lambda \pm i0; H), \qquad \lambda \in R,$$

in the uniform topology as an operator from $L^2_\nu(X)$ into $L^2_{-\nu}(X)$ for $\nu > 1/4$, where the convergence is locally uniform in λ.

As a consequence of Theorems 1 and 2, we can get the following propagation property for $\exp(-itH)$, which follows immediately from the local smoothness theorem due to Kato ([8, **14**]).

COROLLARY 3. *Assume that (V) is fulfilled. Let* $\nu>1/4$ *and let* $f\in C_0^\infty(R)$ *be a smooth function with compact support. Then there exists* $C>0$ *independent of* $\psi\in L^2(X)$ *such that*

$$\int_{-\infty}^{\infty}\|\langle x\rangle^{-\nu}\exp(-itH)f(H)\psi\|_0^2\,dt\leqq C\|\psi\|_0^2,$$

where $\|\cdot\|_0$ *denotes the* L^2 *norm in* $L^2(X)$.

We conclude the section by making a brief comment on the related results which have been obtained in the spectral analysis for Hamiltonians with Stark effect. Many works, for example, [**1**, **5**, **6**, **7**, **12**, **19**, **20**, **22**] have been already done for one or two-particle systems and the two problems which we here consider, including the completeness of wave operators, have been solved for various classes of short-range or long-range potentials. On the other hand, there are only a few works dealing with many-particle systems. The problem of asymptotic completeness has been studied by [**10**, **18**] only for three-particle systems and the non-existence of bound states has been proved by Sigal [**15**] for N-particle systems with fixed centres

$$-\sum_{1\leq j\leq N}\{\Delta/2m_j+e_j\langle\mathcal{E},\,r_j\rangle-V_j(r_j)\}+\sum_{1\leq j<k\leq N}V_{jk}(r_j-r_k),$$

under the main assumptions that the charges e_j take the same sign ($e_j>0$ or $e_j<0$ for all j) and that the interactions V_{jk} between moving particles are repulsive

$$\langle\mathcal{E},\,y\rangle\langle\mathcal{E},\,\nabla V_{jk}(y)\rangle\leqq 0$$

along the direction of electric field $\mathcal{E}$. The result obtained covers an important class of N-particle systems which includes atoms and Born-Oppenheimer molecules as typical examples. In the present work, we have made the strong decay assumption $\nabla V_{jk}(y)=O(|y|^{-\rho})$, $\rho>1/2$, for the first derivatives of pair potentials in place of the repulsive condition.

§ 2. Basic notations in many-particle systems.

We here introduce several basic notations which are often employed in the spectral analysis for many-particle systems.

We denote by letter a or b a partition $a=\{C_1,\cdots,C_m\}$ of the total set $\{1,\cdots,N\}$ into non-empty disjoint subsets (clusters) C_j. Such a partition is called a cluster decomposition. We also denote by $\#(a)$ the number of clusters in a. The one-cluster decomposition $a=\{1, 2,\cdots, N\}$, $\#(a)=1$, is not used throughout the entire discussion. We further write α for pair $(j,\,k)$ with $1\leqq j<k\leqq N$ and V_α for pair potential V_{jk}. The relations $\alpha\subset a$ and $\alpha\not\subset a$ mean, respectively, that j and k are in the same cluster in a and that they are in different clusters

in a. The pair $\alpha=(j, k)$ is sometimes identified with the $N-1$ cluster decomposition $\{(j, k), (1), \cdots, (\hat{j}), \cdots, (\hat{k}), \cdots, (N)\}$. If b is obtained as a refinement of a, then we use the same notation $\subset$ to denote this relation as $b \subset a$.

We define the two subspaces X^a and X_a of X as

$$X^a = \{r \in X : \sum_{j \in C} r_j = 0 \quad \text{for all clusters } C \text{ in } a\},$$

$$X_a = \{r \in X : r^\alpha = r_j - r_k = 0 \quad \text{for all pairs } \alpha \subset a\}.$$

These spaces are mutually orthogonal and span the total space $X = X^a \oplus X_a$, so that $L^2(X)$ is decomposed as the tensor product $L^2(X) = L^2(X^a) \otimes L^2(X_a)$. We also denote by $\pi^a : X \to X^a$ and $\pi_a : X \to X_a$ the orthogonal projections onto X^a and X_a, respectively, and write $x^a = \pi^a x$ and $x_a = \pi_a x$ for a generic point $x \in X$.

The cluster Hamiltonian H_a is defined by

$$H_a = -\Delta/2 - \langle E, x \rangle + V^a \qquad \text{on} \quad L^2(X)$$

as an operator acting on $L^2(X)$, where

$$V^a(r) = \sum_{\alpha \subset a} V_\alpha(r^\alpha), \qquad r^\alpha = r_j - r_k . \tag{2.1}$$

Let $E^a = \pi^a E$ and $E_a = \pi_a E$. Then the operator H_a is decomposed into

$$H_a = H^a \otimes Id + Id \otimes T_a \qquad \text{on} \quad L^2(X^a) \otimes L^2(X_a).$$

Here H^a is the subsystem Hamiltonian defined by

$$H^a = -\Delta/2 - \langle E^a, x^a \rangle + V^a \qquad \text{on} \quad L^2(X^a)$$

and T_a is the free Hamiltonian defined by

$$T_a = -\Delta/2 - \langle E_a, x_a \rangle \qquad \text{on} \quad L^2(X_a).$$

§ 3. Conjugate operator.

We define the direction ω by $\omega = E/|E|$ and denote the coordinate z as $z = \langle x, \omega \rangle \in R$, so that H is rewritten as

$$H = -\Delta/2 - |E| z + V .$$

In the last section, we construct a conjugate operator A with the two properties below. The main body of this work is occupied by the construction of such a conjugate operator.

The first property is that:

(A.1) A is a differential operator of the form

$$A = -i\{\langle c(x), \nabla \rangle + \langle \nabla, c(x) \rangle\} + c_0(x),$$

where $c(x)=\{c_j(x)\}$, $1\leqq j\leqq 3(N-1)$, and $c_0(x)$ are smooth real functions and satisfy the estimates

$$(3.1)\qquad |\partial_x^\beta c_j(x)| \leqq C_\beta\langle z\rangle^{-|\beta|}, \qquad |\partial_x^\beta c_0(x)| \leqq C_\beta\langle z\rangle^{-1-|\beta|}.$$

Since the coefficients of differential operator A are all real functions with bounded derivatives to any order, it admits a unique self-adjoint realization on its natural domain in $L^2(X)$. We denote it by the same notation A. It follows from assumption (V) and (3.1) that the commutator $[H, A]=HA-AH$ satisfies

$$(3.2)\qquad [H, A](H+i)^{-1}\in \mathscr{B}(X), \qquad [[H, A], A](H+i)^{-1} \in \mathscr{B}(X),$$

where $\mathscr{B}(X)$ denotes the set of all bounded operators acting on $L^2(X)$.

We now fix energy $\lambda\in R$ arbitrarily and take a smooth real function $f\in C_0^\infty(R)$ with support in a small interval $(\lambda-\delta, \lambda+\delta)$, $0<\delta\ll 1$, around λ. Then the second property is stated as follows:

(A.2) If $\delta>0$ is chosen small enough, then there exists $d>0$ such that

$$f(H)i[H, A]f(H) \geqq df(H)^2+K$$

for some compact operator K, where δ and d can be taken locally uniformly in λ and the inequality relation is understood in the form sense.

§ 4. Proof of main theorems.

In this section, we first complete the proof of Theorems 1 and 2, accepting the conjugate operator A with properties (A.1) and (A.2) as constructed. The proof of Theorem 1 is based on a modification of the local commutator method in Froese-Herbst [**3**], where the non-existence of positive eigenvalues is verified for N-particle systems without homogeneous electric fields. Once the conjugate operator A has been constructed, the principle of limiting absorption is established as an immediate consequence of the general result in [**2**] (see [**13**] also).

Let $\psi(x)$ be the eigenstate associated with eigenvalue $\lambda\in R$

$$(4.1)\qquad H\psi = \lambda\psi, \qquad \psi\in L^2(X).$$

The proof of Theorem 1 consists of three steps. We show that: (i) the eigenstate ψ has the polynomial decay property; (ii) ψ has the exponential decay property; (iii) ψ vanishes identically. To prove (i) and (ii), we analyze the two commutators $i[H, A]$ and $i[H, \gamma]$, where A is the conjugate operator as in section 3 and γ is defined by

$$(4.2)\qquad \gamma = (D_0+D_0^*)/2, \qquad D_0 = -i\langle x/\langle x\rangle, \nabla\rangle.$$

4.1. We begin by proving the polynomial decay property.

LEMMA 4.1. *For any* $k>0$, $\langle x\rangle^{k}\psi\in L^{2}(X)$.

PROOF. The lemma is verified by contradiction and the proof is divided into several steps. In the proof, we again denote by $\|\cdot\|_{0}$ the L^{2} norm and write $\langle\cdot, \cdot\rangle_{0}$ for the scalar product in $L^{2}(X)$.

(1) Assume that there exists $k>0$ such that

$$\langle x\rangle^{k}\psi\notin L^{2}(X). \tag{4.3}$$

For ε, $0<\varepsilon\ll 1$, small enough, we define the function $F=F(|x|;\varepsilon)$ by

$$F=k(\log\langle x\rangle-\log(1+\varepsilon\langle x\rangle))$$

with $k>0$ as above and set

$$\psi_{\varepsilon}=\langle x\rangle^{k}(1+\varepsilon\langle x\rangle)^{-k}\psi\in L^{2}(X),$$

so that ψ_{ε} is represented as $\psi_{\varepsilon}=e^{F}\psi$. This function obeys the equation

$$H_{F}\psi_{\varepsilon}=\lambda\psi_{\varepsilon}, \tag{4.4}$$

where

$$H_{F}=e^{F}He^{-F}=-2^{-1}(\nabla-\nabla F)^{2}-|E|z+V$$

with $z=\langle x, \omega\rangle$, $\omega=E/|E|$. The operator H_{F} can be also written as

$$H_{F}=H+B_{F}-|\nabla F|^{2}/2$$

with

$$B_{F}=2^{-1}(\langle\nabla F, \nabla\rangle+\langle\nabla, \nabla F\rangle).$$

Let γ be defined by (4.2). We calculate ∇F as $\nabla F=xG/\langle x\rangle$, so that

$$B_{F}=iG\gamma+2^{-1}(|x|/\langle x\rangle)\partial G/\partial|x|, \tag{4.5}$$

where

$$G=G(|x|;\varepsilon)=(\langle x\rangle/|x|)\partial F/\partial|x|=k\langle x\rangle^{-1}(1+\varepsilon\langle x\rangle)^{-1}>0$$

and it behaves like $G=O(|x|^{-1})$, $|x|\rightarrow\infty$, uniformly in ε.

(2) We now normalize $\psi_{\varepsilon}\in L^{2}(X)$ as

$$\varphi_{\varepsilon}(x)=\psi_{\varepsilon}/\|\psi_{\varepsilon}\|_{0}, \qquad \|\varphi_{\varepsilon}\|_{0}=1.$$

By assumption (4.3), it follows that

$$\varphi_{\varepsilon}\longrightarrow 0 \quad \text{weakly in} \quad L^{2}(X) \quad \text{as} \quad \varepsilon\longrightarrow 0. \tag{4.6}$$

By (4.4) and (4.5), φ_{ε} satisfies the equation

$$H\varphi_{\varepsilon}=\lambda\varphi_{\varepsilon}-iG\gamma\varphi_{\varepsilon}+J\varphi_{\varepsilon}, \tag{4.7}$$

where

$$J = J(|x|\,;\varepsilon) = 2^{-1}\{|\nabla F|^2-(|x|/\langle x\rangle)\partial G/\partial|x|\} = O(|x|^{-2}), \quad |x| \to \infty .$$

Hence we see that $\langle x\rangle^{-1/2}\varphi_\varepsilon$ and $\langle x\rangle^{-1}\varphi_\varepsilon$ are bounded in the Sobolev spaces $H^1(X)$ and $H^2(X)$ over X uniformly in ε, respectively. This, together with (4.6), implies that both the terms $G\gamma\varphi_\varepsilon$ and $J\varphi_\varepsilon$ converge to zero strongly in $L^2(X)$ as $\varepsilon\to 0$. Thus we have

(4.8) $$\|(H-\lambda)\varphi_\varepsilon\|_0 \longrightarrow 0, \qquad \varepsilon \longrightarrow 0 .$$

(3) We now use the property (A.2) of A. For $\delta>0$ small enough, we take a smooth real function $f\in C_0^\infty(R)$ to satisfy that f has support in a small interval $(\lambda-2\delta, \lambda+2\delta)$ around λ and that $f=1$ on $[\lambda-\delta, \lambda+\delta]$. Then it follows from (A.2) and (4.6) that

$$\liminf_{\varepsilon\to 0} \langle f(H)i[H, A]f(H)\varphi_\varepsilon, \varphi_\varepsilon\rangle_0 \geqq d \liminf_{\varepsilon\to 0}\|f(H)\varphi_\varepsilon\|_0^2$$

for some $d>0$. Since $[H, A](H+i)^{-1}: L^2(X)\to L^2(X)$ is bounded by (3.2) and since

$$\|(Id-f(H))\varphi_\varepsilon\|_{L^2(X)} \longrightarrow 0, \qquad \varepsilon \longrightarrow 0 ,$$

by (4.8), we have

(4.9) $$\liminf_{\varepsilon\to 0} \langle i[H, A]\varphi_\varepsilon, \varphi_\varepsilon\rangle_0 \geqq d\|\varphi_\varepsilon\|_0^2 = d>0 .$$

(4) Next we calculate the term $\langle i[H, A]\varphi_\varepsilon, \varphi_\varepsilon\rangle_0$ by use of relation (4.7). Let $\chi_R(x)\in C_0^\infty(X)$ be a non-negative function such that χ_R is supported in $|x|<2R$ and $\chi_R=1$ on $|x|\leqq R$. We approximate φ_ε by $\varphi_\varepsilon^R=\chi_R\varphi_\varepsilon$; $\varphi_\varepsilon^R\to\varphi_\varepsilon$, $R\to\infty$, strongly in $L^2(X)$. Then we have

$$\langle i[H, A]\varphi_\varepsilon, \varphi_\varepsilon\rangle_0 = \lim_{R\to\infty} i\{\langle A\varphi_\varepsilon^R, H\varphi_\varepsilon^R\rangle_0-\langle H\varphi_\varepsilon^R, A\varphi_\varepsilon^R\rangle_0\}$$

and hence

(4.10) $$\langle i[H, A]\varphi_\varepsilon, \varphi_\varepsilon\rangle_0 = -2\,\mathrm{Re}\,\langle A\varphi_\varepsilon, G\gamma\varphi_\varepsilon\rangle_0-\langle i[A, J]\varphi_\varepsilon, \varphi_\varepsilon\rangle_0 ,$$

by (4.7). The second term on the right side converges to zero as $\varepsilon\to 0$.

LEMMA 4.2. $\|\langle x\rangle^{\rho/2}G^{1/2}\gamma\varphi_\varepsilon\|_0 = O(1), \qquad \varepsilon \longrightarrow 0 .$

We accept this lemma as proved. The proof of the lemma is given after completing the proof of the lemma. By Lemma 4.2, the first term on the right side of (4.10) also converges to zero as $\varepsilon\to 0$. Thus we have

$$\limsup_{\varepsilon\to 0} \langle i[H, A]\varphi_\varepsilon, \varphi_\varepsilon\rangle_0 = 0 .$$

This contradicts (4.9) and the proof is complete. □

We now prove Lemma 4.2 which has played a basic role in proving the polynomial decay property of eigenstate ψ in (4.1).

PROOF OF LEMMA 4.2. To prove the lemma, we define the operator γ_δ as

$$\gamma_\delta = g_\delta \gamma g_\delta, \qquad g_\delta(x) = \langle x\rangle^{\rho/2}\langle \delta x\rangle^{-(\rho/2+1/4)},$$

for $\delta>0$ small enough and evaluate the term

$$I_{\delta\varepsilon} = \langle i[H, \gamma_\delta]\varphi_\varepsilon, \varphi_\varepsilon\rangle_0.$$

It should be noted that $\gamma_\delta\varphi_\varepsilon \in L^2(X)$ for $\delta>0$. Throughout the proof, we denote by b_k a multiplication operator by $b_k(x;\delta)$ with bound $|b_k(x;\delta)| \leqq C\langle x\rangle^k$ uniformly in δ. According to this notation, the operator γ_δ is related to γ, γ being defined by (4.2), through the relation $\gamma_\delta = g_\delta^2\gamma + b_{\rho-1}$.

We first recall that H is rewritten as

$$H = -\Delta/2 - |E|z + V, \qquad z = \langle x, \omega\rangle.$$

Then the commutator $i[H, \gamma_\delta]$ is calculated as

$$i[H, \gamma_\delta] = i[-\Delta/2, \gamma_\delta] + i[-|E|z, \gamma_\delta] + i[V, \gamma_\delta].$$

The second operator on the right side is equal to

$$i[-|E|z, \gamma_\delta] = g_\delta^2\langle x\rangle^{-1}|E|z = g_\delta^2\langle x\rangle^{-1}(-\Delta/2 - H + V).$$

The first operator is represented in the form

$$i[-\Delta/2, \gamma_\delta] = i[-\Delta/2, g_\delta^2]\gamma + g_\delta^2 i[-\Delta/2, \gamma] + b_{\rho-2}\nabla + b_{\rho-3}.$$

Let D_0 be as in (4.2). Then a simple calculation yields that

$$i[-\Delta/2, \gamma] = \langle x\rangle^{-1}(-\Delta - D_0^* D_0) + b_{-2}\nabla + b_{-3},$$
$$i[-\Delta/2, g_\delta^2] = g_\delta^2\langle x\rangle^{-1}\{\rho - (\rho+1/2)\delta^2\langle\delta x\rangle^{-2}\langle x\rangle^2\}D_0 + b_{\rho-2},$$

so that the first operator takes the form

$$i[-\Delta/2, \gamma_\delta] = g_\delta^2\langle x\rangle^{-1}\{((\rho-1) - (\rho+1/2)\delta^2\langle\delta x\rangle^{-2}\langle x\rangle^2)D_0^* D_0 - \Delta\} + R$$

with $R = b_{\rho-2}\nabla + b_{\rho-3}$. The third operator is written in the form

$$i[V, \gamma_\delta] = b_0,$$

because the first derivatives $\nabla V_{jk}(y)$ behave like $O(|y|^{-\rho})$ at infinity for ρ, $1/2 < \rho < 1$, by assumption (V). Thus we obtain

$$i[H, \gamma_\delta] = B_1 + B_2 + B_3 + g_\delta^2\langle x\rangle^{-1}(-H+V) + b_{\rho-2}\nabla + b_0$$

where

$$B_1 = (1-\rho)g_\delta\langle x\rangle^{-1/2}(-\Delta - D_0^* D_0)\langle x\rangle^{-1/2}g_\delta,$$
$$B_2 = (\rho+1/2)\delta^2 g_\delta\langle x\rangle^{1/2}\langle\delta x\rangle^{-1}(-\Delta - D_0^* D_0)\langle\delta x\rangle^{-1}\langle x\rangle^{1/2}g_\delta,$$
$$B_3 = (\rho+1/2)g_\delta\langle x\rangle^{-1/2}h_\delta^{1/2}(-\Delta)h_\delta^{1/2}\langle x\rangle^{-1/2}g_\delta$$

with $h_\delta(x)=1-\delta^2\langle\delta x\rangle^{-2}\langle x\rangle^2>0$. We should note that the first three operators B_j, $1\leqq j\leqq 3$, on the right side are non-negative.

We now evaluate the term $I_{\delta\varepsilon}$ in question. By relation (4.7), we have

$$\langle H\varphi_\varepsilon,\ g_\delta^2\langle x\rangle^{-1}\varphi_\varepsilon\rangle_0 = O(1)$$

uniformly in δ and ε and hence $I_{\delta\varepsilon}$ is evaluated from below as

$$I_{\delta\varepsilon} \geqq -d \tag{4.11}$$

for some $d>0$ independent of δ and ε. Next we evaluate this term from above. We write it as

$$I_{\delta\varepsilon} = i\{\langle\gamma_\delta\varphi_\varepsilon,\ H\varphi_\varepsilon\rangle_0-\langle H\varphi_\varepsilon,\ \gamma_\delta\varphi_\varepsilon\rangle_0\}.$$

Then it follows again from (4.7) that

$$I_{\delta\varepsilon} = -2\|g_\delta G^{1/2}\gamma\varphi_\varepsilon\|_0+O(1)$$

uniformly in δ and ε. This, together with (4.11), proves the lemma. □

4.2. We proceed to proving the exponential decay property of eigenstate ψ in (4.1).

LEMMA 4.3. *For any* $k>0$, $\exp(k\langle x\rangle)\psi\in L^2(X)$.

PROOF. The lemma is again verified by contradiction. The proof is done by repeated use of the same arguments as in the proof of Lemma 4.1.

(1) Define $k_0\geqq 0$ as

$$k_0 = \sup\{k \geqq 0: \exp(k\langle x\rangle)\psi \in L^2(X)\}.$$

We deny the statement of the lemma and assume that $k_0<\infty$. By this assumption, we can choose k, $0<k<k_0$, so close to k_0 that $k+\kappa>k_0$ for κ, $0<\kappa\ll 1$, small enough. If $k_0=0$, then we take k as $k=0$. Thus we have

$$\exp((k+\kappa)\langle x\rangle)\psi \notin L^2(X). \tag{4.12}$$

For $\eta\gg 1$ large enough, we set

$$\psi_\eta = (1+\kappa\eta^{-1}\langle x\rangle)^\eta \exp(k\langle x\rangle)\psi \in L^2(X)$$

with $\kappa>0$ as above. It should be noted that $\psi_\eta\in L^2(X)$, even if $k=0$, which follows from Lemma 4.1 at once.

We now write ψ_η as $\psi_\eta=e^F\psi$ with

$$F = F(|x|;\eta) = k\langle x\rangle+\eta\log(1+\kappa\eta^{-1}\langle x\rangle)$$

and normalize ψ_η as

$$\varphi_\eta = \psi_\eta/\|\psi_\eta\|_0, \qquad \|\varphi_\eta\|_0 = 1.$$

By (4.12), φ_η converges to zero weakly in $L^2(X)$ as $\eta\to\infty$ and also we see, repeating the same calculation as in the proof of Lemma 4.1, that φ_η obeys the equation

$$H\varphi_\eta = \lambda\varphi_\eta - iG\gamma\varphi_\eta + J\varphi_\eta, \tag{4.13}$$

where

$$G = G(|x|;\eta) = (\langle x\rangle/|x|)\partial F/\partial|x| = (k+\kappa\eta(\eta+\kappa\langle x\rangle)^{-1}) > 0,$$
$$J = J(|x|;\eta) = 2^{-1}\{|\nabla F|^2-(|x|/\langle x\rangle)\partial G/\partial|x|\}.$$

These functions behave like $G=O(1)$ and $J=O(1)$ as $|x|\to\infty$ uniformly in $\eta\gg1$. Hence it follows from equation (4.13) that $\langle x\rangle^{-1/2}\varphi_\eta$ and $\langle x\rangle^{-1}\varphi_\eta$ are bounded in the Sobolev spaces $H^1(X)$ and $H^2(X)$ uniformly in $\eta\gg1$, respectively. Furthermore, J obeys the estimate

$$|J(|x|;\eta)-k^2/2| \leqq C(\kappa+\langle x\rangle^{-1})$$

uniformly in η, so that we may write (4.13) as

$$H\varphi_\eta = (\lambda+k^2/2)\varphi_\eta - iG\gamma\varphi_\eta + J_1\varphi_\eta, \tag{4.14}$$

where

$$J_1 = J_1(|x|;\eta) = J-k^2/2 = O(\kappa)+O(\langle x\rangle^{-1}).$$

(2) We accept the following lemma as proved, the proof of which is done after completing the proof of this lemma.

LEMMA 4.4. *As $\eta\to\infty$, one has:*

(i) $\|\langle x\rangle^{\rho/2}G^{1/2}\gamma\varphi_\eta\|_0 = O(1)$,

(ii) $\|\langle x\rangle^{(\rho-1)/2}\nabla\varphi_\eta\|_0 = O(1)$.

It follows immediately from this lemma that

$$\lim_{\eta\to\infty}\|G\gamma\varphi_\eta\|_0 = 0$$

and hence

$$\limsup_{\eta\to\infty}\|(H-\lambda-k^2/2)\varphi_\eta\|_0 = O(\kappa)$$

by (4.14). We here again make use of property (A.2) and repeat the same arguments as used in the proof of Lemma 4.1. Then we obtain that

$$\liminf_{\eta\to\infty}\langle i[H, A]\varphi_\eta, \varphi_\eta\rangle_0 \geqq d > 0 \tag{4.15}$$

for some d independent of $\kappa>0$ small enough.

(3) We calculate the term on the left side of (4.15) by use of (4.13). Since $\langle x\rangle^L\varphi_\eta\in H^2(X)$ for any $L\gg1$, we have the relation

(4.16) $$\langle i[H, A]\varphi_\eta, \varphi_\eta\rangle_0 = -2\,\mathrm{Re}\,\langle A\varphi_\eta, G\gamma\varphi_\eta\rangle_0 - \langle i[A, J]\varphi_\eta, \varphi_\eta\rangle_0.$$

The second term on the right side converges to zero as $\eta\to\infty$, because $[A, J] = O(|x|^{-1})$, $|x|\to\infty$, uniformly in $\eta\gg 1$. We shall show that the first term also converges to zero as $\eta\to\infty$. To see this, we write it as

$$\langle A\varphi_\eta, G\gamma\varphi_\eta\rangle_0 = \langle\langle x\rangle^{-(\rho-1/2)}G^{1/2}\langle x\rangle^{(\rho-1)/2}A\varphi_\eta, \langle x\rangle^{\rho/2}G^{1/2}\gamma\varphi_\eta\rangle_0.$$

Since $\rho>1/2$ by assumption, it follows from Lemma 4.4 that the first term on the right side of (4.16) is also convergent to zero as $\eta\to\infty$. Thus we have

$$\limsup_{\eta\to\infty}\langle i[H, A]\varphi_\eta, \varphi_\eta\rangle_0 = 0,$$

which contradicts (4.15) and the proof is complete. □

PROOF OF LEMMA 4.4. To prove the lemma, we analyze the term

$$I_\eta = \langle i[H, \gamma_0]\varphi_\eta, \varphi_\eta\rangle_0,$$

where the operator γ_0 is defined by $\gamma_0 = \langle x\rangle^{\rho/2}\gamma\langle x\rangle^{\rho/2}$. The idea is almost the same as in the proof of Lemma 4.2, so we give only a sketch for the proof.

We again denote by b_k a multiplication operator by $b_k(x)$ with bound $|b_k(x)| \leqq C\langle x\rangle^k$. By a calculation similar to that in the proof of Lemma 4.2, we see that the commutator $i[H, \gamma_0]$ takes the form

$$i[H, \gamma_0] = Q_1 + Q_2 + \langle x\rangle^{(\rho-1)}(-H+V) + b_{\rho-2}\nabla + b_0,$$

where

$$Q_1 = (1-\rho)\langle x\rangle^{(\rho-1)/2}(-\Delta - D_0^*D_0)\langle x\rangle^{(\rho-1)/2},$$

$$Q_2 = (\rho+1/2)\langle x\rangle^{(\rho-1)/2}(-\Delta)\langle x\rangle^{(\rho-1)/2}.$$

The operators Q_1 and Q_2 are both non-negative. Hence it follows from (4.13) that

(4.17) $$I_\eta \geqq d\|\langle x\rangle^{(\rho-1)/2}\nabla\varphi_\eta\|_0^2 - 1/d$$

for some $d>0$ independent of η. On the other hand, we again use (4.13) to obtain that

$$I_\eta \leqq -\|\langle x\rangle^{\rho/2}G^{1/2}\gamma\varphi_\eta\|_0^2 + d$$

with another $d>0$. This, together with (4.17), proves the lemma. □

4.3. We complete the proof of Theorem 1 by showing that the eigenstate ψ in (4.1) must vanish identically.

PROOF OF THEOREM 1. Assume that $\psi\neq 0$ and set $\psi_k(x) = \exp(kz)\psi(x)$ for $k\gg 1$ large enough. Then, $\psi_k\in L^2(X)$ by Lemma 4.3. We normalize ψ_k as

$$\varphi_k = \psi_k/\|\psi_k\|_0, \qquad \|\varphi_k\|_0 = 1.$$

As is easily seen, φ_k obeys the equation

$$H\varphi_k = (\lambda + k^2/2)\varphi_k - ikA_\omega\varphi_k \tag{4.18}$$

where $A_\omega = -i\langle\omega, \nabla\rangle$. The commutator $i[H, A_\omega]$ is calculated as

$$i[H, A_\omega] = |E| + i[V, A_\omega].$$

Since the potential V is bounded by assumption (V), we have

$$\langle i[H, A_\omega]\varphi_k, \varphi_k\rangle_0 \geqq 1/d - d\|A_\omega\varphi_k\|_0^2 \tag{4.19}$$

for some $d>1$ independent of $k \gg 1$. On the other hand, we obtain from (4.18) that

$$\langle i[H, A_\omega]\varphi_k, \varphi_k\rangle_0 = -2k\|A_\omega\varphi_k\|_0^2,$$

which, together with (4.19), concludes that $\psi=0$. Thus the theorem is now proved. □

4.4. We shall prove the second main theorem. As previously stated, this theorem is obtained as a direct application of the general result (Theorem 4.9) in [**2**] (see also [**13**]), so we give only a sketch for the proof.

Proof of Theorem 2. Let $\lambda \in R$ be fixed arbitrarily and let $f \in C_0^\infty(R)$ be a smooth real function such that f is supported in $(\lambda-2\delta, \lambda+2\delta)$ and $f=1$ on $[\lambda-\delta, \lambda+\delta]$ for $\delta>0$ small enough. Then it follows from property (A.2) and Theorem 1 that there exists $\delta>0$ small enough such that

$$f(H)i[H, A]f(H) \geqq df(H)^2, \qquad d>0.$$

By this form inequality and (3.2), we know that the conjugate operator A fulfills all the assumptions of Theorem 4.9 in [**2**]. Thus we can prove that the boundary values $R(\lambda \pm i0; H)$ of resolvents to the real axis exist and that

$$(1+|A|)^{-\nu}R(\lambda \pm i0; H)(1+|A|)^{-\nu}: L^2(X) \longrightarrow L^2(X)$$

are bounded for any $\nu>1/2$. Since $\langle x\rangle^{-1/2}(H+i)^{-1}\nabla: L^2(X) \to L^2(X)$ is bounded, it follows by interpolation that

$$\langle x\rangle^{-\nu/2}(H+i)^{-1}(1+|A|)^\nu: L^2(X) \longrightarrow L^2(X)$$

is also bounded for any ν, $0 \leqq \nu \leqq 1$. This enables us to repeat the same argument as in section 8 of [**13**] and the desired result can be obtained. Thus the proof is complete. □

§ 5. Commutator calculus.

We here prepare some commutator calculus which is required to construct the conjugate operator A in the next section.

Let $\mathcal{B}(X)$ again denote the totality of bounded operators acting on $L^2(X)$. We often write Q_k, $k\in R$, for the multiplication operator by $\langle x\rangle^k$. Let $\theta_0\in R$ be fixed arbitrarily and assume that the real parameter θ ranges over the interval $I_0=[\theta_0, \infty)$. We say that a family of bounded operators $B(\theta)$ with parameter $\theta\in I_0$ belongs to $\mathcal{B}(X)$ uniformly in θ, if the norm $\|B(\theta)\|_{L^2(X)}$ as an operator from $L^2(X)$ into itself is bounded uniformly in θ.

LEMMA 5.1. *One has the following statements*:

(i) $Q_{-k}(H+\theta+i)^{-1}Q_k\in\mathcal{B}(X)$, $k\in R$, *uniformly in* $\theta\in I_0$;

(ii) $Q_{-k-1/2}\nabla(H+\theta+i)^{-1}Q_k\in\mathcal{B}(X)$, $k\in R$, *uniformly in* $\theta\in I_0$;

(iii) $\|[\exp(itH), Q_{1/2}](H+\theta+i)^{-1}\|_{L^2(X)}\leqq C|t|, \qquad t\in R,$

for C *independent of* $\theta\in I_0$.

PROOF. The statements (i) and (ii) are easy to prove, so we omit the proof. (iii) can be also easily verified. Since the commutator is written in the integral form

$$[\exp(itH), Q_{1/2}]=\int_0^t \exp(isH)i[H, Q_{1/2}]\exp(i(t-s)H)ds,$$

(iii) follows from (ii) at once. □

LEMMA 5.2. *Let* $f\in C_0^\infty(R)$. *Then*

$$Q_{-k}f(H+\theta)Q_k\in\mathcal{B}(X), \qquad 0\leqq k\leqq 1/2,$$

uniformly in $\theta\in I_0$.

PROOF. It suffices to prove the lemma only for the case $k=1/2$. Denote by $\hat{f}$ the Fourier transform of f;

$$\hat{f}(t)=(2\pi)^{-1/2}\int e^{-its}f(s)ds,$$

where the integration with no domain attached is taken over the whole space. This abbreviation is used throughout. The operator $f(H+\theta)$ is represented as

$$f(H+\theta)=(2\pi)^{-1/2}\int e^{it\theta}\hat{f}(t)\exp(itH)dt.$$

Hence it follows from Lemma 5.1 that

$$[f(H+\theta), Q_{1/2}](H+\theta+i)^{-1}\in\mathcal{B}(X).$$

If we write $f(H+\theta)$ as

$$f(H+\theta)=g(H+\theta)(H+\theta+i)^{-1}, \qquad g\in C_0^\infty(R),$$

then a simple commutator calculation shows that $[f(H+\theta), Q_{1/2}]\in\mathscr{B}(X)$ uniformly in $\theta\in I_0$. This proves the lemma. □

LEMMA 5.3. *Let $f\in C_0^\infty(R)$ and let $q\in C^\infty(X)$ be a smooth function such that*

$$|\partial_x^\beta q(x)|\leqq C_\beta\langle x\rangle^{-|\beta|}.$$

Then one has

$$Q_{1/2}[f(H+\theta),\ q](H+\theta+i)\in\mathscr{B}(X)$$

uniformly in $\theta\in I_0$.

PROOF. The commutator under consideration is represented as

$$[f(H+\theta),\ q]=(2\pi)^{-1/2}\int e^{it\theta}\hat{f}(t)[\exp(itH),\ q]dt.$$

By use of the same argument as in the proof of Lemma 5.1, we can easily prove that

$$(H+\theta+i)^{-1}Q_{1/2}[f(H+\theta),\ q](H+\theta+i)^{-1}\in\mathscr{B}(X)$$

and hence

$$Q_{1/2}(H+\theta+i)^{-1}[f(H+\theta),\ q](H+\theta+i)^{-1}\in\mathscr{B}(X)$$

uniformly in $\theta\in I_0$. If we write

$$f(H+\theta)=(H+\theta+i)^{-1}g(H+\theta)(H+\theta+i)^{-2},\qquad g\in C_0^\infty(R),$$

then the lemma is obtained from Lemmas 5.1 and 5.2 by repeated use of simple commutator calculations and the proof is complete. □

Let H_a be the cluster Hamiltonian defined in section 2. The intercluster potential $I_a(x)$ associated with cluster decomposition a is defined by

$$I_a=H-H_a=\sum_{\alpha\not\subset a}V_\alpha(r^\alpha),\qquad r^\alpha=r_j-r_k.$$

LEMMA 5.4. *Let $f\in C_0^\infty(R)$ and let $q\in C^\infty(X)$ be as in Lemma 5.3. Assume further that*

$$q(x)I_a(x)=O(|x|^{-\rho}),\qquad \rho>1/2,\qquad |x|\longrightarrow\infty,$$

for some cluster decomposition a. Then one has

$$Q_{1/2}(f(H+\theta)-f(H_a+\theta))q(H_a+\theta+i)\in\mathscr{B}(X)$$

uniformly in $\theta\in I_0$.

PROOF. Let $J(\theta)=f(H+\theta)-f(H_a+\theta)$. We represent this difference as

$$J(\theta)=(2\pi)^{-1/2}\int e^{it\theta}\hat{f}(t)(\exp(itH)-\exp(itH_a))dt$$

and also we write

$$\exp(itH)-\exp(itH_a)=\int_0^t \exp(isH)\, iI_a \exp(i(t-s)H_a)ds.$$

Hence, by assumption, it follows from Lemma 5.1 that

$$(H+\theta+i)^{-1}Q_{1/2}J(\theta)q(H_a+\theta+i)^{-1}\in \mathcal{B}(X)$$

so that

$$Q_{1/2}(H+\theta+i)^{-1}J(\theta)q(H_a+\theta+i)^{-1}\in \mathcal{B}(X)$$

uniformly in $\theta\in I_0$. If we write

$$f(H+\theta)=g(H+\theta)(H+\theta+i)^{-1}, \qquad g\in C_0^\infty(R),$$

then the difference $J(\theta)$ equals

$$J(\theta)=(H+\theta+i)^{-1}\{(g(H+\theta)-g(H_a+\theta))-I_a f(H_a+\theta)\}.$$

Hence we obtain by Lemmas 5.1~5.3 that

$$Q_{1/2}J(\theta)q(H_a+\theta+i)^{-1}\in \mathcal{B}(X)$$

uniformly in $\theta\in I_0$. We again write $f(H+\theta)$ as above. Then

$$J(\theta)=\{(g(H+\theta)-g(H_a+\theta))-f(H+\theta)I_a\}(H_a+\theta+i)^{-1}$$

and hence it follows again from Lemmas 5.1~5.3 that $Q_{1/2}J(\theta)q\in\mathcal{B}(X)$ uniformly in $\theta\in I_0$. We repeat the same argument as above to obtain the desired result. Thus the proof is complete. □

LEMMA 5.5. *As $\theta\to\infty$, one has:*
(i) $\|Q_{-k}(H+\theta+i)^{-1}\|_{L^2(X)}=o(1)$, $k>0$, *and, in particular,*

$$\|Q_{-1/2}(H+\theta+i)^{-1}\|_{L^2(X)}=O(\theta^{-1/2}).$$

(ii) $\|Q_{-1/2}\nabla(H+\theta+i)^{-1}\|_{L^2(X)}=O(1)$.

(iii) $\|Q_{-1}\nabla\nabla(H+\theta+i)^{-1}\|_{L^2(X)}=O(1)$.

PROOF. Statements (i) and (ii) can be easily verified. To prove (iii), we consider the equation

$$(H+\theta+i)u=f, \qquad f\in L^2(X),$$

and take the scalar product of $-Q_{-2}\Delta u$ with this equation. Then we obtain by repeated use of partial integration that

$$\|Q_{-1}\Delta u\|_0^2\leqq C\{\|Q_{-1/2}\nabla u\|_0^2+\|Q_{-1}u\|_0^2+\theta\|Q_{-2}u\|_0^2+\|Q_{-1}f\|_0^2\}.$$

This, together with (i) and (ii), proves (iii). □

As an immediate consequence of Lemma 5.5, we obtain the following

LEMMA 5.6. *Let* $f \in C_0^\infty(R)$. *As* $\theta \to \infty$, *one has*:

(i) $\|Q_{-k} f(H+\theta)\|_{L^2(X)} = o(1), \qquad k > 0$.

(ii) $\|Q_{-1/2} \nabla f(H+\theta)\|_{L^2(X)} = O(1)$.

(iii) $\|Q_{-1} \nabla\nabla f(H+\theta)\|_{L^2(X)} = O(1)$.

§ 6. Construction of conjugate operator.

We recall the notation $E_a = \pi_a E$ and define Λ as

$$\Lambda = \{a : E_a = 0,\ 2 \leqq \#(a) \leqq N-1\}.$$

The conjugate operator A with properties (A.1) and (A.2) is constructed by induction on the numbers $\#(a)$ of cluster decompositions $a \in \Lambda$. The construction requires only the assumption $V_{jk}(y) = O(|y|^{-\rho})$ and we do not use the other decay assumptions for derivatives of first and second order as stated in (V).

6.1. We first consider the case of $(N-1)$-cluster decompositions. Let $a \in \Lambda$, $\#(a) = N-1$, be a $(N-1)$-cluster decomposition, so that $E = E^a = \pi^a E \neq 0$. We can identify a with some pair α. The subsystem operator H^a associated with a takes the form

$$H^a = -\Delta/2 - \langle E^a, x^a \rangle + V_\alpha \qquad \text{on} \quad L^2(X^a).$$

We now define the operator A^a as

$$A^a = -i\langle \omega^a, \nabla \rangle, \qquad \omega^a = E^a/|E^a|,$$

which is considered as a differential operator acting over X^a as well as over X. The commutator $i[H^a, A^a]$ is calculated as

$$i[H^a, A^a] = |E^a| + i[V_\alpha, A^a].$$

Let $g_0 \in C_0^\infty(R)$ be a smooth real function with support in the interval $(-1, 1)$. Then, by assumption (V) and Lemma 5.6, we can take θ_∞, $\theta_\infty \gg 1$, so large that

$$g_0(H^a+\theta) i[H^a, A^a] g_0(H^a+\theta) \geqq d g_0(H^a+\theta)^2, \qquad d = |E^a|/2 > 0,$$

for $\theta \in (\theta_\infty, \infty)$, where the inequality relation is understood in the form sense over the space $L^2(X^a)$.

Let θ_∞ be as above. Let $f_0 \in C_0^\infty(R)$ be a smooth real function with support in a small interval $(-\delta, \delta)$, $0 < \delta \ll 1$, around the origin. Note that $(H^a+i)^{-1}[V_\alpha, A^a](H^a+i)^{-1} : L^2(X^a) \to L^2(X^a)$ is a compact operator. Hence, by Theorem 1, H^a has no bound states and also it follows that as an operator acting on $L^2(X^a)$, $f_0(H^a+\theta)$ satisfies

$$\|f_0(H^a+\theta)[V_\alpha,\ A^a]f_0(H^a+\theta)\|_{L^2(X^a)} = o(1), \qquad \delta \longrightarrow 0,$$

uniformly in $\theta \in (\theta_0,\ \theta_\infty)$, θ_0 being again fixed arbitrarily. Thus we can take $\delta > 0$ so small that

$$f_0(H^a+\theta)i[H^a,\ A^a]f_0(H^a+\theta) \geqq d f_0(H^a+\theta)^2, \qquad d = |E^a|/2 > 0,$$

for $\theta \in I_0 = (\theta_0,\ \infty)$.

6.2. In the argument above, we have constructed the operator A^a with the following two properties (P.1) and (P.2) for the case $\#(a) = N-1$. The properties are formulated as follows.

(P.1) A^a is a differential operator of the form

$$A^a = -i\{\langle c^a(x^a),\ \nabla^a\rangle + \langle \nabla^a,\ c^a(x^a)\rangle\} + c_0^a(x^a),$$

where ∇^a denotes the gradient notation over X^a and the coefficients $c^a(x^a) = \{c_j^a(x^a)\}$, $1 \leqq j \leqq 3(N-\#(a))$, and $c_0^a(x^a)$ are smooth real functions obeying the estimates (3.1).

(P.2) Let $f_0 \in C_0^\infty((-\delta,\ \delta))$, $0 < \delta \ll 1$, be as above. If $\delta > 0$ is chosen small enough, then there exists $d > 0$ such that

$$f_0(H^a+\theta)i[H^a,\ A^a]f_0(H^a+\theta) \geqq d f_0(H^a+\theta)^2$$

for $\theta \in I_0$, where the inequality relation is understood in the form sense over the space $L^2(X^a)$.

We now assume that a family of operators A^a with $a \in \Lambda$, $k+1 \leqq \#(a) \leqq N-1$, has been constructed so as to have the two properties above.

6.3. The next task is to construct an operator A^a with properties (P.1) and (P.2) for $a \in \Lambda$ with $\#(a) = k$, assuming the case of m-cluster decompositions with $k+1 \leqq m \leqq N-1$.

Let $a \in \Lambda$, $\#(a) = k$, be a k-cluster decomposition. We define Σ_a as

$$\Sigma_a = \{b :\ b \subset a,\ b \neq a\}$$

and Λ_a and Λ_a^c as

$$\Lambda_a = \Sigma_a \cap \Lambda, \qquad \Lambda_a^c = \Sigma_a \backslash \Lambda_a.$$

We can construct a partition of unity $\{j_b^a(x^a)\}$ over the space X^a with the following property (j), where b ranges over all cluster decompositions in Σ_a. The property is that:

(j.1) $j_b^a \in C^\infty(X^a)$, $j_b^a(x^a) \geqq 0$ and $\sum_b j_b^a(x^a)^2 = 1$;

(j.2) $j_b^a(x^a)$ is homogeneous of degree zero for $|x^a| > 1$;

(j.3) $j_b^a(x^a)V_\alpha(r^\alpha) = O(|x^a|^{-\rho})$ as $|x^a| \to \infty$ for $\alpha \not\subset b$ and $\alpha \subset a$.

By inductive assumption, we may assume that operators A^b with properties

(P.1) and (P.2) have been constructed for $b\in\Lambda_a$. For $b\in\Lambda_a^c$, we define the operator A_b as

$$A_b=-i\langle\omega_b, \nabla\rangle, \qquad \omega_b=E_b/|E_b|. \tag{6.1}$$

Here it should be noted that $E_b\neq 0$ for b as above and it lies in the space $X^a\cap X_b$, so that A_b can be regarded as a differential operator acting over X^a as well as over X. We further define the operator γ^a as

$$\gamma^a=(D_a+D_a^*)/2, \qquad D_a=-i\langle x^a/\langle x^a\rangle, \nabla^a\rangle. \tag{6.2}$$

For the k-cluster decomposition $a\in\Lambda$, we now define the operator A^a as

$$A^a=\sum_{b\in\Lambda_a^c} j_b^a A_b j_b^a+\sum_{b\in\Lambda_a} j_b^a A^b j_b^a+M\gamma^a$$

with $M\gg 1$ large enough to be determined later. Since $z=\langle x, \omega\rangle=\langle x^a, \omega^a\rangle$ for $a\in\Lambda$, it follows by construction that A^a has the property (P.1).

6.4. We shall show that A^a has also the second property (P.2). The present subsection is devoted to a preliminary step for proving this.

Let $a\in\Lambda$ be again a k-cluster decomposition. We denote by X_b^a the space $X_b^a=X^a\cap X_b$ for $b\in\Sigma_a$, so that $X^a=X^b\oplus X_b^a$ and also $L^2(X^a)$ is decomposed as $L^2(X^a)=L^2(X^b)\otimes L^2(X_b^a)$. The cluster Hamiltonian H_b^a obtained from H^a is defined as

$$H_b^a=H^b\otimes Id+Id\otimes T_b^a \qquad \text{on} \quad L^2(X^b)\otimes L^2(X_b^a),$$

where H^b is the subsystem operator associated with b and T_b^a is the free Hamiltonian defined by

$$T_b^a=-\Delta/2-\langle E_b, x^a\rangle \qquad \text{on} \quad L^2(X_b^a).$$

If, in particular, $b\in\Lambda_a$, then T_b^a takes the form $T_b^a=-\Delta/2$. By (j.3), the intercluster potential $I_b^a(x^a)$ with $b\in\Sigma_a$ has the decay property

$$j_b^a(x^a)I_b^a(x^a)=j_b^a(H^a-H_b^a)=\sum_{\alpha\not\subset b,\, \alpha\subset a} j_b^a(x^a)V_\alpha(r^\alpha)=O(|x^a|^{-\rho})$$

as $|x^a|\to\infty$.

Let A_b, $b\in\Lambda_a^c$, be defined by (6.1). Then, by definition,

$$i[H_b^a, A_b]=|E_b|>0$$

and hence we have

$$f_0(H_b^a+\theta)i[H_b^a, A_b]f_0(H_b^a+\theta)=|E_b|f_0(H_b^a+\theta)^2, \qquad b\in\Lambda_a^c. \tag{6.3}$$

If $b\in\Lambda_a$, then $T_b^a=-\Delta/2$ as stated above. By use of the Fourier transformation, we can construct the spectral representation by which T_b^a is transformed into the multiplication operator by σ, $\sigma\geqq 0$, on the space $L^2([0, \infty)\cdot\mathcal{L})$ for some Hilbert space $\mathcal{L}$. By inductive assumption, A^b is a differential operator acting over X^b and hence

$$i[H_b^a, A^b] = i[H^b, A^b].$$

Thus we have

$$f_0(H_b^a+\theta)i[H_b^a, A_b]f_0(H_b^a+\theta) = \int_0^\infty \oplus f_0(H^b+\theta+\sigma)i[H^b, A_b]f_0(H^b+\theta+\sigma)d\sigma$$

by the direct integral. This, together with property (P.2), yields that

$$f_0(H_b^a+\theta)i[H_b^a, A_b]f_0(H_b^a+\theta) \geqq df_0(H_b^a+\theta)^2, \qquad b \in \Lambda_a, \tag{6.4}$$

for $\theta \in I_0$.

Let γ^a and D_a be as in (6.2). We denote by b_k^a a multiplication operator by $b_k^a(x^a)$ with bound $|b_k^a(x^a)| \leqq C\langle x^a\rangle^k$ and, in particular, we write Q_k^a for the multiplication operator by $\langle x^a\rangle^k$. We calculate the commutator $i[H^a, \gamma^a]$ in a way similar to that in the proof of Lemma 4.2. We obtain that

$$G = i[H^a, \gamma^a] - Q_{-1/2}^a(-\Delta/2)Q_{-1/2}^a = G_0+G_1+G_2+R, \tag{6.5}$$

where $R = b_{-2}^a\nabla^a + b_{-3}^a$ and

$$\begin{aligned} G_0 &= Q_{-1/2}^a(-\Delta - D_a^*D_a)Q_{-1/2}^a, \\ G_1 &= Q_{-1/2}^a(-H^a+V^a)Q_{-1/2}^a, \\ G_2 &= i[V^a, \gamma^a], \end{aligned}$$

V^a being defined by (2.1). It should be noted that G_0 is non-negative.

We now set

$$\begin{aligned} F_1^a &= \sum_{b\in\Lambda_a^c} \{i[H^a, j_b^a]A_b j_b^a + j_b^a A_b i[H^a, j_b^a]\}, \\ F_2^a &= \sum_{b\in\Lambda_a} \{i[H^a, j_b^a]A^b j_b^a + j_b^a A^b i[H^a, j_b^a]\}. \end{aligned}$$

Then it follows from (j.2) that

$$F_1^a + F_2^a \geqq -M_1 Q_{-1/2}^a(-\Delta+1)Q_{-1/2}^a$$

for some $M_1>0$. Hence relation (6.5) enables us to take M, $M \geqq 3M_1$, so large that

$$F_1^a + F_2^a + Mi[H^a, \gamma^a] \geqq MG - MQ_{-1}^a. \tag{6.6}$$

The constant M is now determined so as to satisfy the form inequality above.

6.5. We first prove (P.2) for $\theta \gg 1$ large enough. Let $b \in \Lambda_a^c$. Then, by repeated use of Lemmas 5.3~5.6, it follows from (6.3) that for $\varepsilon>0$ small enough, there exists $\theta_\infty \gg 1$ such that

$$f_0(H^a+\theta)j_b^a i[H^a, A_b]j_b^a f_0(H^a+\theta) \geqq df_0(H^a+\theta)\{(j_b^a)^2-\varepsilon\}f_0(H^a+\theta) \tag{6.7}$$

with $d=|E_b|/2>0$ for $\theta\in[\theta_\infty, \infty)$. Similarly, for $b\in\Lambda_a$, we obtain from (6.4) that

(6.8) $f_0(H^a+\theta)j_b^a i[H^a, A^b]j_b^a f_0(H^a+\theta)\geqq d f_0(H^a+\theta)\{(j_b^a)^2-\varepsilon\}f_0(H^a+\theta)$

with another $d>0$ for $\theta\in[\theta_\infty, \infty)$. By assumption (V), the operator G_2 takes the form

$$G_2=i[V^a, \gamma^a]=b_{-\rho}^a\nabla^a+\nabla^a b_{-\rho}^a, \qquad 1/2<\rho<1,$$

and also G_1 is rewritten as

$$G_1=Q_{-1/2}^a(-H^a+V^a)Q_{-1/2}^a=Q_{-1/2}^a\{-(H^a+\theta)+V^a\}Q_{-1/2}^a+\theta Q_{-1}^a.$$

We may assume that $\theta>0$. Since G_0 is non-negative, we have

$$G\geqq -Q_{-1/2}^a(H^a+\theta)Q_{-1/2}^a+R_0$$

with remainder $R_0=b_{-\rho}^a\nabla^a+\nabla^a b_{-\rho}^a+b_{-1}^a$. Hence, by making use of Lemmas 5.3 $\sim$5.6 again, we see that for any $\varepsilon>0$ small enough, there exists $\theta_\infty\gg 1$ such that

$$f_0(H^a+\theta)Gf_0(H^a+\theta)\geqq -\varepsilon f_0(H^a+\theta)^2$$

for $\theta\in[\theta_\infty, \infty)$. This, together with (6.6)$\sim$(6.8), proves that A^a has the property (P.2) for $\theta\gg 1$.

6.6. We now assume that θ lies in a compact interval $[\theta_0, \theta_\infty]$ for $\theta_\infty\gg 1$. Let R_0 be as above. We note that $(H^a+i)^{-1}R_0(H^a+i)^{-1}$ is a compact operator from $L^2(X^a)$ into itself. Furthermore, for $f\in C_0^\infty(R)$,

$$[f(H^a), j_b^a](H^a+i) \quad \text{and} \quad (f(H^a)-f(H_b^a))j_b^a(H^a+i)$$

are also compact operators, which is seen from the proof of Lemmas 5.3 and 5.4. Hence we can obtain the form inequality

$$f_0(H^a+\theta)i[H^a, A^a]f_0(H^a+\theta)\geqq d f_0(H^a+\theta)^2+K^a, \qquad d>0,$$

for some compact operator K^a. According to Theorem 1, this implies that H^a has no bound states and also makes it possible for us to take the support of f_0 around the origin, supp $f_0\subset(-\delta, \delta)$, $\delta>0$, so small that

$$f_0(H^a+\theta)i[H^a, A^a]f_0(H^a+\theta)\geqq d f_0(H^a+\theta)^2$$

with another $d>0$, where δ can be chosen uniformly in $\theta\in[\theta_0, \theta_\infty]$. Thus we have proved that the operator A^a has the second property (P.2).

6.7. We are now in a position to construct the conjugate operator A with properties (A.1) and (A.2). Let γ be as in (4.2) and let

$$\Lambda^c=\{a: E_a\neq 0, 2\leqq \#(a)\leqq N\}.$$

For $a\in\Lambda^c$, we define the operator A_a as

$$A_a = -i\langle \omega_a, \nabla \rangle, \qquad \omega_a = E_a/|E_a|.$$

We further introduce a smooth non-negative partition of unity $\{j_a(x)\}$, $2 \leqq \#(a) \leqq N$, over X. The partition has the same property (j) as before with natural modifications.

The conjugate operator A in question is now defined as

$$A = \sum_{a \in \Lambda^c} j_a A_a j_a + \sum_{a \in \Lambda} j_a A^a j_a + M\gamma$$

with $M \gg 1$. From (P.1), this operator is easily seen to have the property (A.1). On the other hand, property (A.2) is also verified by taking M large enough. In fact, this can be seen by repeating the same arguments as used for proving (P.2) in the previous subsections 6.4~6.6. Thus we have constructed the conjugate operator A with properties (A.1) and (A.2).

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