

Formal neighbourhoods of a toric variety and unirationality of algebraic varieties

By Madoka EBIHARA

(Received July 6, 1992)

§ 0. Introduction.

This paper provides a partial answer to the following question.

QUESTION. *Assume that a nonsingular complete algebraic variety X of dimension $n+2$ ($n \geq 1$) contains a nonsingular rational surface S with $N_{S/X}$ ample. Then is X unirational?*

In the previous paper [2], we have solved this question affirmatively in the case where $n=1$ (i. e., X is three-dimensional) and S is toric. Here we shall generalize the main theorem of [2] to the higher-dimensional case. The main result is the following.

MAIN THEOREM. *Let n be a positive integer. Let X be a nonsingular complete algebraic variety of dimension $n+2$. Assume that X contains a nonsingular projective toric surface S and that the following two conditions (a) and (b) are satisfied:*

(a) $N_{S/X} \cong \bigoplus_{\mu=1}^n A_{\mu}$ where each A_{μ} is an ample line bundle,

(b) $H^1(S, N_{S/X} \otimes S^q(N_{S/X}^{\vee})) = 0$ for each positive integer q .

Then X is unirational.

As is easily seen, this theorem is a generalization of the main theorem of [2]. The following corollary would be a help to understanding of the statement of Main Theorem.

COROLLARY. *Let X be a nonsingular complete algebraic variety of dimension $n+2$ and L a line bundle on X . Assume that there exists a sequence*

$$X = X_0 \supset X_1 \supset \cdots \supset X_n = S$$

of subvarieties of X satisfying the following three conditions:

- (1) X_i is a smooth member of the linear system $|L|_{X_{i-1}}|$ on X_{i-1} ($1 \leq i \leq n$),
- (2) $X_n = S$ is a toric surface,

(3) $L|_S$ is ample on S .

Then X is unirational.

The way of proof heavily depends on the paper [2] and we refer to it for our principle and technique. What we should emphasize here again is that Hironaka-Matsumura's theorem ([3] Theorem (3.3)) plays a central role in our theory. Due to this theorem, we can reduce the problem to the study of formal neighbourhoods of a toric variety.

In §1 we recall how to construct regular formal neighbourhoods of a given smooth variety. Such construction is rather well-known and we have stated it in [2] in the three-dimensional case. Here, in order to fix the notation, we recall the correspondence between transition functions of the coordinates of neighbourhoods and certain Čech cochains without proof.

In §2 we state a key lemma (Lemma 2.3) on formal neighbourhoods of a nonsingular rational curve, which is a slight generalization of Lemma 2.6 of [2]. This lemma provides a sufficient condition for a neighbourhood of \mathbf{P}^1 to be rationally dominated (cf. Definition 2.1).

In §3 we discuss a semi-group, which we shall call a scope, associated to a regular formal neighbourhood (X, S) of a toric variety S such that the normal bundle $N_{S/X}$ is a direct sum of equivariant line bundles. The notion of the scope has been introduced in [2] in the case where X is three-dimensional and S is a toric surface. Here we shall generalize and refine it. The notion of the scope is a technical core of our theory, which makes our arguments on formal neighbourhoods easy to handle.

§'s 4, 5 and 6 are devoted to complete the proof according to the way of proof established in [2]. We estimate the scope of a neighbourhood (X, S) of S by the induction on $\rho(S)$, take a nonsingular rational curve C on S and apply Lemma 2.3 to the neighbourhood $(X, C)^\wedge$ of C . By Hironaka-Matsumura's theorem (loc. cit.), we obtain Main Theorem.

In §7 we shall state some supplementary propositions and problems. First we make some remarks on the condition (b) of Main Theorem. We show an example in which the condition (b) is not satisfied and the scope is so big that we cannot apply our theory itself (cf. Example 7.2). Next we make a remark on the algebrizability of formal neighbourhoods, which we do not discuss in the proof of Main Theorem. We determine all the algebrizable regular formal neighbourhoods of dimension two of \mathbf{P}^1 with ample normal bundle, according to the idea of M. Reid (cf. Proposition 7.4, Example 7.5). Finally we discuss general problems and propose a conjecture (Conjecture 7.8).

The feature of our theory is very similar to that of [2]. The following two points should be distinguished from [2].

(1) We define the scope in arbitrary dimension, though we shall later

restrict ourselves to discussing neighbourhoods of a toric surface.

- (2) We define the scope more intrinsically than [2]. Let S be a nonsingular toric surface and N a direct sum of equivariant line bundles on S . We realize the scope as a semi-group contained in the variety $\text{Spec}(\bigoplus_{q \geq 0} S^q(N^\vee))$, which makes our arguments more perspicuous.

Once the theory of the scope is established in the higher-codimensional case, the remaining problem of generalization of the result of [2] is at most a kind of technical complexity, though the result is quite improved. It suggests that our approach is essentially independent on the codimension of subvarieties, which provides an evidence for Conjecture 7.8.

ACKNOWLEDGEMENT. The author would like to express his thanks to Professors S. Itaka and Y. Kawamata for their advices and warm encouragement. He discussed the results of this paper with Professors I. Nakamura, M. Reid, I. Shimada, T. Usa and many other people. He also expresses his thanks for their stimulating advices and discussions.

§1. Construction of neighbourhoods.

In this section, we discuss how to construct regular neighbourhoods of a given smooth variety with a given vector bundle as the normal bundle. Such construction is rather well-known (cf. [2], [7]). We briefly survey this general theory without proof in order to fix the notation. For the proof we refer to §1 of [2], which is still effective in our case after a slight modification.

Let S be a smooth variety of dimension m and N a vector bundle on S of rank n . We discuss how to construct a regular neighbourhood (X, S) of S with $N_{S/X} \cong N$. For simplicity, we restrict ourselves to the case where S is covered by affine open subsets isomorphic to A_k^n , because we shall later assume that S is a nonsingular toric variety, which satisfies the above condition. Let $\mathcal{U} = (U_i)_{i \in I}$ be an affine open covering of S , and let (X, S) be a regular formal neighbourhood of S with covering $\tilde{U} = (\tilde{V}_i)$ such that $N_{S/X} \cong N$, $U_i = \tilde{U}_i|_S$. We assume that $U_i \cong \text{Spec } k[t_i]$ and $\tilde{U}_i \cong \text{Spf } k[[t_i]][[x_i]]$, where $t_i = (t_i^1, \dots, t_i^m)$ and $x_i = (x_i^1, \dots, x_i^n)$ denote the coordinates. On $\tilde{U}_i \cap \tilde{U}_j$ ($i, j \in I$), the coordinates are related to each other by the following transition relation:

$$(t_i, x_i) = f_{ij}(t_j, x_j),$$

where f_{ij} is a vector-valued formal power series in x_j with the coefficients in $\Gamma(U_i \cap U_j, \mathcal{O}_S)$. We put

$$\begin{aligned} f_{ij} &= (g_{ij}, h_{ij}) \\ &= (g_{ij}^1, \dots, g_{ij}^m, h_{ij}^1, \dots, h_{ij}^n), \end{aligned}$$

that is, $t_i^\lambda = g_{ij}^\lambda(t_j, x_j)$, $x_i^\mu = h_{ij}^\mu(t_j, x_j)$, ($1 \leq \lambda \leq m$, $1 \leq \mu \leq n$). We expand g_{ij}^λ and h_{ij}^μ in the following way :

$$g_{ij}^\lambda = g_{ij10}^\lambda(t_j) + g_{ij11}^\lambda(t_j, x_j) + \dots + g_{ij1q}^\lambda(t_j, x_j) + \dots,$$

$$h_{ij}^\mu = h_{ij10}^\mu(t_j) + h_{ij11}^\mu(t_j, x_j) + \dots + h_{ij1q}^\mu(t_j, x_j) + \dots,$$

where g_{ij1q} and h_{ij1q} are homogeneous polynomials of degree q in x_j . Note that $h_{ij10} = 0$. We put :

$$g_{ij1q} = (g_{ij1q}^1, \dots, g_{ij1q}^m),$$

$$h_{ij1q} = (h_{ij1q}^1, \dots, h_{ij1q}^n),$$

$$f_{ij1q} = (g_{ij1q}, h_{ij1q}).$$

We also use the following notation :

$$g_{ij[q]}^\lambda = g_{ij10}^\lambda + g_{ij11}^\lambda + \dots + g_{ij1q}^\lambda,$$

$$h_{ij[q]}^\mu = h_{ij11}^\mu + \dots + h_{ij1q}^\mu,$$

$$g_{ij[q]} = (g_{ij[q]}^1, \dots, g_{ij[q]}^m),$$

$$h_{ij[q]} = (h_{ij[q]}^1, \dots, h_{ij[q]}^n),$$

$$f_{ij[q]} = (g_{ij[q]}, h_{ij[q]}).$$

The collection $\{g_{ij10}\}$ of the terms of degree zero is nothing but the transition functions that determine the variety S , which is already given. The collection $\{h_{ij11}\}$ is nothing but the transition functions that determine the vector bundle N , which is also given.

To construct (X, S) , we have to give a collection $\{f_{ij}\}$ satisfying the following :

- (1) $\{g_{ij10}\}$ determines S ;
- (2) $\{h_{ij11}\}$ determines N ;
- (3) $f_{ij}(f_{jk}(t_k, x_k)) = f_{ik}(t_k, x_k)$ for $i, j, k \in I$.

We successively construct the q -th infinitesimal neighbourhoods. We introduce another notation. For $f, g \in \Gamma(U_i \cap U_j, \mathcal{O}_S)[[x_j]]$, we write $f \equiv_q g$ if $f \equiv g \pmod{(x_j)^{q+1}}$.

To construct the first infinitesimal neighbourhood, we have to give a collection $\{f_{ij11}\}$ and determine $\{f_{ij[1]}\}$ satisfying the following condition $(*)_1$:

$$(*)_1 \quad f_{ij[1]}(f_{jk[1]}(t_k, x_k)) \equiv_1 f_{ik[1]}(t_k, x_k) \quad \text{for } i, j, k \in I.$$

To the vector-valued function g_{ij11} , we attach an element $G_{ij11} \in \Gamma(U_i \cap U_j, \mathcal{O}_S \otimes N^\vee)$ in the following way :

$$\begin{aligned}
 G_{ij11} &= \sum_{\lambda=1}^m \left(\frac{\partial}{\partial t_i^\lambda}\right)' \otimes g_{ij11}^\lambda(g_{ji10}, h_{ji11}) \bmod (x_i)^2 \\
 &= \sum_{\lambda=1}^m \sum_{\mu=1}^n \left(\frac{\partial}{\partial t_i^\lambda}\right)' \otimes \tilde{g}_{ij11}^{\lambda,\mu}(t_i)x_i^\mu \bmod (x_i)^2,
 \end{aligned}$$

where $\{(\partial/\partial t_i^\lambda)' | 1 \leq \lambda \leq m\}$ denote the local basis of the sheaf Θ_S on U_i and $\{x_i^\mu \bmod (x_i)^2\}$ is the local basis of the sheaf N^\vee on U_i . Since $\tilde{g}_{ij11}^{\lambda,\mu}$ belongs to $\Gamma(U_i \cap U_j, \Theta_S)$, we can consider G_{ij11} to be an element of $\Gamma(U_i \cap U_j, \Theta_S \otimes N^\vee)$. We put $\underline{g}_1 = \Theta_S \otimes N^\vee$. Thus we often identify a collection $\{g_{ij11}\}_{i,j \in I}$ with $G_1 = (G_{ij11})_{i,j \in I} \in C^1(\mathcal{U}, \underline{g}_1)$.

CLAIM 1.1. *A collection $\{g_{ij11}\}$ determines $\{g_{ij[1]}\}$ satisfying the condition $(*)_1$ if and only if the corresponding Čech cochain G_1 satisfies the cocycle condition, i.e., $G_1 \in Z^1(\mathcal{U}, \underline{g}_1)$.*

CLAIM 1.2. *Let $G, G' \in Z^1(\underline{g}_1)$. Assume that $G' - G \in B^1(\mathcal{U}, \underline{g}_1)$. Then the first infinitesimal neighbourhoods determined by G and G' are isomorphic to each other.*

Suppose that a description of the $(q-1)$ -th infinitesimal neighbourhood is given ($q \geq 2$), that is, a collection $\{f_{ij[q-1]}\}$ of the transition functions is determined up to degree $q-1$ with respect to the coordinates x_j 's. We have $f_{ij[q-1]}(f_{jk[q-1]}) \equiv_{q-1} f_{ik[q-1]}$. We put

$$\begin{aligned}
 \phi_{ijk1q} &= (f_{ij[q-1]}(f_{jk[q-1]}) - f_{ik[q-1]})_{[q]} \\
 &= (a_{ijk1q}, b_{ijk1q}) \\
 &= (a_{ijk1q}^1, \dots, a_{ijk1q}^m, b_{ijk1q}^1, \dots, b_{ijk1q}^n).
 \end{aligned}$$

To the function ϕ_{ijk1q} , we attach an element $\Psi_{ijk1q} \in \Gamma(U_i \cap U_j \cap U_k, \Theta_X|_S \otimes S^q(N^\vee))$ in the following way:

$$\begin{aligned}
 \Psi_{ijk1q} &= \sum_{\mu=1}^m \frac{\partial}{\partial t_i^\mu} \otimes a_{ijk1q}^\mu(g_{ki10}, h_{ki11}) \\
 &\quad + \sum_{\mu=1}^n \frac{\partial}{\partial x_i^\mu} \otimes b_{ijk1q}^\mu(g_{ki10}, h_{ki11}) \bmod (x_i)^{q+1} \\
 &= \sum_{\lambda=1}^m \sum_{\substack{\vec{q} \geq \vec{0} \\ 1 \cdot \vec{q} = q}} \frac{\partial}{\partial t_i^\lambda} \otimes \tilde{a}_{ijk1q}^{\lambda, \vec{q}}(x_i)^{\vec{q}} \\
 &\quad + \sum_{\mu=1}^n \sum_{\substack{\vec{q} \geq \vec{0} \\ 1 \cdot \vec{q} = q}} \frac{\partial}{\partial x_i^\mu} \otimes \tilde{b}_{ijk1q}^{\mu, \vec{q}}(x_i)^{\vec{q}} \bmod (x_i)^{q+1},
 \end{aligned}$$

where $\partial/\partial t_i^\lambda$ ($1 \leq \lambda \leq m$) and $\partial/\partial x_i^\mu$ ($1 \leq \mu \leq n$) denote in this time the local basis of the sheaf $\Theta_X|_S$ on U_i , $(x_i)^{\vec{q}} \bmod (x_i)^{q+1}$ ($1 \cdot \vec{q} = q, \vec{q} \geq \vec{0}$) are the local basis of

the sheaf $S^q(N^\vee)$ on U_i with $\vec{q}=(q_1, \dots, q_n)$, $\vec{0}=(0, \dots, 0)$, $\vec{1}=(1, \dots, 1)$, $(x_i)^{\vec{q}}=(x_1^{q_1} \cdots x_n^{q_n})$ and $\vec{1} \cdot \vec{q}=q_1 + \dots + q_n$ denoting the usual inner product.

From now on, we use the following notation about vectors unless otherwise mentioned.

DEFINITION 1.3. Let $\vec{a}=(a_1, \dots, a_n)$, $\vec{b}=(b_1, \dots, b_n)$. We denote $\vec{a} > \vec{b}$ (resp. $\vec{a} \geq \vec{b}$) if $a_\mu > b_\mu$ (resp. $a_\mu \geq b_\mu$) for each μ with $1 \leq \mu \leq n$. We denote by $\vec{0}$ the vector $(0, \dots, 0)$. We denote by $\vec{1}$ the vector $(1, \dots, 1)$. We denote by \vec{e}_μ the vector of which the μ -th coefficient is equal 1 and the others are 0. We define $\vec{a} \cdot \vec{b}=a_1 b_1 + \dots + a_n b_n$ and $(\vec{a})^{\vec{b}}=(a_1)^{b_1} \cdots (a_n)^{b_n}$.

We put $\mathcal{F}_q=\mathcal{O}_X|_S \otimes S^q(N^\vee)$. Thus we often identify a collection $\{\phi_{ijk|q}\}_{i,j,k \in I}$ with $\Psi_q=(\Psi_{ijk|q}) \in C^2(\mathcal{U}, \mathcal{F}_q)$.

CLAIM 1.4. $\Psi_q \in Z^2(\mathcal{U}, \mathcal{F}_q)$.

To construct the q -th infinitesimal neighbourhood, we have to add a collection $\{f_{ij|q}\}$ of the terms of degree q to $\{f_{ij|q-1}\}$ which is already determined and determine $\{f_{ijk|q}\}$ satisfying the following condition $(*)_q$:

$$(*)_q \quad f_{ijk|q}(f_{jk|q}(t_k, x_k)) \equiv_q f_{ik|q}(t_k, x_k) \quad \text{for } i, j, k \in I.$$

To the function $f_{ij|q}=(g_{ij|q}, h_{ij|q})$, we attach an element $F_{ij|q} \in \Gamma(U_i \cap U_j, \mathcal{F}_q)$ in the following way:

$$F_{ij|q} = \sum_{\lambda=1}^m \frac{\partial}{\partial t_i^\lambda} \otimes g_{ij|q}^\lambda(g_{ji|0}, h_{ji|1}) + \sum_{\mu=1}^n \frac{\partial}{\partial x_i^\mu} \otimes h_{ij|q}^\mu(g_{ji|0}, h_{ji|1}) \pmod{(x_i)^{q+1}}.$$

Thus we often identify a collection $\{f_{ij|q}\}_{i,j \in I}$ with $F_q=(F_{ij|q})_{i,j \in I} \in C^1(\mathcal{U}, \mathcal{F}_q)$.

CLAIM 1.5.

(1) In order to satisfy $(*)_q$, F_q must satisfy $dF_q = -\Psi_q$, where d denotes the coboundary map. In particular, if $\Psi_q \notin B^2(\mathcal{U}, \mathcal{F}_q)$, there does not exist such a cochain F_q .

(2) Assume that F_q and F'_q determine two q -th infinitesimal neighbourhoods and that $F'_q - F_q \in B^1(\mathcal{U}, \mathcal{F}_q)$. Then these two neighbourhoods are isomorphic to each other.

§ 2. RD Lemma.

In this section we state a lemma which plays a key role in this paper and which is a slight modification of Lemma 2.6 in [2].

DEFINITION 2.1. Let (X, C) be a regular formal neighbourhood of a non-singular rational curve C , that is, X is a regular formal scheme with the reduced subscheme C . The neighbourhood (X, C) is said to be rationally dominated if there exists a dominant morphism $\varphi: (\mathbf{P}^N, l)^\wedge \rightarrow (X, C)$, where l denotes a line in \mathbf{P}^N and $(\mathbf{P}^N, l)^\wedge$ the formal completion of \mathbf{P}^N along l .

PROPOSITION 2.2 (cf. [2] Prop. 2.5). *Let X be a nonsingular complete algebraic variety. Assume that there exists a nonsingular rational curve C such that $(X, C)^\wedge$ is rationally dominated. Then X is unirational.*

PROOF. We refer to [2]. This proposition is an easy corollary of Theorem (3.3) of [3].

The following is a key lemma in this paper, which we shall call the RD Lemma, and which we have proved in the three-dimensional case in [2].

LEMMA 2.3 (RD Lemma). *Let*

$$(X, C) = \text{Spf}(k[t_0][[x_0^1, \dots, x_0^N]]) \cup \text{Spf}(k[t_1][[x_1^1, \dots, x_1^N]])$$

be a formal neighbourhood of a nonsingular rational curve C with the following transition relation of the coordinates:

$$t_0 = (t_1)^{-1} + \sum_{\substack{q_1, \dots, q_N \geq 0 \\ q_1 + \dots + q_N \geq 1}} a_{\alpha q_1 \dots q_N} (t_1)^{-\alpha} (x_1^1)^{q_1} \dots (x_1^N)^{q_N},$$

$$x_0^\mu = \sum_{\substack{q_1, \dots, q_N \geq 0 \\ q_1 + \dots + q_N \geq 1}} b_{\alpha q_1 \dots q_N}^\mu (t_1)^{-\alpha} (x_1^1)^{q_1} \dots (x_1^N)^{q_N},$$

$(1 \leq \mu \leq N)$. *Assume that there exists a positive integer r satisfying the following condition: If $\alpha < (q_1 + \dots + q_N)/r$, then $a_{\alpha q_1 \dots q_N} = 0$ and $b_{\alpha q_1 \dots q_N}^\mu = 0$ for all μ .*

Then the neighbourhood (X, C) is rationally dominated.

PROOF. Let l be a line in \mathbf{P}^{N+1} . Then we have

$$(\mathbf{P}^{N+1}, l)^\wedge \cong \text{Spf}(k[u_0][[z_0^1, \dots, z_0^N]]) \cup \text{Spf}(k[u_1][[z_1^1, \dots, z_1^N]])$$

with $u_0 = (u_1)^{-1}$ and $z_0^\mu = (u_1)^{-1} z_1^\mu$ ($1 \leq \mu \leq N$). We can explicitly construct a dominant morphism $\varphi: (\mathbf{P}^{N+1}, l)^\wedge \rightarrow (X, C)$ by the following two homomorphisms ϕ_0 and ϕ_1 of rings:

$$\phi_0: k[t_0][[x_0^1, \dots, x_0^N]] \longrightarrow k[u_0][[z_0^1, \dots, z_0^N]],$$

$$t_0 \longmapsto (u_0)^r + \sum a_{\alpha q_1 \dots q_N} (u_0)^{r - \alpha - q_1 - \dots - q_N} (z_0^1)^{q_1} \dots (z_0^N)^{q_N},$$

$$x_0^\mu \longmapsto \sum b_{\alpha q_1 \dots q_N}^\mu (u_0)^{r - \alpha - q_1 - \dots - q_N} (z_0^1)^{q_1} \dots (z_0^N)^{q_N},$$

and

$$\phi_1: k[t_1][[x_1^1, \dots, x_1^N]] \longrightarrow k[u_1][[z_1^1, \dots, z_1^N]],$$

$$t_1 \longmapsto (u_1)^r,$$

$$x_1^\mu \longmapsto z_1^\mu \quad (1 \leq \mu \leq N).$$

§ 3. Semi-groups associated to neighbourhoods.

This section provides a generalization and refinement of § 3 of [2]. We introduce certain semi-groups, which we shall call scopes, in order to describe regular formal neighbourhoods of a smooth toric variety whose normal bundle is isomorphic to a direct sum of equivariant line bundles. We use the same notation as in § 1. Let S be a nonsingular projective toric variety of dimension m . Let A_1, \dots, A_n be equivariant line bundles on S and $N = \bigoplus_{\mu=1}^n A_\mu$. Let (X, S) be a regular formal neighbourhood of S with $N_{S/X} \cong N$.

We use the following notation for $q > 0$ and $\vec{q} = (q_1, \dots, q_n)$ with $\bar{1} \cdot \vec{q} = q_1 + \dots + q_n = q$:

$$\mathcal{F}_q = \Theta_X|_S \otimes S^q(N^\vee),$$

$$\mathcal{G}_q = \Theta_S \otimes S^q(N^\vee),$$

$$\mathcal{H}_q = N \otimes S^q(N^\vee),$$

$$\mathcal{F}_{\vec{q}} = \Theta_X|_S \otimes (A_1)^{-q_1} \otimes \dots \otimes (A_n)^{-q_n},$$

$$\mathcal{G}_{\vec{q}} = \Theta_S \otimes (A_1)^{-q_1} \otimes \dots \otimes (A_n)^{-q_n},$$

$$\mathcal{H}_{\mu; \vec{q}} = A_\mu \otimes (A_1)^{-q_1} \otimes \dots \otimes (A_n)^{-q_n},$$

$$\mathcal{H}_{\vec{q}} = \bigoplus_{\mu=1}^n \mathcal{H}_{\mu; \vec{q}}.$$

Directly from the definition, we have the following:

$$\mathcal{F}_q = \bigoplus_{\bar{1} \cdot \vec{q} = q} \mathcal{F}_{\vec{q}},$$

$$\mathcal{G}_q = \bigoplus_{\bar{1} \cdot \vec{q} = q} \mathcal{G}_{\vec{q}},$$

$$\mathcal{H}_q = \bigoplus_{\mu=1}^n \bigoplus_{\bar{1} \cdot \vec{q} = q} \mathcal{H}_{\mu; \vec{q}}.$$

We have the following exact sequences:

$$0 \longrightarrow \mathcal{G}_q \xrightarrow{\iota_q} \mathcal{F}_q \xrightarrow{\tau_q} \mathcal{H}_q \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{G}_{\vec{q}} \xrightarrow{\iota_{\vec{q}}} \mathcal{F}_{\vec{q}} \xrightarrow{\tau_{\vec{q}}} \mathcal{H}_{\vec{q}} \longrightarrow 0.$$

To construct transition functions of (X, S) , we have to give collections $\{g_{i,j|1}\}$ and $\{(g_{i,j|q}, h_{i,j|q})\}$ ($q \geq 2$), or equivalently Čech cochains $G_1 \in Z^1(\mathcal{U}, \mathcal{G}_1)$ and $F_q \in$

$C^1(\mathcal{U}, \mathcal{F}_q)$ ($q \geq 2$) (cf. § 1). We also use the following notation throughout this paper. We may assume that the top terms $\{g_{i,j,0}\}$ and $\{h_{i,j,1}\}$ of the transition functions are monomials and that $\{h_{i,j,1}^\mu\}$ determines the line bundle A_μ ($1 \leq \mu \leq n$). For $1 \leq \lambda \leq m$ and $1 \leq \mu \leq n$, we put

$$g_{i,j,0}^\lambda = \prod_{\nu=1}^m (t_j^\nu)^{g(\lambda, \nu; i, j)},$$

$$h_{i,j,1}^\mu = \left(\prod_{\nu=1}^m (t_j^\nu)^{h(\mu, \nu; i, j)} \right) x_j^\mu.$$

We define $T(i, j) \in M(m+n, m+n; \mathbf{Z})$ in the following way:

$$T(i, j) = \begin{pmatrix} G(i, j) & 0 \\ H(i, j) & E_n \end{pmatrix}$$

with

$$G(i, j) = (g(\lambda, \nu; i, j))_{\substack{1 \leq \lambda \leq m \\ 1 \leq \nu \leq m}} \in M(m, m; \mathbf{Z}),$$

$$H(i, j) = (h(\mu, \nu; i, j))_{\substack{1 \leq \mu \leq n \\ 1 \leq \nu \leq m}} \in M(n, m; \mathbf{Z}).$$

We put

$$\vec{g}(\lambda; i, j) = (g(\lambda, 1; i, j), \dots, g(\lambda, m; i, j)),$$

$$\vec{h}(\mu; i, j) = (h(\mu, 1; i, j), \dots, h(\mu, m; i, j)).$$

It is easy to see the following.

CLAIM 3.1.

- (1) $T(i, j) \in GL(m+n, \mathbf{Z})$ for any $i, j \in I$.
- (2) $T(i, j) \cdot T(j, k) = T(i, k)$ for any $i, j, k \in I$.
- (3) $(g_{i,j,0})^{\vec{\alpha}(i)} (h_{i,j,1})^{\vec{\beta}(i)} = (t_j)^{\vec{\alpha}(j)} (x_j)^{\vec{\beta}(j)}$ if

$$(\vec{\alpha}(j), \vec{\beta}(j)) = (\vec{\alpha}(i), \vec{\beta}(i)) T(i, j),$$

where $\vec{\alpha}(i) = (\alpha^1(i), \dots, \alpha^m(i))$, $\vec{\beta}(i) = (\beta^1(i), \dots, \beta^n(i))$, etc..

Let us denote by \tilde{M} the group of the characters of $\text{Spec}(\bigoplus_{q \geq 0} S^q(N^\vee))$ of dimension $m+n$, which is also toric. We denote the characters in \tilde{M} induced by the coordinates t_i^λ and x_j^μ ($i \in I, 1 \leq \lambda \leq m, 1 \leq \mu \leq n$) of (X, S) by $[t_i^\lambda]$ and $[x_j^\mu]$, respectively. We put $[t_i] = ([t_i^1], \dots, [t_i^m])$ and $[x_j] = ([x_j^1], \dots, [x_j^n])$. We define the scope of a description $f = \{f_{ij}\} = \{(g_{ij}, h_{ij})\}$ of (X, S) .

DEFINITION 3.2. The scope of a description $f = \{f_{ij}\}_{i, j \in I}$ of (X, S) is the semi-group contained in \tilde{M} generated by the following elements:

- (a) $\vec{\alpha} \cdot [t_j] + \vec{\beta} \cdot [x_j] - [t_i^\lambda]$ with $(t_j)^{\vec{\alpha}} (x_j)^{\vec{\beta}}$ appearing in the function g_{ij}^λ ($1 \leq \lambda \leq m$);

(b) $\vec{\alpha} \cdot [t_j] + \vec{\beta} \cdot [x_j] - [x_i^\mu]$ with $(t_j)^{\vec{\alpha}}(x_j)^{\vec{\beta}}$ appearing in the function h_{ij}^μ ($1 \leq \mu \leq n$),

where i and j run all over the index set I . We denote it by $\text{Scope}(f)$.

Since it is convenient to fix an integral basis of $\tilde{M} \cong \mathbf{Z}^{m+n}$, we make another definition.

DEFINITION 3.3. Let $0 \in I$. The scope of a description $f = \{f_{ij}\}_{i,j \in I}$ of (X, S) with respect to the coordinates (t_0, x_0) is the semi-group contained in \mathbf{Z}^{m+n} generated by the following elements:

(a) $(\vec{\alpha} - \vec{g}(\lambda; i, j), \vec{\beta})T(j, 0)$ with $(t_j)^{\vec{\alpha}}(x_j)^{\vec{\beta}}$ appearing in the function g_{ij}^λ ($1 \leq \lambda \leq m$);

(b) $(\vec{\alpha} - \vec{h}(\mu; i, j), \vec{\beta} - \vec{e}_\mu)T(j, 0)$ with $(t_j)^{\vec{\alpha}}(x_j)^{\vec{\beta}}$ appearing in the function h_{ij}^μ ($1 \leq \mu \leq n$),

where i and j run all over the index set I . We denote it by $\text{Scope}(f; 0)$.

REMARK 3.4. Let $0 \in I$. It is easy to see that $\text{Scope}(f; 0)$ is the representation of $\text{Scope}(f)$ in \mathbf{Z}^{m+n} with respect to the integral basis $\{[t_0], [x_0]\}$ of \tilde{M} . Note that the matrix $T(i, j)$ is nothing but the transition matrix of the bases $\{[t_i], [x_i]\}$ and $\{[t_j], [x_j]\}$ and that

$$\text{Scope}(f; j) = \text{Scope}(f; i)T(i, j) \quad \text{for } i, j \in I.$$

We also define the scopes of elements or subsets of $C^p(\mathcal{U}, \mathcal{F}_q)$, $C^p(\mathcal{U}, \mathcal{G}_q)$ and $C^p(\mathcal{U}, \mathcal{H}_q)$.

Let $F = (F_{i_0 \dots i_p}) \in C^p(\mathcal{U}, \mathcal{F}_q)$ with $F_{i_0 \dots i_p} \in \Gamma(U_{i_0 \dots i_p}, \mathcal{F}_q)$. We write

$$F_{i_0 \dots i_p} = \sum_{\lambda=1}^m \sum_{\substack{\vec{q} \geq \vec{0} \\ \mathbf{1} \cdot \vec{q} = q}} G_{i_0 \dots i_p}^{\lambda, \vec{q}}(t_{i_0}) \frac{\partial}{\partial t_{i_0}^\lambda} \otimes (x_{i_0})^{\vec{q}} \\ + \sum_{\mu=1}^n \sum_{\substack{\vec{q} \geq \vec{0} \\ \mathbf{1} \cdot \vec{q} = q}} H_{i_0 \dots i_p}^{\mu, \vec{q}}(t_{i_0}) \frac{\partial}{\partial x_{i_0}^\mu} \otimes (x_{i_0})^{\vec{q}} \pmod{(x_{i_0})^{q-1}},$$

where $\partial/\partial t_{i_0}^\lambda \otimes (x_{i_0})^{\vec{q}} \pmod{(x_{i_0})^{q+1}}$ and $\partial/\partial x_{i_0}^\mu \otimes (x_{i_0})^{\vec{q}} \pmod{(x_{i_0})^{q+1}}$ ($1 \leq \lambda \leq m, 1 \leq \mu \leq n, \mathbf{1} \cdot \vec{q} = q$) denote the local basis of \mathcal{F}_q on U_{i_0} .

DEFINITION 3.5.

(1) The scope of the above element $F \in C^p(\mathcal{U}, \mathcal{F}_q)$ is the semi-group contained in \tilde{M} generated by the following elements:

(a) $\vec{\alpha} \cdot [t_{i_0}] + \vec{q} \cdot [x_{i_0}] - [t_{i_0}^\lambda]$ with $(t_{i_0})^{\vec{\alpha}}$ appearing in $G_{i_0 \dots i_p}^{\lambda, \vec{q}}$ ($1 \leq \lambda \leq m$);

(b) $\vec{\alpha} \cdot [t_{i_0}] + \vec{q} \cdot [x_{i_0}] - [x_{i_0}^\mu]$ with $(t_{i_0})^{\vec{\alpha}}$ appearing in $H_{i_0 \dots i_p}^{\mu, \vec{q}}$ ($1 \leq \mu \leq n$),

where i_0, \dots, i_p run all over the set I . We denote it by $\text{Scope}(F)$.

(2) Let $0 \in I$. We denote the representation of $\text{Scope}(F)$ in \mathbf{Z}^{m+n} with respect to the basis $\{[t_0], [x_0]\}$ by $\text{Scope}(F; 0)$ and we call it the scope of F with respect to the coordinates (t_0, x_0) . More explicitly, we define $\text{Scope}(F; 0)$

to be the semi-group in \mathbf{Z}^{m+n} generated by the following elements :

- (a) $(\vec{\alpha}-\vec{e}_\lambda, \vec{q})T(i_0, 0)$ with $(t_{i_0})^{\vec{\alpha}}$ appearing in $G_{i_0 \dots i_p}^{\lambda, \vec{q}}$ ($1 \leq \lambda \leq m$);
- (b) $(\vec{\alpha}, \vec{q}-\vec{e}_\mu)T(i_0, 0)$ with $(t_{i_0})^{\vec{\alpha}}$ appearing in $H_{i_0 \dots i_p}^{\mu, \vec{q}}$ ($1 \leq \mu \leq n$).

(3) Since $\mathcal{F}_{\vec{q}}$ is a subsheaf of \mathcal{F}_q ($q=\vec{1} \cdot \vec{q}$), we naturally define the scope of an element F of $C^p(\mathcal{U}, \mathcal{F}_{\vec{q}})$. That is, we define $Scope(F)$ in the above way regarding F as an element of $C^p(\mathcal{U}, \mathcal{F}_q)$.

(4) Let Y be a subset of $C^p(\mathcal{U}, \mathcal{F}_q)$ (resp. $C^p(\mathcal{U}, \mathcal{F}_{\vec{q}})$). We define $Scope(Y)$ and $Scope(Y; 0)$ in the following way :

$$Scope(Y) = \sum_{F \in Y} Scope(F),$$

$$Scope(Y; 0) = \sum_{F \in Y} Scope(F; 0).$$

Let $G=(G_{i_0 \dots i_p}) \in C^p(\mathcal{U}, \mathcal{G}_q)$ with $G_{i_0 \dots i_p} \in \Gamma(U_{i_0 \dots i_p}, \mathcal{G}_q)$. We write

$$G_{i_0 \dots i_p} = \sum_{\lambda=1}^m \sum_{\substack{\vec{\alpha} \geq \vec{0} \\ \vec{1} \cdot \vec{q} = q}} G_{i_0 \dots i_p}^{\lambda, \vec{q}}(t_{i_0}) \left(\frac{\partial}{\partial t_{i_0}^{\lambda}} \right)' \otimes (x_{i_0})^{\vec{\alpha}} \pmod{(x_{i_0})^{q+1}}.$$

DEFINITION 3.6.

(1) The scope of the above element $G \in C^p(\mathcal{U}, \mathcal{G}_q)$ is the semi-group contained in \hat{M} generated by the following elements :

$$\vec{\alpha} \cdot [t_{i_0}] + \vec{q} \cdot [x_{i_0}] - [t_{i_0}^{\vec{\alpha}}] \text{ with } (t_{i_0})^{\vec{\alpha}} \text{ appearing in } G_{i_0 \dots i_p}^{\lambda, \vec{q}} \text{ (} 1 \leq \lambda \leq m \text{),}$$

where i_0, \dots, i_p run all over set I . We denote it by $Scope(G)$.

(2) Let $0 \in I$. We denote the representation of $Scope(G)$ in \mathbf{Z}^{m+n} with respect to the basis $\{[t_0], [x_0]\}$ by $Scope(G; 0)$ and we call it the scope of G with respect to the coordinates (t_0, x_0) . That is, it is the semi-group in \mathbf{Z}^{m+n} generated by the following elements :

$$(\vec{\alpha}-\vec{e}_\lambda, \vec{q})T(i_0, 0) \text{ with } (t_{i_0})^{\vec{\alpha}} \text{ appearing in } G_{i_0 \dots i_p}^{\lambda, \vec{q}} \text{ (} 1 \leq \lambda \leq m \text{).}$$

(3) Since $\mathcal{G}_{\vec{q}}$ is a subsheaf of \mathcal{G}_q ($q=\vec{1} \cdot \vec{q}$), we naturally define the scope of an element G of $C^p(\mathcal{U}, \mathcal{G}_{\vec{q}})$ by regarding F as an element of $C^p(\mathcal{U}, \mathcal{G}_q)$.

(4) Let V be a subset of $C^p(\mathcal{U}, \mathcal{G}_q)$ (resp. $C^p(\mathcal{U}, \mathcal{G}_{\vec{q}})$). We define $Scope(V)$ and $Scope(V; 0)$ in the following way :

$$Scope(V) = \sum_{G \in V} Scope(G),$$

$$Scope(V; 0) = \sum_{G \in V} Scope(G; 0).$$

Let $H=(H_{i_0 \dots i_p}) \in C^p(\mathcal{U}, \mathcal{H}_q)$ with $H_{i_0 \dots i_p} \in \Gamma(U_{i_0 \dots i_p}, \mathcal{H}_q)$. We write

$$H_{i_0 \dots i_p} = \sum_{\mu=1}^n \sum_{\substack{\vec{\alpha} \geq \vec{0} \\ \vec{1} \cdot \vec{q} = q}} H_{i_0 \dots i_p}^{\mu, \vec{q}}(t_{i_0}) \left(\frac{\partial}{\partial x_{i_0}^{\mu}} \right)'' \otimes (x_{i_0})^{\vec{\alpha}} \pmod{(x_{i_0})^{q+1}},$$

where $(\partial/\partial x_{i_0}^{\mu}) \otimes (x_{i_0})^{\vec{\alpha}} \pmod{(x_{i_0})^{q+1}}$ ($1 \leq \mu \leq n, \vec{1} \cdot \vec{q} = q$) denote the local basis of \mathcal{H}_q on U_{i_0} .

DEFINITION 3.7.

(1) The scope of the above element $H \in C^p(\mathcal{U}, \mathcal{A}_q)$ is the semi-group contained in \tilde{M} generated by the following elements:

$\tilde{\alpha} \cdot [t_{i_0}] + \tilde{q} \cdot [x_{i_0}] - [x_{i_0}^q]$ with $(t_{i_0})^{\tilde{\alpha}}$ appearing in $H_{i_0 \dots i_p}^{\mu, \tilde{q}}$ ($1 \leq \mu \leq n$), where i_0, \dots, i_p run all over the set I . We denote it by $Scope(H)$.

(2) Let $0 \in I$. We denote the representation of $Scope(H)$ in \mathbf{Z}^{m+n} with respect to the basis $\{[t_0], [x_0]\}$ by $Scope(H; 0)$ and we call it the scope of H with respect to the coordinates (t_0, x_0) . That is, $Scope(F; 0)$ is the semi-group in \mathbf{Z}^{m+n} generated by the following elements:

$(\tilde{\alpha}, \tilde{q} - \tilde{\nu}_\mu)T(i_0, 0)$ with $(t_{i_0})^{\tilde{\alpha}}$ appearing in $H_{i_0 \dots i_p}^{\mu, \tilde{q}}$ ($1 \leq \mu \leq n$).

(3) Since $\mathcal{A}_{\mu; \tilde{q}}$ is a subsheaf of \mathcal{A}_q ($q = \tilde{1} \cdot \tilde{q}$, $1 \leq \mu \leq n$), we naturally define the scope of an element H of $C^p(\mathcal{U}, \mathcal{A}_{\mu; \tilde{q}})$ by regarding F as an element of $C^p(\mathcal{U}, \mathcal{A}_q)$.

(4) Let W be a subset of $C^p(\mathcal{U}, \mathcal{A}_q)$ (resp. $C^p(\mathcal{U}, \mathcal{A}_{\mu; \tilde{q}})$), we define $Scope(W)$ and $Scope(W; 0)$ in the following way:

$$Scope(W) = \sum_{H \in W} Scope(H),$$

$$Scope(W; 0) = \sum_{H \in W} Scope(H; 0).$$

As for cochains of the sheaves \mathcal{G}_q and \mathcal{A}_q , we can interpret their scopes in another way. We first remark that the m -dimensional algebraic torus T acts on S and that $(m+n)$ -dimensional algebraic torus \tilde{T} acts on the variety $\text{Spec}(\bigoplus_{q \geq 0} S^q(N^\sim))$. Then $t_i^\lambda (\partial/\partial t_i^\lambda)'$ and $x_i^\mu (\partial/\partial x_i^\mu)''$ ($i \in I$, $1 \leq \lambda \leq m$, $1 \leq \mu \leq n$) are semi-invariant under the action of \tilde{T} . In other words, we have $[(\partial/\partial t_i^\lambda)'] = -[t_i^\lambda]$ and $[(\partial/\partial x_i^\mu)'] = -[x_i^\mu]$ in \tilde{M} , where $[(\partial/\partial t_i^\lambda)']$ and $[(\partial/\partial x_i^\mu)']$ denote the characters corresponding to $(\partial/\partial t_i^\lambda)'$ and $(\partial/\partial x_i^\mu)''$, respectively.

DEFINITION 3.8. For any rational function $F \in k(S)$, we define the set of lattice points $S(F)$ of F in the following way:

$$S(F) = \{m \in M \mid F_m \neq 0\},$$

where $F = \sum_{m \in M} F_m$ denotes the expansion of F corresponding to the decomposition into the eigenspaces associated to the characters, that is, F_m is the monomial part in F corresponding to $m \in M$. Since \tilde{M} naturally includes M , we regard $S(F)$ as a subset of \tilde{M} .

Let $G = (G_{i_0 \dots i_p}) \in C^p(\mathcal{U}, \mathcal{G}_q)$ and $H = (H_{i_0 \dots i_p}) \in C^p(\mathcal{U}, \mathcal{A}_q)$. We write:

$$G_{i_0 \dots i_p} = \sum_{\lambda=1}^m \sum_{\substack{\tilde{q} \geq \tilde{0} \\ \tilde{1} \cdot \tilde{q} = q}} G_{i_0 \dots i_p}^{\lambda, \tilde{q}}(t_{i_0}) \frac{\partial}{\partial t_{i_0}^\lambda} (x_{i_0})^{\tilde{q}},$$

$$H_{i_0 \dots i_p} = \sum_{\mu=1}^n \sum_{\substack{\bar{q} \geq \bar{0} \\ \bar{1} \cdot \bar{q} = q}} H_{i_0 \dots i_p}^{\mu, \bar{q}}(t_{i_0}) \frac{\partial}{\partial x_{i_0}^\mu} \otimes (x_{i_0})^{\bar{q}} \pmod{(x_{i_0})^{q+1}}.$$

Then $Scope(G)$ (resp. $Scope(H)$) is the semi-group in \tilde{M} generated by the following subset :

$$\begin{aligned} & \bigcup_{i_0, \dots, i_p \in I} \bigcup_{\lambda=1}^m \bigcup_{\substack{\bar{q} \geq \bar{0} \\ \bar{1} \cdot \bar{q} = q}} \left(S(G_{i_0 \dots i_p}^{\lambda, \bar{q}}) + \left[\left(\frac{\partial}{\partial t_{i_0}^\lambda} \right)' \right] + \bar{q} \cdot [x_{i_0}] \right), \\ & \left(\text{resp. } \bigcup_{i_0, \dots, i_p \in I} \bigcup_{\mu=1}^n \bigcup_{\substack{\bar{q} \geq \bar{0} \\ \bar{1} \cdot \bar{q} = q}} \left(S(H_{i_0 \dots i_p}^{\mu, \bar{q}}) + \left[\left(\frac{\partial}{\partial x_{i_0}^\mu} \right)'' \right] + \bar{q} \cdot [x_{i_0}] \right) \right). \end{aligned}$$

We have the following proposition on the behaviour of the coboundary maps.

PROPOSITION 3.9. *The coboundary maps of the Čech complexes $C^*(\mathcal{U}, \mathcal{G}_q)$ and $C^*(\mathcal{U}, \mathcal{H}_q)$ preserve the scope as follows: $Scope(df) \subset Scope(f)$, where f is an element of $C^*(\mathcal{U}, \mathcal{G}_q)$ (resp. $C^*(\mathcal{U}, \mathcal{H}_q)$) and d denotes the coboundary map.*

PROOF. It immediately follows from the fact that $t_i^\lambda (\partial/\partial t_i^\lambda)'$ and $x_i^\mu (\partial/\partial x_i^\mu)''$ ($i \in I, 1 \leq \lambda \leq m, 1 \leq \mu \leq n$) are semi-invariant under the action of \tilde{T} . Or equivalently, we can also directly prove it by the following transition relations of local bases of Θ_S and N :

$$\begin{aligned} t_j^\lambda \left(\frac{\partial}{\partial t_j^\lambda} \right)' &= \sum_{\nu=1}^m g(\nu, \lambda; i, j) t_i^\nu \left(\frac{\partial}{\partial t_i^\nu} \right)', \\ x_j^\mu \left(\frac{\partial}{\partial x_j^\mu} \right)'' &= x_i^\mu \left(\frac{\partial}{\partial x_i^\mu} \right)''. \end{aligned}$$

REMARK 3.10. The coboundary maps of the complex $C^p(\mathcal{U}, \mathcal{F}_q)$ are not scope-preserving (cf. Lemma 3.17).

We recall definitions of H^p -slice and H^p -basis in [2].

DEFINITION 3.11. Let \mathcal{F} be any sheaf on S .

(1) A finite-dimensional vector subspace V of $Z^p(\mathcal{U}, \mathcal{F})$ is said to be an H^p -slice of the sheaf \mathcal{F} if V satisfies $\pi(V) = H^p(S, \mathcal{F})$, where $\pi : Z^p(\mathcal{U}, \mathcal{F}) \rightarrow H^p(S, \mathcal{F})$ denotes the canonical projection.

(2) Let V be an H^p -slice of \mathcal{F} . We call a basis $\{v_1, \dots, v_k\}$ of V an H^p -basis of the sheaf \mathcal{F} .

The following is the main result of this section, which is a generalization of Theorem 3.8 in [2].

THEOREM 3.12 (FUNDAMENTAL THEOREM ON SCOPES). *Let S and N be as before. Let $V_{\bar{q}}$ be an H^1 -slice of the sheaf $\mathcal{G}_{\bar{q}}$ ($\bar{1} \cdot \bar{q} = q \geq 1, \bar{q} \geq \bar{0}$) and $W_{\mu; \bar{q}}$ an H^1 -slice of $\mathcal{H}_{\mu; \bar{q}}$. We put*

$$\begin{aligned}
 V_q &= \bigoplus_{\substack{\hat{q} \geq \hat{0} \\ i \cdot \hat{q} = q}} V_{\hat{q}}, \\
 W_q &= \bigoplus_{\mu=1}^n \bigoplus_{\substack{\hat{q} \geq \hat{0} \\ i \cdot \hat{q} = q}} W_{\mu; \hat{q}}, \\
 \Omega &= \sum_{q \geq 1} \text{Scope}(V_q) + \sum_{q \geq 1} \text{Scope}(W_q) \\
 &= \sum_{\substack{\hat{q} \geq \hat{0} \\ i \cdot \hat{q} = q \geq 1}} \text{Scope}(V_{\hat{q}}) + \sum_{\mu=1}^n \sum_{\substack{\hat{q} \geq \hat{0} \\ i \cdot \hat{q} = q \geq 1}} \text{Scope}(W_{\mu; \hat{q}}).
 \end{aligned}$$

Then any formal neighbourhood (X, S) of S with $N_{S|X}$ isomorphic to N admits a description $f = \{f_{ij}\}_{i,j \in I}$ by the transition functions such that $\text{Scope}(f) \subset \Omega$.

COROLLARY 3.13. *Under the same situation as in Theorem 3.12, we further assume that N is ample and that $H^1(S, \mathcal{H}_q) = 0$ for each $q \geq 1$. Then any formal neighbourhood (X, S) of S with $N_{S|X} \cong N$ admits a description f such that $\text{Scope}(f)$ is finitely generated and that*

$$\text{Scope}(f) \subset \sum_{q \geq 1} \text{Scope}(V_q).$$

PROOF OF COROLLARY 3.13. Straightforward.

We prove Theorem 3.12 in such a way as the proof of Theorem 3.8 of [2]. First we consider the first infinitesimal neighbourhoods.

LEMMA 3.14. *Any first infinitesimal neighbourhood (X_1, S) of S with $N_{S|X_1} \cong N$ has a description $f_{[1]} = \{f_{ij[1]}\}_{i,j \in I}$ such that $\text{Scope}(f_{[1]}) \subset \text{Scope}(V_1)$.*

PROOF. It immediately follows from the definition of the scope, Claims 1.1 and 1.2.

The definition of the scope of $C(\mathcal{F}_q)$ depends on the description $f_{[1]}$ of the first infinitesimal neighbourhood. From now on, we always assume that $\text{Scope}(f_{[1]}) \subset \text{Scope}(V_1)$. Next we discuss the scope of the ambiguity ϕ_q (cf. § 1).

LEMMA 3.15. *Let $q \geq 2$. Suppose that some $(q-1)$ -th infinitesimal neighbourhood (X_{q-1}, S) of S is described by $f_{[q-1]} = \{f_{ij[q-1]}\}_{i,j \in I}$. Let $\Psi_q = \{\mu_{ijk[q]}\}_{i,j,k \in I} \in \mathbb{Z}^3(\mathcal{U}, \mathcal{F}_q)$ corresponds to $\phi_q = \{\phi_{ijk[q]}\}$ with*

$$\phi_{ijk[q]} = (f_{ij[q-1]}(f_{jk[q-1]}) - f_{ik[q-1]})_{[q]}$$

(cf. § 1). Then $\text{Scope}(\Psi_q) \subset \text{Scope}(f_{[q-1]})$.

We make a preliminary definition before we prove Lemma 3.15.

DEFINITION 3.16. Let Ω be a semi-group contained in $\mathbb{Z}^{m+n} = \mathbb{Z}^m \times \mathbb{Z}^n$. For $t = (t^1, \dots, t^m)$ and $x = (x^1, \dots, x^n)$, we define subsets $P(\Omega; t, x)$ and $UP(\Omega; t, x)$

of $k((t, x))$ in the following way :

$$P(\Omega; t, x) = \left\{ \sum_{(\vec{\alpha}, \vec{\beta}) \in \Omega} a_{\vec{\alpha}\vec{\beta}}(t)^{\vec{\alpha}}(x)^{\vec{\beta}} \mid a_{\vec{\alpha}\vec{\beta}} \in k \right\},$$

$$UP(\Omega; t, x) = \left\{ \sum a_{\vec{\alpha}\vec{\beta}}(t)^{\vec{\alpha}}(x)^{\vec{\beta}} \in P(\Omega; t, x) \mid a_{\vec{0}\vec{0}} = 1 \right\},$$

where $\vec{\alpha}=(\alpha_1, \dots, \alpha_m)$ and $\vec{\beta}=(\beta_1, \dots, \beta_n)$.

It is easy to see the following.

CLAIM 3.17.

- (1) $P(\Omega; t, x)$ is a subring of $k((t, x))$.
- (2) $UP(\Omega; t, x)$ is a group with respect to the natural multiplication.

PROOF OF LEMMA 3.15. We put $\Omega_i = \text{Scope}(f_{[q-1]}; i)$ for $i \in I$. Then $f_{i_j[q-1]} = (g_{i_j[q-1]}, h_{i_j[q-1]})$ is written in the following form :

$$g_{i_j[q-1]}^{\lambda}(t_j, x_j) = (t_j)^{\vec{g}(\lambda; i, j)} \left(1 + \sum_{(\vec{\alpha}, \vec{\beta}) \in \Omega_j} G_{i_j; \vec{\alpha}\vec{\beta}}^{\lambda}(t_j)^{\vec{\alpha}}(x_j)^{\vec{\beta}} \right),$$

$$h_{i_j[q-1]}^{\mu}(t_j, x_j) = (t_j)^{\vec{h}(\mu; i, j)} x_j^{\mu} \left(1 + \sum_{(\vec{\alpha}, \vec{\beta}) \in \Omega_j} H_{i_j; \vec{\alpha}\vec{\beta}}^{\mu}(t_j)^{\vec{\alpha}}(x_j)^{\vec{\beta}} \right),$$

with $G_{i_j; \vec{\alpha}\vec{\beta}}^{\lambda}, H_{i_j; \vec{\alpha}\vec{\beta}}^{\mu} \in k$. If we put $g_{j_k[q-1]}^{\lambda} = (t_k)^{\vec{g}(\lambda; j, k)} \tilde{g}_{j_k}^{\lambda}$ and $h_{j_k[q-1]}^{\mu} = (t_k)^{\vec{h}(\mu; j, k)} \cdot x_k^{\mu} \tilde{h}_{j_k}^{\mu}$, then $\tilde{g}_{j_k}^{\lambda}, \tilde{h}_{j_k}^{\mu} \in UP(\Omega_k; t_k, x_k)$. We put

$$\begin{aligned} f_{i_j[q-1]}(f_{j_k[q-1]}) &= (g_{i_j k}, h_{i_j k}) \\ &= (g_{i_j k}^1, \dots, g_{i_j k}^m, h_{i_j k}^1, \dots, h_{i_j k}^n). \end{aligned}$$

After a similar calculation to that in [2], we have :

$$\begin{aligned} &(t_k)^{-\vec{g}(\lambda; i, k)} g_{i_j k}^{\lambda} \\ &= (\tilde{g}_{j_k}^{\lambda})^{\vec{g}(\lambda; i, j)} \left(1 + \sum_{(\vec{\alpha}, \vec{\beta}) \in \Omega_j} G_{i_j; \vec{\alpha}\vec{\beta}}^{\lambda}(t_k)^{\vec{\alpha}(k)}(x_k)^{\vec{\beta}(k)} (\tilde{g}_{j_k}^{\lambda})^{\vec{\alpha}}(\tilde{h}_{j_k}^{\mu})^{\vec{\beta}} \right), \\ &(t_k)^{-\vec{h}(\mu; i, k)} (x_k^{\mu})^{-1} h_{i_j k}^{\mu} \\ &= (\tilde{g}_{j_k}^{\lambda})^{\vec{h}(\mu; i, j)} \tilde{h}_{j_k}^{\mu} \left(1 + \sum_{(\vec{\alpha}, \vec{\beta}) \in \Omega_j} H_{i_j; \vec{\alpha}\vec{\beta}}^{\mu}(t_k)^{\vec{\alpha}(k)}(x_k)^{\vec{\beta}(k)} (\tilde{g}_{j_k}^{\lambda})^{\vec{\alpha}}(\tilde{h}_{j_k}^{\mu})^{\vec{\beta}} \right), \end{aligned}$$

where $(\vec{\alpha}(k), \vec{\beta}(k)) = (\vec{\alpha}, \vec{\beta})T(j, k) \in \Omega_k$. Then

$$(t_k)^{-\vec{g}(\lambda; i, k)} g_{i_j k}^{\lambda}, (t_k)^{-\vec{h}(\mu; i, k)} (x_k^{\mu})^{-1} h_{i_j k}^{\mu} \in UP(\Omega_k; t_k, x_k).$$

Next, we consider exact sequences

$$0 \longrightarrow \mathcal{G}_q \xrightarrow{\iota_q} \mathcal{F}_q \xrightarrow{\tau_q} \mathcal{H}_q \longrightarrow 0.$$

We write $\iota = \iota_q$ and $\tau = \tau_q$ for simplicity. We also denote the morphism

$$C^p(\mathcal{U}, \mathcal{G}_q) \longrightarrow C^p(\mathcal{U}, \mathcal{F}_q)$$

$$(\text{resp. } C^p(\mathcal{U}, \mathcal{F}_q) \longrightarrow C^p(\mathcal{U}, \mathcal{H}_q)),$$

which is induced by $\iota: \mathcal{G}_q \rightarrow \mathcal{F}_q$ (resp. $\tau: \mathcal{F}_q \rightarrow \mathcal{H}_q$) by the same symbol ι (resp. τ). The following lemma is a generalization of Sublemma 3.16 of [2]. It enables us to estimate the scopes of elements appearing in diagram chasing on the above exact sequences.

LEMMA 3.18.

- (1) $Scope(\iota(x)) \subset Scope(x) + Scope(V_1)$ for $x \in C^p(\mathcal{U}, \mathcal{G}_q)$.
- (2) For an element $y \in C^p(\mathcal{U}, \mathcal{H}_q)$, there exists an element $z \in C^p(\mathcal{U}, \mathcal{F}_q)$ such that $\tau(z) = y$ and that $Scope(z) \subset Scope(y) + Scope(V_1)$.
- (3) $Scope(dw) \subset Scope(w) + Scope(V_1)$ for $w \in C^p(\mathcal{U}, \mathcal{F}_q)$.

PROOF. Let $\Omega_j = Scope(V_1; j)$. Then $f_{ij[1]} = (g_{ij[1]}, h_{ij[1]})$ is written in the following form :

$$g_{ij[1]}^\lambda = (t_j)^{\tilde{k}(\lambda; i, j)} \left(1 + \sum_{\nu=1}^n G(\lambda, \nu; i, j) x_j^\nu \right),$$

$$h_{ij[1]}^\mu = (t_j)^{\tilde{h}(\mu; i, j)} x_j^\mu,$$

where $G(\lambda, \nu; i, j) \in \Gamma(U_i \cap U_j, \mathcal{O}_S)$ and $G(\lambda, \nu; i, j) x_j^\nu \in P(\Omega_j; y_j, x_j)$. Then we have the following transition relation between the local basis of $\Theta_x|_S$:

$$t_j^\lambda \frac{\partial}{\partial t_j^\lambda} = \sum_{\nu=1}^m g(\nu, \lambda; i, j) t_i^\nu \frac{\partial}{\partial t_i^\nu},$$

$$x_j^\mu \frac{\partial}{\partial x_j^\mu} = \sum_{\nu=1}^m G(\nu, \mu; i, j) x_i^\nu t_i^\nu \frac{\partial}{\partial t_i^\nu} + x_i^\mu \frac{\partial}{\partial x_i^\mu}.$$

Let Σ be a semi-group in \mathbf{Z}^{m+n} and $P_1, \dots, P_m, Q_1, \dots, Q_n \in P(\Sigma; t_j, x_j)$. Then we have the following relation :

$$\begin{aligned} & \sum_{\lambda=1}^m t_j^\lambda \frac{\partial}{\partial t_j^\lambda} \otimes P_\lambda + \sum_{\mu=1}^n x_j^\mu \frac{\partial}{\partial x_j^\mu} \otimes Q_\mu \\ &= \sum_{\nu=1}^m t_i^\nu \frac{\partial}{\partial t_i^\nu} \otimes \left(\sum_{\lambda=1}^m g(\nu, \lambda; i, j) P_\lambda + \sum_{\mu=1}^n G(\nu, \mu; i, j) x_j^\mu Q_\mu \right) \\ &+ \sum_{\mu=1}^n x_i^\mu \frac{\partial}{\partial x_i^\mu} \otimes Q_\mu. \end{aligned}$$

Note that $\sum_{\lambda=1}^m g(\nu, \lambda; i, j) P_\lambda + \sum_{\mu=1}^n G(\nu, \mu; i, j) x_j^\mu Q_\mu$ belongs to the ring $P(\Omega_j + \Sigma; t_j, x_j)$. The assertions (1), (2) and (3) immediately follow from the above relation. Thus Lemma 3.18 is proved.

To complete the proof of Theorem 3.12, we prove the following two lemmas.

LEMMA 3.19. *Let $q \geq 2$. For an element $\varphi \in B^2(\mathcal{U}, \mathfrak{F}_q)$, there exists $\psi \in C^1(\mathcal{U}, \mathfrak{F}_q)$ such that $d\psi = \varphi$ and that*

$$\text{Scope}(\psi) \subset \text{Scope}(\varphi) + \text{Scope}(W_q) + \text{Scope}(V_1).$$

LEMMA 3.20. *Let $q \geq 2$. There exists an H^1 -slice Y of \mathfrak{F}_q such that*

$$\text{Scope}(Y) \subset \text{Scope}(V_q) + \text{Scope}(W_q) + \text{Scope}(V_1).$$

PROOF OF LEMMAS 3.19 AND 3.20. We refer to Lemmas 3.12 and 3.13 of [2]. Once Lemma 3.18 is proved, the same arguments as in [2] are effective.

PROOF OF THEOREM 3.12. We construct neighbourhoods in such a way as in §1. Theorem 3.12 immediately follows from Lemmas 3.14, 3.15, 3.19 and 3.20.

§4. Further properties on scopes.

From now on, we restrict ourselves to the case where S is a nonsingular toric surface. In this section we estimate the scope of an H^1 -slice of \mathcal{A}_q by the induction on the Picard number $\rho(S)$ of S . The way of arguments is a slight modification of §4 of [2], using an interpretation of the edge sequence of the Leray spectral sequence in terms of Čech cochains (cf. §1.B of [2]). We use similar notation to that in [2] (cf. [6] for detail). Let S be a nonsingular projective toric surface, on which the algebraic torus $T \cong G_m^2$ acts. We denote the T -invariant prime divisors by D_1, \dots, D_s and we put $D = D_1 + \dots + D_s$. We denote by G_S the weighted dual graph of D ; G_S is a circular graph with s vertices with weights a_1, \dots, a_s , where $a_i = (D_i)^2$ ($i=1, \dots, s$).

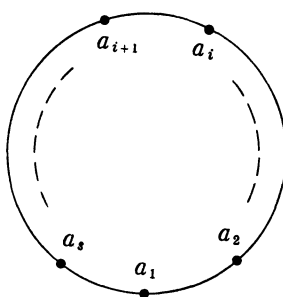


Figure A.

We may assume that the weights a_1, \dots, a_s lie counter-clockwise such as in Figure A. Conventionally, we put $D_{ls+i} = D_i$ and $a_{ls+i} = a_i$ for $l \in \mathbb{Z}$. We also put $p_i = D_i \cap D_{i+1}$.

Conversely, the weighted dual graph G_S uniquely determines S up to iso-

morphism. Moreover, we can construct from G_s an affine open covering $S = \cup_{i=0}^{s-1} U_i$ of S with $U_i \cong \text{Spec } k[t_i^1, t_i^2]$, and determine the transition functions between the coordinates (t_i^1, t_i^2) ($i=0, \dots, s-1$) in such a way that the following conditions are satisfied: $t_i^1 = (t_{i+1}^2)^{-1}$ and $t_i^2 = t_{i+1}^1 (t_{i+1}^2)^{-a_{i+1}}$. The equation $t_i^1 = 0$ determines D_i on U_i , $t_i^2 = 0$ determines D_{i+1} on U_i , and $t_i^1 = t_i^2 = 0$ determines p_i on U_i . From now on, we always take the coordinates (t_i, u_i) as above unless otherwise mentioned.

DEFINITION 4.1. We call the above affine open covering $\{U_i\}$ the canonical open covering of S determined by the weighted dual graph. We also call the coordinates (t_i^1, t_i^2) the canonical coordinates on U_i .

Let $B_\mu = \sum_{i=1}^s b_i^\mu D_i$ ($\mu=1, \dots, n$) be invariant divisors on S , $A_\mu = \mathcal{O}(B_\mu)$ and $N = \bigoplus_{\mu=1}^n A_\mu$. We put $\vec{b}_i = (b_i^1, \dots, b_i^n)$. In this paper, we describe the pair (S, N) by the multi-weighted circular graph in Figure B.

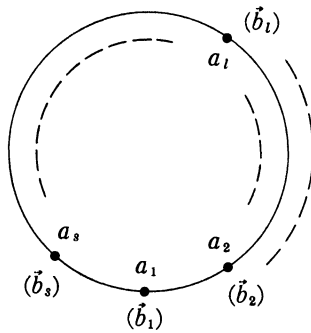


Figure B.

The following claim is well-known.

CLAIM 4.2. *The vector bundle N is ample if and only if the following inequalities are satisfied for $i=1, \dots, s$: $\vec{b}_{i-1} + a_i \vec{b}_i + \vec{b}_{i+1} > \vec{0}$.*

From the multi-weighted circular graph corresponding to N , we can recover the coordinates (t_i, x_i) ($i \in \{1, \dots, s\}$) and the top terms $\{g_{i,j|0}\}$ and $\{h_{i,j|1}\}$ of the transition functions describing a formal neighbourhood (X, S) of S with $N_{S/X} \cong N$. We can determine the matrices $T(i, j) \in GL(2+n, \mathbf{Z})$ ($i, j \in \{1, \dots, s\}$) which are defined in § 3 in such a way that the following conditions are satisfied:

$$T(i, i+1) = \begin{pmatrix} 0 & -1 & \vec{0} \\ 1 & -a_{i+1} & \vec{0} \\ {}^t\vec{0} & -{}^t\vec{d}_{i+1} & E_n \end{pmatrix},$$

where $\vec{d}_{i+1} = \vec{b}_i + a_{i+1} \vec{b}_{i+1} + \vec{b}_{i+2}$.

Suppose that a nonsingular projective toric surface S and a vector bundle $N = \bigoplus_{\mu=1}^n A_\mu$ are determined by the following multi-weighted circular graph in Figure C with $(D_i)^2 = a_i$ and $A_\mu = \mathcal{O}(\sum b_i^\mu D_i)$:

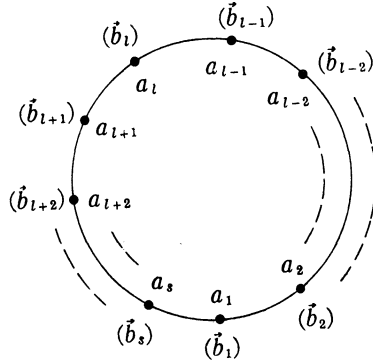


Figure C.

Let $f: \tilde{S} \rightarrow S$ be the equivariant blowing-up of S along $p_l = D_l \cap D_{l+1}$. We put $\tilde{A}_\mu = f^* A_\mu \otimes \mathcal{O}(-c^\mu E)$ and $\tilde{N} = \bigoplus_{\mu=1}^n \tilde{A}_\mu$, where E denotes the exceptional divisor of f . We put $\tilde{c} = (c^1, \dots, c^n)$. Then \tilde{S} and \tilde{N} are determined by the double-weighted dual graph in Figure D.

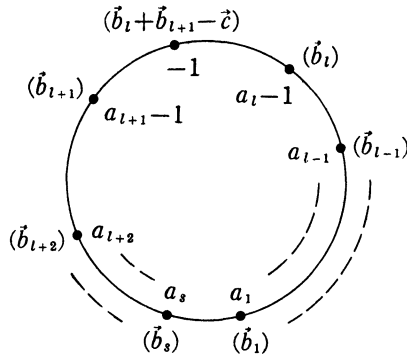


Figure D.

Let $T(i, j)$ ($i, j \in \{1, \dots, s\}$) be the transition matrices with respect to S and N . To get an affine open covering of \tilde{S} , we replace U_i by \tilde{U}_i ($i \neq l$) and U_l by $\tilde{U}_{l-\varepsilon} \cup \tilde{U}_{l+\varepsilon}$ with $U_i \cong \tilde{U}_i$ ($i \neq l$), $\tilde{U}_{l-\varepsilon} \cong \text{Spec } k[t_{l-\varepsilon}^1, t_{l-\varepsilon}^2]$ and $\tilde{U}_{l+\varepsilon} \cong \text{Spec } k[t_{l+\varepsilon}^1, t_{l+\varepsilon}^2]$, using a symbol ε . That is, \tilde{S} is covered by open subsets \tilde{U}_i ($i=1, \dots, l-\varepsilon, l+\varepsilon, \dots, s$). The transition matrices $\tilde{T}(i, j)$ with respect to \tilde{S} and \tilde{N} are calculated in the following way: $\tilde{T}(i, j) = T(i, j)$ if $i \neq l$ and $j \neq l$,

$\tilde{T}(i, l+\varepsilon)=T(i, l)T(l, l+\varepsilon; \tilde{c})$ and $\tilde{T}(i, l-\varepsilon)=T(i, l)T(l, l-\varepsilon; \tilde{c})$, where

$$T(l, l+\varepsilon; \tilde{c}) = \begin{pmatrix} 1 & 0 & \bar{0} \\ 1 & 1 & \bar{0} \\ {}^t\tilde{c} & {}^t\bar{0} & E_n \end{pmatrix}$$

and

$$T(l, l-\varepsilon; \tilde{c}) = \begin{pmatrix} 1 & 1 & \bar{0} \\ 0 & 1 & \bar{0} \\ {}^t\bar{0} & {}^t\tilde{c} & E_n \end{pmatrix}.$$

We put $T(l+\varepsilon, l; \tilde{c})=T(l, l+\varepsilon; \tilde{c})^{-1}$ and $T(l-\varepsilon, l; \tilde{c})=T(l, l-\varepsilon; \tilde{c})^{-1}$. Note that the matrices $T(l, l+\varepsilon; \tilde{c})$ and $T(l, l-\varepsilon; \tilde{c})$ do not depend on l . We put $p_{l+\varepsilon}=\{t_{l+\varepsilon}^i=t_{l+\varepsilon}^0=0\} \in U_{l+\varepsilon}$ and $p_{l-\varepsilon}=\{t_{l-\varepsilon}^i=t_{l-\varepsilon}^0=0\} \in U_{l-\varepsilon}$. Then we have $p_{l-\varepsilon}=D_l \cap E$ and $p_{l+\varepsilon}=D_{l+1} \cap E$. The following theorem is the aim of this section, which estimates the scopes of H^1 -slices of $\mathcal{G}_{\tilde{q}}$ after blowing-up.

THEOREM 4.3. *Let $f: \tilde{S} \rightarrow S$ be an equivariant blowing-up of S along $p_l \in S$ as above, and let A_μ (resp. \tilde{A}_μ) line bundles on S (resp. \tilde{S}) with $\tilde{A}_\mu = f^*A_\mu \otimes \mathcal{O}(-c^\mu E)$, where E denotes the exceptional curve of f and $\tilde{c}=(c^1, \dots, c^n) > \bar{0}$. Let $\{U_i\}_{i \in I}$ be the canonical open covering of S determined by the weighted dual graph. Let $N = \bigoplus_{\mu=1}^n A_\mu$, $\tilde{N} = \bigoplus_{\mu=1}^n \tilde{A}_\mu$, $\mathcal{G}_{\tilde{q}} = \theta_S \otimes (A_1)^{-q_1} \otimes \dots \otimes (A_n)^{-q_n}$, $\tilde{\mathcal{G}}_{\tilde{q}} = \theta_{\tilde{S}} \otimes (\tilde{A}_1)^{-q_1} \otimes \dots \otimes (\tilde{A}_n)^{-q_n}$ for $\tilde{q}=(q_1, \dots, q_n) \geq \bar{0}$ with $\tilde{q} \neq \bar{0}$. Let $V_{\tilde{q}}$ be any H^1 -slice of $\mathcal{G}_{\tilde{q}}$. Then there exists an H^1 -slice $\tilde{V}_{\tilde{q}}$ of $\tilde{\mathcal{G}}_{\tilde{q}}$ which satisfies the following condition:*

$$\text{Scope}(\tilde{V}_{\tilde{q}}) \subset \text{Scope}(V_{\tilde{q}}) + \sum_{\substack{\alpha \leq -1, \beta \leq -1 \\ \alpha + \beta + \tilde{c} \cdot \tilde{q} \geq -1}} \mathbf{Z}_{\geq 0}(\alpha[t_i^1] + \beta[t_i^2] + \tilde{q} \cdot [x_i]).$$

Or equivalently,

$$\text{Scope}(\tilde{V}_{\tilde{q}}; 0) \subset \text{Scope}(V_{\tilde{q}}; 0) + \sum_{\substack{\alpha \leq -1, \beta \leq -1 \\ \alpha + \beta + \tilde{c} \cdot \tilde{q} \geq -1}} \mathbf{Z}_{\geq 0}(\alpha, \beta, \tilde{q})T(l, 0)$$

for $0 \in I \setminus \{l\}$.

REMARK 4.4. Since the group \tilde{M} of the characters is common to both varieties $\text{Spec}(\bigoplus_{q \geq 0} S^q(N^\vee))$ and $\text{Spec}(\bigoplus_{q \geq 0} S^q(\tilde{N}^\vee))$, the above inclusion of semi-groups is reasonable.

We make some preparation before we prove Theorem 4.3. We put:

$$\begin{aligned} \mathcal{P} &= \text{Coker}(\theta_{\tilde{S}} \rightarrow f^*\theta_S), \\ \mathcal{P}_{\tilde{q}} &= \mathcal{P} \otimes (\tilde{A}_1)^{-q_1} \otimes \dots \otimes (\tilde{A}_n)^{-q_n}, \\ \mathcal{Q}_{\tilde{q}} &= f^*\theta_S \otimes (\tilde{A}_1)^{-q_1} \otimes \dots \otimes (\tilde{A}_n)^{-q_n} \end{aligned}$$

for $\bar{q}=(q_1, \dots, q_n)$. Then the exact sequence

$$0 \longrightarrow \check{g}_{\bar{q}} \longrightarrow Q_{\bar{q}} \longrightarrow \mathcal{P}_{\bar{q}} \longrightarrow 0$$

induces the exact sequence

$$(4-1) \quad H^0(\check{S}, \mathcal{P}_{\bar{q}}) \longrightarrow H^1(\check{S}, \check{g}_{\bar{q}}) \longrightarrow H^1(\check{S}, Q_{\bar{q}}) \longrightarrow H^1(\check{S}, \mathcal{P}_{\bar{q}}).$$

On the other hand, we consider the Leray spectral sequence on the sheaf $Q_{\bar{q}}$ and the morphism $f: \check{S} \rightarrow S$. Since $\check{c} > \bar{0}$, we easily see $f_*Q_{\bar{q}} \cong \mathcal{G}_{\bar{q}}$ and $R^1f_*Q_{\bar{q}} \cong \mathcal{G}_{\bar{q}} \otimes R^1f_*\mathcal{O}((\check{c} \cdot \bar{q})E)$. Thus we have the following exact sequence:

$$(4-2) \quad 0 \longrightarrow H^1(S, \mathcal{G}_{\bar{q}}) \longrightarrow H^1(\check{S}, Q_{\bar{q}}) \longrightarrow H^0(S, \mathcal{G}_{\bar{q}} \otimes R^1f_*\mathcal{O}((\check{c} \cdot \bar{q})E)) \longrightarrow H^2(S, \mathcal{G}_{\bar{q}}).$$

We introduce the notion of the scope on the Čech complexes of sheaves appearing in the sequence (4-1) and (4-2).

Suppose $i \in I$. If $i \neq l$, then $f^*(\partial/\partial t_i^{\lambda}) \otimes (x_i)^{\check{q}} \bmod (x_i)^{q+1}$ ($\lambda=1, 2$) are considered to be the local basis of the sheaf $Q_{\bar{q}}$ on \check{U}_i . On the open set $\check{U}_{l+\varepsilon}$ (resp. $\check{U}_{l-\varepsilon}$), $f^*(\partial/\partial t_i^{\lambda}) \otimes (x_{l+\varepsilon})^{\check{q}} \bmod (x_{l+\varepsilon})^{q+1}$ (resp. $f^*(\partial/\partial t_i^{\lambda}) \otimes (x_{l-\varepsilon})^{\check{q}} \bmod (x_{l-\varepsilon})^{q+1}$) ($\lambda=1, 2$) are the local basis of $Q_{\bar{q}}$. Let $\mathcal{U}=(U_i)_{i \in I}$ be the canonical open covering of S and let $\check{\mathcal{U}}=(\check{U}_i)_{i \in I}=(\check{U}_i)_{i \in I \setminus \{l\}}, \check{U}_{l-\varepsilon}, \check{U}_{l+\varepsilon})$ the canonical open covering of \check{S} . We can define the scopes of elements of $C^p(\check{\mathcal{U}}, Q_{\bar{q}})$ and $C^p(\mathcal{U}, f_*C^r(\check{\mathcal{U}}, Q_{\bar{q}}))$ as follows.

DEFINITION 4.5.

(1) Let $Q \in \Gamma_{\text{rat}}(\check{S}, Q_{\bar{q}})$ be any rational section of $Q_{\bar{q}}$ and $0 \in \check{I}$. We can write

$$Q = \sum_{\lambda=1}^2 Q_{\lambda}(t_i) f^* \frac{\partial}{\partial t_i^{\lambda}} \otimes (x_j)^{\check{q}}$$

with $i \in I, j \in \check{I}$. (If $j \neq l$, we usually take $i=j$. If $j=l+\varepsilon$ or $j=l-\varepsilon$, we take $i=l$.) We define the scope $\text{Scope}(Q)$ of Q to be the semi-group in \check{M} generated by the following subsets: $S(Q_{\lambda}) - [t_i^{\lambda}] + \bar{q} \cdot [x_j]$ ($\lambda=1, 2$).

(2) Let $Q \in \Gamma_{\text{rat}}(\check{S}, Q_{\bar{q}})$ and $0 \in \check{I}$. We define the scope $\text{Scope}(Q; 0)$ of Q with respect to $0 \in \check{I}$ to be the representation of $\text{Scope}(Q)$ in \mathbf{Z}^{2+n} with respect to the integral basis $\{[t_0], [x_0]\}$. It is also explicitly defined in the following way.

(A) If $0 \neq l-\varepsilon, l+\varepsilon$, we can write

$$Q = \sum_{\lambda=1}^2 Q_{\lambda}(t_0) f^* \frac{\partial}{\partial t_0^{\lambda}} \otimes (x_0)^{\check{q}} \bmod (x_0)^{q+1}.$$

Then we define $\text{Scope}(Q; 0)$ to be the semi-group contained in \mathbf{Z}^{2+n} generated by the following elements:

(A-1): $(\alpha_1-1, \alpha_2, \bar{q})$ with $(t_0^1)^{\alpha_1}(t_0^2)^{\alpha_2}$ appearing in $Q_1(t_0)$,

and

(A-2): $(\alpha_1, \alpha_2-1, \bar{q})$ with $(t_0^1)^{\alpha_1}(t_0^2)^{\alpha_2}$ appearing in $Q_2(t_0)$.

(B) If $0=l+\varepsilon$, we write

$$Q = \sum_{\lambda=1}^2 Q_{\lambda}(t_{l+\varepsilon}) f^* \frac{\partial}{\partial t_{\lambda}^i} \otimes (x_{l+\varepsilon})^{\bar{q}} \bmod (x_{l+\varepsilon})^{q+1}.$$

Then we define $Scope(Q; 0) = Scope(Q; l+\varepsilon)$ to be the semi-group contained in \mathbb{Z}^{2+n} generated by the following elements:

(B-1): $(\alpha_1-1, \alpha_2, \bar{q})$ with $(t_{l+\varepsilon}^1)^{\alpha_1} (t_{l+\varepsilon}^2)^{\alpha_2}$ appearing in $Q_1(t_{l+\varepsilon})$,

and

(B-2): $(\alpha_1-1, \alpha_2-1, \bar{q})$ with $(t_{l+\varepsilon}^1)^{\alpha_1} (t_{l+\varepsilon}^2)^{\alpha_2}$ appearing in $Q_2(t_{l+\varepsilon})$.

(C) If $0=l-\varepsilon$, we write

$$Q = \sum_{\lambda=1}^2 Q_{\lambda}(t_{l-\varepsilon}) f^* \frac{\partial}{\partial t_{\lambda}^i} \otimes (x_{l-\varepsilon})^{\bar{q}} \bmod (x_{l-\varepsilon})^{q+1}.$$

Then we define $Scope(Q; 0) = Scope(Q; l-\varepsilon)$ to be the semi-group contained in \mathbb{Z}^{2+n} generated by the following elements:

(B-1): $(\alpha_1-1, \alpha_2-1, \bar{q})$ with $(t_{l-\varepsilon}^1)^{\alpha_1} (t_{l-\varepsilon}^2)^{\alpha_2}$ appearing in $Q_1(t_{l-\varepsilon})$,

and

(B-2): $(\alpha_1, \alpha_2-1, \bar{q})$ with $(t_{l-\varepsilon}^1)^{\alpha_1} (t_{l-\varepsilon}^2)^{\alpha_2}$ appearing in $Q_2(t_{l-\varepsilon})$.

(3) Let $Q = (Q_{i_0 \dots i_p}) \in C^p(\check{\mathcal{U}}, Q_{\check{q}})$ with $Q_{i_0 \dots i_p} \in \Gamma(\check{U}_{i_0 \dots i_p}, Q_{\check{q}})$. We define $Scope(Q)$ and $Scope(Q; 0)$ as follows:

$$Scope(Q) = \sum_{i_0, \dots, i_p} Scope(Q_{i_0 \dots i_p}),$$

$$Scope(Q; 0) = \sum_{i_0, \dots, i_p} Scope(Q_{i_0 \dots i_p}; 0).$$

(4) Let $Q \in C^p(\mathcal{U}, f_* C^r(\check{\mathcal{U}}, Q_{\check{q}}))$. We write $Q = (Q_{i_0 \dots i_p; j_0 \dots j_r})$ with

$$Q_{i_0 \dots i_p; j_0 \dots j_r} \in \Gamma(f^{-1}(U_{i_0 \dots i_p}) \cap \check{U}_{j_0 \dots j_r}, Q_{\check{q}}),$$

(cf. [2] § 1.B). We define $Scope(Q)$ and $Scope(Q; 0)$ as follows:

$$Scope(Q) = \sum_{\substack{i_0, \dots, i_p \\ j_0, \dots, j_r}} Scope(Q_{i_0 \dots i_p; j_0 \dots j_r}),$$

$$Scope(Q; 0) = \sum_{\substack{i_0, \dots, i_p \\ j_0, \dots, j_r}} Scope(Q_{i_0 \dots i_p; j_0 \dots j_r}; 0).$$

REMARK 4.6. (1) The natural map $C^p(\check{\mathcal{U}}, \check{Q}_{\check{q}}) \rightarrow C^p(\check{\mathcal{U}}, Q_{\check{q}})$ is scope-preserving.

(2) The differential maps of the double complex $C^*(\mathcal{U}, f_* C^*(\check{\mathcal{U}}, Q_{\check{q}}))$ are scope-preserving.

For an element $P \in C^p(\check{\mathcal{U}}, \mathcal{P}_{\check{q}})$, we define $Scope(P)$ and $Scope(P; 0)$ in such a way that the natural map $C^p(\check{\mathcal{U}}, Q_{\check{q}}) \rightarrow C^p(\check{\mathcal{U}}, \mathcal{P}_{\check{q}})$ is scope-preserving. Let $\pi_{\varepsilon} : \Gamma(U_{l+\varepsilon}, Q_{\check{q}}) \rightarrow \Gamma(U_{l+\varepsilon}, \mathcal{P}_{\check{q}})$ and $\pi_{-\varepsilon} : \Gamma(U_{l-\varepsilon}, Q_{\check{q}}) \rightarrow \Gamma(U_{l-\varepsilon}, \mathcal{P}_{\check{q}})$ be the canonical projections. If we put

$$\pi_\varepsilon\left(f^* \frac{\partial}{\partial t_i^2} \otimes (x_{l+\varepsilon})^{\tilde{q}} \bmod (x_{l+\varepsilon})^{q+1}\right) = \xi_{l+\varepsilon, \tilde{q}}$$

and

$$\pi_{-\varepsilon}\left(f^* \frac{\partial}{\partial t_i^1} \otimes (x_{l-\varepsilon})^{\tilde{q}} \bmod (x_{l-\varepsilon})^{q+1}\right) = \xi_{l-\varepsilon, \tilde{q}},$$

we have the isomorphisms

$$\Gamma(U_{l+\varepsilon}, \mathcal{P}_{\tilde{q}}) \cong k[t_{l+\varepsilon}^2] \cdot \xi_{l+\varepsilon, \tilde{q}}$$

and

$$\Gamma(U_{l-\varepsilon}, \mathcal{P}_{\tilde{q}}) \cong k[t_{l-\varepsilon}^1] \cdot \xi_{l-\varepsilon, \tilde{q}}.$$

The morphism π_ε and $\pi_{-\varepsilon}$ are determined by the following :

$$\begin{aligned} \pi_\varepsilon\left(\sum_{\lambda=1}^2 F_\lambda(t_{l+\varepsilon}^1, t_{l+\varepsilon}^2) f^* \frac{\partial}{\partial t_i^\lambda} \otimes (x_{l+\varepsilon})^{\tilde{q}} \bmod (x_{l+\varepsilon})^{q+1}\right) \\ = \{F_2(0, t_{l+\varepsilon}^2) - t_{l+\varepsilon}^2 F_1(0, t_{l+\varepsilon}^2)\} \xi_{l+\varepsilon, \tilde{q}}, \\ \pi_{-\varepsilon}\left(\sum_{\lambda=1}^2 F_\lambda(t_{l-\varepsilon}^1, t_{l-\varepsilon}^2) f^* \frac{\partial}{\partial t_i^\lambda} \otimes (x_{l-\varepsilon})^{\tilde{q}} \bmod (x_{l-\varepsilon})^{q+1}\right) \\ = \{F_1(t_{l-\varepsilon}^1, 0) - t_{l-\varepsilon}^1 F_2(t_{l-\varepsilon}^1, 0)\} \xi_{l-\varepsilon, \tilde{q}}. \end{aligned}$$

Note that the following transition relation are satisfied :

$$\xi_{l+\varepsilon, \tilde{q}} = -(t_{l-\varepsilon}^1)^{1-\tilde{c} \cdot \tilde{q}} \xi_{l-\varepsilon, \tilde{q}}.$$

DEFINITION 4.7. Let $0 \in \tilde{I}$.

(1) For an rational section $P \in \Gamma_{\text{rat}}(E, \mathcal{P}_{\tilde{q}})$, we define the scope $\text{Scope}(P)$ (resp. $\text{Scope}(P; 0)$) in the following two ways which are equivalent to each other :

(A) If we write $P = F(t_{l+\varepsilon}^1) \xi_{l+\varepsilon, \tilde{q}}$, $\text{Scope}(P)$ (resp. $\text{Scope}(P; 0)$) is the semi-group in \tilde{M} (resp. \mathbf{Z}^{2+n}) generated by $\beta[t_{l+\varepsilon}^1] - [t_i^1] + \tilde{q} \cdot [x_{l+\varepsilon}]$ (resp. $(-1, \beta-1, \tilde{q}) \cdot \tilde{T}(l+\varepsilon, 0)$) with $(t_{l+\varepsilon}^1)^\beta$ appearing in $F(t_{l+\varepsilon}^1)$.

(B) If we write $P = G(t_{l-\varepsilon}^1) \xi_{l-\varepsilon, \tilde{q}}$, $\text{Scope}(P)$ (resp. $\text{Scope}(P; 0)$) is the semi-group in \tilde{M} (resp. \mathbf{Z}^{2+n}) generated by $\alpha[t_{l-\varepsilon}^1] - [t_i^1] + \tilde{q} \cdot [x_{l-\varepsilon}]$ (resp. $(\alpha-1, -1, \tilde{q}) \cdot \tilde{T}(l-\varepsilon, 0)$) with $(t_{l-\varepsilon}^1)^\alpha$ appearing in $G(t_{l-\varepsilon}^1)$.

(2) We naturally induce the scope of an element $C^p(\tilde{\mathcal{U}}, \mathcal{P}_{\tilde{q}})$ by (1).

Then it is easy to see that the following exact sequence of complexes are scope-preserving :

$$0 \longrightarrow C^*(\tilde{\mathcal{U}}, \tilde{\mathcal{Q}}_{\tilde{q}}) \longrightarrow C^*(\tilde{\mathcal{U}}, Q_{\tilde{q}}) \longrightarrow C^*(\tilde{\mathcal{U}}, \mathcal{P}_{\tilde{q}}) \longrightarrow 0.$$

PROOF OF THEOREM 4.3. The proof is done by the same arguments as the proof of Theorem 4.2 of [2] after a slight modification as follows. We refer

to §1.B of [2] for the interpretation of the edge sequence of the Leray spectral sequence in terms of Čech cochains.

Let $A_{\tilde{q}}$ be a vector subspace of the space $C^0(\mathcal{U}, f_*C^1(\tilde{\mathcal{U}}, Q_{\tilde{q}}))$ which represents $H^0(S, \mathcal{G}_{\tilde{q}} \otimes R^1 f_* \mathcal{O}((\tilde{c} \cdot \tilde{q})E))$. Let $C_{\tilde{q}}$ be an H^0 -slice of $\mathcal{P}_{\tilde{q}}$. Then there exists an H^1 -slice $\tilde{V}_{\tilde{q}}$ of $\tilde{\mathcal{G}}_{\tilde{q}}$ such that $Scope(\tilde{V}_{\tilde{q}}) \subset Scope(V_{\tilde{q}}) + Scope(A_{\tilde{q}}) + Scope(C_{\tilde{q}})$. Let V be the vector subspace of $\Gamma(U_{l+\varepsilon} \cap U_{l-\varepsilon}, Q_{\tilde{q}})$ generated by the elements $(t_l^\varepsilon)^\alpha (t_l^\varepsilon)^\beta f^*(\partial/\partial t_l^\varepsilon) \otimes (x_l)_{\tilde{q}} \bmod (x_l)^{q+1}$ and $(t_l^\varepsilon)^\alpha (t_l^\varepsilon)^\beta f^*(\partial/\partial t_l^\varepsilon) \otimes (x_l)_{\tilde{q}} \bmod (x_l)^{q+1}$ with $\alpha < 0, \beta < 0$ and $\alpha + \beta + \tilde{c} \cdot \tilde{q} \geq 0$. Then we have $\pi(V) = H^1(U_{l+\varepsilon} \cup U_{l-\varepsilon}, Q_{\tilde{q}})$, where $\pi: \Gamma(U_{l+\varepsilon} \cap U_{l-\varepsilon}, Q_{\tilde{q}}) \rightarrow H^1(U_{l+\varepsilon} \cup U_{l-\varepsilon}, Q_{\tilde{q}})$ denotes the canonical projection. In fact, the above elements are considered to be elements of $\Gamma(U_{l+\varepsilon} \cap U_{l-\varepsilon}, Q_{\tilde{q}})$ by the following equations:

$$\begin{aligned} & (t_l^\varepsilon)^\alpha (t_l^\varepsilon)^\beta f^* \frac{\partial}{\partial t_l^\varepsilon} \otimes (x_l)_{\tilde{q}} \bmod (x_l)^{q+1} \\ &= (t_{l+\varepsilon}^\varepsilon)^{\alpha+\beta+\tilde{c} \cdot \tilde{q}} (t_{l+\varepsilon}^\varepsilon)^\beta f^* \frac{\partial}{\partial t_l^\varepsilon} \otimes (x_{l+\varepsilon})_{\tilde{q}} \bmod (x_{l+\varepsilon})^{q+1} \\ &= (t_{l-\varepsilon}^\varepsilon)^\alpha (t_{l-\varepsilon}^\varepsilon)^{\alpha+\beta+\tilde{c} \cdot \tilde{q}} f^* \frac{\partial}{\partial t_l^\varepsilon} \otimes (x_{l-\varepsilon})_{\tilde{q}} \bmod (x_{l-\varepsilon})^{q+1}. \end{aligned}$$

By the same argument as [2], we can take $A_{\tilde{q}}$ satisfying $Scope(A_{\tilde{q}}) = Scope(V)$, whence we have

$$Scope(A_{\tilde{q}}) = \sum_{\substack{\alpha, \beta \leq -1 \\ \alpha + \beta + \tilde{c} \cdot \tilde{q} \geq -1}} \mathbf{Z}_{\geq 0}(\alpha[t_l^\varepsilon] + \beta[t_l^\varepsilon] + \tilde{q} \cdot [x_l]).$$

Next, we calculate $Scope(C_{\tilde{q}})$. Since $\mathcal{P}_{\tilde{q}} \cong \iota_* \mathcal{O}_{P^1}(1 - \tilde{c} \cdot \tilde{q})$, where $\iota: E \rightarrow \tilde{S}$ denotes the natural inclusion, we have $H^0(\tilde{S}, \mathcal{P}_{\tilde{q}}) = 0$ unless the following condition is satisfied: $q_\mu = 1$ for some $1 \leq \mu \leq n, q_\nu = 0$ for $\nu \neq \mu$ and $c_\mu = 1$. Suppose $\tilde{q} = \tilde{e}_\mu$ and $c_\mu = 1$. Then $\dim H^0(\tilde{S}, \mathcal{P}_{\tilde{q}}) = 1$ and we can take as $C_{\tilde{q}}$ the set consisting of the elements $a \xi_{l+\varepsilon, \tilde{q}} = -a \xi_{l-\varepsilon, \tilde{q}}$ with $a \in k$. Thus we have:

$$\begin{aligned} Scope(C_{\tilde{q}}) &= \mathbf{Z}_{\geq 0}(-[t_l^\varepsilon] + [x_{l+\varepsilon}^\mu]) \\ &= \mathbf{Z}_{\geq 0}(-[t_l^\varepsilon] + [x_{l-\varepsilon}^\mu]) \\ &= \mathbf{Z}_{\geq 0}(-[t_l^\varepsilon] - [t_l^\varepsilon] + [x_l^\mu]). \end{aligned}$$

Thus Theorem 4.3 is proved.

§ 5. Reduction to scopes.

This section is a modification of § 5 of [2]. First, we fix the notation concerning the Hirzebruch surface $\Sigma_e = P_{P^1}(\mathcal{O} \oplus \mathcal{O}(-e))$. The surface Σ_e is described by the weighted dual graph with four vertices. Let $D = D_1 + D_2 + D_3 + D_4$ be the

corresponding invariant Cartier divisor with $(D_1)^2=e$, $(D_2)^2=0$, $(D_3)^2=-e$ and $(D_4)^2=0$, and let $p_0=D_4 \cap D_1$, $p_1=D_1 \cap D_2$, $p_2=D_2 \cap D_3$ and $p_3=D_3 \cap D_4$. Then Σ_e is covered by four sheets U_i ($i=0, 1, 2, 3$) of affine open subsets with $U_i = \text{Spec } k[t_1^i, t_2^i]$. Then the following relations are satisfied: $t_0^1=(t_1^2)^{-1}$, $t_0^2=t_1^1(t_1^2)^{-e}$, $t_1^1=(t_2^2)^{-1}$, $t_1^2=t_2^1$, $t_2^1=(t_3^2)^{-1}$, $t_2^2=t_3^1(t_3^2)^e$, $t_3^1=(t_0^2)^{-1}$ and $t_3^2=t_0^1$.

DEFINITION 5.1. Let $f: \tilde{S} \rightarrow S$ be a proper birational morphism of non-singular projective surfaces. We define the set $Fund(f)$ of the fundamental points of f as follows:

$$Fund(f) = \{x \in S \mid f^{-1} \text{ is not defined at } x\}.$$

Using the above notation, we state the following lemma.

LEMMA 5.2. Let S be a nonsingular projective toric surface. Assume that S is not isomorphic to \mathbf{P}^2 . Then S is one of the following three types:

(Type I): There exists a proper birational morphism $f: S \rightarrow \Sigma_e$ which is a succession of equivariant blowing-ups such that $e \geq 2$ and that $Fund(f) \subset \{p_2, p_3\}$.

(Type II): There exists a proper birational morphism $f: S \rightarrow \Sigma_0 = \mathbf{P}^1 \times \mathbf{P}^1$ which is a succession of equivariant blowing-ups such that $Fund(f) \subset \{p_0, p_2\}$.

(Type III): There exists a proper birational morphism $f: S \rightarrow \Sigma_1$ which is a succession of equivariant blowing-ups such that $Fund(f) \subset \{p_0, p_2, p_3\}$.

PROOF. We refer to Lemma 5.2 of [2].

REMARK 5.3. The above three types of surfaces are not exclusive. For example, there exists a surface S of type II and type III at once. Precisely speaking, we consider the pair (S, f) of the surface S and the above morphism f when we say that S is of type A ($A=I, II, III$).

DEFINITION 5.4. (1) Let (S, f) be a toric surface of type I. Let D_1 denote the invariant curve on Σ_e with $(D_1)^2=e$ as is stated before, that is, D_1 is determined by the equations $t_0^2=0$ on the open subset U_0 and $t_1^1=0$ on U_1 . We call the strict transform C of D_1 with respect to f the reference curve of type I.

(2) Let (S, f) be a toric surface of type II. Let Γ be the diagonal curve on $\mathbf{P}^1 \times \mathbf{P}^1$ defined by the equations $t_1^1=t_2^1$ on U_1 and $t_3^1=t_3^2$ on U_3 . We call the strict transform C of Γ with respect to f the reference curve of type II.

(3) Let (S, f) be a toric surface of type III. Let D'_1 be a displacement of the curve D_1 on Σ_1 defined by the equations $t_0^2=1$ on U_0 and $t_1^1=t_2^1$ on U_1 . We call the strict transform C of D'_1 with respect to f the reference curve of type III.

Moreover, we fix the following notation. Let (S, f) be a toric surface of Type A ($A=I, II, III$). Since $Fund(f) \not\ni p_1$, there exists an open subset U of S

such that $f|_U: U \rightarrow U_1$ is an isomorphism. This open set U admits the natural coordinates which is induced by the coordinate (t_1^1, t_1^2) on U_1 . We denote the open set U by U_1 . We also denote the coordinates induced by (t_1^1, t_1^2) by the same symbols (t_1^1, t_1^2) . Let (X, S) be a formal neighbourhood of S described by a collection $\Phi = \{\Phi_{ij}\}_{i,j \in I}$ of the transition functions. Since $X|_{U_1} \cong \text{Spf}(k[t_1^1, t_1^2][[x_1]])$ for some coordinates $x_1 = (x_1^1, \dots, x_1^n)$, we can define the scope $\text{Scope}(\Phi; 1)$ of the description Φ with respect to the coordinates (t_1, x_1) .

DEFINITION 5.5. We define a semi-group Ω_{RD} contained in $\mathbf{Z}^2 \times (\mathbf{Z}_{\geq 0})^n$ in the following way :

$$\Omega_{RD} = \{(\alpha, \beta, \bar{q}) \in \mathbf{Z}^2 \times (\mathbf{Z}_{\geq 0})^n \mid \alpha + \beta + \bar{1} \cdot \bar{q} \leq 0\}.$$

PROPOSITION 5.6. Let S be a toric surface of type A ($A = \text{I, II or III}$) and C the reference curve of type A on S . Let (X, S) be a formal neighbourhood of S such that $N \cong N_{S|X}$ and that $N \otimes_{\mathcal{O}_S} \mathcal{O}_C$ is ample on C . Assume that (X, S) is described by a collection Φ of the transition functions such that $\text{Scope}(\Phi; 1) \subset \Omega_{RD}$. Then the induced formal neighbourhood $(X, C)^\wedge$ of the curve C in X is rationally dominated. More precisely, $(X, C)^\wedge$ admits a description by the transition functions satisfying the assumption of Lemma 2.3 for $r=1$.

PROOF. The proof is done by the same arguments as the proof of Proposition 5.6 of [2] after a slight modification as follows. First, we assume that (S, f) is a toric surface of type I. Since $\text{Fund}(f) \cong p_0, p_1, f^{-1}(U_0)$ and $f^{-1}(U_1)$ are isomorphic to A^2 , which we denote by U_0 and U_1 , respectively. Then we have

$$T(0, 1) = \begin{pmatrix} 0 & -1 & \bar{0} \\ 1 & -e & \bar{0} \\ \bar{0} & -\bar{a} & E_n \end{pmatrix},$$

where $\bar{a} = (a_1, \dots, a_n)$ and $A_\mu \otimes_{\mathcal{O}_S} \mathcal{O}_C \cong \mathcal{O}_{P^1}(a_\mu)$ with $a_\mu > 0$ ($1 \leq \mu \leq n$). By the assumption the transition relation between the coordinates (t_0, x_0) on $X|_{U_0}$ and (t_1, x_1) on $X|_{U_1}$ is written in the following way :

$$\begin{aligned} t_0^1 &= (t_1^1)^{-1} (1 + \sum a_{\alpha\beta\bar{q}} (t_1^1)^\alpha (t_1^2)^\beta (x_1)^{\bar{q}}), \\ t_0^2 &= t_1^2 (t_1^1)^{-e} (1 + \sum b_{\alpha\beta\bar{q}} (t_1^1)^\alpha (t_1^2)^\beta (x_1)^{\bar{q}}), \\ x_0^\mu &= (t_1^1)^{-a_\mu} x_1^\mu (1 + \sum c_{\alpha\beta\bar{q}}^\mu (t_1^1)^\alpha (t_1^2)^\beta (x_1)^{\bar{q}}), \end{aligned}$$

where $a_{\alpha\beta\bar{q}}, b_{\alpha\beta\bar{q}}$ or $c_{\alpha\beta\bar{q}}^\mu \neq 0$ implies $\alpha + \beta + \bar{1} \cdot \bar{q} \leq 0$. Since the reference curve C is defined by the equations $t_0^2 = 0$ on U_0 and $t_1^1 = 0$ on U_1 , we can consider the above equations to be a transition relation describing the neighbourhood $(X, C)^\wedge$ of C in X . We apply Lemma 2.3 to this description.

We now assume that (S, f) is a toric surface of type II. Since $p_1, p_3 \notin \text{Fund}(f)$, $f^{-1}(U_1)$ and $f^{-1}(U_3)$ are isomorphic to A^2 , which we denote by U_1 and U_3 , respectively. Then we have

$$T(3, 1) = \begin{pmatrix} -1 & 0 & \bar{0} \\ 0 & -1 & \bar{0} \\ -{}^t\bar{b} & -{}^t\bar{a} & E_n \end{pmatrix}$$

for some integer $\bar{a}=(a_1, \dots, a_n)$ and $\bar{b}=(b_1, \dots, b_n)$. The transition relation between the coordinates (t_3, x_3) on $X|_{U_3}$ and (t_1, x_1) on $X|_{U_1}$ is written in the following way:

$$\begin{aligned} t_3^1 &= (t_1^1)^{-1}(1 + \sum a_{\alpha\beta\bar{q}}(t_1^1)^\alpha (t_1^2)^\beta (x_1)^{\bar{q}}), \\ t_3^2 &= (t_1^2)^{-1}(1 + \sum b_{\alpha\beta\bar{q}}(t_1^2)^\alpha (t_1^1)^\beta (x_1)^{\bar{q}}), \\ x_3^\mu &= (t_1^1)^{-b_\mu} (t_1^2)^{-a_\mu} (x_1^\mu (1 + \sum c_{\alpha\beta\bar{q}}^\mu (t_1^1)^\alpha (t_1^2)^\beta (x_1)^{\bar{q}})), \end{aligned}$$

where $a_{\alpha\beta\bar{q}}$, $b_{\alpha\beta\bar{q}}$ or $c_{\alpha\beta\bar{q}}^\mu \neq 0$ implies $\alpha + \beta + \bar{1} \cdot \bar{q} \leq 0$. To obtain a transition relation of $(X, C)^\wedge$, we change the coordinates near the curve C in the following way: We put $T_3=t_3^2$, $X_3^\mu=x_3^\mu$ ($1 \leq \mu \leq n$) and $Y_3=t_3^2-t_3^1$ near $C \cap U_3$, and $T_1=t_1^1$, $X_1^\mu=x_1^\mu$ ($1 \leq \mu \leq n$) and $Y_1=t_1^2-t_1^1$ near $C \cap U_1$. Then the curve C is defined by the equation $X_3=Y_3=0$ and $X_1=Y_1=0$. The transition relation between the new coordinates (T_3, X_3, Y_3) and (T_1, X_1, Y_1) is easily calculated as follows:

$$\begin{aligned} T_3 &= (T_1)^{-1}(1+(T_1)^{-1}Y_1)^{-1}\{1+\sum b_{\alpha\beta\bar{q}}\psi_{\alpha\beta\bar{q}}\}, \\ X_3^\mu &= (T_1)^{-a_\mu-b_\mu}(1+(T_1)^{-1}Y_1)^{-a_\mu}X_1^\mu\{1+\sum c_{\alpha\beta\bar{q}}^\mu\psi_{\alpha\beta\bar{q}}\}, \\ Y_3 &= (T_1)^{-1}(1+(T_1)^{-1}Y_1)^{-1}\{1+\sum b_{\alpha\beta\bar{q}}\psi_{\alpha\beta\bar{q}}\} \\ &\quad - (T_1)^{-1}\{1+\sum a_{\alpha\beta\bar{q}}\psi_{\alpha\beta\bar{q}}\}. \end{aligned}$$

where $\psi_{\alpha\beta\bar{q}}=(T_1)^{\alpha+\beta}(1+(T_1)^{-1}Y_1)^\beta(X_1)^{\bar{q}}$. Since $N_{S/X} \otimes \mathcal{O}_C$ is ample, we have $\bar{a}+\bar{b}>0$. We apply Lemma 2.3 to this description.

Finally, we assume that (S, f) is a toric surface of type III. As was stated before, the surface Σ_1 is covered by four affine open subsets U_0, U_1, U_2 and U_3 . Corresponding to an equivariant blowing-up, we replace an affine open subset by two sheets of affine open sets. By replacing open covering in such a way as we stated before, we get an affine open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of S . The curve D_4 in Σ_1 is defined by the equations $t_0=0$ on U_0 and $t_3^2=0$ on U_3 . There exists an element $\delta \in \Lambda$ such that $f(U_\delta) \subset U_0$ and that U_δ intersects with the strict transform of D_4 with respect to f . Then the transition matrix $T(\delta, 1)$ is written in the following way (cf. § 4). First, we formally put $T(\delta, 1)=T(\delta, 0)T(0, 1)$. Then

$$T(0, 1) = \begin{pmatrix} 0 & -1 & \vec{0} \\ 1 & -1 & \vec{0} \\ {}^t\vec{0} & -{}^t\vec{a} & E_n \end{pmatrix}$$

for some $\vec{a}=(a_1, \dots, a_n)$, and $T(\delta, 0)$ is a product of matrices of the form

$$\begin{pmatrix} 1 & 1 & \vec{0} \\ 0 & 1 & \vec{0} \\ {}^t\vec{0} & {}^t\vec{c} & E_n \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & \vec{0} \\ 0 & 1 & \vec{0} \\ {}^t\vec{0} & -{}^t\vec{c} & E_n \end{pmatrix}.$$

Hence $T(\delta, 0)$ and $T(\delta, 1)$ is written in the following form :

$$T(\delta, 0) = \begin{pmatrix} 1 & -l & \vec{0} \\ 0 & 1 & \vec{0} \\ {}^t\vec{0} & -{}^t\vec{p} & E_n \end{pmatrix},$$

$$T(\delta, 1) = \begin{pmatrix} -l & l-1 & \vec{0} \\ 1 & -1 & \vec{0} \\ -{}^t\vec{p} & {}^t(\vec{p}-\vec{a}) & E_n \end{pmatrix}$$

for some $l \in \mathbf{Z}$ and $\vec{p}=(p_1, \dots, p_n)$. The transition relation between the coordinates (t_δ, x_δ) on U_δ and (t_1, x_1) on U_1 is written in the following form :

$$t_\delta^1 = (t_1^1)^{-l}(t_1^2)^{l-1}(1 + \sum a_{\alpha\beta\vec{q}}(t_1^1)^\alpha (t_1^2)^\beta (x_1)^{\vec{q}}),$$

$$t_\delta^2 = t_1^1(t_1^2)^{-1}(1 + \sum b_{\alpha\beta\vec{q}}(t_1^1)^\alpha (t_1^2)^\beta (x_1)^{\vec{q}}),$$

$$x_\delta^\mu = (t_1^1)^{-p_\mu}(t_1^2)^{p_\mu - a_\mu} x_1^\mu (1 + \sum c_{\alpha\beta\vec{q}}(t_1^1)^\alpha (t_1^2)^\beta (x_1)^{\vec{q}}),$$

where $a_{\alpha\beta\vec{q}}, b_{\alpha\beta\vec{q}}$ or $c_{\alpha\beta\vec{q}} \neq 0$ implies $\alpha + \beta + \vec{1} \cdot \vec{q} \leq 0$. In order to describe the formal neighbourhood $(X, C)^\wedge$ of C in X , we change coordinates near the curve C . We take coordinates $(T_\delta, X_\delta, Y_\delta)$ near $C \cap U_\delta$ as follows: $T_\delta = t_\delta^1(t_\delta^2)^l, X_\delta^\mu = (t_\delta^2)^{p_\mu} x_\delta^\mu$ and $Y_\delta = t_\delta^2 - 1$. Note that we can take such coordinates around $C \cap U_\delta$, since $t_\delta^2 \neq 0$ near $C \cap U_\delta$. We take coordinates (T_1, X_1, Y_1) near $C \cap U_1$ as follows: $T_1 = t_1^1, X_1^\mu = x_1^\mu$ and $Y_1 = t_1^1 - t_1^2$. Then the reference curve C is defined by the equations $X_\delta = Y_\delta = 0$ and $X_1 = Y_1 = 0$. The transition relation between the coordinates $(T_\delta, X_\delta, Y_\delta)$ and (T_1, X_1, Y_1) is calculated as follows :

$$T_\delta = (T_1)^{-1}(1 + \sum a_{\alpha\beta\vec{q}}\varphi_{\alpha\beta\vec{q}})(1 + \sum b_{\alpha\beta\vec{q}}\varphi_{\alpha\beta\vec{q}})^l,$$

$$X_\delta^\mu = (T_1)^{-a_\mu} X_1^\mu (1 + \sum b_{\alpha\beta\vec{q}}\varphi_{\alpha\beta\vec{q}})^{p_\mu} (1 + \sum c_{\alpha\beta\vec{q}}^\mu\varphi_{\alpha\beta\vec{q}}),$$

$$Y_\delta = -1 + (1 + (T_1)^{-1}Y_1)(1 + \sum b_{\alpha\beta\vec{q}}\varphi_{\alpha\beta\vec{q}}),$$

where $\varphi_{\alpha\beta\vec{q}} = (T_1)^{\alpha+\beta}(1 + (T_1)^{-1}Y_1)^\alpha (X_1)^{\vec{q}}$. We apply Lemma 2.3 to this description,

noting that $\bar{a} > \bar{0}$.

Thus Proposition 5.6 is proved.

As for \mathbf{P}^2 , we fix the covering $\mathbf{P}^2 = U_0 \cup U_1 \cup U_2$ with $U_i \cong \text{Spec } k[t_i^1, t_i^2]$ ($i = 0, 1, 2$) such that the following transition relations are satisfied: $t_0^1 = (t_1^1)^{-1}$ and $t_0^2 = t_1^2(t_1^1)^{-1}$ on $U_0 \cap U_1$, and $t_1^1 = (t_2^1)^{-1}$ and $t_1^2 = t_2^2(t_2^1)^{-1}$ on $U_1 \cap U_2$.

DEFINITION 5.7. We define a semi-group Ω'_{RD} contained in $\mathbf{Z}^2 \times (\mathbf{Z}_{\neq 0})^n$ in the following way:

$$\Omega'_{RD} = \left\{ (\alpha, \beta, \bar{q}) \in \mathbf{Z}^2 \times (\mathbf{Z}_{\neq 0})^n \mid -\beta \geq \frac{1}{2}(\alpha + \bar{1} \cdot \bar{q}) \right\}.$$

PROPOSITION 5.8. Let $S = \mathbf{P}^2$ and let C a nonsingular rational curve on S defined by the equations $t_0^2 = 0$ on U_0 and $t_1^1 = 0$ on U_1 . Let (X, S) be a formal neighbourhood of S with $N_{S/X} \cong N$. Assume that (X, S) is described by a collection Φ of the transition functions such that $\text{Scope}(\Phi, 1) \subset \Omega'_{RD}$. Then the induced formal neighbourhood $(X, C)^\wedge$ of the curve C in X admits a description by the transition functions satisfying the assumption of Lemma 2.3 for $r=2$.

PROOF. The transition relation between the coordinates (t_0, x_0) and (t_1, x_1) is written in the following way:

$$\begin{aligned} t_0^1 &= (t_1^1)^{-1} (1 + \sum a_{\alpha\beta\bar{q}} (t_1^1)^\alpha (t_1^2)^\beta (x_1)^{\bar{q}}), \\ t_0^2 &= t_1^2 (t_1^1)^{-1} (1 + \sum b_{\alpha\beta\bar{q}} (t_1^1)^\alpha (t_1^2)^\beta (x_1)^{\bar{q}}), \\ x_0^\mu &= (t_1^1)^{-\alpha} x_1^\mu (1 + \sum c_{\alpha\beta\bar{q}}^\mu (t_1^1)^\alpha (t_1^2)^\beta (x_1)^{\bar{q}}), \end{aligned}$$

with $a_\mu > 0$, where $a_{\alpha\beta\bar{q}}, b_{\alpha\beta\bar{q}}$ or $c_{\alpha\beta\bar{q}}^\mu \neq 0$ implies $-\beta \geq (\alpha + \bar{1} \cdot \bar{q})/2$. We can consider it to be a transition relation describing $(X, C)^\wedge$ as it is. We apply Lemma 2.3.

§ 6. The proof of Main Results.

In this section, we prove Main Theorem. We use the same notation as in § 5.

THEOREM 6.1. Let S be a nonsingular projective toric surface and $N = \bigoplus_{\mu=1}^n A_\mu$ a direct sum of ample line bundles on S .

(1) If $S = \mathbf{P}^2$, then, for each \bar{q} with $\bar{q} \geq \bar{0}$ and $\bar{1} \cdot \bar{q} = q \geq 1$, there exists an H^1 -slice $V_{\bar{q}}$ of $\mathcal{Q}_{\bar{q}}$ such that the following condition is satisfied:

$$\text{Scope}(V_{\bar{q}}; 1) \subset \Omega'_{RD}.$$

(2) Assume that there exists a morphism $f: S \rightarrow \Sigma_e$ such that the pair (S, f) is of type A ($A = \text{I, II or III}$). Then, for each $\bar{q} > \bar{0}$ with $\bar{q} \geq \bar{0}$ and $\bar{1} \cdot \bar{q} = q \geq 1$, there exists an H^1 -slice $V_{\bar{q}}$ of $\mathcal{Q}_{\bar{q}}$ such that the following condition is satisfied:

$$\text{Scope}(V_{\tilde{q}}; 1) \subset \Omega_{RD}.$$

COROLLARY 6.2. *Let S be any nonsingular projective toric surface. Then there exists a nonsingular rational curve C on S satisfying the following condition: If $N = \bigoplus_{\mu=1}^n A_{\mu}$ is a direct sum of ample line bundles on S such that $H^1(S, N \otimes S^q(N^{\vee})) = 0$ for each $q > 0$, then, for any formal neighbourhood (X, S) of S with $N_{S/X} \cong N$, the neighbourhood $(X, C)^{\wedge}$ of C on X is rationally dominated.*

COROLLARY 6.3 (MAIN THEOREM). *Let n be a positive integer. Let X be a nonsingular complete algebraic variety of dimension $n+2$. Assume that X contains a nonsingular projective toric surface S and that the following two conditions (a) and (b) are satisfied:*

- (a) $N_{S/X} \cong \bigoplus_{\mu=1}^n A_{\mu}$, where each A_{μ} is an ample line bundle,
- (b) $H^1(S, N_{S/X} \otimes S^q(N_{S/X}^{\vee})) = 0$ for each $q > 0$.

Then X is unirational.

REMARK 6.4. (1) If $n=1$, the condition (b) is always satisfied for an ample line bundle N .

(2) The condition (b) is equivalent to the following condition (b').

(b'): For any $\mu \in \{1, \dots, n\}$ and any

$$(q_1, \dots, q_{\mu-1}, q_{\mu+1}, \dots, q_n) \in (\mathbf{Z}_{\geq 0})^{n-1} \setminus \{(0, \dots, 0)\},$$

we have

$$H^1(S, A_{\mu} \otimes (A_1)^{-q_1} \otimes \dots \otimes (A_{\mu-1})^{-q_{\mu-1}} \otimes (A_{\mu+1})^{-q_{\mu+1}} \otimes \dots \otimes (A_n)^{-q_n}) = 0.$$

In fact, we have $H^1(S, \mathcal{O}_S) = 0$ and $H^1(S, (A)^{-1}) = 0$ for an ample line bundle A .

(3) In particular, the conditions (a) and (b) are satisfied if $N = A^{\oplus n}$ for an ample line bundle A .

(4) In the case where $S = \mathbf{P}^2$, then the condition (a) implies the condition (b).

COROLLARY 6.5. *Let X be a nonsingular complete algebraic variety of dimension $n+2$ and let L be a line bundle on X . Assume that there exists a sequence*

$$X = X_0 \supset X_1 \supset \dots \supset X_n = S$$

of subvarieties of X satisfying the following three conditions:

- (1) X_i is a smooth member of the linear system $|L|_{X_{i-1}}$ on X_{i-1} ($1 \leq i \leq n$),
- (2) $X_n = S$ is a toric surface,
- (3) $L|_S$ is ample on S .

Then X is unirational.

PROOF OF COROLLARY 6.2. It immediately follows from Theorems 6.1, 3.12, Propositions 5.6, 5.8, 2.5, Corollaries 3.13 and 6.2.

PROOF OF COROLLARY 6.3. It immediately follows from Corollary 6.2 and

Proposition 2.2.

PROOF OF COROLLARY 6.5. We put $A = L \otimes \mathcal{O}_S$. Since $H^1(S, \mathcal{O}_S) = 0$, the following exact sequence splits for $1 \leq i \leq n-1$:

$$0 \longrightarrow N_{S/X_i} \longrightarrow N_{S/X_{i-1}} \longrightarrow N_{X_i/X_{i-1}} \otimes \mathcal{O}_S \longrightarrow 0.$$

Noting that $N_{X_i/X_{i-1}} \otimes \mathcal{O}_S \cong A$, we have $N_{S/X} \cong A^{\oplus n}$. Then Corollary 6.5 follows from Remark 6.4. (3).

PROOF OF THEOREM 6.1. First, we prove Theorem 6.1 in the case where S is the projective space \mathbf{P}^2 . As is easily seen, the cohomology group $H^1(S, \Theta_{\mathbf{P}^2} \otimes \mathcal{O}_{\mathbf{P}^2}(-a))$ vanishes unless $a=3$. We also have $\dim H^1(S, \Theta_{\mathbf{P}^2} \otimes \mathcal{O}_{\mathbf{P}^2}(-3))=1$. After elementary Čech cohomological calculation, we have the element $\psi = (\psi_{i_0 i_1}) \in Z^1(\mathcal{U}, \Theta_{\mathbf{P}^2} \otimes \mathcal{O}_{\mathbf{P}^2}(-3))$ with $\psi_{i_0 i_1} \in \Gamma(U_{i_0 i_1}, \Theta_{\mathbf{P}^2} \otimes \mathcal{O}_{\mathbf{P}^2}(-3))$ as an H^1 -basis of the sheaf $\Theta_{\mathbf{P}^2} \otimes \mathcal{O}_{\mathbf{P}^2}(-3)$ as follows:

$$\begin{aligned} \psi_{01} &= -(t_0^1)^{-1} \frac{\partial}{\partial t_0^2} \otimes \eta_0, \\ \psi_{02} &= (t_0^2)^{-1} \frac{\partial}{\partial t_0^1} \otimes \eta_0, \\ \psi_{12} &= (t_0^1)^{-1} \frac{\partial}{\partial t_0^2} \otimes \eta_0 + (t_0^2)^{-1} \frac{\partial}{\partial t_0^1} \otimes \eta_0 \\ &= -(t_1^1)^{-1} \frac{\partial}{\partial t_1^2} \otimes \eta_1, \end{aligned}$$

where η_i denotes the local basis of the sheaf $\Theta_{\mathbf{P}^2} \otimes \mathcal{O}_{\mathbf{P}^2}(-3)$ on U_i . Thus we can take an H^1 -basis of $\mathcal{G}_{\vec{q}}$ in the following way. We put $N = \bigoplus_{\mu=1}^n A_\mu$, $A_\mu = \mathcal{O}(a_\mu)$ and $\vec{a} = (a_1, \dots, a_n)$. We easily see $H^1(\mathbf{P}^2, \mathcal{G}_{\vec{q}}) = 0$ unless $\vec{a} \cdot \vec{q} = 3$. We have the following element $G = (G_{i_0 i_1}) \in Z^1(\mathcal{U}, \mathcal{G}_{\vec{q}})$ with $G_{i_0 i_1} \in \Gamma(U_{i_0 i_1}, \mathcal{G}_{\vec{q}})$ as an H^1 -basis of $\mathcal{G}_{\vec{q}}$:

$$\begin{aligned} G_{01} &= -(t_0^1)^{-1} \frac{\partial}{\partial t_0^2} \otimes (x_0)^{\vec{q}} \pmod{(x_0)^{q+1}}, \\ G_{02} &= (t_0^2)^{-1} \frac{\partial}{\partial t_0^1} \otimes (x_0)^{\vec{q}} \pmod{(x_0)^{q+1}}, \\ G_{12} &= -(t_1^1)^{-1} \frac{\partial}{\partial t_1^2} \otimes (x_1)^{\vec{q}} \pmod{(x_1)^{q+1}}. \end{aligned}$$

The scope $\text{Scope}(G; 1)$ is generated by $(-1, -1, \vec{q})$, which belongs to Ω'_{RD} , since $q = \vec{1} \cdot \vec{q} \leq \vec{a} \cdot \vec{q} = 3$. Thus Theorem 6.1.(1) is proved.

Next, we prove Theorem 6.1 in the case where S is the Hirzebruch surface Σ_e . We make some preparations before we state the proof.

DEFINITION 6.6. Let $S = \Sigma_e$, $\pi: S \rightarrow C \cong \mathbf{P}^1$ the natural projection, F a fiber

of π and s_0 the section with $s_0^2 = -e$. For $a, b \in \mathbf{Z}$, we denote the invertible sheaf $\mathcal{O}(aF + bs_0)$ by the symbol $\mathcal{O}(a, b)$.

LEMMA 6.7 (cf. [2] LEMMA 6.6). *The cohomology group $H^1(\Sigma_e, \mathcal{O}(p, r))$ vanishes unless one of the following two conditions is satisfied:*

- (a) $p \geq e(r+1)$ and $r \leq -2$
- (b) $p \leq er - 2$ and $r \geq 0$.

In the case (a), we can take the following elements as an H^1 -basis of $\mathcal{O}(p, r)$:

$$\begin{aligned} \Psi^{\alpha, \beta} &= (0, \varphi^{\alpha, \beta}, \varphi^{\alpha, \beta}, \varphi^{\alpha, \beta}, \varphi^{\alpha, \beta}, 0) \\ &\in \Gamma(U_{01}, \mathcal{O}(p, r)) \times \Gamma(U_{02}, \mathcal{O}(p, r)) \times \Gamma(U_{03}, \mathcal{O}(p, r)) \\ &\quad \times \Gamma(U_{12}, \mathcal{O}(p, r)) \times \Gamma(U_{13}, \mathcal{O}(p, r)) \times \Gamma(U_{23}, \mathcal{O}(p, r)) \end{aligned}$$

with $\alpha \geq 0$, $\beta < 0$, $p - \alpha - e\beta \geq 0$ and $r - \beta < 0$, where $\varphi^{\alpha, \beta} = (t_0^1)^\alpha (t_0^2)^\beta \eta_0$ and η_0 denotes the local basis of $\mathcal{O}(p, r)$ on U_0 .

In the case (b), we can take the following elements as an H^1 -basis of $\mathcal{O}(p, r)$:

$$\begin{aligned} \Phi^{\alpha, \beta} &= (\varphi^{\alpha, \beta}, \varphi^{\alpha, \beta}, 0, 0, -\varphi^{\alpha, \beta}, -\varphi^{\alpha, \beta}) \\ &\in \Gamma(U_{01}, \mathcal{O}(p, r)) \times \Gamma(U_{02}, \mathcal{O}(p, r)) \times \Gamma(U_{03}, \mathcal{O}(p, r)) \\ &\quad \times \Gamma(U_{12}, \mathcal{O}(p, r)) \times \Gamma(U_{13}, \mathcal{O}(p, r)) \times \Gamma(U_{23}, \mathcal{O}(p, r)) \end{aligned}$$

with $\alpha < 0$, $\beta \geq 0$, $p - \alpha - e\beta < 0$ and $r - \beta \geq 0$.

We put $L = \text{Ker}(\Theta_S \rightarrow \pi^* L_C)$. Then we have the exact sequence

$$0 \longrightarrow L \longrightarrow \Theta_S \longrightarrow \pi^* \Theta_C \longrightarrow 0.$$

Then $L \cong \mathcal{O}(e, 2)$ and $\pi^* \Theta_C \cong \mathcal{O}(2, 0)$ (cf. [2]). We denote $L \otimes (A_1)^{-q_1} \otimes \cdots \otimes (A_n)^{-q_n}$ by $L_{\vec{q}}$ and $\pi^* \Theta_C \otimes (A_1)^{-q_1} \otimes \cdots \otimes (A_n)^{-q_n}$ by $M_{\vec{q}}$. Then we have the following exact sequence

$$0 \longrightarrow L_{\vec{q}} \longrightarrow \mathcal{G}_{\vec{q}} \longrightarrow M_{\vec{q}} \longrightarrow 0.$$

Since $L_{\vec{q}}$ is a subsheaf of $\mathcal{G}_{\vec{q}}$, we can naturally define the scope of an element of $C(\mathcal{U}, L_{\vec{q}})$ so that the natural map $C(\mathcal{U}, L_{\vec{q}}) \rightarrow C(\mathcal{U}, \mathcal{G}_{\vec{q}})$ is scope-preserving. We can also define the scope of an element of the group $C(\mathcal{U}, M_{\vec{q}})$ so that the natural map $C(\mathcal{U}, \mathcal{G}_{\vec{q}}) \rightarrow C(\mathcal{U}, M_{\vec{q}})$ is scope-preserving. Thus we have only to calculate the scopes of H^1 -slices of $L_{\vec{q}}$ and $M_{\vec{q}}$ in order to calculate the scope of an H^1 -slice of $\mathcal{G}_{\vec{q}}$.

If we put $A_\mu \cong \mathcal{O}(a_\mu, b_\mu)$, $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b} = (b_1, \dots, b_n)$, then we obtain the isomorphisms $L_{\vec{q}} \cong \mathcal{O}(e - \vec{q} \cdot \vec{a}, 2 - \vec{q} \cdot \vec{b})$ and $M_{\vec{q}} \cong \mathcal{O}(2 - \vec{q} \cdot \vec{a}, -\vec{q} \cdot \vec{b})$. Since N is ample, the following inequalities are satisfied: $\vec{a} > e\vec{b}$, $\vec{b} > \vec{0}$. Since

$$T(0, 1) = \begin{pmatrix} 0 & -1 & \bar{0} \\ 1 & -e & \bar{0} \\ {}^t\bar{0} & -{}^t\bar{a} & E_n \end{pmatrix}$$

in this case, a vector (α, β, \bar{q}) belongs to $\Omega_{RD}T(1, 0)$ if and only if $\alpha+(e-1)\beta+(\bar{a}-\bar{1})\cdot\bar{q}\geq 0$.

First, we calculate the scope of an H^1 -slice of $L_{\bar{q}}$. We put $p=e-\bar{q}\cdot\bar{a}$ and $r=2-\bar{q}\cdot\bar{b}$. Since $\bar{a}>e\bar{b}$, we obtain the inequality $p-e(r+1)<0$. Thus the case (a) in Lemma 6.7 does not occur. Hence we have $\bar{q}\cdot\bar{b}\leq 2$ if $H^1(S, L_{\bar{q}})\neq 0$. Noting that $\partial/\partial t_0^i \otimes (x_0)^{\bar{q}} \bmod (x_0)^{q+1}$ is the local basis of $L_{\bar{q}}$ on U_0 , we can take an H^1 -slice $V_{\bar{q}}$ of $L_{\bar{q}}$ such that $Scope(V_{\bar{q}}; 0)$ is generated by the vectors $(\alpha, \beta-1, \bar{q})$ with $\alpha<0, \beta\geq 0, p-\alpha-e\beta<0$ and $r-\beta\geq 0$.

Suppose $\bar{q}\cdot\bar{b}=2$. Then $\beta\geq 0$ and $r-\beta=2-\bar{q}\cdot\bar{b}-\beta\geq 0$ imply $\beta=0$. Thus $Scope(V_{\bar{q}}; 0)$ is generated by $(\alpha, -1, \bar{q})$ with $e-\bar{q}\cdot\bar{a}+1\leq\alpha\leq-1$. Then we have

$$\alpha+(e-1)\cdot(-1)+(\bar{a}-\bar{1})\cdot\bar{q}\geq 2-\bar{1}\cdot\bar{q}\geq 0,$$

whence $Scope(V_{\bar{q}}; 1)\subset\Omega_{RD}$. Note that $\bar{1}\cdot\bar{q}\leq\bar{q}\cdot\bar{b}=2$, since $\bar{b}\geq\bar{1}$.

Suppose $\bar{q}\cdot\bar{b}=1$. Then $\beta\geq 0$ and $r-\beta=2-\bar{q}\cdot\bar{b}-\beta\geq 0$ imply $\beta=0$ or 1 . Thus $Scope(V_{\bar{q}}; 0)$ is generated by $(\alpha, -1, \bar{q})$ with $e-\bar{q}\cdot\bar{a}+1\leq\alpha\leq-1$ and $(\alpha, 0, \bar{q})$ with $-\bar{q}\cdot\bar{a}+1\leq\alpha\leq-1$. For (α, β) with $e-\bar{q}\cdot\bar{a}+1\leq\alpha\leq-1$ and $\beta=0$, we have

$$\alpha+(e-1)(\beta-1)+(\bar{a}-\bar{1})\cdot\bar{q}\geq 2-\bar{1}\cdot\bar{q}\geq 0.$$

For (α, β) with $-\bar{q}\cdot\bar{a}+1\leq\alpha\leq-1$ and $\beta=1$, we have

$$\alpha+(e-1)(\beta-1)+(\bar{a}-\bar{1})\cdot\bar{q}\geq 1-\bar{1}\cdot\bar{q}\geq 1-\bar{b}\cdot\bar{q}=0.$$

Hence we have $Scope(V_{\bar{q}}; 1)\subset\Omega_{RD}$.

Next, we calculate the scope of an H^1 -slice of $M_{\bar{q}}$. We now put $p=2-\bar{q}\cdot\bar{a}$ and $r=-\bar{q}\cdot\bar{b}$. Since $r<0$, the case (b) in Lemma 6.7 does not occur. Noting that the image of $\partial/\partial t_0^i \otimes (x_0)^{\bar{q}} \bmod (x_0)^{q+1}$ is the local basis of $M_{\bar{q}}$ on U_0 , we can take an H^1 -slice $W_{\bar{q}}$ of $M_{\bar{q}}$ such that $Scope(W_{\bar{q}}; 0)$ is contained in the semi-group generated by the vectors $(\alpha-1, \beta, \bar{q})$ with $\alpha\geq 0, \beta<0, p-\alpha-e\beta=2-\bar{q}\cdot\bar{a}-\alpha-e\beta\geq 0$ and $r-\beta=-\bar{q}\cdot\bar{b}-\beta<0$. Since

$$0\leq p-e(r+1)=2-e-\bar{q}\cdot(\bar{a}-e\bar{b})\leq 2-e-\bar{q}\cdot\bar{1}\leq 1-e,$$

we have $e\leq 1$. For such vectors $(\alpha-1, \beta, \bar{q})$ as above, we have

$$\alpha-1+(e-1)\beta+(\bar{a}-\bar{1})\cdot\bar{q}\geq -1+(1-e)+e\bar{b}\cdot\bar{q}\geq 0,$$

whence such vectors belong to $\Omega_{RD}\cdot T(1, 0)$. Thus Theorem 6.1.(2) is partially proved in the case where $S=\Sigma_e$.

Suppose $\rho(S) \geq 5$. Then the morphism $f: S \rightarrow \Sigma_e$ is written as the composition $S = S_m \xrightarrow{f_m} S_{m-1} \xrightarrow{f_{m-1}} S_{m-2} \rightarrow \dots \rightarrow S_1 \xrightarrow{f_1} S_0 = \Sigma_e$ of equivariant blowing-ups. Let $A_{\mu;j} = ((f_{j+1} \circ \dots \circ f_m)_* A_\mu)^{\vee}$ for $0 \leq j \leq m-1$ and $A_{\mu,m} = A_\mu$, ($1 \leq \mu \leq n$). Let $Fund(f_j) = q_j \in S_{j-1}$ and $E_j = f_j^{-1}(q_j) \subset S_j$ ($1 \leq j \leq m$). Then, for each j with $1 \leq j \leq m$, there exists a positive vector $\tilde{c}_j = (c_j^1, \dots, c_j^n)$ such that $A_{\mu,j} \cong f_j^* A_{\mu,j-1} \otimes \mathcal{O}(-c_j^\mu E_j)$. We construct an affine open covering $\mathcal{U}^{(j)} = \{U_i\}_{i \in I(j)}$ of S_j in the following way. We take the open covering $\mathcal{U}^{(0)} = \{U_i\}_{i \in I(0)}$ of $S_0 = \Sigma_e$ with $I(0) = \{0, 1, 2, 3\}$ as before. Suppose that the open covering $\mathcal{U}^{(j-1)} = \{U_i\}_{i \in I(j-1)}$ of S_{j-1} with $U_i = \text{Spec } k[t_i^1, t_i^2]$ is determined. For $i \in I(j-1)$, we denote by p_i the point on S_{j-1} determined by the equation $t_i^1 = t_i^2 = 0$. Using this notation, we can write $q_j = p_{s(j)}$ for some $s(j) \in I(j-1)$. Then we put

$$I(j) = (I(j-1) \setminus \{s(j)\}) \cup \{s(j) + \varepsilon, s(j) - \varepsilon\},$$

with ε the symbol as is used in § 4. For $i \in I(j-1) \setminus \{s(j)\}$, we denote $f_j^{-1}(U_i)$ the same symbol U_i and we use the same coordinates (t_i^1, t_i^2) . We have $f_j^{-1}(U_{s(j)}) = U_{s(j)+\varepsilon} \cup U_{s(j)-\varepsilon}$ with the coordinates satisfying the following: $t_{s(j)+\varepsilon}^1 = t_{s(j)}^1$, $t_{s(j)+\varepsilon}^2 = (t_{s(j)}^1)^{-1} t_{s(j)}^2$, $t_{s(j)-\varepsilon}^1 = t_{s(j)}^1 (t_{s(j)}^2)^{-1}$ and $t_{s(j)-\varepsilon}^2 = t_{s(j)}^2$. By the induction on $\rho(S)$ and Theorem 4.3, it is enough to show

$$\{(\alpha, \beta, \tilde{q}) \mid \alpha \leq -1, \beta \leq -1, \alpha + \beta + \tilde{q} \cdot \tilde{c}_m \geq -1\} \cdot T(s(m), 1) \subset \mathcal{Q}_{RD}.$$

By the same argument as in the proof of Theorem 6.1 of [2], we may assume that f is a succession of equivariant blowing-ups along successive infinitely near points, that is, $s(j+1) = s(j) + \varepsilon$ or $s(j+1) = s(j) - \varepsilon$ for $1 \leq j \leq m-1$. We put $A_{\mu;0} = \mathcal{O}(a_\mu, b_\mu)$, $\tilde{a} = (a_1, \dots, a_n)$ and $\tilde{b} = (b_1, \dots, b_n)$. Then we have $\tilde{a} > e\tilde{b}$ and $\tilde{b} > \tilde{0}$. We also put

$$T(s(j), 1) = \begin{pmatrix} a_{11}(j) & a_{12}(j) & 0 & \dots & 0 \\ a_{21}(j) & a_{22}(j) & 0 & \dots & 0 \\ a_{31}(j) & a_{32}(j) & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ a_{2+n,1}(j) & a_{2+n,2}(j) & 0 & \dots & 1 \end{pmatrix}.$$

We divide the proof into three cases.

Case I: $s(1) = 0$. In this case, (S, f) is of type II or type III. Thus $e \leq 1$. We put $r_1(j) = -a_{11}(j) - a_{12}(j)$, $r_2(j) = a_{21}(j) + a_{22}(j)$ and $r_{2+\mu}(j) = -a_{2+\mu,1}(j) - a_{2+\mu,2}(j) - 1$ for $1 \leq \mu \leq n$ and $1 \leq j \leq m$.

CLAIM 6.8. $r_i(j) \geq 0$ for $i = 1, 2$.

PROOF. We have

$$T(s(1), 1) = T(0, 1) = \begin{pmatrix} 0 & -1 & \vec{0} \\ 1 & -e & \vec{0} \\ {}^t\vec{0} & -{}^t\vec{a} & E_n \end{pmatrix}.$$

Thus $r_1(1)=1$, $r_2(1)=1-e \geq 0$. If $s(j+1)=s(j)+\varepsilon$, we have

$$\begin{aligned} T(s(j+1), 1) &= T(s(j)+\varepsilon, s(j); \check{c}_j)T(s(j), 1) \\ &= \begin{pmatrix} 1 & 0 & \vec{0} \\ -1 & 1 & \vec{0} \\ -{}^t\check{c}_j & {}^t\vec{0} & E_n \end{pmatrix} T(s(j), 1). \end{aligned}$$

Thus we have $r_1(j+1) = r_1(j)$, $r_2(j+1) = r_1(j) + r_2(j)$ and $r_{2+\mu}(j+1) = -c_j^\mu r_1(j) + r_{2+\mu}(j)$, ($1 \leq \mu \leq n$).

If $s(j+1)=s(j)-\varepsilon$, then we have

$$\begin{aligned} T(s(j+1), 1) &= T(s(j)-\varepsilon, s(j); \check{c}_j)T(s(j), 1) \\ &= \begin{pmatrix} 1 & -1 & \vec{0} \\ 0 & 1 & \vec{0} \\ {}^t\vec{0} & -{}^t\check{c}_j & E_n \end{pmatrix} T(s(j), 1). \end{aligned}$$

Thus we have $r_1(j+1)=r_1(j)+r_2(j)$, $r_2(j+1)=r_2(j)$ and $r_{2+\mu}(j+1)=c_j^\mu r_2(j)+r_{2+\mu}(j)$. Thus Claim 6.8 is proved.

CLAIM 6.9. $r_{2+\mu}(j) - c_j^\mu r_1(j) \geq 0$ for $1 \leq \mu \leq n$ and $1 \leq j \leq m$.

PROOF. First, we assume that $s(j+1)=s(j)+\varepsilon$ for all j . Since N is ample, the following inequalities are satisfied: $\vec{a} > \check{c}_1 + \check{c}_2 + \dots + \check{c}_m$, $\check{c}_m > 0$, $\check{c}_{m-1} > \check{c}_m$, \dots , $\check{c}_1 > \check{c}_2$, $\vec{b} > \check{c}_1$, $\vec{a} > e\vec{b}$. Then we have

$$\begin{aligned} T(s(j), 1) &= \begin{pmatrix} 1 & -(j-1) & \vec{0} \\ 0 & 1 & \vec{0} \\ {}^t\vec{0} & -{}^t(\check{c}_1 + \dots + \check{c}_{j-1}) & E_n \end{pmatrix} T(0, 1) \\ &= \begin{pmatrix} & -(j-1) & e(j-1)-1 & \vec{0} \\ & 1 & -e & \vec{0} \\ -{}^t(\check{c}_1 + \dots + \check{c}_{j-1}) & e \cdot {}^t(\check{c}_1 + \dots + \check{c}_{j-1}) - {}^t\vec{a} & E_n \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} &r_{2+\mu}(j) - c_j^\mu r_1(j) \\ &= a_\mu - c_j^\mu - 1 + (1-e)\{c_1^\mu + \dots + c_{j-1}^\mu - (j-1)c_j^\mu\} \geq 0. \end{aligned}$$

Assume that $r_{2+\mu}(j) - c_j^\mu r_1(j) \geq 0$ for some j . Moreover, we assume that the

morphisms $f_j, f_{j+1}, \dots, f_{j+t}(t \geq 1)$ are chosen such that the following are satisfied: $s(j+1) = s(j) - \varepsilon$, $s(j+\lambda) = s(j+\lambda-1) + \varepsilon$ for $2 \leq \lambda \leq t$. Note that t may be equal to one. Since N is ample, the following inequalities are satisfied: $\check{c}_j > \check{c}_{j+1} + \dots + \check{c}_{j+t}$, $\check{c}_{j+t} > \check{0}$, $\check{c}_{j+t-1} > \check{c}_{j+t}$, \dots , $\check{c}_{j+1} > \check{c}_{j+2}$. Then we have:

$$\begin{aligned} r_1(j+1) &= r_1(j) + r_2(j), \\ r_2(j+1) &= r_2(j), \\ r_{2+\mu}(j+1) &= c_j^\mu r_2(j) + r_{2+\mu}(j); \\ r_1(j+t) &= r_1(j+1), \\ r_2(j+t) &= (t-1)r_1(j+1) + r_2(j+1), \\ r_{2+\mu}(j+t) &= -(c_{j+1}^\mu + \dots + c_{j+t-1}^\mu)r_1(j+1) + r_{2+\mu}(j+1). \end{aligned}$$

We obtain:

$$\begin{aligned} r_{2+\mu}(j+t) - c_{j+t}^\mu r_1(j+t) \\ = r_{2+\mu}(j) - c_j^\mu r_1(j) + (c_j^\mu - c_{j+1}^\mu - \dots - c_{j+t}^\mu)r_1(j+1) \\ \geq 0. \end{aligned}$$

Thus Claim 6.9 is proved.

Let $(\alpha, \beta, \bar{q}) \in \mathbf{Z}^3 \times (\mathbf{Z}_{\neq 0})^n$ with $\alpha \leq -1$, $\beta \leq -1$ and $\alpha + \beta + \check{c}_m \cdot \bar{q} \geq -1$. Putting

$$(\alpha, \beta, \bar{q})T(s(m), 1) = (A, B, \bar{q}),$$

we have:

$$\begin{aligned} -A - B - \bar{1} \cdot \bar{q} &= r_1(m)\alpha - r_2(m)\beta + \sum_{\mu=1}^n r_{2+\mu}(m)q_\mu \\ &= -(r_1(m) + r_2(m))\beta + r_1(m)(\alpha + \beta) + \sum_{\mu=1}^n r_{2+\mu}(m)q_\mu \\ &\geq r_2(m) + \sum_{\mu=1}^n (r_{2+\mu}(m) - c_m^\mu r_1(m))q_\mu \geq 0. \end{aligned}$$

Case II: $s(1)=2$. In this case, we put $r_1(j) = a_{11}(j) + a_{12}(j)$, $r_2(j) = -a_{21}(j) - a_{22}(j)$ and $r_{2+\mu}(j) = -a_{2+\mu,1}(j) - a_{2+\mu,2}(j) - 1$. We have

$$T(s(1), 1) = T(2, 1) = \begin{pmatrix} 0 & 1 & \check{0} \\ -1 & 0 & \check{0} \\ -{}^t\bar{b} & {}^t\check{0} & E_n \end{pmatrix}.$$

If $s(j+1) = s(j) + \varepsilon$, we have $r_1(j+1) = r_1(j)$, $r_2(j+1) = r_1(j) + r_2(j)$ and $r_{2+\mu}(j+1) = c_j^\mu r_1(j) + r_{2+\mu}(j)$. If $s(j+1) = s(j) - \varepsilon$, we have $r_1(j+1) = r_1(j) + r_2(j)$, $r_2(j+1) = r_2(j)$ and $r_{2+\mu}(j+1) = -c_j^\mu r_2(j) + r_{2+\mu}(j)$. It is easy to see that $r_i(j) \geq 0$ for $i=1, 2$.

CLAIM 6.10. $r_{2+\mu}(j) - c_\mu^j r_2(j) \geq 0$.

PROOF. First, we assume that $s(j+1) = s(j) - \varepsilon$ for all j . Since N is ample, the following inequalities are satisfied: $\vec{a} > e\vec{b} + \vec{c}_1, \vec{c}_1 > \vec{c}_2, \dots, \vec{c}_{m-1} > \vec{c}_m, \vec{c}_m > 0$, and $\vec{b} > \vec{c}_1 + \vec{c}_2 + \dots + \vec{c}_m$. Then we have $r_{2+\mu}(j) - c_\mu^j r_2(j) = b_\mu - 1 - (c_1^j + \dots + c_m^j) \geq 0$.

Assume that $r_{2+\mu}(j) - c_\mu^j r_2(j) \geq 0$ for some j and that, for $t \geq 1$, the following are satisfied: $s(j+1) = s(j) + \varepsilon, s(j+\lambda) = s(j+\lambda-1) - \varepsilon$ for $2 \leq \lambda \leq t$. Since N is ample, the following inequalities are satisfied: $\vec{c}_{j+1} > \vec{c}_{j+2}, \dots, \vec{c}_{j+t-1} > \vec{c}_{j+t}, \vec{c}_{j+t} > \vec{0}, \vec{c}_j > \vec{c}_{j+1} + \dots + \vec{c}_{j+t}$. Then we have

$$\begin{aligned} r_{2+\mu}(j+t) - c_{j+t}^\mu r_2(j+t) &= r_{2+\mu}(j) - c_j^\mu r_2(j) + (c_j^\mu - c_{j+1}^\mu - \dots - c_{j+t}^\mu) r_2(j+1) \\ &\geq 0. \end{aligned}$$

Thus Claim 6.10 is proved.

Let $(\alpha, \beta, \vec{q}) \in \mathbf{Z}^2 \times (\mathbf{Z}_{\neq 0})^n$ with $\alpha \leq -1, \beta \leq -1$ and $\alpha + \beta + \vec{c}_m \cdot \vec{q} \geq -1$. Putting

$$(\alpha, \beta, \vec{q})T(s(m), 1) = (A, B, \vec{q}),$$

we have:

$$\begin{aligned} -A - B - \vec{1} \cdot \vec{q} &= -r_1(m)\alpha + r_2(m)\beta + \sum_{\mu=1}^n r_{2+\mu}(m)q_\mu \\ &= -(r_1(m) + r_2(m))\alpha + r_2(m)(\alpha + \beta) + \sum_{\mu=1}^n r_{2+\mu}(m)q_\mu \\ &\geq r_1(m) + \sum_{\mu=1}^n (r_{2+\mu}(m) - c_m^\mu r_2(m))q_\mu \geq 0. \end{aligned}$$

Case III: $s(1) = 3$. In this case, (S, f) is of type I or type III. Thus $e \geq 1$. We now put $r_1(j) = a_{11}(j) + a_{12}(j), r_2(j) = -a_{21}(j) - a_{22}(j)$ and $r_{2+\mu}(j) = -a_{2+\mu,1}(j) - a_{2+\mu,2}(j) - 1$ as in Case II. We have

$$T(s(1), 1) = T(3, 1) = \begin{pmatrix} -1 & e & \vec{0} \\ 0 & -1 & \vec{0} \\ -\vec{t}\vec{b} & e^t\vec{b} - \vec{t}\vec{a} & E_n \end{pmatrix}.$$

By the same argument as before, we see that $r_j(j) \geq 0$ for $i = 1, 2$. To prove Theorem 6.1.(2) in this case, it is sufficient to show that $r_{2+\mu}(j) - c_\mu^j r_2(j) \geq 0$ for $1 \leq \mu \leq n$.

First, we assume that $s(j+1) = s(j) - \varepsilon$ for all j . Then the following inequalities are satisfied: $\vec{b} > \vec{c}_1, \vec{c}_1 > \vec{c}_2, \dots, \vec{c}_{m-1} > \vec{c}_m, \vec{c}_m > 0, \vec{a} > e\vec{b} + \vec{c}_1 + \vec{c}_2 + \dots + \vec{c}_m$. Then we have

$$\begin{aligned} r_{2+\mu}(j) - c_\mu^j r_2(j) &= r_{2+\mu}(1) - (c_1^j + \dots + c_{j-1}^\mu) r_2(j) - c_j^\mu r_2(j) \\ &= r_{2+\mu}(1) - (c_1^j + \dots + c_j^\mu) r_2(1) \\ &= a_\mu - (e-1)b_\mu - 1 - (c_1^j + \dots + c_j^\mu) \geq 0. \end{aligned}$$

Assume that $r_{2+\mu}(j) - c_j^\mu r_2(j) \geq 0$ for some j and that, for $t \geq 1$, the following are satisfied: $s(j+1) = s(j) + \varepsilon$, $s(j+\lambda) = s(j+\lambda-1) - \varepsilon$ for $2 \leq \lambda \leq t$. Then we obtain $r_{2+\mu}(j+t) - c_{j+t}^\mu r_2(j+t) \geq 0$ by the same argument as in Case II.

Thus Theorem 6.1.(2) is proved.

§ 7. Supplements.

A. A note on the assumption of Main Theorem.

In the statement of Main Theorem (Corollary 6.3) we have assumed the two conditions (a) and (b). But the condition (b) seems to be superfluous for the conclusion that X is unirational. We shall later conjecture a more general statement (cf. Corollary 7.8). But, so far as we insist on our technique of scopes and RD Lemma, which does not seem sharp enough, these assumptions (a) and (b) are necessary for our calculation.

First, we state the following proposition which suggests what the condition (b) means.

PROPOSITION 7.1. *Let S be a nonsingular algebraic variety and A_1, \dots, A_n line bundles on S . Let $N = \bigoplus_{\mu=1}^n A_\mu$. Assume that $H^q(S, N \otimes S^q(N^\vee)) = 0$ for each $q > 0$. Let (X, S) be any regular formal neighbourhood of dimension $m+n$ with $N_{S|X} \cong N$. Then there exist regular formal subschemes X_1, \dots, X_n of X of codimension one such that $N_{S|X_\mu} \cong \bigoplus_{\lambda \neq \mu} A_\lambda$ and that $X_1 \cap \dots \cap X_n = S$.*

PROOF. Let $S = \bigcup_{i \in I} U_i$ be an open covering of S by the open sets U_i with the coordinates (t_i^1, \dots, t_i^m) . Let $(t_i^1, \dots, t_i^m, x_i^1, \dots, x_i^n)$ be the coordinates on $X|_{U_i}$ such that $x_i^\mu \bmod (x_i)^\mu$ is the local basis of A_μ^\vee on U_i . By the assumption we have

$$H^1(S, A_\mu \otimes (A_1)^{-q_1} \otimes \dots \otimes (A_{\mu-1})^{-q_{\mu-1}} \otimes (A_{\mu+1})^{-q_{\mu+1}} \otimes \dots \otimes (A_n)^{-q_n}) = 0$$

for each μ with $1 \leq \mu \leq n$ and each $(q_1, \dots, q_{\mu-1}, q_{\mu+1}, \dots, q_n) \in (\mathbf{Z}_{\geq 0})^{n-1} \setminus \{(0, \dots, 0)\}$, which implies the following: We can take the transition relation

$$(t_i, x_i) = f_{ij}(t_j, x_j) = (g_{ij}^1, \dots, g_{ij}^m, h_{ij}^1, \dots, h_{ij}^n)$$

on $U_i \cap U_j$ ($i, j \in I$) such that a term $(x_j^1)^{q_1} \dots (x_j^{\mu-1})^{q_{\mu-1}} (x_j^{\mu+1})^{q_{\mu+1}} \dots (x_j^n)^{q_n}$ does not appear in the function h_{ij}^μ . It follows that the equations $x_i^\mu = 0$ ($i \in I$) patch together and define a regular formal subscheme of X , which we denote by X_μ . Then X_1, \dots, X_n satisfy the required property.

Next, we show an example in which the scope is too big to satisfy the assumption of RD Lemma (Lemma 2.3).

EXAMPLE 7.2. Let $S = \Sigma_2$, $A_1 = \mathcal{O}(10, 1)$, $A_2 = \mathcal{O}(5, 2)$ and $N = A_1 \oplus A_2$. Then

$$H^1(S, \mathcal{A}_{1;(0,2)}) = H^1(S, A_1 \otimes (A_2)^{-2}) \neq 0.$$

We have an H^1 -slice $W_{1;(0,2)}$ with

$$\text{Scope}(W_{1;(0,2)}; 1) = \sum_{\substack{\alpha = -1, -2 \\ 0 \leq \beta \leq -2\alpha}} Z_{=0}(\alpha, \beta, -1, 2).$$

Take the reference curve of type I on S , apply Lemma 2.3. Then the description of the second infinitesimal neighbourhood determined by a general element of the above H^1 -slice $W_{1;(0,2)}$ fails to satisfy the assumption of Lemma 2.3 for any covering index $r > 0$ of \mathbf{P}^1 .

This example suggests that our arguments on scopes and RD Lemma are not sharp enough. The following two points are the weak points of our theory.

(1) RD Lemma provides a sufficient condition for a neighbourhood of \mathbf{P}^1 to be rationally dominated, which is not a necessary one at all. We have started a special type of cyclic covering of \mathbf{P}^1 , which seems too easy, though it is not easy to prove another RD Lemma which is useful for our later arguments and which comes from a more general covering of \mathbf{P}^1 .

(2) If we are given transition functions of a neighbourhood of a toric variety, we can calculate the scope of this description. But the scope does not recover the transition relation as it is. In particular, we neglect the discussion of coefficients of monomial terms and obstructions in the second cohomologies when we discuss scopes.

PROBLEM 7.3. Remove the assumption (b) from the statement of Main Theorem. More generally, develop similar arguments in the case where the normal bundle is an ample equivariant vector bundle which does not a direct sum of line bundles and generalize our results.

B. A remark on algebrizability.

A formal neighbourhood (X, S) of S is said to be algebrizable if there exists an algebraic variety Y containing S such that the completion \hat{Y} of Y along S is isomorphic to (X, S) . In Corollary 6.2, we discuss all the regular formal neighbourhood (X, S) of S with $N_{S/X} \cong N$ which might not be algebrizable, though we have only to discuss algebrizable neighbourhoods in order to prove Main Theorem. How many algebrizable neighbourhoods are there among all the neighbourhoods? We have few answers to this question. As for two-dimensional neighbourhood of \mathbf{P}^1 , we have the following proposition, which suggests that the algebrizability would be a somewhat strong condition. This proposition is essentially due to the idea of M. Reid, who informed the author of its prototype.

PROPOSITION 7.4. Let (X, C) be a two-dimensional regular neighbourhood of

$C \cong \mathbf{P}^1$ with ample normal bundle. Assume that (X, C) is algebrizable. Then (X, C) is rationally dominated.

PROOF. Suppose that a nonsingular projective surface Y contains $C \cong \mathbf{P}^1$ with ample normal bundle. Since $(K_Y + C)C = -2$, the divisor $K_Y + C$ is not nef, whence there exists a curve R with $(K_Y + C)R < 0$ (cf. [4] for terminology). Since $(C)_Y^2 > 0$, C is nef and big. Thus we have $K_Y \cdot R < 0$, which implies that R determines an extremal ray. If R is an exceptional curve of the first kind, then we have $C \cdot R = 0$, because $(K_Y + C)R < 0$, $C \cdot R \geq 0$ and $K_Y \cdot R = -1$. Let $f: Y \rightarrow Y'$ be the contraction of R . In this case $(Y, C)^\wedge$ is isomorphic to $(Y', C)^\wedge$ via f . In the case where $Y = \mathbf{P}^2$, then C is a line or a conic. If C is a line, then we have:

$$(Y, C)^\wedge \cong \text{Spf}(k[t_0][[x_0]]) \cup \text{Spf}(k[t_1][[x_1]])$$

with $t_0 = (t_1)^{-1}$ and $x_0 = (t_1)^{-1}x_1$.

If C is a conic, then we have the following transition relation of $(Y, C)^\wedge$ after elementary calculation:

$$\begin{aligned} t_0 &= (t_1)^{-1}(1 + (t_1)^{-2}x_1)^{-1}, \\ x_0 &= (t_1)^{-4}x_1(1 + (t_1)^{-2}x_1)^{-2}. \end{aligned}$$

If the contraction of R determines a \mathbf{P}^1 -bundle over a curve, then we have $C \cdot R = 1$. In fact, $K_Y \cdot R = -2$, $(K_Y + C)R < 0$ and $C \cdot R \geq 0$ imply $C \cdot R = 0$ or 1. If $C \cdot R = 0$, then C is a fiber of the contraction morphism, a contradiction. Thus C is a section of a \mathbf{P}^1 -bundle over \mathbf{P}^1 . In this case we have the following transition relation of $(Y, C)^\wedge$:

$$\begin{aligned} t_0 &= (t_1)^{-1}, \\ x_0 &= (t_1)^{-p}x_1(1 + \varphi((t_1)^{-1}x_1))^{-1}, \end{aligned}$$

where $p > 0$ and φ is a polynomial with $\deg(\varphi) \leq p - 1$ and $\varphi(0) = 0$.

All the above neighbourhoods turn out to be rationally dominated. Thus Proposition 7.4 is proved.

By the above proposition we can also construct examples of neighbourhoods which are not algebrizable as follows.

EXAMPLE 7.5. Let (X, C) be a regular formal neighbourhood of $C \cong \mathbf{P}^1$ with $N_{C/X} \cong \mathcal{O}(1)$. Assume that (X, C) is not isomorphic to the neighbourhood $\mathbf{1}$ of the zero section of the normal bundle. Then (X, C) is not algebrizable. Note that such neighbourhoods do exist and that they are not isomorphic to neighbourhoods appearing above (cf. [2] Proposition 2.7).

C. Problems.

The origin of Main Theorem is the following theorem due to M. Noether.

THEOREM 7.6 (M. NOETHER). *Let X be a nonsingular projective surface. Assume that X contains a nonsingular rational curve C with $(C)_X^2 > 0$. Then X is a rational surface.*

We consider the following questions which are generalizations of Theorem 7.6.

QUESTION $R(n, m)$ (resp. $UR(n, m), RC(n, m)$). *Let $n > m > 0$. Let X be a nonsingular projective algebraic variety of dimension n and Y a nonsingular subvariety of dimension m . Assume that $N_{Y/X}$ is an ample vector bundle and that Y is rational (resp. unirational, connected). Then so is X ?*

An algebraic variety X is said to be rationally connected if the following is satisfied: For general points x and y of X , there exists a rational curve C passing through both x and y . As is easily seen, a rational variety is unirational and one is rationally connected. But a unirational variety is not rational in general. It is an open problem whether any rationally connected variety is unirational or not.

Since any nonsingular rational surface contains a nonsingular rational curve with ample normal bundle, $R(n, 1)$ (resp. $UR(n, 1), RC(n, 1)$) implies $R(n, 2)$ (resp. $UR(n, 2), RC(n, 2)$). As for the question $R(3, 2)$, we have a counter-example due to [1] as follows. Let X be a nonsingular cubic hypersurface in \mathbf{P}^4 and $Y = X \cap H$, where H denotes a general hyperplane. Then Y is a rational surface, $N_{Y/X}$ is ample and X is unirational, but X is not rational. This example also provides a counter-example against $R(3, 1)$.

On the other hand, $RC(n, 1)$ holds true for any $n \geq 2$, which is proved by applying the following theorem due to [5] for the inclusion morphism $Y \hookrightarrow X$.

THEOREM 7.7 (KOLLÁR-MIYAOKA-MORI). *For a nonsingular algebraic variety X , the following are equivalent to each other.*

- (1) X is rationally connected;
- (2) There exists a morphism $f: \mathbf{P}^1 \rightarrow X$ such that $f^* \theta_X$ is ample.

$RC(n, m)$ is also true for any $n > m$.

Our Main Theorem tells that $UR(n, 2)$ holds true under certain assumption. Whether $UR(n, 1)$ holds true or not is as much doubtful as whether any rationally connected variety is unirational or not. There seems to be essential difference between the problems $UR(n, m)$ with $m \geq 2$ and the problem $UR(n, 1)$. We shall end this paper with the following conjecture.

CONJECTURE 7.8. *$UR(n, m)$ holds true for $n > m \geq 2$.*

As we have mentioned in §0 of [2], we need essentially finite parameters to describe neighbourhoods of a variety of dimension greater than or equal to two, which is essentially different from neighbourhoods of a curve. Such finiteness seems to be deeply related to the unirationality of algebraic varieties.

References

- [1] C. Clemens and P. Griffiths, The intermediate Jacobian of the cubic threefold, *Ann. of Math.*, **95** (1972), 281-356.
- [2] M. Ebihara, On unirationality of threefolds which contain toric surfaces with ample normal bundles, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **39** (1992), 87-139.
- [3] H. Hironaka and H. Matsumura, Formal functions and formal embeddings, *J. Math. Soc. Japan*, **20** (1968), 52-82.
- [4] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model program, *Proc. Sympos. Algebraic Geometry, Sendai, 1985*, *Adv. Stud. Pure Math.*, **10**, North-Holland, 1987, pp. 283-360.
- [5] J. Kollár, Y. Miyaoka and S. Mori, Rationally connected varieties, *RIMS*, preprint.
- [6] T. Oda, *Convex bodies and algebraic geometry: An introduction to the theory of toric varieties*, Springer-Verlag, 1987.
- [7] *Revêtements étales et groupe fondamentale*, *Lecture Notes in Math.*, **224**, Springer-Verlag.

Madoka EBIHARA
Department of Mathematics
Faculty of Science
Gakushuin University
1-5-1 Mejiro, Toshima-ku
Tokyo 171
Japan