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Bounds on the fundamental group of a manifold with almost nonnegative Ricci curvature

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Several recent papers have given generalizations of the following classical theorems relating the curvature of a compact Riemannian manifold to its fundamental group:

THEOREM 1 (Bochner). *Let M be a compact Riemannian manifold. If M has Ricci curvature nonnegative everywhere and strictly positive somewhere, then* $H^1(M; R)=0$.

THEOREM 2 (Myers). *A complete Riemannian manifold with Ricci curvature bounded from below by a positive constant is compact and has finite fundamental group.*

THEOREM 3 (Milnor). *The fundamental group of a compact Riemannian manifold with nonnegative Ricci curvature has polynomial growth.*

For example, Berard [**2, 3**] and Elworthy-Rosenberg [**5**] extended Bochner's theorem to the case where the Ricci curvature is arbitrarily negative, as long as the set where Ricci is negative has small enough volume; we refer to such a set as a deep well of negative Ricci of small volume. Wu [**16**] generalized Myers' theorem to the case where Ricci is negative on a set of small diameter (a deep well of small diameter), and Elworthy-Rosenberg gave an extension of Myers' theorem to the case where Ricci is allowed to be a little bit negative on a set of small volume (a shallow well of small volume). Finally, Shen and Wei [**13, 15**] generalized Milnor's theorem, allowing the lower bound on Ricci curvature to be sufficiently small in relation to the diameter and either the volume or first systole of the manifold (a shallow, wide well).

In this paper, we show that all these extensions for compact manifolds can be essentially obtained by placing integral bounds on the negative part of the Ricci curvature (or the part of the Ricci curvature lying below a fixed constant), although in some cases there are technical differences in the hypotheses. From

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the analytic viewpoint, these L^p bounds on the negative part of Ricci are a more natural definition of the notion of a deep well of small volume than the hypotheses used in [2, 5]. Specifically, we show:

THEOREM 4 (cf. Theorem 2.2). *Let M be a complete Riemannian n-manifold. If there exists $w>0$ satisfying*

$$\|(\rho-w)_-\|_{n/2} < \min(A^{-1},\ wB^{-1})$$

then M has no nontrivial L^2 harmonic 1-forms. In particular, if M is compact, $H^1(M'; R)=0$ for every finite cover M' of M.

Here $\rho(x)$ denotes the lowest eigenvalue of the Ricci tensor at $x\in M$, $f_-(x) = \min(0, f(x))$ for a function f, and A and B are constants in the Sobolev inequality (2.1); for compact M these constants depend only on the usual volume, diameter and Ricci bounds.

THEOREM 5 (cf. Corollary 3.3). *Let $\mathcal{M}(n, K, D, V)$ denote the class of Riemannian n-manifolds with sectional curvature bounded below by K, diameter bounded above by D, and volume bounded below by V. Then given $w>0$, there exists $\varepsilon=\varepsilon(\mathcal{M}, w)$ such that any $M\in\mathcal{M}$ with $\|(\rho-w)_-\|_1<\varepsilon$ has finite fundamental group.*

THEOREM 6 (cf. Theorem 1.3). *Given $n\geqq 2$, $q>n$, $\sigma, \alpha>0$, there exists $\varepsilon(n, p, \sigma, \alpha)>0$ such that a compact Riemannian n-manifold M satisfying*

$$\mathrm{sys}_1(M) \geqq 2\sigma \operatorname{diam}(M)$$

$$\operatorname{vol}(M) \geqq \alpha^n \operatorname{diam}(M)^n$$

$$\|\rho_-\|_{q/2} \leqq \varepsilon(n, q, \sigma, \alpha)^2 \operatorname{diam}(M)^{-2(1-n/q)}$$

has fundamental group with polynomial growth.

There is also a companion result to Theorem 1.3 given L^1 bounds on the negative part of Ricci.

The final result is a more geometric argument showing that for compact manifolds, Wu's hypothesis on the deep well of small diameter can be weakened to a pointwise hypothesis on a deep well of small volume.

THEOREM 7 (cf. Corollary 11.8). *Let $\mathcal{N}(n, \kappa, D, V)$ denote the class of Riemannian n-manifolds with Ricci curvature bounded below by $-(n-1)\kappa^2$, diameter bounded above by D, and volume bounded below by V. Then given $H>0$, there exists $\varepsilon=\varepsilon(\mathcal{N}, H)$ such that any $M\in\mathcal{N}$ with*

$$\operatorname{vol}(\{x : \rho(x) < (n-1)H^2\}) < \varepsilon$$

has finite fundamental group.

§§ 1-3 state the main analytic theorems precisely. In § 4, various properties of the Gromov-Hausdorff distance and length spaces are recalled. § 5 is devoted to the observation that if a sequence of compact Riemannian manifolds with first systole bounded from below by a positive constant converges with respect to Gromov-Hausdorff distance, then the limiting metric space is a compact length space with fundamental group isomorphic to the fundamental group of almost every manifold in the sequence. This yields in § 7 a new proof of Shen and Wei's theorem using the Bishop-Gromov volume comparison theorem and its corollary, the Gromov precompactness theorem. Applying the volume estimate and isoperimetric inequality obtained in [**17**] gives extensions of Shen and Wei's theorem such as Theorem 6 in § 9. In § 10, sufficient conditions for the nonexistence of L^2 harmonic sections of a Schrödinger operator $-\nabla^p\nabla_p+V$ on a vector bundle over a complete Riemannian manifold are given; again, only integral bounds on the potential V are needed. Applying this to the Laplace-Beltrami operator for 1-forms gives Theorem 4. As explained in § 3, combining this theorem with Theorem 6 and [**5**, §§ 1-2] yields Theorem 5. Finally, the proof of Theorem 7 is given in § 11; this section is independent of the rest of the paper.

As indicated by this outline, Theorem 5 is the most involved result of the paper; its proof combines techniques from [**5**, **9**, **17**]. In contrast, the proof of Theorem 7 involves only a careful use of the Bishop-Gromov comparison theorem, but imposes stronger pointwise rather than integral conditions on the well of negative curvature. We would like to thank the referee for suggesting that a result like Theorem 7 should exist.

1. Theorems on the growth of the fundamental group.

Let M be a Riemannian manifold with sectional curvature $K(M)$, Ricci curvature $\mathrm{Rc}(M)$, diameter $\mathrm{diam}(M)$, and volume $\mathrm{vol}(M)$. Let $\rho(x)$ be the lowest eigenvalue of $\mathrm{Rc}(x)$, let $\rho=\min_{x\in M}(\rho(x))$, and set $\rho_-(x)=\min(0, \rho(x))$ for $x\in M$. The first systole of M, $\mathrm{sys}_1(M)$, is defined to be the infimum of the lengths of homotopically nontrivial loops in M.

The following is a theorem of Shen and Wei [**13**]; a new proof is given in § 7:

THEOREM 1.1. *Given $n\geqq 2$ and $\sigma>0$, there exists $\varepsilon(n, \sigma)>0$ such that any compact Riemannian manifold M satisfying*

$$\dim M = n$$

$$\mathrm{sys}_1(M) \geqq 2\sigma \operatorname{diam}(M)$$

$$\rho \geqq -\varepsilon(n, \sigma) \operatorname{diam}(M)^{-2}$$

has a fundamental group with polynomial growth.

The following theorems generalize Theorem 1.1 and results of Elworthy-Rosenberg [**5**]. First, if a pointwise lower bound on Ricci curvature is assumed, it suffices to assume that the L^1 norm of the negative part of Ricci is small:

THEOREM 1.2. *Given $n \geqq 2$, σ, $\kappa > 0$, there exists $\varepsilon(n, \sigma, \kappa) > 0$ such that any compact Riemannian manifold M satisfying*

$$\dim M = n$$

$$\mathrm{sys}_1(M) \geqq 2\sigma \operatorname{diam}(M), \qquad \rho \geqq -\kappa^2 \operatorname{diam}(M)^{-2}$$

$$\|\rho_-\|_1 \leqq \varepsilon(n, \sigma, \kappa)^2 \operatorname{diam}(M)^{-2} \operatorname{vol}(M)$$

has a fundamental group with polynomial growth.

Next, if a lower bound on the volume of the manifold is assumed, then it suffices to assume that the $L^{q/2}$ bound of ρ_-, $q > n$, is small:

THEOREM 1.3. *Given $n \geqq 2$, $q > n$, σ, $\alpha > 0$, there exists $\varepsilon(n, p, \sigma, \alpha) > 0$ such that a compact Riemannian manifold M satisfying*

$$\dim M = n$$

$$\mathrm{sys}_1(M) \geqq 2\sigma \operatorname{diam}(M)$$

$$\operatorname{vol}(M) \geqq \alpha^n \operatorname{diam}(M)^n$$

$$\|\rho_-\|_{q/2} \leqq \varepsilon(n, q, \sigma, \alpha)^2 \operatorname{diam}(M)^{-2(1-n/q)}$$

has a fundamental group with polynomial growth.

The proofs of Theorems 1.2 and 1.3 are given in § 9.

Using the Klingenberg-Cheeger estimate for the length of the shortest closed geodesic [**10**], the lower bound on the first systole can be replaced by a lower bound on sectional curvature:

COROLLARY 1.4. *Given $n \geqq 2$, α, $\kappa > 0$, there exists $\varepsilon(n, \alpha, \kappa) > 0$ such that a compact Riemannian manifold M satisfying*

$$\dim M = n$$

$$\operatorname{vol}(M) \geqq \alpha^n \operatorname{diam}(M)^n$$

$$K(M) \geqq -\kappa^2 \operatorname{diam}(M)^{-2}$$

$$\|\rho_-\|_1 \leqq \varepsilon(n, \alpha, \kappa)\,\mathrm{diam}\,(M)^{-2}\,\mathrm{vol}\,(M)$$

has fundamental group with polynomial growth.

2. Theorems on the vanishing of the first Betti number.

We now state two generalizations of Bochner's theorem (Theorem 0.1) for complete Riemannian manifolds M. We shall assume that there exist positive constants A and B such that for any $f \in C_0^\infty(M)$, the following Sobolev inequality holds:

$$\|f\|_{2n/(n-2)}^2 \leqq A\|\nabla f\|_2^2 + B\|f\|_2^2 . \tag{2.1}$$

This holds, for example, if M is compact or if the injectivity radius of M is positive. While a noncompact finite volume hyperbolic manifold cannot satisfy (2.1), for manifolds with certain curvature assumptions and sufficiently large volume growth at infinity, B may be set equal to zero. When M is compact, B may be chosen to equal $(2\,\mathrm{vol}\,(M))^{-2/n}$ and A to equal the isoperimetric constant of M (cf. [7]), in which case Croke's work [4] shows that A depends only on $\dim(M)$, an upper bound for $\mathrm{diam}\,(M)$, and lower bounds for $\mathrm{vol}\,(M)$ and $\mathrm{Rc}\,(M)$.

The following theorems improve results obtained by Elworthy-Rosenberg [5] and Berard [2, 3]:

THEOREM 2.2. *If there exists $w>0$ satisfying*

$$\|(\rho-w)_-\|_{n/2} < \min(A^{-1},\ wB^{-1})$$

then M has no nontrivial L^2 harmonic 1-forms. In particular, if M is compact, $H^1(M'; \boldsymbol{R})=0$ for every finite cover M' of M.

THEOREM 2.3. *Given $3 \leqq n < q \leqq \infty$, there exists $\varepsilon(n, q)>0$ such that if*

$$\|(\rho-w)_-\|_1 \leqq \varepsilon(n, q)w[B+(A\|\rho_-\|_{q/2})^{q/(q-n)}]^{-n/2}$$

then M has no nontrivial L^2 harmonic 1-forms. In particular, if M is compact, $H^1(M'; \boldsymbol{R})=0$ for every finite cover M' of M.

The exponent $q/(q-n)$ is taken to be one if $q=\infty$. The proofs of these results are given in § 10.

3. Corollaries on the finiteness of the fundamental group.

In [5] the following is shown:

PROPOSITION 3.1. *Let M be a compact manifold with $H^1(M'; \boldsymbol{R})=0$ for* *every finite cover M' of M. If $\pi_1(M)$ is almost solvable, then $\pi_1(M)$ is finite. In particular, if $\pi_1(M)$ has polynomial growth, then $\pi_1(M)$ is finite.*

Here a group is almost solvable if it contains a solvable subgroup of finite index. While the proof of the first statement is elementary, the last statement depends on Gromov's theorem that a group of polynomial growth is almost nilpotent and hence almost solvable [8].

The results of §§1-2 and Proposition 3.1 imply various generalizations of Myers' theorem, including the following:

COROLLARY 3.2. *Given $n\geqq 3$, σ, α, κ, $w>0$, there exists $\delta(n, \sigma, \kappa)$, $\varepsilon(n, \alpha, \sigma, \kappa)>0$ such that any compact manifold M satisfying*

$$\dim M = n$$

$$\mathrm{sys}_1(M) \geqq 2\sigma \operatorname{diam}(M)$$

$$\mathrm{vol}(M) \geqq \alpha^n \operatorname{diam}(M)^n$$

$$\rho \geqq -\kappa^2 \operatorname{diam}(M)^{-2}$$

$$\|\rho_-\|_1 \leqq \delta(n, \sigma, \kappa) \operatorname{diam}(M)^{-2} \mathrm{vol}(M)$$

$$\|(\rho-w)_-\|_1 \leqq \varepsilon(n, \sigma, \alpha, \kappa) w \,\mathrm{vol}(M)$$

has finite fundamental group.

Here we have used Theorem 1.2 and Theorem 2.3 with $q=\infty$. In particular, applying the Klingenberg-Cheeger estimate [**10**] yields a proof that a manifold with a deep, narrow well of negative Ricci curvature has finite fundamental group:

COROLLARY 3.3. *Let $\mathcal{M}(n, K, D, V)$ denote the class of Riemannian n-manifolds with sectional curvature bounded below by K, diameter bounded above by D, and volume bounded below by V. Then given $w>0$ there exists $\varepsilon=\varepsilon(\mathcal{M}, w)$ such that any $M\in\mathcal{M}$ with $\|(\rho-w)_-\|_1<\varepsilon(\mathcal{M}, w)$ has finite fundamental group.*

Note that to derive the corollary we use the bound on $\operatorname{diam}(M)$ and $\|\rho_-\|_1 \leqq \|(\rho-w)_-\|_1$.

4. Gromov-Hausdorff distance and length spaces.

We recall the definition of Gromov-Hausdorff distance [**8**, **9**]. Let X and Y be metric spaces and Z their disjoint union. Consider the set of all metrics δ on Z such that δ restricted to X and Y equals the original metrics on X and Y. The Gromov-Hausdorff distance $d_H(X, Y)$ is defined to be the infimum of numbers $\varepsilon>0$ such that X is contained in an ε-neighborhood of Y and Y is contained in an ε-neighborhood of X with respect to one such metric δ.

In particular, if $d_H(X, Y)\leqq\varepsilon$, then for any $\varepsilon'>\varepsilon$ there exists a distance function δ on the disjoint union such that given any $x\in X$ there exists $y\in Y$ such that $\delta(x, y)<\varepsilon'$ and *vice versa*.

A sequence (X_i, x_i) of pointed, locally compact metrics spaces converges to a pointed metric space (X, x) with respect to pointed Gromov-Hausdorff distance if for any $R>0$, the sequence $B(x_i, R)$ converges to $B(x, R)$ in Gromov-Hausdorff distance. Here $B(x, R)=\{y \mid d(x, y)<R\}$ is the open ball of radius R centered at x in (X, d).

Given $x\in X$ and $0<r<R$, let $N(x, r, R)$ be the maximum number of disjoint balls of radius r with centers contained in $B(x, R)$.

LEMMA 4.1. *Let X and Y be metric spaces, $x\in X$, $y\in Y$, and $R>0$ such that*

$$d_H(B(x, R), B(y, R))=\varepsilon\,.$$

Then given $r>\varepsilon$

$$N(x, r, R)\leqq N(y, r-\varepsilon, R)\,.$$

PROOF. Given $\varepsilon'>\varepsilon$, let δ be the distance function on $B(x, R)\amalg B(y, R)$ described above. Let $B(x_i, r)$, $1\leqq i\leqq N(x, r, R)$, be a maximal set of disjoint balls of radius r such that each $x_i\in B(x, R)$. For each i, there exists $y_i\in B(y, R)$ such that $\delta(x_i, y_i)\leqq\varepsilon'$. By the triangle inequality and by letting $\varepsilon'\to\varepsilon$, $d(y_i, y_j)\geqq 2r-2\varepsilon$. $\square$

This gives:

PROPOSITION 4.2. *Let (X_i, x_i) be a sequence of pointed metric spaces converging with respect to pointed Hausdorff distance to a pointed metric space (X, x). If there exists a continuous function $N(r, R)$ such that for each $0<r<R$,*

$$\limsup_{i\to\infty} N(x_i, r, R)\leqq N(r, R)$$

then $N(x, r, R)\leqq N(r, R)$.

We now review some basic facts about length spaces [**9**]. Let (X, d) be a metric space. Given a continuous map $\gamma:[0, 1]\to X$, the length of γ is defined to be

$$l(\gamma)=\lim_{N\to\infty}\sup_{0=t_0<t_1<\cdots<t_N=1}\sum_{i=1}^{N} d(\gamma(t_{i-1}), \gamma(t_i))\,.$$

DEFINITION. If for any $x, y\in X$, the distance $d(x, y)$ is equal to the infimum of the lengths of all curves joining x to y, then X is a *length space*. X is a *complete length space* if it is also complete as a metric space.

PROPOSITION 4.3 ([**9**, Proposition 3.8]). *The pointed Gromov-Hausdorff limit of a convergent sequence of complete, locally compact length spaces is itself a complete, locally compact length space.*

THEOREM 4.4 ([**9**, Theorem 1.10]). *Given x and y in a locally compact, complete length space X, there exists a curve γ joining x to y such that*

$$d(x, y)=l(\gamma)\,.$$

DEFINITION. The curve γ is called a *minimizing geodesic.*

LEMMA 4.5. *The universal cover $\tilde{X}$ of a connected, locally compact, complete length space X is itself a connected, locally compact, complete length space. Moreover, if the first systole of X is positive, then the natural projection $\tilde{X}\rightarrow X$ is a local isometry.*

The proof of Lemma 4.5 follows from the argument in [**9**, §1.12]. Note that the length of a curve in $\tilde{X}$ is simply the length of the curve projected into X.

5. Fundamental groups of Hausdorff close length spaces.

The next result is essentially a special case of a homotopy finiteness theorem for convergent sequences of $LGC^n(\rho)$ spaces [**14**].

THEOREM 5.1. *Let X and Y be locally compact, complete length spaces such that*

$$\mathrm{sys}_1(X),\quad \mathrm{sys}_1(Y)\geqq 2\sigma>0\,.$$

If the Gromov-Hausdorff distance between X and Y is strictly less than $\sigma/9$, then $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic.

PROOF. (Cf. [**14**, §2].) Let $d_H(X, Y)=\varepsilon<\sigma/9$, and let δ be the distance function on $X\amalg Y$ described in §4 associated to some $\varepsilon'>\varepsilon$. From now on we do not distinguish between ε and ε'. Pick $x_0\in X$ and $y_0\in Y$ be such that $\delta(x_0, y_0)<\varepsilon$. Let $\eta>0$ satisfy $9\varepsilon+4\eta<\sigma$.

Given a loop $\gamma:[0, 1]\rightarrow X$ in X based at x_0, we construct a loop ϕ based at y_0 as follows. Choose $0=t_0<t_1<\cdots<t_N=1$ so that $l(\gamma([t_{i-1}, t_i]))<\eta$, $1\leqq i\leqq N$. For each $i=1, \cdots, N-1$, there exists y_i such that $\delta(\gamma(t_i), y_i)<\varepsilon$. Let $\phi:[0, 1]\rightarrow Y$ be the loop based at y_0 obtained by joining y_{i-1} to y_i by a minimizing geodesic and with $\phi(t_i)=y_i$.

First, we show that this construction associates to γ a unique homotopy class of loops in Y based at y_0. Let ϕ be a loop in Y obtained from γ using a partition $t_0, \cdots, t_N$ and ϕ' a loop obtained from γ using a partition $t'_0, \cdots, t'_{N'}$.

Given $0 \leqq t \leqq 1$, there exist i and j such that

$$\begin{aligned} d_Y(\phi(t), \phi'(t)) &\leqq \delta(\phi(t), \phi(t_i)) + \delta(\phi(t_i), \gamma(t_i)) + \delta(\gamma(t_i), \gamma(t'_j)) \\ &\quad + \delta(\gamma(t'_j), \phi'(t'_j)) + \delta(\phi'(t'_j), \phi'(t)) \\ &\leqq (2\varepsilon+\eta)+\varepsilon+\eta+\varepsilon+(2\varepsilon+\eta) \\ &= 6\varepsilon+3\eta < \sigma . \end{aligned}$$

Now apply the following:

LEMMA 5.2. *Let $\gamma_1, \gamma_2 : [0, 1] \to X$ be continuous curves such that $\gamma_1(0)=\gamma_2(0)$ and $\gamma_1(1)=\gamma_2(1)$ and such that $d(\gamma_1(t), \gamma_2(t))<\sigma$, $0 \leqq t \leqq 1$. Then the two curves are homotopic relative to their endpoints.*

PROOF. Let $\mu>0$ satisfy $d(\gamma_1(t), \gamma_2(t))+\mu<\sigma$, $0 \leqq t \leqq 1$. Choose a partition $t_0=0<t_1< \cdots <t_N=1$ such that $l(\gamma_\alpha([t_{i-1}, t_i]))<\mu$, $\alpha=0, 1$. Join each $\gamma_1(t_i)$ to $\gamma_2(t_i)$ by a geodesic τ_i. Each loop formed by $\gamma_1([t_{i-1}, t_i])$, τ_i, $-\gamma_2([t_{i-1}, t_i])$, and $-\tau_{i-1}$ has length strictly less than 2σ and is therefore null-homotopic. The homotopy between γ_1 and γ_2 is obtained by "filling in" the small loops. □

Next, we show that two homotopic loops γ_0 and γ_1 based at x_0 map to the same homotopy class in Y. Let $\Gamma: [0, 1]\times[0, 1] \to X$ be a homotopy satisfying

$$\Gamma(\alpha, t) = \gamma_\alpha(t)$$

$$\Gamma(s, 0) = \Gamma(s, 1) = x_0$$

for $\alpha=0, 1$ and $0 \leqq s, t \leqq 1$. Choose partitions $0=s_0<s_1< \cdots <s_N=1$ and $0=t_0<t_1< \cdots <t_N=1$ so that for any $0 \leqq i \leqq N$, $1 \leqq j \leqq N$, $l(\Gamma([s_{j-1}, s_j], t_i))$, $l(\Gamma(s_i, [t_{j-1}, t_j]))<\eta$.

We construct a corresponding homotopy $\Phi: [0, 1]\times[0, 1] \to Y$ as follows. Let $\Phi(s, 0)=\Phi(s, 1)=y_0$. Choose $\Phi(s_i, t_j)$ so that $\delta(\Gamma(s_i, t_j), \Phi(s_i, t_j))<\varepsilon$, and join $\Phi(s_i, t_j)$ to $\Phi(s_{i+1}, t_j)$ and $\Phi(s_i, t_{j+1})$ by minimizing geodesics. Each loop formed by $\Phi(s_{i-1}, [t_{j-1}, t_j])$, $\Phi([s_{i-1}, s_i], t_j)$, $-\Phi(s_i, [t_{j-1}, t_j])$, $-\Phi([s_{i-1}, s_i], t_{j-1})$ has length less than $8\varepsilon+4\eta<2\sigma$. Since they are all homotopically trivial, they can be filled in, yielding the homotopy Φ.

This construction gives a homomorphism $\alpha: \pi_1(X) \to \pi_1(Y)$ and a corresponding map $\beta: \pi_1(Y) \to \pi_1(X)$. We now show that $\beta\alpha$ is the identity; the proof that $\alpha\beta$ is the identity is similar.

Recall that $Z=X \amalg Y$. As above, there exist i and j such that

$$\begin{aligned} d_Z(\phi(t), \gamma(t)) &< d_Y(\phi(t_i), \phi(t_{i-1})) + d_Z(\phi(t_{i-1}), \gamma(t_{i-1})) + d_X(\gamma(t_{i-1}), \gamma(t_i)) \\ &< (2\varepsilon+\eta)+\varepsilon+\eta \\ &= 3\varepsilon+2\eta . \end{aligned}$$

It follows that if $\psi(t)$ is a loop in $\beta\alpha([\gamma(t)])$, then

$$\begin{aligned} d_X(\gamma(t), \psi(t)) &< d_Z(\gamma(t), \phi(t)) + d_Z(\phi(t), \psi(t)) \\ &< (3\varepsilon+2\eta) + d_X(\psi(t_i), \psi(t_{i-1})) + d_Z(\psi(t_{i-1}), \phi(t_{i-1})) + d_Y(\phi(t_{i-1}), \phi(t_i)) \\ &< (3\varepsilon+2\eta) + (2\varepsilon+(2\varepsilon+\eta)) + \varepsilon + (2\varepsilon+\eta) \\ &= 9\varepsilon+4\eta < \sigma . \end{aligned}$$

Thus by Lemma 5.2, $\beta\alpha([\gamma(t)])$ equals $[\gamma(t)]$. □

REMARK. The argument above generalizes easily to higher homotopy groups, yielding the following weak version of the results of Petersen [**14**] and Ferry [**6**] (see these papers for definitions, details, and stronger results):

THEOREM 5.3. *Given a contractibility function ρ and connected metric spaces X and Y in $LGC^n(\rho)$, there exists ε depending only on n and ρ such that if $d_H(X, Y)<\varepsilon$, $\pi_k(X)\cong\pi_k(Y)$ for $1\leqq k\leqq n$.*

6. Growth of the fundamental group.

This section is devoted to a slight extension of results of Milnor [**11**] and Gromov [**8**]. In the next lemma, $|S|$ denotes the number of elements in a finite set S, and (X, d) is a compact connected length space whose universal cover $\tilde{X}$ is given the induced length space structure. For a finitely generated group Γ, $B_\Gamma(R)$ denotes the number of words in Γ of length less than R with respect to some fixed finite set of generators.

LEMMA 6.1. *If* $\mathrm{sys}_1(X)\geqq 2\sigma>0$, *then the fundamental group $\Gamma=\pi_1(X)$ is discrete and finitely generated. Moreover, given $R>0$ and $\tilde{x}\in\tilde{X}$*

$$\left|B_\Gamma\left(\frac{R}{2D}\right)\right| \leqq N(\tilde{x}, \sigma, R)$$

where D is the diameter of X. In particular, $\pi_1(X)$ has polynomial growth if there exists constants C and d such that

$$N(\tilde{x}, \sigma, R) \leqq CR^d .$$

PROOF. We follow the proof in [**9**, Lemma 5.19]. Choose a set of loops $\{\gamma_i\}$ that generate $\Gamma=\pi_1(X, x)$, where x is the projection of $\tilde{x}$. Given $\varepsilon>0$, there exists a new set of generators with length at most $2D+\varepsilon$ as follows. Take each generator $\gamma=\gamma_i$ and choose a partition $0=t_0<t_1<\cdots<t_N=1$ such that $l(\gamma([t_{k-1}, t_k]))\leqq\varepsilon$. Let τ_k be a minimal geodesic joining x to $\gamma(t_k)$. Replace γ by the N loops consisting of τ_{k-1}, $\gamma([t_{k-1}, t_k])$, $-\tau_k$, where $k=1, \cdots, N$.

We can therefore assume that the generators $\{\gamma_i\}$ have length less than

$2D+\varepsilon$. If Γ is viewed as a group of covering transformations of $\tilde{X}$, it follows that $\gamma_i\tilde{x}$ is contained in the closure of $B(\tilde{x}, 2D+\varepsilon)$, which is compact. Moreover, the lower bound on the first systole implies

$$d(\gamma_i x_0, \gamma_j x_0) \geqq 2\sigma .$$

Thus $\{\gamma_i\tilde{x}\}$ is finite, i. e. Γ is finitely generated. Moreover,

$$\left|B_\Gamma\left(\frac{R}{2D}\right)\right| \leqq |B(\tilde{x}, R)\cap\Gamma\tilde{x}| \leqq N(\tilde{x}, \sigma, R). \qquad \square$$

REMARK. The example of an infinite bouquet of circles with lengths converging to zero shows that the positivity of $\mathrm{sys}_1(X)$ is necessary.

7. Proof of Theorem 1.1.

We shall need the following consequences of the Bishop-Gromov volume comparison theorem:

THEOREM 7.1 ([**9**, Theorem 5.3]). *Given $n\geqq 2$, D, $\kappa>0$, the set of all compact Riemannian manifolds such that*

$$\dim M = n$$

$$\operatorname{diam}(M) \leqq D$$

$$\operatorname{Rc}(M) \geqq -\kappa^2$$

is precompact with respect to Gromov-Hausdorff distance.

PROPOSITION 7.2. *Given $n\geqq 2$, R, $\varepsilon>0$, there exists $\kappa>0$ such that if*

$$\operatorname{Rc} \geqq -\kappa^2$$

on a complete Riemannian n-manifold M, then given any $0<r<R$ and $x\in M$,

$$\frac{\operatorname{vol}(B(x, R))}{\operatorname{vol}(B(x, r))} \leqq (1+\varepsilon)\left(\frac{R}{r}\right)^n .$$

PROOF OF PROPOSITION 7.2. According to Bishop-Gromov, we have

$$\frac{\operatorname{vol}(B(x, R))}{\operatorname{vol}(B(x, r))} \leqq \frac{\int_0^R (\sinh \kappa\sqrt{t})^{n-1} dt}{\int_0^r (\sinh \kappa\sqrt{t})^{n-1} dt} .$$

The proposition follows by plugging in the Taylor series for sinh. $\square$

Recall that $\tilde{Y}$ denotes the universal cover of a space Y.

PROOF OF THEOREM 1.1. Suppose that the theorem is not true. Then given n and σ there exists a sequence of n-dimensional compact Riemannian manifolds M_i where each manifold has diameter equal to 1, first systole bounded from below by 2σ, the lower bound on Ricci curvature converging to zero as $i\to\infty$, but fundamental group of larger than polynomial growth. By Theorem 7.1, there is a subsequence that converges with respect to Gromov-Hausdorff distance to a compact length space X with first systole easily seen to be bounded from below by 2σ. Moreover, given $\tilde{x}_i\in\tilde{M}_i$, by passing to a subsequence we may assume that $(\tilde{M}_i, \tilde{x}_i)\to(\tilde{X}, \tilde{x})$ in pointed Gromov-Hausdorff distance for some $\tilde{x}\in\tilde{X}$. By Theorem 5.1 a further subsequence contains only manifolds with fundamental group isomorphic to the fundamental group of X.

Given $R>0$ it follows from Proposition 7.2 that for $0<r<R$,

$$\limsup_{i\to\infty} N(\tilde{x}_i, r, R) \leqq \left(\frac{R}{r}\right)^n .$$

Proposition 4.2 and Lemma 6.1 imply that the fundamental group of X has polynomial growth. This contradicts the assumption that none of the M_i has fundamental group with polynomial growth. □

8. Volume growth of geodesic balls.

The following elementary lemma replaces the Bishop-Gromov volume comparison theorem when only L^p bounds on curvature are assumed (cf. [**1**]):

LEMMA 8.1. *Let X be an n-dimensional complete Riemannian manifold on which there exist constants a, b, $r>0$ such that for any $x\in X$ and $0<s<4r$,*

$$as^n \leqq \operatorname{vol}(B(s)) \leqq bs^n .$$

Then there exist constants $C(n, a, b, r)$, $k(n, a, b, r)>0$ such that for any $x\in X$ and $R>0$,

$$\operatorname{vol}(B(x, R)) \leqq Ce^{kR} .$$

PROOF. It suffices to prove the bound for $R=Nr$ by induction on $N\in\boldsymbol{Z}^+$. The bound clearly holds for $N=1$.

We use a standard covering argument to obtain an upper bound on the volume of a ball with radius $(N+1)r$ in terms of the volume of a ball with radius Nr. Let $R=Nr$. Given $x\in X$, let $B(x_1, r), \cdots, B(x_K, r)$ be a maximal collection of disjoint balls contained in $B(x, R)$. For any $y\in B(x, R-r)$ there exists x_i such that $d(x_i, y)<2r$; if not, $B(y, r)$ would be a ball contained in $B(x, R)$ that is disjoint from the given collection of balls. For any $z\in B(x, R+r)$, there exists $y\in B(x, R-r)$ such that $d(y, z)<2r$. Thus there exists x_i such that $d(x_i, z)<4r$. It follows that the balls $B(x_1, 4r), \cdots, B(x_K, 4r)$ cover

$B(x, R+r)$. This cover consists of K balls, where K is at most

$$\operatorname{vol}(B(x, R))/\min_i \operatorname{vol}(B(x_i, r)).$$

Therefore, the volume of $B(x, R+r)$ is bounded from above by

$$K \max_i \operatorname{vol}(B(x_i, 4r)) \leqq \frac{b}{a} 4^n \operatorname{vol}(B(x, R)).$$

By induction it follows that for $N \geqq 1$

$$\operatorname{vol}(B(x, Nr)) \leqq \left(\frac{4^n b}{a}\right)^{N-1} br^n.$$

Thus, given any $R>0$

$$\operatorname{vol}(B(x, R)) \leqq Ce^{kR}$$

where $C=ar^n 4^{-n}$ and $k=[\log(4^n b/a)]/r$. □

The following generalization of Gromov's precompactness theorem (Theorem 7.1) follows directly from the lemma (cf. [**9**], Proposition 5.2):

COROLLARY 8.2. *Given n, a, b, and $r>0$, the set of n-dimensional complete Riemannian manifolds satisfying the assumptions of Lemma* 8.1 *is precompact with respect to pointed Gromov-Hausdorff distance.*

We also need a result which shows that manifolds with small negative Ricci curvature in the L^q sense have a packing estimate similar to that of manifolds with positive Ricci curvature.

PROPOSITION 8.3. *Given $n \geqq 2$, $q>n$, $\alpha>0$, there exists $\kappa(n, q, \alpha)$, $C(n, q, \alpha)$ >0 such that if a complete Riemannian n-manifold M, $x \in M$, and $R>0$ satisfy*

$$\operatorname{vol}(B(x, 2R)) \geqq (2\alpha R)^n$$

$$\left(\int_{B(x,2R)} |\rho_-|^{q/2}\right)^{2/q} \leqq \kappa(n, q, \alpha)^2 R^{-2(1-n/q)}$$

then given any $0<r<R$,

$$\frac{\operatorname{vol}(B(x, R))}{\operatorname{vol}(B(x, r))} \leqq C(n, q, \alpha)\left(\frac{R}{r}\right)^n.$$

PROOF. The estimate $\operatorname{vol}(B(x, r)) \geqq C' r^n$ follows from the isoperimetric inequality of [**17**, Theorem 7.4] (cf. [**17**, Lemma 4.1]). The estimate $\operatorname{vol}(B(x, R)) \leqq C'' R^n$ follows from the volume bound of [**17**, Theorem 7.1]. □

9. Proof of Theorems 1.2 and 1.3.

PROOF OF THEOREM 1.3. The proof is analogous to that of Theorem 1.1. If the theorem is not true, then given n, $q>n$, σ, $\alpha>0$, there exists a sequence of n-dimensional compact manifolds M_i with diameter 1, volume bounded from below by α^n, first systole bounded from below by 2σ, the $L^{q/2}$ norm of ρ_- converging to 0 as $i\to\infty$, but with fundamental group of larger than polynomial growth.

By setting $R=1/2$ in Proposition 8.3, the assumptions of Lemma 8.1 hold with $r=1/2$. By Corollary 8.2 there exists a subsequence that converges in Gromov-Hausdorff distance to a compact length space X with first systole bounded from below by 2σ. By Theorem 5.1 there exists a further subsequence such that the fundamental group of each manifold is isomorphic to $\pi_1(X)$. Using Corollary 8.2 again, we can restrict to a subsequence, also called M_i, such that there exist $\tilde{x}_i\in\tilde{M}_i$ with $(\tilde{M}_i, \tilde{x}_i)\to(\tilde{M}, \tilde{x})$ in pointed Gromov-Hausdorff distance.

By Proposition 8.3 there exists a constant $C>0$ such that for any $R>0$,

$$\limsup_i N(\tilde{x}_i, \sigma, R) \leqq CR^n . \tag{9.1}$$

As before, this implies that $\pi_1(X)$ has polynomial growth, which is a contradiction. $\square$

The proof of Theorem 1.2 is similar. In the argument by contradiction, the convergence of a subsequence of manifolds is obtained by picking an upper bound for diam(M) (i.e. breaking the scale invariance of Theorem 1.2) and applying Theorem 7.1. The volume growth of a large geodesic ball in the universal cover is estimated as in the proof to Theorem 1.3. Finally, for $q>n$

$$\begin{aligned}\|\rho_-\|_{q/2} &\leqq \|\rho_-\|_1^{2/q}\|\rho_-\|_\infty^{(q-2)/q}\\ &\leqq \varepsilon^{4/q}\kappa^{2(q-2)/q}\operatorname{diam}(M)^{-2}\operatorname{vol}(M)^{2/q}\\ &\leqq C(n, \kappa, \operatorname{diam}(M))\varepsilon^{4/q}\end{aligned}$$

where the last inequality follows from Bishop-Gromov. This reduces the proof of Theorem 1.2 to that of Theorem 1.3.

10. Positivity of Laplacian plus potential.

Let M be an n-dimensional complete Riemannian manifold. As in § 2, we assume that there exist constants A, $B>0$ such that for any $f\in C_0^\infty(M)$,

$$\left(\int |f|^{2n/(n-2)}\right)^{(n-2)/n} \leqq A\int|\nabla f|^2+B\int|f|^2 .$$

Let E be a hermitian vector bundle over M with compatible connection ∇ and associated Bochner Laplacian $\nabla^*\nabla = -\nabla^p\nabla_p$. Given a symmetric endomorphism $R: E \to E$, let $\rho(x)$ be the lowest eigenvalue of $R(x)$.

THEOREM 10.1. *Let $R: E \to E$ and $w>0$ satisfy*

$$\|(\rho - w)_-\|_{n/2} < \min(A^{-1}, wB^{-1}).$$

Then the operator $L = -\nabla^p\nabla_p + R$ is strictly positive on the space of L^2 sections of E. In addition, if M is compact, $H^1(M'; R) = 0$ for every finite cover M' of M.

Let $\mathfrak{N} = \mathfrak{N}(n, \kappa, D, V)$ be the class of Riemannian n-manifolds with Ricci curvature bounded below by $-(n-1)\kappa^2$, diameter bounded above by D, and volume bounded below by V. The theorem implies that given $w>0$ there exists a positive constant $C = C(\mathfrak{N}, w)$ such that if L acts on a bundle over a manifold $M \in \mathfrak{N}$ and satisfies $\|(\rho - w)_-\|_{n/2} < C$, then L is strictly positive.

PROOF. By semigroup domination/Kato's inequality, L will be a strictly positive operator provided $\Delta + \rho > \varepsilon$ on L^2 functions for some $\varepsilon > 0$ (cf. [**2**] for a quick proof). In particular, it suffices to show that $\int \Delta f \cdot f + \int \rho f^2 > \varepsilon \|f\|_2^2$ for every $f \in L^2(M)$. Now

$$\begin{aligned}
(10.2) \qquad \int \Delta f \cdot f + \int \rho f^2 &\geqq \|\nabla f\|_2^2 + w\int f^2 + \int (\rho - w)_- f^2 \\
&\geqq \|\nabla f\|_2^2 + w\int f^2 - \|(\rho - w)_-\|_{n/2} \|f\|_{2n/(n-2)}^2 \\
&\geqq \|\nabla f\|_2^2 (1 - \|(\rho - w)_-\|_{n/2} A) + \|f\|_2^2 (w - \|(\rho - w)_-\|_{n/2} B) \\
&\geqq \varepsilon \|f\|_2^2
\end{aligned}$$

if $0 < \varepsilon < w - \|(\rho - w)_-\|_{n/2} B$.

The statement about finite covers follows from the fact that the lowest eigenvalue of $\Delta + \rho$ is unchanged under passage to a finite cover [**5**, Proposition 1.11]. $\square$

THEOREM 10.3. *Given $q > n$, there exists $\varepsilon(n, q) > 0$ such that if $R: E \to E$ and $w > 0$ satisfy*

$$\|(\rho - w)_-\|_1 < \varepsilon(n, q) w [(A\|\rho_-\|_{q/2})^{q/(q-n)} + B]^{-n/2}$$

then the operator $-\nabla^p\nabla_p + R$ is strictly positive on the space of L^2 sections of E. In addition, if M is compact, $H^1(M'; R) = 0$ for every finite cover M' of M.

PROOF. Assume that $\Delta + \rho$ is not strictly positive. Define a bounded domain $\Omega\subset M$ as follows. If M is compact, set $\Omega=M$. For M noncompact, by assumption for any $e>0$ there exists a function f on M such that

$$\frac{\int|\nabla f|^2+\rho f^2}{\int f^2}<e\,.$$

Since $C_0^\infty(M)$ is dense in the domains of (the closure of) Δ and ρ, we may assume that f has compact support. Let Ω be a bounded domain with smooth boundary containing the support of f.

It follows that the lowest eigenvalue λ of the operator $\Delta+\rho$ acting on functions satisfying Dirichlet conditions on Ω is less than e. Let ϕ be a non-negative function satisfying

$$\Delta+(\rho-\lambda)\phi=0 \tag{10.4}$$

on Ω and vanishing on $\partial\Omega$. Then

$$\begin{aligned}\lambda\int\phi^2&=\int|\nabla\phi|^2+\rho\phi^2\\&\geqq w\int\phi^2+\int(\rho-w)_-\phi^2\\&\geqq w\int\phi^2-\|(\rho-w)_-\|_1\|\phi\|_\infty^2\,.\end{aligned} \tag{10.5}$$

Applying Moser iteration [12] to (10.4) yields

$$\begin{aligned}\|\phi\|_\infty^2&\leqq C(n,q)[(A\|(\rho-\lambda)_-\|_{q/2})^{q/(q-n)}+B]^{n/2}\|\phi\|_2^2\\&\leqq C(n,q)[(A\|\rho_-\|_{q/2})^{q/(q-n)}+B]^{n/2}\|\phi\|_2^2\end{aligned}$$

since by assumption $\lambda\leqq 0$ for Ω large enough.
Substituting this into (10.5) gives

$$\lambda\int\phi^2\geqq[w-C(n,q)[(A\|\rho_-\|_{q/2})^{q/(q-n)}+B]^{n/2}\|(\rho-w)_-\|_1]\int\phi^2\,.$$

Therefore, given any $e>0$

$$e>\lambda\geqq w-C(n,q)[(A\|\rho_-\|_{q/2})^{q/(q-n)}+B]^{n/2}\|(\rho-w)_-\|_1\,.$$

Setting $\varepsilon(n,p)=(2C(n,q))^{-1}$ and letting e approach zero, we obtain a contradiction. Thus $-\nabla^p\nabla_p+R$ is a positive operator on L^2 sections of E. □

Theorems 2.2 and 2.3 follow directly from Theorems 10.1 and 10.3 with $E=T^*M$ and $L=dd^*+d^*d=-\nabla^p\nabla_p+\mathrm{Rc}$, the Laplace-Beltrami operator for 1-forms.

11. A geometric generalization of Myers' theorem.

Given a Riemannian manifold M and a smooth curve $\gamma:[a, b]\to M$, parametrized by arclength, and a subset $S\subset\gamma([a, b])$, let $|S|$ denote the Lebesgue measure of $\gamma^{-1}(S)\subset[a, b]$. This is equivalent to the 1-dimensional Hausdorff measure of S.

THEOREM 11.1. *Given v, D, $\kappa>0$, let X be a complete, n-dimensional, Riemannian manifold such that any ball of radius D in X has volume greater than v and such that the Ricci curvature is bounded from below by $-(n-1)\kappa^2<0$.*

Then given $H>0$ and $\delta>\pi/H$, there exists $\varepsilon>0$, depending on n, v, κ, H, D, δ such that if for some $p\in X$,

$$\mathrm{vol}\,(A\cap B(p, \delta)) < \varepsilon$$

where

$$A = \{x : \rho(x) < (n-1)H^2\}$$

then X is compact and has diameter less than $2(\delta+D)$.

PROOF. We begin by describing Wu's improvement of Myers' theorem. Recall the following index lemma for geodesics:

LEMMA 11.2. *Let $\gamma:[0, l]\to X$ be a geodesic parameterized by arclength. If there exists a function $f:[0, l]\to \boldsymbol{R}$ such that $f(0)=f(l)=0$ and*

$$\int_0^l (n-1)f'(s)^2-\mathrm{Rc}(\gamma', \gamma')f^2ds < 0$$

then γ is not minimal geodesic and there is a point in $\gamma(0, l)$ that is conjugate to $\gamma(0)$.

Myers' theorem follows from the lemma by setting

$$f(s) = \sin\frac{\pi s}{l}.$$

Wu [**16**] observed that Myers' theorem could be generalized as follows: Let

$$\phi(\tau) = 2\int_{1/2-\tau/2}^{1/2+\tau/2}(\sin \pi t)^2dt.$$

Given $\eta>\pi$ and $\xi>0$, there exists a unique $0<\tau^*(\eta, \xi)<1$ such that

$$\eta^2(1-\phi(\tau^*))-\xi^2\phi(\tau^*) = \pi^2.$$

LEMMA 11.3. *Assume that the Ricci curvature is bounded from below by $-(n-1)\kappa^2$. Given $H>0$, and $l>\pi/H$, any geodesic $\gamma:[0, l]\to X$, parameterized by arclength, satisfying*

$$|\{\gamma(s): \rho(\gamma(s))<(n-1)H^2\}| < \tau^*(H|\gamma|, \kappa|\gamma|)|\gamma|$$

has at least one conjugate point.

Again, the proof of this lemma follows from Lemma 11.2 by setting $f(s)=\sin \pi s/l$. We shall use the contrapositive:

LEMMA 11.4. *A constant speed minimal geodesic $\gamma: [a, b]\to X$ of length greater than π/H must satisfy*

$$|\{\gamma(s): \rho(\gamma(s)) < (n-1)H^2\}| \geqq \tau^*(H|\gamma|, \kappa|\gamma|)|\gamma|. \tag{11.5}$$

We shall prove the theorem by showing that

$$B(p, \delta+D) = X. \tag{11.6}$$

Suppose (11.6) does not hold. Then there exists $x\in X$ such that $d(p, x)=\delta+D$. Let S_p denote the set of unit tangent vectors at p and Θ the set of unit tangent vectors of minimal geodesics joining p to a point in $B(x, D)$. Let Δ be defined as follows:

$$\exp_p^* dV_g = \Delta(r, \theta)^{n-1} dr d\theta$$

where $d\theta$ is the volume form on S_p and r is the arclength parameter along geodesics radiating from p. By the Bishop-Gromov inequality, given any $0\leqq r'\leqq r$,

$$\frac{\Delta(r, \theta)}{\Delta(r', \theta)} \leqq \frac{\Delta_\kappa(r)}{\Delta_\kappa(r')}$$

where

$$\Delta_\kappa(r) = \kappa^{-1}\sinh \kappa r$$

corresponds to constant sectional curvature $-\kappa^2$.

We use the Bishop-Gromov inequality to observe that

$$\begin{aligned} v &\leqq \operatorname{vol}(B(x, D)) \\ &= \int_\Theta \int_{\gamma_\theta \cap B(x, D)} \Delta(r, \theta)^{n-1} dr d\theta \\ &\leqq \int_\Theta \int_\delta^{\delta+2D} \Delta(r, \theta)^{n-1} dr d\theta \\ &\leqq \left(\int_\Theta \Delta(\delta, \theta)^{n-1} d\theta\right) \int_\delta^{\delta+2D} \frac{\Delta_\kappa(r)^{n-1}}{\Delta_\kappa(\delta)^{n-1}} dr. \end{aligned}$$

Here γ_θ is the minimal geodesic associated to $\theta\in\Theta$. We therefore obtain

$$\Delta_\kappa(\delta)^{-n+1}\int_\Theta \Delta(\delta, \theta)^{n-1} d\theta \geqq v\left(\int_\delta^{\delta+2D} \Delta_\kappa(r)^{n-1} dr\right)^{-1}. \tag{11.7}$$

On the other hand, using the Bishop-Gromov inequality, inequality (11.7), and Lemma 11.4,

$$\begin{aligned}\operatorname{vol}(A\cap B(p,\delta)) &\geqq \int_{\Theta}\int_{\gamma_\theta((0,\delta))\cap A}\Delta(r,\theta)^{n-1}dr\,d\theta \\ &\geqq \Delta_\kappa(\delta)^{-n+1}\int_{\Theta}\Delta(\delta,\theta)^{n-1}\Big(\int_{\gamma_\theta((0,\delta))\cap A}\Delta_\kappa(r)^{n-1}dr\Big)d\theta \\ &\geqq \Delta_\kappa(\delta)^{-n+1}\int_{\Theta}\Delta(\delta,\theta)^{n-1}\Big(\int_0^{|\gamma_\theta((0,\delta))\cap A|}\Delta_\kappa(r)^{n-1}dr\Big)d\theta \\ &\geqq \Delta_\kappa(\delta)^{-n+1}\Big(\int_{\Theta}\Delta(\delta,\theta)^{n-1}d\theta\Big)\int_0^{\tau^*(H\delta,\kappa\delta)\delta}\Delta_\kappa(r)^{n-1}dr \\ &\geqq v\frac{f_n(\tau^*(H\delta,\kappa\delta)\kappa\delta)}{f_n(\kappa(\delta+2D))-f_n(\kappa\delta)}\end{aligned}$$

where

$$f_n(\sigma)=\int_0^\sigma(\sinh t)^{n-1}dt\,.$$

If we now choose

$$\varepsilon=v\frac{f_n(\tau^*(H\delta,\kappa\delta)\kappa\delta)}{f_n(\kappa(\delta+2D))-f_n(\kappa\delta)}$$

then we have a contradiction. □

The following generalizes the Main Theorem of [**16**], assuming that the manifold is compact:

COROLLARY 11.8. *Given n, v, D, $\kappa>0$, let M be an n-dimensional compact Riemannian manifold such that* $\operatorname{vol}(M)>v$, $\operatorname{diam}(M)<D$, *and* $\rho>-(n-1)\kappa^2$. *Then given $H>0$, there exists $\varepsilon>0$, depending on n, v, D, κ, H such that if*

$$\operatorname{vol}(\{x:\rho(x)<(n-1)H^2\})<\varepsilon$$

then M has finite fundamental group.

PROOF. Let X be the universal cover of M with the induced Riemannian metric and $\pi: X\to M$ the canonical covering map. We show that X has finite diameter.

Given $p\in X$, $\operatorname{vol}(B(p,D))\geqq\operatorname{vol}(M)>v$. Also, given any $x\in M$, the number of elements in $\pi^{-1}(x)\cap B(p,D)$ is at most $\operatorname{vol}(B(p,2D))/v$. Therefore, if we let

$$A=\{x\in X:\rho(x)<(n-1)H^2\}$$

then

$$\operatorname{vol}(A \cap B(p, D)) \leqq \varepsilon \frac{\operatorname{vol}(B(p, 2D))}{v}$$
$$\leqq \varepsilon \frac{f_n(2\kappa D)}{\kappa^n v}.$$

Therefore, if we set $\delta = \max(D, \pi/H)$ and

$$\varepsilon = \kappa^n v^2 \frac{f_n(\tau^*(H\delta, \kappa\delta)\kappa\delta)}{f_n(2\kappa D)(f_n(\kappa(\delta+2D)) - f_n(\kappa\delta))}$$

the corollary follows from the theorem. □

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