Spectrum of the Laplacian on a complete Riemannian manifold with nonnegative Ricci curvature which possess a pole

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Let M be a complete noncompact Riemannian manifold of dimension n. The Laplacian of M is denoted by Δ. The spectrum of Δ has been studied by many authors. Especially Donnelly [D] showed that the Laplacian of a constant negative curvature $(-k^2)$ space form has purely continuous spectrum $\sigma_c(\Delta)=(-\infty, 0]$. Donnelly and Li [D-L] and Chen and author [C-L] observed that changing the metric of a constant negative curvature space form on a compact domain we can obtain a complete manifold with a rotational invariant metric of strongly negatively sectional curvature on which Δ has discrete spectrum. Escobar [E] proved that if M is a complete manifold with a rotational invariant metric of nonnegative curvature then $\sigma_c(\Delta)=(-\infty, 0]$.

In this paper we prove the following theorem.

THEOREM. *Let M be an n-dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature. If M possess a pole then Δ has essential spectrum $\sigma_{ess}(\Delta)=(-\infty, 0]$.*

A pole is by definition [G-W] a point $x_0 \in M$ such that the exponential map $\exp_{x_0}: M_{x_0} \to M$ is a diffeomorphism.

If x_0 is a pole in M, $r(x)=\mathrm{dist}(x, x_0)$ which is the geodesic distance from x to x_0 is clearly differentiable. Let $C_t = \{x \in M \mid r(x)=t\}$. We claim that $\eta(x) = \lim_{t \to \infty}(t - \mathrm{dist}(x, C_t)) = r(x)$.

In fact, for $t > r(x)$ we have $r(x)+\mathrm{dist}(x, C_t) \geq \mathrm{dist}(x_0, C_t)=t$, and so $t - \mathrm{dist}(x, C_t) \leq r(x)$. On the other hand since x_0 is a pole $q = \exp_{x_0} s(\exp_{x_0}^{-1}x)/r(x)|_{s=t} \in C_t$ and thus $r(x)+\mathrm{dist}(x, q)=t$, we therefore have $t - \mathrm{dist}(x, C_t)=r(x)$ for $t > r(x)$. The claim follows.

By Theorem A in [W] we know that if the Ricci curvature of M is nonnegative then $\Delta r(x) \geq 0$. The laplacian comparison theorem [G-W] yields the following lemma.

LEMMA. *Suppose M satisfies the hypotheses in Theorem. If x_0 is a pole in M then*

$$0 \leq \Delta r(x) \leq \frac{n-1}{r(x)}. \tag{1}$$

PROOF OF THEOREM. Let x_0 be a pole in M. $r(x)=\text{dist}(x, x_0)$, $B_{x_0}(R)=\{x \in M | r(x) < R\}$, $V_{x_0}(R)=\text{Vol}(B_{x_0}(R))$ which is the volume of $B_{x_0}(R)$. By comparison theorem [C-G-T] one has

$$\frac{V_{x_0}(R)}{V_{x_0}(r)} \leq \left(\frac{R}{r}\right)^n \quad \text{for } R \geq r > 0 \tag{2}$$

$$\frac{V_{x_0}(R)}{V_{x_0}(r)} \geq \frac{(R-2r)^n}{R^n-(R-2r)^n} \quad \text{for } R \geq 2r > 0. \tag{3}$$

We choose $\phi(r) \in C_0^\infty(M)$ satisfying

$$\phi(r) = \begin{cases} 1 & a_n \leq r \leq b_n \\ 0 & r \leq c_n, \quad r \geq d_n \end{cases} \tag{4}$$

$0 \leq \phi(r) \leq 1$, $|\phi'(r)| \leq A_n$, $|\phi''(r)| \leq A_n$ where $c_n < a_n < b_n < d_n$ are constants to be fixed, A_n depends on a_n, b_n, c_n and d_n.

For any $\lambda \geq 0$, we consider the sequence of function

$$\phi_k^\lambda(x) = \eta_k^{1/2}\phi(\varepsilon_k r(x))e^{i\sqrt{\lambda}r(x)} \in C_0^\infty(M) \tag{5}$$

where ε_k is a monotone sequence which tends to zero, η_k is a sequence to be fixed.

Computing directly we have

$$\nabla\phi_k^\lambda(x) = \eta_k^{1/2}\phi'(\varepsilon_k r)e^{i\sqrt{\lambda}r}\varepsilon_k\nabla r+i\eta_k^{1/2}\sqrt{\lambda}\phi(\varepsilon_k r)e^{i\sqrt{\lambda}r}\nabla r \tag{6}$$

$$\Delta\phi_k^\lambda(x) = \eta_k^{1/2}\varepsilon_k^2\phi''(\varepsilon_k r)e^{i\sqrt{\lambda}r}+2i\eta_k^{1/2}\sqrt{\lambda}\varepsilon_k\phi'(\varepsilon_k r)e^{i\sqrt{\lambda}r}$$

$$+\eta_k^{1/2}\varepsilon_k\phi'(\varepsilon_k r)e^{i\sqrt{\lambda}r}\Delta r+i\eta_k^{1/2}\sqrt{\lambda}\phi(\varepsilon_k r)e^{i\sqrt{\lambda}r}\Delta r-\lambda\phi_k^\lambda(x). \tag{7}$$

So

$$|\Delta\phi_k^\lambda(x)+\lambda\phi_k^\lambda(x)| \leq \eta_k^{1/2}\varepsilon_k^2|\phi''(\varepsilon_k r)|+2\eta_k^{1/2}\varepsilon_k\sqrt{\lambda}|\phi'(\varepsilon_k r)|$$

$$+\eta_k^{1/2}\varepsilon_k|\phi'(\varepsilon_k r)|\Delta r+\eta_k^{1/2}\sqrt{\lambda}|\phi(\varepsilon_k r)|\Delta r.$$

By Lemma one has

$$\frac{1}{32}\int_M |\Delta\phi_k^\lambda(x)+\lambda\phi_k^\lambda(x)|^2 dx$$

$$\leq \eta_k\varepsilon_k^4\int_M |\phi''(\varepsilon_k r)|^2 dx+\eta_k\varepsilon_k^4\int_M |\phi'(\varepsilon_k r)|^2\frac{(n-1)^2}{(\varepsilon_k r)^2}dx$$

$$+\eta_k\varepsilon_k^2\lambda\int_M |\phi'(\varepsilon_k r)|^2 dx+\eta_k\varepsilon_k^2\lambda\int_M |\phi(\varepsilon_k r)|^2\frac{(n-1)^2}{(\varepsilon_k r)^2}dx$$

$$\leq A_n^2 \eta_k \varepsilon_k^4 \Big|_{c_n \leq \varepsilon_k r \leq d_n} dx + \frac{A_n^2}{c_n^2} \eta_k \varepsilon_k^4 (n-1)^2 \Big|_{c_n \leq \varepsilon_k r \leq d_n} dx$$

$$+ A_n^2 \eta_k \varepsilon_k^2 \lambda \Big|_{c_n \leq \varepsilon_k r \leq d_n} dx + \frac{(n-1)^2}{c_n^2} \eta_k \varepsilon_k^2 \lambda \Big|_{c_n \leq \varepsilon_k r \leq d_n} dx$$

$$\leq A_n^2 \Big(1 + \frac{(n-1)^2}{c_n^2}\Big) \eta_k \varepsilon_k^4 \Big[V_{x_0}\Big(\frac{d_n}{\varepsilon_k}\Big) - V_{x_0}\Big(\frac{c_n}{\varepsilon_k}\Big) \Big]$$

$$+ \Big(A_n^2 + \frac{(n-1)^2}{c_n^2}\Big) \eta_k \varepsilon_k^2 \lambda \Big[V_{x_0}\Big(\frac{d_n}{\varepsilon_k}\Big) - V_{x_0}\Big(\frac{c_n}{\varepsilon_k}\Big) \Big].$$

We choose

$$\eta_k = \Big[V_{x_0}\Big(\frac{d_n}{\varepsilon_k}\Big) - V_{x_0}\Big(\frac{c_n}{\varepsilon_k}\Big) \Big]^{-1} \tag{8}$$

then

$$\lim_{k \to \infty} \int_M |\Delta \phi_k^\lambda(x) + \lambda \phi_k^\lambda(x)|^2 dx = 0. \tag{9}$$

Obviously

$$\int_M |\phi_k^\lambda(x)|^2 dx \geq \frac{V_{x_0}(b_n/\varepsilon_k) - V_{x_0}(a_n/\varepsilon_k)}{V_{x_0}(d_n/\varepsilon_k) - V_{x_0}(c_n/\varepsilon_k)} \geq \frac{V_{x_0}(b_n/\varepsilon_k)}{V_{x_0}(d_n/\varepsilon_k)} - \frac{V_{x_0}(a_n/\varepsilon_k)}{V_{x_0}(d_n/\varepsilon_k)}.$$

Using (2) and (3) we have

$$\int_M |\phi_k^\lambda(x)|^2 dx \geq \Big(\frac{b_n}{d_n}\Big)^n - \frac{d_n^n - (d_n - 2a_n)^n}{(d_n - 2a_n)^n} \geq 1 + \Big(\frac{b_n}{d_n}\Big)^n - \Big(\frac{d_n}{d_n - 2a_n}\Big)^n.$$

Choosing a_n sufficiently small we have

$$\int_M |\phi_k^\lambda(x)|^2 dx \geq \alpha_n > 0.$$

So

$$\liminf_{k \to \infty} \|\phi_k^\lambda(x)\|_{L^2(M)} \geq \alpha_n > 0. \tag{10}$$

(3) implies

$$\frac{V_{x_0}(d_n/\varepsilon_k)}{V_{x_0}(c_n/\varepsilon_k)} \geq \frac{(d_n - 2c_n)^n}{d_n^n - (d_n - 2c_n)^n}.$$

By choosing c_n sufficiently small we have

$$\frac{V_{x_0}(d_n/\varepsilon_k)}{V_{x_0}(c_n/\varepsilon_k)} \geq \beta_n > 1$$

which implies $V_{x_0}(d_n/\varepsilon_k) - V_{x_0}(c_n/\varepsilon_k) \geq (\beta_n - 1) V_{x_0}(c_n/\varepsilon_k) \to \infty$ as $k \to \infty$. We therefore have

$$\phi_k^\lambda(x) \longrightarrow 0 \quad \text{weakly in } L^2(M) \text{ as } k \to \infty. \tag{11}$$

(9), (10), (11) imply $-\lambda \in \sigma_{ess}(\Delta)$. This completes the proof of Theorem.

References

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