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Asymptotic behavior of positive solutions of singular Emden-Fowler type equations

Dedicated to Professor Takaŝi Kusano on his 60th birthday

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1. Introduction.

This paper is concerned with positive solutions of the singular Emden-Fowler type equation

$$x'' = p(t)x^{-\lambda}, \qquad t \geqq t_0 > 0, \tag{1.1}$$

where $\lambda>0$, and p is a positive continuously differentiable function on $[t_0, \infty)$. By a positive solution of (1.1) we mean a positive function $x(t)$ of class C^2 which solves (1.1) for $t\geqq t_0$.

Let $x(t)$ be a positive solution of (1.1). Then, it is easily seen that $x(t)$ has exactly one of the next properties:

$$x'(t) < 0 \qquad \text{for all } t, \text{ and } \lim_{t\to\infty} x(t) \in [0, \infty), \tag{1.2}$$

or

$$x'(t) > 0 \qquad \text{for all large } t, \text{ and } \lim_{t\to\infty} \frac{x(t)}{t} \in (0, \infty]. \tag{1.3}$$

A positive solution $x(t)$ of (1.1) is called a positive decaying solution if (1.2) is satisfied with $\lim_{t\to\infty} x(t)=0$, and is called a positive increasing solution if (1.3) is satisfied.

Singular equations of this kind appear in many branches of mathematical physics. We refer the reader to the papers [**3**, **4**, **5**] for physical aspect of equation (1.1). Interesting results have been obtained for these equations; see [**1**, **7**, **8**]. Sufficient, or necessary conditions for the existence of positive solutions satisfying (1.2), in particular, positive decaying solutions, were discussed in [**8**, **9**]. However, it seems that very little is known about the asymptotic behavior as well as the uniqueness of positive decaying solutions. Our first objective is to investigate these problems. We discuss in Section 2 the asymptotic behavior of positive decaying solutions. Uniqueness criteria for positive decaying solutions are established in Section 3. Another important problem is to determine the asymptotic behavior of positive increasing solutions. Section 4 is devoted to the study of this topic. The proofs require some transformation of variables motivated by Bellman [**2**].

Our results mainly concern the case for which $p(t)$ behaves like a positive constant multiple of t^{α}, $\alpha \in \boldsymbol{R}$. By employing the results below, we can completely determine the asymptotic behavior of all positive solutions of the singular Emden-Fowler equation $x''=t^{\alpha}x^{-\lambda}$. (See Example 5.1.1.) Other related results appear in the papers [**9**, **10**].

2. Asymptotic behavior of positive decaying solutions.

We will begin with the investigation of asymptotic behavior of positive decaying solutions of (1.1). Let us consider another equation of the form (1.1):

$$y'' = q(t)y^{-\lambda}, \qquad t \geqq t_0, \tag{2.1}$$

where q is positive and continuously differentiable on $[t_0, \infty)$. Our first result asserts that, if $p(t)$ and $q(t)$ have the same asymptotic behavior in some sense, then so do the positive decaying solutions of these equations as $t\to\infty$. Notice that, when (2.1) admits a positive decaying solution,

$$q_0(t) \equiv \int_t^{\infty}(s-t)q(s)ds$$

converges for $t \geqq t_0$ (see [**8**, **9**]).

THEOREM 2.1. *Let $x(t)$ and $y(t)$ be positive decaying solutions of equations* (1.1) *and* (2.1), *respectively. Suppose that*

$$\lim_{t\to\infty}\frac{p(t)}{q(t)} = 1, \tag{2.2}$$

$$\limsup_{t\to\infty}[q_0(t)]^{-1/(\lambda+1)}\int_t^{\infty}(s-t)q(s)[q_0(s)]^{-\lambda/(\lambda+1)}ds < \infty, \tag{2.3}$$

$$\begin{aligned} 0 &< \liminf_{t\to\infty}[(q(t)[y(t)]^{3-\lambda})^{-1/2}]'[y(t)]^2 \\ &\leqq \limsup_{t\to\infty}[(q(t)[y(t)]^{3-\lambda})^{-1/2}]'[y(t)]^2 < \infty, \end{aligned} \tag{2.4}$$

and either

$$\int^{\infty}[q(t)]^{1/2}[y(t)]^{-(1+\lambda)/2}\left|\frac{p(t)}{q(t)}-1\right|dt < \infty \tag{2.5}$$

or

$$\int^{\infty}\left|\left(\frac{p(t)}{q(t)}\right)'\right|dt < \infty \tag{2.6}$$

hold. Then $x(t)\sim y(t)$ *as* $t\rightarrow\infty$.

Henceforth the notation "$f(t)\sim g(t)$ as $t\rightarrow\infty$" means that $\lim_{t\to\infty}f(t)/g(t)=1$. The following simple result due to Bellman [**2**, p. 155] is often employed in this paper.

LEMMA 2.1. *If* $z\in C^1[T, \infty)$ *satisfies*

$$\int^{\infty}[z(t)]^2dt<\infty, \quad \text{and} \quad |z'(t)|\leqq C, \qquad t\geqq T,$$

for some $C>0$, *then* $\lim_{t\to\infty}z(t)=0$.

PROOF OF THEOREM 2.1. First we note under our conditions that $x(t)$ and $y(t)$ satisfy for $t\geqq t_0$

$$\begin{aligned}(2.7)\qquad C_1[q_0(t)]^{1/(\lambda+1)}&\leqq x(t),\ y(t)\\ &\leqq C_2\int_t^{\infty}(s-t)q(s)[q_0(s)]^{-\lambda/(\lambda+1)}ds\end{aligned}$$

for some $C_i=C_i(\lambda)>0$. The first inequality of (2.7) is known by [**9**, Remark]. Then, the second inequality of (2.7) follows immediately from

$$y(t)=\int_t^{\infty}\left(\int_s^{\infty}q(r)[y(r)]^{-\lambda}dr\right)ds, \qquad t\geqq t_0.$$

Define the new function $v(t)$ by $v(t)=x(t)/y(t)$, $t\geqq t_0$. Then we see that $v(t)$ satisfies the equation

$$(2.8)\qquad v''+\frac{2y'(t)}{y(t)}v'+q(t)[y(t)]^{-\lambda-1}v=p(t)[y(t)]^{-\lambda-1}v^{-\lambda}, \qquad t\geqq t_0.$$

By introducing the new independent variable

$$\tau=\int_{t_0}^{t}[y(r)]^{-2}dr,$$

this equation is transformed into

$$\frac{d^2v}{d\tau^2}+q(t)[y(t)]^{3-\lambda}v=p(t)[y(t)]^{3-\lambda}v^{-\lambda}, \qquad \tau\geqq 0.$$

Moreover, since

$$\int_0^{\infty}(q(t(\tau))[y(t(\tau))]^{3-\lambda})^{1/2}d\tau=\infty,$$

by (2.4), the change of variable

$$s=\int_0^{\tau}(q(t(\xi))[y(t(\xi))]^{3-\lambda})^{1/2}d\xi$$

transforms this equation into

$$\ddot{v}-f(s)\dot{v}+v = g(s)v^{-\lambda}, \qquad s \geqq 0, \tag{2.9}$$

where a dot $\cdot$ denotes differentiation with respect to s,

$$f(s) = [(q(t)[y(t)]^{3-\lambda})^{-1/2}]'[y(t)]^2,$$

and $g(s)=p(t)/q(t)$, $s \geqq 0$. Notice that (2.2), (2.4), (2.5) and (2.6) are equivalent to the conditions

$$\lim_{s\to\infty} g(s) = 1, \tag{2.10}$$

$$0 < \liminf_{s\to\infty} f(s) \leqq \limsup_{s\to\infty} f(s) < \infty, \tag{2.11}$$

$$\int^{\infty} |g(s)-1|\, ds < \infty \tag{2.12}$$

and

$$\int^{\infty} |\dot{g}(s)|\, ds < \infty, \tag{2.13}$$

respectively. It follows from (2.3) and (2.7) that

$$m \leqq v(s) \leqq M, \qquad s \geqq 0 \tag{2.14}$$

for some $m, M>0$. Condition (2.11) implies that

$$k_1 \leqq f(s) \leqq k_2, \qquad s \geqq s_0 \tag{2.15}$$

for large s_0 and some $k_1, k_2>0$. We suppose $s_0=0$ for simplicity. The proof will be complete if we can show that $v(s)\to 1$ as $s\to\infty$. In the sequel (of this paper) it will be assumed for simplicity that $\lambda \neq 1$.

From (2.10) and (2.14) there is $M_1>0$ satisfying $|g(s)[v(s)]^{-\lambda}-v(s)| \leqq M_1$, $s \geqq 0$. Let L be a constant satisfying $L>M_1/k_1$. We assert that $\dot{v}(s)$ is bounded in $[0, \infty)$, i.e.,

$$|\dot{v}(s)| \leqq L, \qquad \text{for large } s. \tag{2.16}$$

To see this, assume to the contrary that (2.16) fails to hold. Since, clearly (2.14) implies $\liminf_{s\to\infty}|\dot{v}(s)|=0$, we can find a sequence of intervals $\{[a_n, b_n]\}$, $n \in \boldsymbol{N}$, such that

$$b_n < a_{n+1}, \qquad \lim_{n\to\infty} a_n = \infty,$$

$$|\dot{v}(a_n)| = |\dot{v}(b_n)| = L,$$

$$|\dot{v}(s)| > L, \qquad a_n < s < b_n.$$

Then, Rolle's theorem shows that there exists $c_n \in (a_n, b_n)$ satisfying $\ddot{v}(c_n)=0$, $|\dot{v}(c_n)|>L$. However by putting $s=c_n$ in (2.9) we reach a contradiction for large n:

$$k_1 L \leqq f(c_n)|\dot{v}(c_n)| = |g(c_n)[v(c_n)]^{-\lambda} - v(c_n)| \leqq M_1 .$$

Hence (2.16) follows. Note that $\ddot{v}(s)$ is also bounded on $[0, \infty)$ from the boundedness of $\dot{v}(s)$ and our conditions.

First suppose that (2.5) holds. By rewriting (2.9) as

$$\left(\frac{\dot{v}^2}{2}\right)^{\cdot} - f(s)\dot{v}^2 + \left(\frac{v^2}{2}\right)^{\cdot} = g(s)\left(\frac{v^{1-\lambda}}{1-\lambda}\right)^{\cdot}, \qquad s \geqq 0,$$

and integrating over $[0, s]$, we have

$$\frac{[\dot{v}(s)]^2}{2} - \int_0^s f(r)[\dot{v}(r)]^2 dr + \left(\frac{[v(s)]^2}{2} - \frac{[v(s)]^{1-\lambda}}{1-\lambda}\right) = \int_0^s [g(r)-1][v(r)]^{-\lambda}\dot{v}(r)dr + c_1, \qquad s \geqq 0, \tag{2.17}$$

where $c_1 \in \boldsymbol{R}$. Since (2.12), (2.14) and (2.16) show that

$$\int_0^s |[g(r)-1][v(r)]^{-\lambda}\dot{v}(r)| dr \leqq C\int_0^\infty |g(s)-1| ds < \infty ,$$

for some $C>0$, (2.17) together with (2.14), implies that

$$\int^\infty f(s)[\dot{v}(s)]^2 ds < \infty ,$$

and hence,

$$\int^\infty [\dot{v}(s)]^2 ds < \infty , \tag{2.18}$$

by (2.15). From the boundedness of $\ddot{v}(s)$ and (2.18), we find that $\lim_{s\to\infty} \dot{v}(s)=0$ by Lemma 2.1. Accordingly it follows from (2.17) that the limit

$$\lim_{s\to\infty}\left(\frac{[v(s)]^2}{2} - \frac{[v(s)]^{1-\lambda}}{1-\lambda}\right)$$

must exist as a finite value, that is, $v(s)$ has a finite limit l as $s\to\infty$. Letting $s\to\infty$ in (2.9), we have $\lim_{s\to\infty} \ddot{v}(s) = l^{-\lambda} - l$. If $l^{-\lambda} - l \neq 0$, then the boundedness of $\dot{v}(s)$ is violated. Hence $l^{-\lambda} - l = 0$, i. e., $l=1$ as desired.

Next, Let (2.6) be satisfied. We notice that (2.17) is equivalent to

$$\frac{[\dot{v}(s)]^2}{2} - \int_0^s f(r)[\dot{v}(r)]^2 dr + \left(\frac{[v(s)]^2}{2} - \frac{g(s)[v(s)]^{1-\lambda}}{1-\lambda}\right) = -\frac{1}{1-\lambda}\int_0^s \dot{g}(r)[v(r)]^{1-\lambda} dr + c_2, \qquad s \geqq 0, \tag{2.19}$$

for some $c_2 \in \boldsymbol{R}$. Condition (2.13) ensures the convergence of the right hand side of (2.19). Hence we have (2.18) and $\lim_{s\to\infty} \dot{v}(s)=0$, which in turn show that the limit

$$\lim_{s\to\infty}\left(\frac{[v(s)]^2}{2}-\frac{g(s)[v(s)]^{1-\lambda}}{1-\lambda}\right)$$

exists in $\boldsymbol{R}$ by (2.19). The remainder of the proof can be completed by the same method as in the preceding discussion. This completes the proof of Theorem 2.1.

REMARK. Let $0<\lambda<1$ and suppose that there exists a nonincreasing function $q^*\in C[t_0, \infty)$ such that $q(t)\leqq q^*(t)$, $t\geqq t_0$. Then, (2.3) may be replaced by the condition

$$\limsup_{t\to\infty}\,[q_0(t)]^{-1/2}\int_t^\infty [q^*(s)]^{1/2}ds < \infty .$$

This is a simple consequence of [**9**, Proof of Theorem 2].

Consider the case in which $q(t)$ behaves like ct^α with $c>0$ and $\alpha+2<0$, that is, assume

$$0<\liminf_{t\to\infty}\frac{q(t)}{t^\alpha}\leqq\limsup_{t\to\infty}\frac{q(t)}{t^\alpha}<\infty\,, \tag{2.20}$$

$$0<\liminf_{t\to\infty}\frac{-q'(t)}{t^{\alpha-1}}\leqq\limsup_{t\to\infty}\frac{-q'(t)}{t^{\alpha-1}}<\infty\,. \tag{2.21}$$

Then, for any positive decaying solution $y(t)$ of (2.1), (2.7) shows that

$$C_1t^{(\alpha+2)/(\lambda+1)}\leqq y(t)\leqq C_2t^{(\alpha+2)/(\lambda+1)}\,,\qquad t\geqq t_0$$

for some C_1, $C_2>0$. Moreover, the equation

$$-y'(t)=\int_t^\infty q(s)[y(s)]^{-\lambda}ds\,,\qquad t\geqq t_0\,,$$

gives the estimates

$$C_3t^{(\alpha+2)/(\lambda+1)-1}\leqq -y'(t)\leqq C_4t^{(\alpha+2)/(\lambda+1)-1}\,,\qquad t\geqq t_0$$

for some C_3, $C_4>0$. Therefore we can see that (2.4) is surely satisfied if $\lambda\leqq 3$. The following corollary is derived from this observation.

COROLLARY 2.1. *Let $\lambda\leqq 3$ and $\alpha+2<0$, and let $x(t)$ and $y(t)$ be positive decaying solutions of equations* (1.1) *and* (2.1), *respectively. Suppose that* (2.2), (2.20), (2.21) *and either*

$$\int^\infty\frac{1}{t}\left|\frac{p(t)}{q(t)}-1\right|dt<\infty$$

or

$$\int^\infty\left|\left(\frac{p(t)}{q(t)}\right)'\right|dt<\infty$$

hold. Then $x(t) \sim y(t)$ *as* $t \to \infty$.

The next corollary will be employed in proving the uniqueness of positive decaying solutions in Theorem 3.2.

COROLLARY 2.2. *Let* $\alpha+2<0$. *Suppose that* $p(t)$ *is of the form* $p(t)=ct^{\alpha}[1+\varepsilon(t)]$, *where* c *is a positive constant,* $\varepsilon(t) \to 0$ *as* $t \to \infty$ *and either*

$$\int^{\infty} \frac{|\varepsilon(t)|}{t} dt < \infty \tag{2.22}$$

or

$$\int^{\infty} |\varepsilon'(t)| dt < \infty \tag{2.23}$$

holds. Then any positive decaying solution $x(t)$ *of* (1.1) *satisfies*

$$x(t) \sim \left[\frac{c(\lambda+1)^2}{(\alpha+2)(\alpha-\lambda+1)}\right]^{1/(\lambda+1)} t^{(\alpha+2)/(\lambda+1)} \quad \text{as } t \to \infty .$$

For the proof of this result, it is sufficient to notice that the singular Emden-Fowler equation

$$y'' = ct^{\alpha} y^{-\lambda}, \qquad t \geqq t_0,$$

with $c>0$ and $\alpha+2<0$, has a positive decaying solution $y(t)$ explicitly given by

$$y(t) = \left[\frac{c(\lambda+1)^2}{(\alpha+2)(\alpha-\lambda+1)}\right]^{1/(\lambda+1)} t^{(\alpha+2)/(\lambda+1)}, \qquad t \geqq t_0,$$

for which (2.4) (with $q(t)=ct^{\alpha}$) is satisfied.

3. Uniqueness of positive decaying solutions.

We now turn to the question of the uniqueness of positive decaying solutions of (1.1).

THEOREM 3.1. *Let* $\lambda \leqq 3$ *and* $\alpha+2<0$. *Suppose that* $p(t)$ *satisfies*

$$0 < \liminf_{t\to\infty} \frac{p(t)}{t^{\alpha}} \leqq \limsup_{t\to\infty} \frac{p(t)}{t^{\alpha}} < \infty,$$

and

$$0 < \liminf_{t\to\infty} \frac{-p'(t)}{t^{\alpha-1}} \leqq \limsup_{t\to\infty} \frac{-p'(t)}{t^{\alpha-1}} < \infty .$$

Then (1.1) *has a unique positive decaying solution.*

PROOF. Suppose to the contrary that we have two distinct positive decaying solutions $x(t)$ and $y(t)$ of (1.1). If the difference $x(t)-y(t)$ is of constant sign for all sufficiently large t, then the simple observation developed in [**9**, Theorem 3] implies that $x(t)\equiv y(t)$. Therefore we treat the case where $x(t)-y(t)$ changes sign infinitely many times. The proof will be carried out by employing the same transformations used in the proof of Theorem 2.1.

Define the new functions $v(t)$ and τ by $v(t)=x(t)/y(t)$ and $\tau=\int_{t_0}^{t}[y(r)]^{-2}dr$, $t\geqq t_0$, respectively. Then calculations give

$$\frac{d^2v}{d\tau^2}+p(t)[y(t)]^{3-\lambda}(v-v^{-\lambda})=0\,, \qquad \tau\geqq 0\,.$$

Moreover the change of variable

$$s=\int_0^{\tau}(p(t(\xi))[y(t(\xi))]^{3-\lambda})^{1/2}d\xi$$

transforms this equation into

$$\ddot{v}-f(s)\dot{v}+v=v^{-\lambda}\,, \qquad s\geqq 0\,, \tag{3.1}$$

where a dot $\cdot$ denotes differentiation with respect to s, and

$$f(s)=[(p(t)[y(t)]^{3-\lambda})^{-1/2}]'[y(t)]^2\,, \qquad s\geqq 0\,.$$

Then, the proofs of Theorem 2.1 and Corollary 2.1 show that $\lim_{s\to\infty}\dot{v}(s)=0$. But, we can get a contradiction immediately. To see this, let s_n, $n\in \boldsymbol{N}$, be the points at which $v(s)$ cuts the horizontal line $v=1$ and $s_n\uparrow\infty$ as $n\to\infty$. The existence of such points is guaranteed by the fact that $v(s)$ oscillates around $v=1$. Then, an integration of (3.1) multiplied by $\dot{v}(s)$ over $[s_n, s_{n+1}]$ yields

$$[\dot{v}(s_{n+1})]^2-[\dot{v}(s_n)]^2=2\int_{s_n}^{s_{n+1}}f(s)[\dot{v}(s)]^2ds\,,$$

which contradicts the fact that $\lim_{s\to\infty}\dot{v}(s)=0$ unless $v(s)\equiv 1$. The proof is complete.

THEOREM 3.2. *Let $\alpha+2<0$. Suppose that $p(t)$ is of the form $p(t)=ct^{\alpha}[1+\varepsilon(t)]$, where c is a positive constant, $\varepsilon(t)$, $t\varepsilon'(t)\to 0$ as $t\to\infty$, and either* (2.22) *or* (2.23) *holds. Then* (1.1) *has a unique positive decaying solution.*

PROOF. Let $x(t)$ and $y(t)$ be two positive decaying solutions of (1.1). We may suppose that $x(t)-y(t)$ changes sign infinitely many times, as before. We notice from Corollary 2.2 and L'Hospital's rule that $x(t)$ and $y(t)$ satisfy

$$x(t),\ y(t)\sim Dt^{\sigma} \quad\text{and}\quad x'(t),\ y'(t)\sim \sigma Dt^{\sigma-1} \qquad \text{as } t\to\infty\,, \tag{3.2}$$

where

$$D=\left[\frac{c(\lambda+1)^2}{(\alpha+2)(\alpha-\lambda+1)}\right]^{1/(\lambda+1)} \quad\text{and}\quad \sigma=\frac{\alpha+2}{\lambda+1}\,.$$

Then, as in the proof of Theorem 3.1, we obtain equation (3.1). By taking account of (3.2), it is easily seen that $\lim_{s\to\infty} f(s)=c^{-1/2}D^{(\lambda+1)/2}(\lambda-2\alpha-3)(\lambda+1)^{-1}>0$. Hence, The remainder of the proof can be completed as in that of Theorem 3.1. This completes the proof.

4. Asymptotic behavior of positive increasing solutions.

This section concerns asymptotic behavior of positive increasing solutions of equation (1.1). Recall that a positive increasing solution of (1.1) is defined to be a positive C^2-solution $x(t)$ having the asymptotic behavior

$$\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} \frac{x(t)}{t} = \text{const} \in (0, \infty] . \tag{4.1}$$

It is proved in [8] that equation (1.1) always admits positive increasing solutions under our basic assumptions $p \in C[t_0, \infty)$ and $p(t)>0$, $t \geqq t_0$.

Suppose that $p(t)$ is sufficiently small in the sense that

$$\int^{\infty} t^{-\lambda} p(t)dt < \infty . \tag{4.2}$$

Then we can show with ease that any positive increasing solution $x(t)$ of (1.1) satisfies

$$\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} \frac{x(t)}{t} = \text{const} < \infty .$$

In fact, the definition (4.1) shows that $x(t) \geqq kt$, $t \geqq t_0$ for some $k>0$. Then an integration of (1.1) gives

$$\begin{aligned} x'(t) &= x'(t_0) + \int_{t_0}^{t} p(s)[x(s)]^{-\lambda} ds \\ &\leqq x'(t_0) + k^{-\lambda} \int_{t_0}^{\infty} s^{-\lambda} p(s) ds < \infty , \end{aligned}$$

$t \geqq t_0$. Since $x'(t)$ is an increasing function, it must have a finite limit $\lim_{t\to\infty} x'(t)$. Thus our assertion follows. From this observation, we can restrict our attention to the case that (4.2) fails to hold.

The following additional notation will be used:

$$P(t) = \int_{t_0}^{t} s^{-\lambda} p(s) ds , \qquad t \geqq t_0 ;$$

$$R(t) = \int_{t_0}^{t} [P(s)]^{1/(\lambda+1)} ds , \qquad t \geqq t_0 ;$$

$$\rho(t) = \int_{t}^{\infty} [R(s)]^{-2} ds , \qquad t \geqq t_0 .$$

LEMMA 4.1. *Suppose that*

$$\int^{\infty} t^{-\lambda} p(t)dt = \infty . \tag{4.3}$$

Then, any positive increasing solution $x(t)$ *of* (1.1) *satisfies*

$$\liminf_{t\to\infty} \frac{x(t)}{R(t)} \geqq (\lambda+1)^{1/(\lambda+1)} .$$

PROOF. Let $x(t)$ be a positive increasing solution of (1.1). Since $x''(t)>0$, we have

$$x(t) = x(t_0)+\int_{t_0}^{t} x'(s)ds \leqq x(t_0)+(t-t_0)x'(t) , \qquad t\geqq t_0 .$$

Hence

$$[x'(t)]^{\lambda} \geqq t^{-\lambda}[x(t)]^{\lambda}(1+o(1)) \qquad \text{as } t\to\infty ,$$

which is equivalent to

$$[x'(t)]^{\lambda} x''(t) \geqq t^{-\lambda} p(t)(1+o(1)) \qquad \text{as } t\to\infty .$$

Rewriting this inequality as

$$\left(\frac{[x'(t)]^{\lambda+1}}{\lambda+1}\right)' \geqq t^{-\lambda} p(t)(1+o(1)) ,$$

and integrating, we have

$$x'(t) \geqq (\lambda+1)^{1/(\lambda+1)}\left(\int_{t_0}^{t} s^{-\lambda} p(s)ds\right)^{1/(\lambda+1)}(1+o(1)).$$

Then, one more integration completes the proof.

The next simple lemma is useful in establishing our results.

LEMMA 4.2. *Let* $f(t)$ *and* $g(t)$ *be continuously differentiable functions defined near infinity such that* $g'(t)\neq 0$ *and* $\lim_{t\to\infty} g(t)=\infty$. *Then*

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \leqq \liminf_{t\to\infty} \frac{f(t)}{g(t)} \leqq \limsup_{t\to\infty} \frac{f(t)}{g(t)} \leqq \limsup_{t\to\infty} \frac{f'(t)}{g'(t)} .$$

In the case that (4.3) is satisfied, it is worthwhile noticing that the function

$$z(t) = CR(t) , \qquad t\geqq t_0 , \tag{4.4}$$

is a positive increasing solution of the equation

$$z'' = \frac{C^{\lambda+1}}{\lambda+1} p(t)\left[\frac{R(t)}{t[P(t)]^{1/(\lambda+1)}}\right]^{\lambda} z^{-\lambda} , \qquad t\geqq t_1>t_0 , \tag{4.5}$$

where C is a positive constant. Hence it is natural to expect that the positive increasing solutions of (1.1) behave like $z(t)$ (defined by (4.4)) when the coefficient function of (4.5) is asymptotic to $p(t)$. This observation leads to the theorems below.

THEOREM 4.1. *Let* $0<\lambda<1$. *Suppose that* (4.3) *and*

$$\lim_{t\to\infty}\frac{R(t)}{t[P(t)]^{1/(\lambda+1)}}=a\in(0,\infty) \tag{4.6}$$

hold. Then, any positive increasing solution $x(t)$ *of* (1.1) *satisfies*

$$x(t)\sim(\lambda+1)^{1/(\lambda+1)}a^{-\lambda/(\lambda+1)}R(t) \qquad as\ t\to\infty. \tag{4.7}$$

THEOREM 4.2. *Suppose that* (4.3) *and* (4.6) *hold. Suppose moreover that*

$$\lim_{t\to\infty} p(t)[R(t)]^{3-\lambda}[\rho(t)]^2=b\in(0,\infty), \tag{4.8}$$

$$\int^\infty |(t^{-\lambda}p(t)[P(t)]^{-\lambda/(\lambda+1)}[R(t)]^3[\rho(t)]^2)'|\,dt<\infty$$

and either

$$\int^\infty[\rho(t)]^{-1}[R(t)]^{-2}\left|\frac{R(t)}{t[P(t)]^{1/(\lambda+1)}}-a\right|dt<\infty \tag{4.9}$$

or

$$\int^\infty\left|\left[\frac{R(t)}{t[P(t)]^{1/(\lambda+1)}}\right]'\right|dt<\infty$$

hold. Then any positive increasing solution $x(t)$ *of* (1.1) *has the asymptotic form* (4.7).

THEOREM 4.3. *Suppose that* (4.3) *and* (4.6) *hold. Suppose moreover that* (4.8), (4.9) *and*

$$\int^\infty[\rho(t)]^{-1}[R(t)]^{-2}|\,p(t)[R(t)]^{3-\lambda}[\rho(t)]^2-b\,|\,dt<\infty$$

hold. Then any positive increasing solution $x(t)$ *of* (1.1) *has the asymptotic form* (4.7).

Lemma 4.1 shows under assumption (4.3) that any positive increasing solution $x(t)$ of (1.1) satisfies $x(t)\geqq mR(t)$, $t\geqq t_0$, for some $m>0$. Therefore, to prove these theorems, it suffices to show the next theorems.

THEOREM 4.4. *Let* $0<\lambda<1$ *and* $x(t)$ *and* $y(t)$ *be positive increasing solutions of equations* (1.1) *and* (2.1), *respectively, such that*

$$x(t)\geqq my(t),\quad t\geqq t_0,\qquad for\ some\ m>0. \tag{4.10}$$

Suppose that (4.3) *and*

$$\lim_{t\to\infty}\frac{p(t)}{q(t)}=1, \tag{4.11}$$

hold. Then $x(t)\sim y(t)$ *as* $t\to\infty$.

THEOREM 4.5. *Let* $x(t)$ *and* $y(t)$ *be positive increasing solutions of equations* (1.1) *and* (2.1), *respectively, satisfying* (4.10). *Suppose that* (4.3), (4.11),

$$\lim_{t\to\infty} q(t)[y(t)]^{3-\lambda}\left(\int_t^\infty [y(s)]^{-2}ds\right)^2 = l \in (0, \infty), \tag{4.12}$$

$$\int^\infty \left|\left[q(t)[y(t)]^{3-\lambda}\left(\int_t^\infty [y(s)]^{-2}ds\right)^2\right]'\right| dt < \infty, \tag{4.13}$$

and either

$$\int^\infty \left(\int_t^\infty [y(s)]^{-2}ds\right)^{-1}[y(t)]^{-2}\left|\frac{p(t)}{q(t)}-1\right| dt < \infty \tag{4.14}$$

or

$$\int^\infty \left|\left(\frac{p(t)}{q(t)}\right)'\right| dt < \infty \tag{4.15}$$

hold. Then $x(t)\sim y(t)$ *as* $t\to\infty$.

THEOREM 4.6. *Let* $x(t)$ *and* $y(t)$ *be positive increasing solutions of equations* (1.1) *and* (2.1), *respectively, satisfying* (4.10). *Suppose that* (4.3), (4.11), (4.12),

$$\int^\infty \left(\int_t^\infty [y(s)]^{-2}ds\right)^{-1}[y(t)]^{-2}\left|q(t)[y(t)]^{3-\lambda}\left(\int_t^\infty [y(s)]^{-2}ds\right)^2 - l\right| dt < \infty,$$

and either (4.14) *or* (4.15) *hold. Then* $x(t)\sim y(t)$ *as* $t\to\infty$.

PROOF OF THEOREM 4.4. Put

$$\underline{L} = \liminf_{t\to\infty} \frac{x(t)}{y(t)}, \qquad \bar{L} = \limsup_{t\to\infty} \frac{x(t)}{y(t)}.$$

By our assumptions we know that $\underline{L} \geqq m$. Then, Lemma 4.2 yields

$$\bar{L} \leqq \limsup_{t\to\infty} \frac{x'(t)}{y'(t)} \leqq \limsup_{t\to\infty} \frac{p(t)}{q(t)}\left(\frac{y(t)}{x(t)}\right)^\lambda \leqq m^{-\lambda}.$$

This, in turn, implies that

$$\underline{L} \geqq \liminf_{t\to\infty} \frac{x'(t)}{y'(t)} \geqq \liminf_{t\to\infty} \frac{p(t)}{q(t)}\left(\frac{y(t)}{x(t)}\right)^\lambda \geqq m^{\lambda^2}$$

from Lemma 4.2, again. By continuing the above procedure, we obtain inductively two sequences $\{\underline{a}_n\}$ and $\{\bar{a}_n\}$ which satisfy

$$\underline{a}_n \leqq \underline{L} \leqq \bar{L} \leqq \bar{a}_n, \tag{4.16}$$
$$\underline{a}_{n+1} = \bar{a}_n^{-\lambda}, \qquad \bar{a}_n = \underline{a}_n^{-\lambda},$$

for all $n\in\boldsymbol{N}$, with $\underline{a}_1 = m$ and $\bar{a}_1 = m^{-\lambda}$. Since $0<\lambda<1$, an elementary computation shows that

$$\lim_{n\to\infty} \underline{a}_n = \lim_{n\to\infty} \bar{a}_n = 1 .$$

Thus (4.16) proves the validity of our theorem by letting $n\to\infty$. The proof is complete.

PROOF OF THEOREM 4.5. It is evident from the proof of Theorem 4.4 that

$$my(t) \leqq x(t) \leqq My(t), \qquad t \geqq t_0, \tag{4.17}$$

for some $M>0$. Set $v(t)=x(t)/y(t)$, $t\geqq t_0$. Since $v(t)$ satisfies (2.8), the change of variable

$$\tau = \left(\int_t^\infty [y(r)]^{-2} dr\right)^{-1}$$

implies that $v(\tau)$ satisfies

$$\frac{d^2v}{d\tau^2}+\frac{2}{\tau}\frac{dv}{d\tau}+q(t)[y(t)]^{3-\lambda}\tau^{-4}v = p(t)[y(t)]^{3-\lambda}\tau^{-4}v^{-\lambda}, \qquad \tau\geqq\tau_0,$$

for some $\tau_0>0$. Furthermore, by introducing the new independent variable $s=\log\tau$, this equation is reduced to

$$\ddot{v}+\dot{v}+\bar{q}(s)v = \bar{p}(s)v^{-\lambda}, \qquad s\geqq s_0=\log\tau_0, \tag{4.18}$$

where a dot $\cdot$ denotes differentiation with respect to s, and

$$\bar{q}(s) = q(t)[y(t)]^{3-\lambda}\tau^{-2},$$

$$\bar{p}(s) = p(t)[y(t)]^{3-\lambda}\tau^{-2}$$

for $s\geqq s_0$. Our conditions (4.11), (4.12), (4.13), (4.14) and (4.15) become

$$\lim_{s\to\infty}\frac{\bar{p}(s)}{\bar{q}(s)} = 1, \tag{4.19}$$

$$\lim_{s\to\infty}\bar{q}(s) = l, \tag{4.20}$$

$$\int^\infty |\dot{\bar{q}}(s)|\,ds < \infty, \tag{4.21}$$

$$\int^\infty \left|\frac{\bar{p}(s)}{\bar{q}(s)}-1\right| ds < \infty, \tag{4.22}$$

and

$$\int^\infty \left|\left(\frac{\bar{p}(s)}{\bar{q}(s)}\right)^{\cdot}\right| ds < \infty,$$

respectively. By (4.17) we have

$$m \leqq v(s) \leqq M, \qquad s\geqq s_0. \tag{4.23}$$

Since (4.19) and (4.20) prove that $\lim_{s\to\infty}\bar{p}(s)=l$, the argument used in the proof of Theorem 2.1 shows that $\dot{v}(s)$ is bounded on $[s_0, \infty)$. Therefore it follows from equation (4.18) that $\ddot{v}(s)$ is also bounded on $[s_0, \infty)$ by our conditions. Multiplying (4.18) by $\dot{v}(s)$, we get

$$\left(\frac{\dot{v}^2}{2}\right)^{\cdot}+\dot{v}^2+\bar{q}(s)\left(\frac{v^2}{2}-\frac{v^{1-\lambda}}{1-\lambda}\right)^{\cdot}=\bar{q}(s)\left(\frac{\bar{p}(s)}{\bar{q}(s)}-1\right)v^{-\lambda}\dot{v}\,, \qquad s\geqq s_0\,. \tag{4.24}$$

To complete the proof that $\lim_{s\to\infty} v(s)=1$ in the case of hypothesis (4.14), we integrate (4.24) over $[s_0, s]$ to obtain

$$\begin{aligned}\frac{[\dot{v}(s)]^2}{2}+\int_{s_0}^{s}[\dot{v}(r)]^2dr+\bar{q}(s)\left(\frac{[v(s)]^2}{2}-\frac{[v(s)]^{1-\lambda}}{1-\lambda}\right)&\\ -\int_{s_0}^{s}\dot{\bar{q}}(r)\left(\frac{[v(r)]^2}{2}-\frac{[v(r)]^{1-\lambda}}{1-\lambda}\right)dr&\\ =c_1+\int_{s_0}^{s}\bar{q}(r)\left(\frac{\bar{p}(r)}{\bar{q}(r)}-1\right)[v(r)]^{-\lambda}\dot{v}(r)dr\,, &\qquad s\geqq s_0\,,\end{aligned} \tag{4.25}$$

where $c_1\in \boldsymbol{R}$. Conditions (4.21) and (4.23) ensure the convergence of the second integral on the left hand side of (4.25) as $s\to\infty$. Similarly, (4.20), (4.22), and the boundedness of $\dot{v}(s)$ imply the convergence of the integral on the right hand side, from which we have

$$\int^{\infty}[\dot{v}(s)]^2ds<\infty\,.$$

Hence $\dot{v}(s)\to 0$ as $s\to\infty$ by Lemma 2.1. The remainder of the proof is completed in the same way as that of Theorem 2.1.

The proof under hypothesis (4.15) is the same as above. Therefore the detailed verification is left to the reader. The proof is finished.

Theorem 4.6 can be proved similarly, and hence the verification will be left to the reader.

Since the function

$$z(t)=(\lambda+1)^{1/(\lambda+1)}R(t)\,, \qquad t\geqq t_0\,,$$

solves the equation

$$z''=p(t)\left[\frac{R(t)}{t[P(t)]^{1/(\lambda+1)}}\right]^{\lambda}z^{-\lambda}\,, \qquad t\geqq t_1>t_0\,,$$

Lemma 4.1 and the proof of Theorem 4.4 show that the assumption $0<\lambda<1$ of Theorem 4.1 is superfluous if $a=1$ in (4.6). Therefore, we get the following result.

COROLLARY 4.1. *If* (4.3) *and* (4.6) *with* $a=1$ *hold, then any positive increasing solution* $x(t)$ *of* (1.1) *satisfies*

$$x(t)\sim(\lambda+1)^{1/(\lambda+1)}R(t) \qquad as\ t\to\infty\,.$$

Consider the case in which $p(t)$ behaves like a positive constant multiple of t^{α}, $\alpha-\lambda+1>0$. Lemma 4.1 shows that, for any positive increasing solution $x(t)$ of (1.1), $x(t)\geqq mt^{(\alpha+2)/(\lambda+1)}$, $t\geqq t_0$, for some $m>0$. On the other hand, it is easily seen that the singular Emden-Fowler equation

$$y''=ct^{\alpha}y^{-\lambda}, \qquad t\geqq t_0,$$

with $c>0$, admits a positive increasing solution $y(t)$ explicitly given by

$$y(t)=\left[\frac{c(\lambda+1)^2}{(\alpha+2)(\alpha-\lambda+1)}\right]^{1/(\lambda+1)}t^{(\alpha+2)/(\lambda+1)}, \qquad t\geqq t_0.$$

Clearly this $y(t)$ and $q(t)=ct^{\alpha}$, $t\geqq t_0$, satisfy (4.12). Therefore Theorems 4.4 and 4.5 give the next result, which can be regarded as an analogue to Corollary 2.2.

COROLLARY 4.2. *Let* $\alpha>\lambda-1$. *Suppose that* $p(t)$ *has the form* $p(t)=ct^{\alpha}[1+\varepsilon(t)]$, *with* c *a positive constant,* $\varepsilon(t)\to 0$ *as* $t\to\infty$.

(i) *Let* $0<\lambda<1$. *Then any positive increasing solution* $x(t)$ *of* (1.1) *satisfies*

$$x(t)\sim\left[\frac{c(\lambda+1)^2}{(\alpha+2)(\alpha-\lambda+1)}\right]^{1/(\lambda+1)}t^{(\alpha+2)/(\lambda+1)} \quad \text{as } t\to\infty. \tag{4.26}$$

(ii) *Let* λ *be arbitrary. Suppose moreover that either* (2.22) *or* (2.23) *holds. Then any positive increasing solution* $x(t)$ *of* (1.1) *behaves like* (4.26).

5. Examples.

Finally we present some examples illustrating the results obtained in the paper.

EXAMPLE 5.1. The results developed here and the known results due to Kusano and Swanson [8] enable us to classify all the positive solutions of the singular Emden-Fowler equation

$$x''=t^{\alpha}x^{-\lambda}, \qquad t\geqq 1, \quad \alpha\in\boldsymbol{R},$$

by means of their asymptotic behavior.

(i) Let $\alpha-\lambda+1<0$ and $\alpha+2<0$. Then each positive solution $x(t)$ has the asymptotic form either

$$x(t)\sim c,$$

or

$$x(t)\sim ct \tag{5.1}$$

as $t\to\infty$, for some $c>0$, or is given explicitly by

$$x(t)=\left[\frac{(\lambda+1)^2}{(\alpha+2)(\alpha-\lambda+1)}\right]^{1/(\lambda+1)}t^{(\alpha+2)/(\lambda+1)}.$$

(ii) Let $\alpha-\lambda+1<0$ and $\alpha+2\geqq 0$. Then any positive solution $x(t)$ has the asymptotic form (5.1) for some $c>0$.

(iii) Let $\alpha-\lambda+1=0$ and $\alpha+2>0$. Then any positive solution $x(t)$ has the asymptotic form

$$x(t) \sim (\lambda+1)^{1/(\lambda+1)} t(\log t)^{1/(\lambda+1)} \qquad \text{as } t \to \infty .$$

(This directly follows from Corollary 4.1.)

(iv) Let $\alpha-\lambda+1>0$ and $\alpha+2>0$. Then any positive solution $x(t)$ has the asymptotic form

$$x(t) \sim \left[\frac{(\lambda+1)^2}{(\alpha+2)(\alpha-\lambda+1)}\right]^{1/(\lambda+1)} t^{(\alpha+2)/(\lambda+1)} \qquad \text{as } t \to \infty .$$

EXAMPLE 5.2. This example is related to [**6**, Theorem 7] and [**7**].

Let $0<\lambda<1$ and $T \in (t_0, \infty)$ be fixed arbitrarily. We say that a positive function $x \in C^2[t_0, T)$ is a positive singular solution of (1.1) at T if it solves (1.1) on $[t_0, T)$ and satisfies

$$\lim_{t\to T-} x(t) = \lim_{t\to T-} x'(t) = 0. \tag{5.2}$$

It is proved in [**7**] that, under our basic conditions, for each $T>t_0$ (1.1) has a positive singular solution at T. We shall study the asymptotic representation of positive singular solutions near T with the help of the results in Section 2. In fact, we can show that the positive singular solution $x(t)$ at T satisfies

$$x(t) \sim \left[\frac{p(T)(\lambda+1)^2}{2(1-\lambda)}\right]^{1/(\lambda+1)} (T-t)^{2/(\lambda+1)} \qquad \text{as } t \to T- . \tag{5.3}$$

To see this we set

$$s = (T-t)^{-1} \quad \text{and} \quad x(t) = sv(s) .$$

Then (1.1) and the boundary condition (5.2) reduce to

$$\ddot{v} = s^{\lambda-3} p(t) v^{-\lambda} \qquad \text{for large } s \tag{5.4}$$

and

$$\lim_{s\to\infty} v(s) = \lim_{s\to\infty} \dot{v}(s) = 0 ,$$

respectively, where $\cdot = d/ds$. Since the coefficient function of (5.4) can be rewritten as

$$p(T) s^{\lambda-3}\left(1+\frac{p(t)-p(T)}{p(T)}\right),$$

our desired conclusion immediately follows from Corollary 2.1. It is to be noted that the above transformation is motivated by [**6**].

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