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Ideal points and incompressible surfaces in two-bridge knot complements

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§0. Introduction.

An ideal point will be a limit of representations of a fundamental group Γ of a three manifold in $PSL_2\boldsymbol{C}$. In [**6**] Culler and Shalen constructed incompressible surfaces of a three manifold from ideal points via Serre's tree. However, it is hard to understand Serre's tree from topological viewpoint because its definition is purely algebraic.

In this paper we construct Serre's trees concretely for two-bridge knot complements. We regard Serre's tree as the way how the geodesics in $\boldsymbol{H}^3$ fixed by γ ($\in\Gamma$) converge, as in Lemma 3.1. With this observation we can guess a rough shape of Serre's tree by describing geodesic in $\boldsymbol{H}^3$ with computer graphics for representations near an ideal point. This observation is also useful in determining shapes of trees with fine prospects. However, to give a precise proof we cannot use Lemma 3.1 and we need to construct trees step by step using Γ-tree's arguments. As the results, we classify the ideal points for two-bridge knot complements, and determine the complete correspondence between the ideal points and incompressible surfaces which are classified in [**7**].

In §1 we show that $PSL_2\boldsymbol{C}$ representation space of a two-bridge knot group is a punctured Riemann surface, as is studied by Riley in [**9**]. We use here the method of Burde [**2**], in which $SO(3)$ representation spaces are discussed. Section 2 recalls the definition of Serre's tree and the classification of incompressible surfaces of a two-bridge knot complement. In sections 3 and 4 we examine Serre's trees for the ideal points of a two-bridge knot complement. In §5 we summarize the main results. We apply the results to obtain a new proof of the fact that two-bridge knots have property P, which was proved by Takahashi [**12**] in 1981 and by Burde [**2**] in 1987. In §6 we give proofs of lemmas used in §4.

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§1. PSL_2C representation space.

Let K be a two-bridge knot of type (α, β) where α and β are co-prime odd integers and $0<\beta<\alpha$. We denote by R the union of $\dim_C \geqq 4$ components of the variety of all the representations of $\Gamma=\pi_1(S^3-K)$ in $PSL_2\boldsymbol{C}$. Since the space of abelian representations is three dimensional, each component of R contains an irreducible representation. The space of irreducible representations is Zariski open in R.

We shall study the space X which is defined as the set of conjugacy classes of representations in R. Γ is expressed by two generators and one relation, that is,

$$\Gamma = \langle u, v \mid wu = vw \rangle$$

with $w=u^{\varepsilon_1}v^{\varepsilon_2}\cdots u^{\varepsilon_{\alpha-2}}v^{\varepsilon_{\alpha-1}}$, $\varepsilon_j=(-1)^{[j\beta/\alpha]}$. This suggests that an element $[\rho]$ of X is identified with a pair of an eigenvalue of $\rho(u)$ and 'co-relation' of $U=\rho(u)$ and $V=\rho(v)$. We mean by the co-relation the *complex distance* defined below.

Let $U, V \in PSL_2\boldsymbol{C}$ be conjugate to each other and suppose that eigenvalues are not equal to ± 1, $\pm\sqrt{-1}$. There are conjugate lifts $\tilde{U}$, $\tilde{V} \in SL_2\boldsymbol{C}$, and they can be expressed as

$$\tilde{U} = c+sP$$
$$\tilde{V} = c+sQ$$

with matrices P, $Q \in SL_2\boldsymbol{C}$ such that trace P = trace $Q=0$ and with scalars $c=(\tilde{\mu}+\tilde{\mu}^{-1})/2$, $s=(\tilde{\mu}-\tilde{\mu}^{-1})/(2\sqrt{-1})$ where $\tilde{\mu}$ is an eigenvalue of $\tilde{U}$.

DEFINITION 1.1. The complex distance τ between U and V is defined by $\tau=-(1/2)\operatorname{trace} PQ$.

Though P and Q have the ambiguity of sign ± 1, the complex distance τ depends only on U and V. τ is also characterized by $\tau=(1/2)(\operatorname{trace} g)^2-1$ where g is a matrix such that $V=gUg^{-1}$ and the geodesic fixed by g in $\boldsymbol{H}^3$ is vertical to the geodesics fixed by U and V. Therefore, τ decides the mutual position of U and V.

Some calculations prove the following lemma.

LEMMA 1.2.
(i) $P^2=Q^2=-1$
(ii) $PQ+QP=-2\tau$
(iii) $PQP=-2\tau P+Q, \qquad QPQ=-2\tau Q+P$
(iv) $c^2+s^2=1$
(v) $\tilde{U}^{-1}=c-sP, \qquad \tilde{V}^{-1}=c-sQ$

PROOF. (i) is concluded by the fact that trace P=trace $Q=0$ and $P, Q \in SL_2C$. In a similar way we have

$$(PQ)^2+2\tau PQ+1=0$$

because trace $PQ=-2\tau$ and $PQ \in SL_2C$. Using (i) we obtain (ii) and (iii). The other formulas are easily checked. ■

X is identified with the set

$$\{(U, V) \in PSL_2\boldsymbol{C} \times PSL_2\boldsymbol{C} \mid WU=VW\}/\text{conj}$$

with $W=U^{\varepsilon_1}V^{\varepsilon_2}\cdots U^{\varepsilon_{\alpha-2}}V^{\varepsilon_{\alpha-1}}$, and this set is parametrized by the pair of $\xi = -2/((\text{trace}\, U)^2-4)$ and the complex distance τ between U and V. We want to know the relation between τ and ξ coming from the equation $WU=VW$. The calculation can be done by the method of Burde [**2**] who calculated $SU(2)$ representation spaces of two-bridge knot complements.

Next lemma will be sufficient for the purpose.

LEMMA 1.3 ([**2**]). *Put* $\tilde{W}=\tilde{U}^{\varepsilon_1}\tilde{V}^{\varepsilon_2}\cdots\tilde{U}^{\varepsilon_{\alpha-2}}\tilde{V}^{\varepsilon_{\alpha-1}}$, $k=(\alpha-1)/2$ *then*

$$\tilde{W}\tilde{U}-\tilde{V}\tilde{W}=s^\alpha z(\tau, \xi)(P-Q)$$

with

$$z(\tau, \xi)=(2\xi)^k+a_1(\tau)(2\xi)^{k-1}+\cdots+a_k(\tau)$$

$$a_j(\tau) \in \boldsymbol{Z}[\tau], \qquad \deg a_j(\tau) \leqq j, \qquad \deg a_k(\tau)=k.$$

PROOF. From the definition of $\tilde{W}$,

$$\tilde{W}=s^{\alpha-1}(\gamma+\varepsilon_1 P)(\gamma+\varepsilon_2 Q)\cdots(\gamma+\varepsilon_{\alpha-1}Q)$$

with $\gamma=c/s$. Since $\tilde{W}\tilde{U}=\tilde{V}\tilde{W}=s(\tilde{W}P-Q\tilde{W})$, it is sufficient to show $LP-QL=z(\tau, \xi)(P-Q)$ with $L=(\gamma+\varepsilon_1 P)(\gamma+\varepsilon_2 Q)\cdots(\gamma+\varepsilon_{\alpha-1}Q)$. Using Lemma 1.2 (i), LP is expanded to the sum of the form $\pm\gamma^{\alpha-1-r}\varepsilon_{i_1}\cdots\varepsilon_{i_r}(PQ)^jP\cdot P$ (resp. $\pm\gamma^{\alpha-1-r}\varepsilon_{i_1}\cdots\varepsilon_{i_r}(PQ)^jP$) for r odd (resp. even). To these there are corresponding terms of QL: $\pm\gamma^{\alpha-1-r}\varepsilon_{\alpha-i_1}\cdots\varepsilon_{\alpha-i_r}Q(QP)^jQ$ (resp. $\pm\gamma^{\alpha-1-r}\varepsilon_{\alpha-i_1}\cdots\varepsilon_{\alpha-i_r}Q(QP)^j$). Symmetry $\varepsilon_j=\varepsilon_{\alpha-j}$ shows that the terms for r odd cancel. Since $\gamma^2=2\xi-1$, $LP-QL$ is expressed by the sum of $(PQ)^jP-Q(PQ)^j$ with polynomial coefficients in 2ξ. Using Lemma 1.2 (iii) inductively, $(PQ)^jP-Q(PQ)^j$ becomes $P-Q$ with polynomial coefficient in τ. Hence $LP-QL=z(\tau, \xi)(P-Q)$ is shown, and by some more consideration, additional conditions are confirmed. ■

REMARK 1.4. When α is even, K is a two-bridge link, and in general the space of conjugacy classes of representations is not a Riemann surface. However, consider only representations for which $\rho(u)$ is conjugate to $\rho(v)$, and with slight changes Lemma 1.3 remains valid for X which is defined to be the set of such representations in the same way as α odd.

Looking over the proof of Lemma 1.3, Γ is replaced by $\Gamma=\langle u, v \mid wu=uw\rangle$ with $w=v^{\varepsilon_1}u^{\varepsilon_2}\cdots u^{\varepsilon_{\alpha-2}}v^{\varepsilon_{\alpha-1}}$, $\varepsilon_j=(-1)^{[j\beta/\alpha]}$. Hence $LP-PL$ is expressed by the sum of $(PQ)^j-(QP)^j$ with polynomial coefficients in 2ξ. Therefore

$$\widetilde{W}\widetilde{U}-\widetilde{U}\widetilde{W} = s^{\alpha}z(\tau, \xi)(PQ-QP)$$

and remaining statements is also valid for $k=(\alpha-2)/2$. ∎

For each $(p, q)\in \boldsymbol{Z}\times\boldsymbol{Z}$, there is a regular function $I_{p,q}\colon X\to \boldsymbol{C}$ defined by $I_{p,q}([\rho])=\{\text{trace}\,\rho(m^p l^q)\}^2-4$ where m is meridian and l is longitude of K. Since X is a Riemann surface defined by $z(\tau, \xi)=0$, the degree of $I_{p,q}$ is defined. We denote this degree by $\Phi(p, q)$.

After [4], we call elements of $\bar{X}-X$ *ideal points*, and denote by $\phi_x(p, q)$ the order of the pole of $I_{p,q}$ at x. Then $\Phi(p, q)=\sum_x \phi_x(p, q)$, and as a result of Lemma 1.3, $\Phi(1, 0)=(\alpha-1)/2$.

§ 2. Serre's trees and incompressible surfaces.

As in [11] we shall define Serre's tree T. Let F be a field with a valuation $v\colon F^*\to\boldsymbol{Z}$. In this paper we assume that F is complete with respect to v.

Let $\mathcal{O}$ denote the valuation ring of F, $\mathfrak{m}$ the maximal ideal, and $\mathfrak{k}$ the residue field. We define a *lattice* to be any rank 2 $\mathcal{O}$-submodule of $F\oplus F$. F^* acts on the set of lattices. Define the vertex set of T to be the orbit space of this action. Two vertices Λ, Λ' are joined by an edge if and only if there exist lattices L and L' in Λ and Λ' such that $L'\subset L$ and $L/L'\approx\mathfrak{k}$. Let a length of each edge be one.

Since some argument shows connectedness and simply connectedness of T, T is a tree. The natural $GL_2(F)$ action on T is transitive. We call $\mathcal{O}\oplus\mathcal{O}$ the *origin* of T where $\mathcal{O}\oplus\mathcal{O}$ is the standard lattice of $F\oplus F$. The stabilizer of $GL_2(F)$ action at the origin is $GL_2(\mathcal{O})$.

Next we study the shape of T in more detail. Let $\mathcal{L}_n$ be the set of vertices the distance to which from the origin is equal to n. We construct a $GL_2(\mathcal{O})$ equivariant bijection from $\mathcal{L}_n$ to $P^1(\mathcal{O}/\mathfrak{m}^n)$: the set of direct factors of $\mathcal{O}/\mathfrak{m}\oplus\mathcal{O}/\mathfrak{m}$ of rank 1, as follows. Let Λ be an element of $\mathcal{L}_n$. We can choose a lattice L in Λ such that $(\mathcal{O}\oplus\mathcal{O})/L\approx\mathcal{O}/\mathfrak{m}^n$. The image of L by the projection $\phi\colon \mathcal{O}\oplus\mathcal{O}\to\mathcal{O}/\mathfrak{m}^n\oplus\mathcal{O}/\mathfrak{m}^n$ is an element of $P^1(\mathcal{O}/\mathfrak{m}^n)$. Conversely, given an element of $P^1(\mathcal{O}/\mathfrak{m}^n)$, we can obtain a vertex in $\mathcal{L}_n$ by pulling back of the element by ϕ.

The path from a vertex in $\mathcal{L}_n$ to the origin is obtained by natural map

$$P^1(\mathcal{O}/\mathfrak{m}^n)\longrightarrow P^1(\mathcal{O}/\mathfrak{m}^{n-1})\longrightarrow\cdots\longrightarrow P^1(\mathcal{O}/\mathfrak{m}).$$

It follows that an *end* of T (i.e. an infinite path from the origin) corresponds to an element of $\varprojlim P^1(\mathcal{O}/\mathfrak{m}^n)=P^1(\mathcal{O})$.

Now, let s be a holomorphic local section of $R\to X$ in a neighbourhood of an ideal point x. We denote by δ a local coordinate around x such that $\delta=0$ at x. By lifting s and δ, we obtain a local section $\tilde{s}$ of $\mathrm{Hom}(\Gamma, SL_2\boldsymbol{C})\to \mathrm{Hom}(\Gamma, SL_2\boldsymbol{C})/\mathrm{conj}$ and a coordinate $\tilde{\delta}$ in $\mathrm{Hom}(\Gamma, SL_2\boldsymbol{C})/\mathrm{conj}$ such that $\tilde{\delta}^2=\delta$. Put $\rho_{\tilde{\delta}}=\tilde{s}(\tilde{\delta})$, then $\rho_{-\tilde{\delta}}(\gamma)=\pm\rho_{\tilde{\delta}}(\gamma)$ holds for each $\gamma\in\Gamma$. Therefore, either $\rho_{\tilde{\delta}}$ or $\tilde{\delta}\cdot\rho_{\tilde{\delta}}(\gamma)$ belongs to $GL_2(F)$ where $F=\boldsymbol{C}((\delta))$. Regarding them as elements of $PGL_2(F)$, we obtain the canonical representation $\Gamma\to PGL_2(F)$.

A valuation is determined by the order of zero at the ideal point. The tree T is constructed as above. Since the action of F^* is trivial, the group $PGL_2(F)$ acts on T. Composing the above tautological representation, we obtain an action of Γ on T.

Through this action, Culler and Shalen [**6**] constructs an incompressible surface of S^3-K as follows. To begin with, if the action has an inversion (i. e. there exists an invariant edge which change its orientation), we reform T, changing the midpoints of all edges to vertices. Then we may assume that there exists a subtree T' which contains exactly one edge from each orbit of the action of Γ on the edges of T. Given an edge e of T/Γ, there exists the edge $\tilde{e}$ of T' which projects to e under the quotient map. For each edge e, let X_e be the space of type $K(\Gamma_{\tilde{e}}, 1)$ where $\Gamma_{\tilde{e}}$ is the stabilizer of $\tilde{e}$. For each vertex v, we choose a vertex $\tilde{v}$ of T' which projects to v, and let X_v be the space of type $K(\Gamma_{\tilde{v}}, 1)$. Consider all the spaces $X_e\times[0, 1]$ and X_v. If $\tilde{e}$ is an edge with vertex $\gamma\tilde{v}$, we embed $\Gamma_{\tilde{e}}$ into $\Gamma_{\tilde{v}}$ by sending x to $\gamma x\gamma^{-1}$, and attach $X_e\times\{0\}$ (or $\{1\}$) to X_v by the induced map. We denote by K_T the resulting space. The natural isomorphism $\Gamma\to\pi_1(K_T)$ is called the *splitting* of Γ. A stabilizer of an edge (resp. vertex) is called an *edge group* (resp. a *vertex group*). The splitting will be non-trivial if all the vertex groups are proper subgroups.

When there exists a non-trivial splitting of $\Gamma=\pi_1(S^3-K)$, we can construct a nonempty system $\Sigma=\Sigma_1\cup\Sigma_2\cup\cdots\cup\Sigma_m$ of incompressible surfaces in S^3-K by pulling back midpoints of the edges by a map $S^3-K\to K_T$, such that $\mathrm{im}(\pi_1(\Sigma_j)\to\Gamma)$ is contained in an edge group for $j=1, \cdots, m$ and $\mathrm{im}(\pi_1(C)\to\Gamma)$ is contained in a vertex group for each component C of $(S^3-K)-\Sigma$.

Hatcher and Thurston [**7**] classified incompressible surfaces in two-bridge knot complements. In order to state their result, we shall give some notations.

We express a continued fraction expansion of β/α as

$$\frac{\beta}{\alpha}=r+[n_1, n_2, \cdots, n_N]=r+\cfrac{1}{n_1+\cfrac{1}{n_2+\cfrac{}{\ddots+\cfrac{1}{n_N}}}}, \qquad r,\ n_j\in \boldsymbol{Z}.$$

In this paper we consider only a continued fraction expansion with $|n_j|\geqq 2$, $(j=1, \cdots, N)$. As is well-known, the two-bridge knot K of type (α, β) is the boundary of the surface obtained by plumbing together N bands as shown in Fig. 2.1.

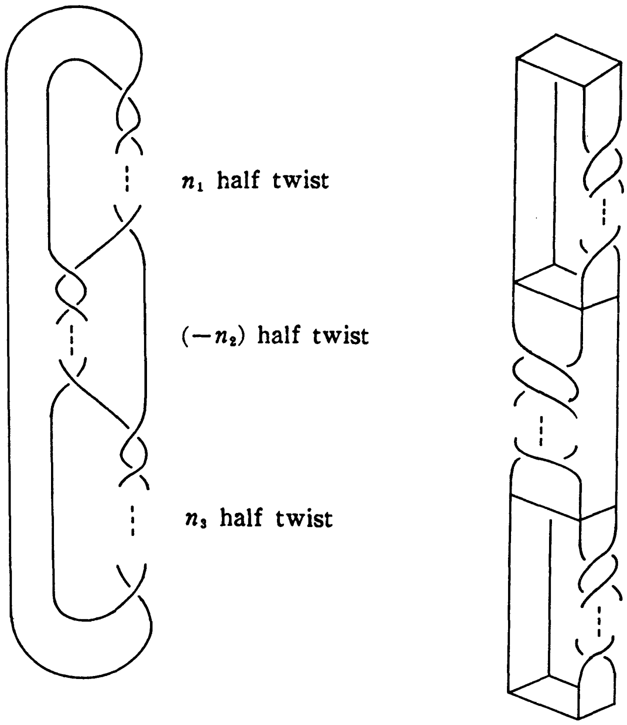


Fig. 2.1.

Now we shall define the surface labelled $S_n(m_1, \cdots, m_{N-1})$, where $n\geqq 1$ and $0\leqq m_j\leqq n$. Let $S_n(m_1, \cdots, m_{N-1})$ be a surface which consists of n parallel sheets running close to the vertical portions of each band and bifurcates into m_j outer sheets and $n-m_j$ inner sheets at j-th plumbing square as shown in Fig. 2.2.

We regard S^3 as the two points compactification of $S^2\times \boldsymbol{R}$ with the spheres $S^2\times\{*\}$ being horizontal, and the outer sheets will be compactified at infinity.

THEOREM 2.1 (weak form of [7]).

(i) *Each* $S_n(m_1, \cdots, m_{N-1})$ *is incompressible.*

(ii) *Any orientable incompressible surface in* S^3-K *is isotopic to either one of the surfaces* $S_n(m_1, \cdots, m_{N-1})$ *or a* ∂*-parallel annulus.*

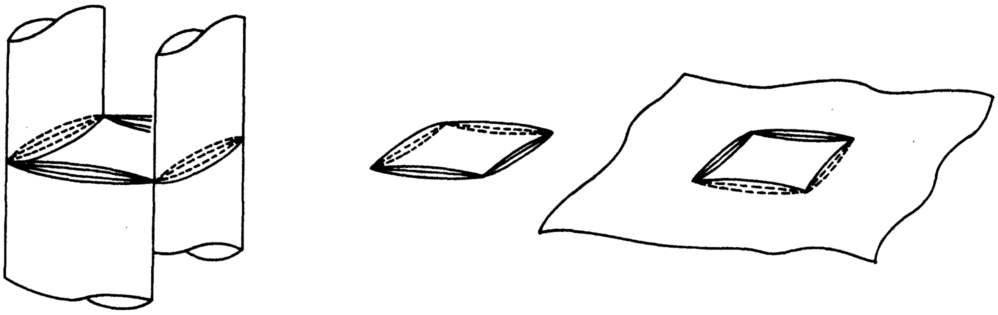


Fig. 2.2.

REMARK 2.2. For two-bridge links, Theorem 2.1 applies to those surfaces meeting both components of the link in the same number of sheets.

Each boundary slope of a surface $S_n(m_1, \cdots, m_{N-1})$ is $m^{N_{[n_j]}}l$ with meridian m and longitude l. Function $N_{[n_j]}$ is given by

$$N_{[n_j]} = 2\{(n^+ - n^-) - (n_0^+ - n_0^-)\}$$

where $n^+ = \#\{(-1)^{j+1}n_j > 0\}$, $n^- = \#\{(-1)^{j+1}n_j < 0\}$ and n_0^+ and n_0^- are the corresponding numbers for the unique continued fraction expansion $\beta/\alpha = r' + [n'_1, \cdots, n'_{N'}]$ with each n'_j even.

§ 3. Trees of two-bridge knot complements.

In § 2, we have Serre's tree T on which Γ acts. As is known in the theory of Γ-trees, if $\|\gamma\| > 0$ then we can find unique invariant axis C_γ in T, where translation function $\|\gamma\|$ is defined by

$$\|\gamma\| = \inf_{x \in T} \operatorname{dist}(x, \gamma x)$$

with $\gamma \in \Gamma$. The axis C_γ is unique invariant set of γ which is homeomorphic to $\boldsymbol{R}$. Further in § 2 we have a family of representations $\rho_{\tilde{\delta}}$ of Γ into SL_2C, such that $\rho_{-\tilde{\delta}}(\gamma) = \pm \rho_{\tilde{\delta}}(\gamma)$ for each $\gamma \in \Gamma$.

If $\|\gamma\| > 0$, matrix $\rho_{\tilde{\delta}}(\gamma)$ has eigenvalues $\tilde{\delta}^n$ and $\tilde{\delta}^{-n}$ with $n = \|\gamma\|$. Hence for sufficiently small fixed $\tilde{\delta}$, $\rho_{\tilde{\delta}}(\gamma)$ is hyperbolic in $SL_2\boldsymbol{C}$ and has unique geodesic in the three dimensional hyperbolic space $\boldsymbol{H}^3$. We denote by $\rho_{\tilde{\delta}}(\gamma)^+$ and $\rho_{\tilde{\delta}}(\gamma)^-$ ends of this geodesic in $P^1(\boldsymbol{C})$. Since $\rho_{-\tilde{\delta}}(\gamma) = \pm \rho_{\tilde{\delta}}(\gamma)$, these ends depend on $\delta = \tilde{\delta}^2$ and these are regarded as in $P^1(\boldsymbol{C}[[\delta]])$.

By definition of the action of Γ on Serre's tree T, we naturally have the next lemma.

LEMMA 3.1. *The following formulas hold.*

$$\gamma^+ = \rho_{\tilde{\delta}}(\gamma)^+$$

$$\gamma^- = \rho_{\bar{\delta}}(\gamma)^-$$

PROOF. Matrix $\rho_{\bar{\delta}}(\gamma)$ has a fixed point $[f, g]$ in $P^1(\boldsymbol{C}[[\delta]])$. For fixed δ, this is $[f_\delta, g_\delta]$ in $P^1(\boldsymbol{C})$. These are equal to $\gamma^\pm$ and $\rho_{\bar{\delta}}(\gamma)^\pm$ respectively by definition. Further both γ^+ and $\rho_{\bar{\delta}}(\gamma)^+$ are corresponding to the greater eigenvalue of $\rho_{\bar{\delta}}(\gamma)$. So we obtain this lemma. ■

Now we consider the case $\beta=1$ i.e. the case of $(\alpha, 1)$ two-bridge knot complement, before discussing the general case. In this case K is the torus knot of type $(\alpha, 2)$, and we can construct concrete representations in a neighbourhood of an ideal point.

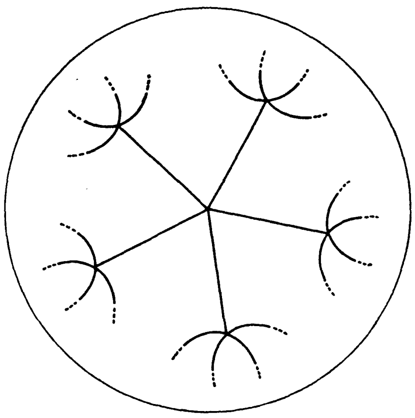


Fig. 3.1.

To begin with, we give a tree parametrized by complex number ν in $\boldsymbol{H}^3$, as shown in Fig. 3.1. Each edge is geodesic of length $|\nu|$. α edges meet at

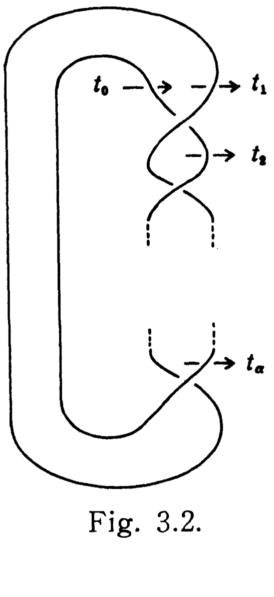


Fig. 3.2.

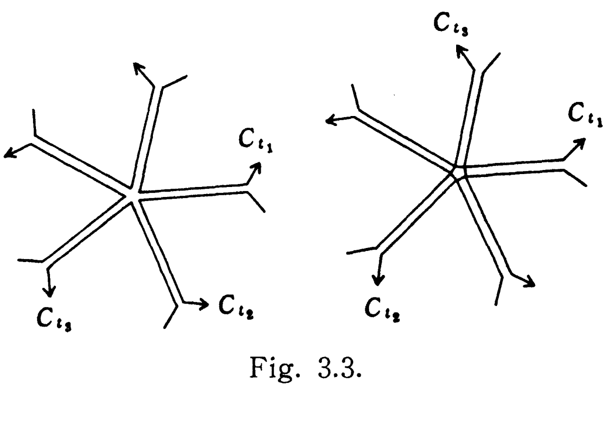


Fig. 3.3.

each vertex, α tangent vectors of which belong to the same plane in the tangent space of the vertex and any two adjoining vectors make an angle $2\pi/\alpha$. When $\nu\in\boldsymbol{R}$, this tree is contained in $\boldsymbol{H}^2$, and for general ν, any two tangent spaces at adjoining vertices make an angle equal to the argument of ν.

Let generators t_i of Γ be as illustrated in Fig. 3.2. There exist $(\alpha-1)/2$ actions of Γ on the tree shown in Fig. 3.3, where C_{t_i} is the invariant axis of t_i. Each C_{t_i} bends at each vertex with an angle $2k\pi/\alpha$. Each of these actions on the tree induces isometries on a neighbourhood of the tree and extends to the action on $\boldsymbol{H}^3$. Finally, depending on k, we obtain $(\alpha-1)/2$ families of representations parametrized by ν. Further each ideal point is a simple pole i. e. $\phi(1, 0)=1$. Because an eigenvalue of $\rho(m)$ has a pole which has the same order as $\sqrt{\nu}$ and $\phi(1, 0)$ is an order of $I_{1,0}=(\text{trace}\,\rho(m))^2-4$. Since we have $\Phi(1, 0)=(\alpha-1)/2$ in §1, all the ideal points are obtained above.

Put a coordinate $\delta=\nu^{-1}$ in a neighbourhood of an ideal point, then the ends of C_{t_i} are in $\partial\boldsymbol{H}^3=P^1(\boldsymbol{C})$ depending on δ. Regarding these as elements of $P^1(\boldsymbol{C}[[\delta]])$, above trees are just Serre's trees obtained from the ideal points.

Now we consider the general case. With Lemma 3.1 we can guess a rough shape of T by calculating concrete representations near an ideal point. However, to give a precise proof we need to proceed using Γ-tree's argument. Let a_j, b_j, c_j, $d_j(\in\Gamma)$ be as illustrated in Fig. 3.4, which shows a plumbing representation of the two-bridge knot of type (α, β). We call j-th twisting part T_j.

We shall search for the ideal points which produce incompressible surfaces of this plumbing representation. As is shown in Fig. 3.5, b_ja_j and b_1d_{j-1} do not cross incompressible surfaces. Therefore, when an incompressible surface associated with $[n_j]$ is constructed from Γ-tree, b_ja_j and b_1d_{j-1} belong to the vertex group.

Here, in a similar way for the proof of Lemma 2.6 in [**5**], we have the following lemma.

LEMMA 3.2. *If* $\|\gamma_1\|=\|\gamma_2\|>0$ *and* $\|\gamma_1\gamma_2^{-1}\|=0$, *then* $|C_{\gamma_1}\cap C_{\gamma_2}|\geqq\|\gamma_1\|$ *where we denote by* $|\cdot|$ *the length of the segment.*

PROOF. Suppose both $\|\gamma_1\|=\|\gamma_2\|>0$ and $|C_{\gamma_1}\cap C_{\gamma_2}|<\|\gamma_1\|$. By pursuing the images of $C_{\gamma_1}\cap C_{\gamma_2}$ by $(\gamma_1\gamma_2^{-1})^n$ as shown in Fig. 3.6, we obtain $\|\gamma_1\gamma_2^{-1}\|=2(\|\gamma_1\|-|C_{\gamma_1}\cap C_{\gamma_2}|)>0$. ■

In the present case, meridian m satisfies $\|m\|>0$. This is because if m has a fixed point in T, replacing it by its conjugation if necessary, we can assume the fixed point is the standard lattice $\mathcal{O}\oplus\mathcal{O}$, then $\{\text{trace}\,\rho(m)\}^2$ does not have a pole at the ideal point and τ can be obtained as in §1. This contradicts for the definition of ideal points.

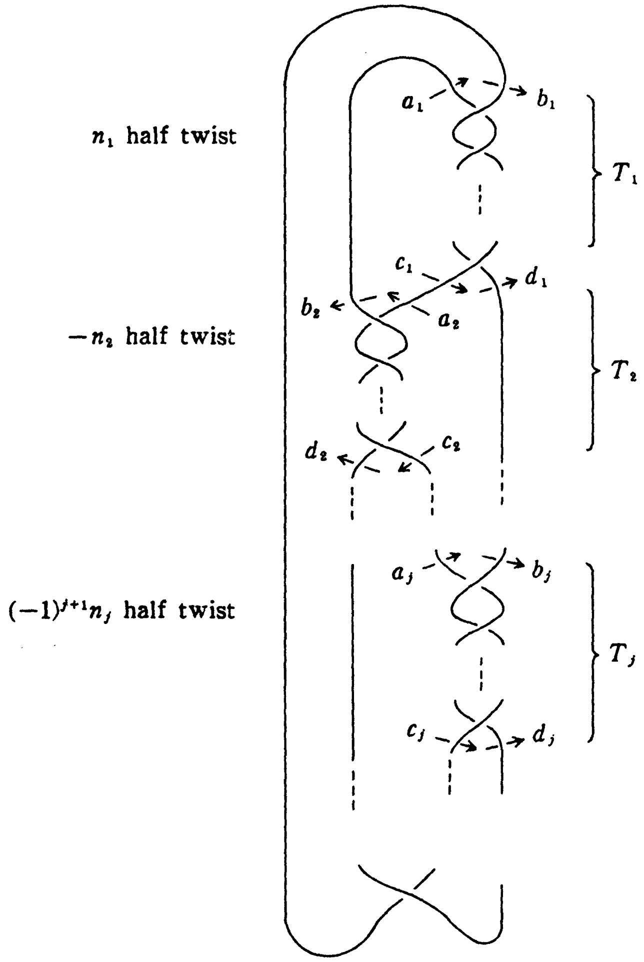


Fig. 3.4.

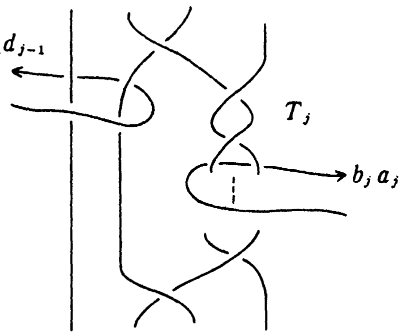


Fig. 3.5.

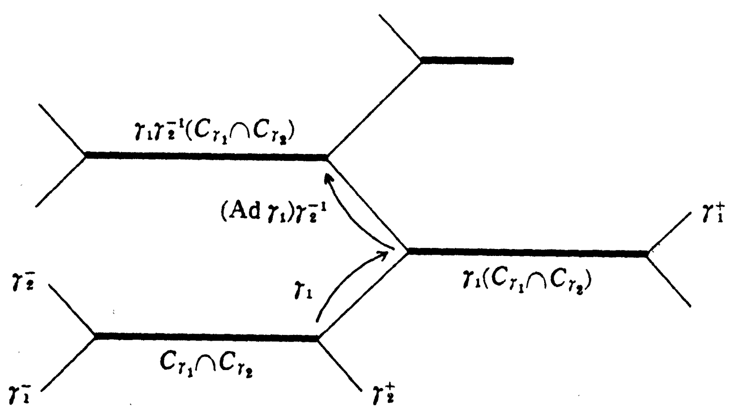


Fig. 3.6.

Since a pair b_j and a_j^{-1} satisfies the assumptions of Lemma 3.2, C_{a_j} and C_{b_j} have an intersection of length $\geqq \|m\|$. Furthermore we have the following lemma.

LEMMA 3.3. *Let τ be the complex distance between $\rho(a)$ and $\rho(b)$. If $\tau-1\in O(\delta^n)$ and $\notin o(\delta^n)$ i.e. $v(\tau-1)=n$, then $|C_a \cap C_b|=n$ where δ is the coordinate around the ideal point.*

To prove Lemma 3.3 we prepare the next lemma.

LEMMA 3.4. *Let τ be the complex distance between $\rho(a)$ and $\rho(b)$. Then the following formulas hold, where we regard an end as power series in δ through a map φ: $\boldsymbol{C}[[\delta]]\to P^1(\boldsymbol{C}[[\delta]])$ which maps f to $[f, 1]$.*

$$(a^+-a^-)(b^+-b^-)\frac{\tau+1}{2}=-(a^+-b^-)(a^--b^+)$$

$$(a^+-a^-)(b^+-b^-)\frac{-\tau+1}{2}=-(a^+-b^+)(a^--b^-)$$

PROOF.

$$P=\frac{\sqrt{-1}}{a^+-a^-}\begin{pmatrix} a^+ \\ 1 \end{pmatrix}(1-a^-)-\frac{\sqrt{-1}}{a^+-a^-}\begin{pmatrix} a^- \\ 1 \end{pmatrix}(1-a^+)$$

is the matrix with eigenvalues $\pm\sqrt{-1}$ and its fixed geodesic in $\boldsymbol{H}^3$ has ends a^+, a^-. About b^+, b^-, the following matrix plays a role similar to P.

$$Q=\frac{\sqrt{-1}}{b^+-b^-}\begin{pmatrix} b^+ \\ 1 \end{pmatrix}(1-b^-)-\frac{\sqrt{-1}}{b^+-b^-}\begin{pmatrix} b^- \\ 1 \end{pmatrix}(1-b^+)$$

By definition, $\tau=-(\operatorname{trace} PQ)/2$. These give the required formulas. ∎

PROOF OF LEMMA 3.3. We may assume a^+ is different from each of a^-, b^+ and b^- in the link of the origin $O=[\mathcal{O}\oplus\mathcal{O}]$ i.e. in the image of the projection $P^1(\boldsymbol{C}[[\delta]])\to P^1(\boldsymbol{C})$ (which corresponds the projection $\boldsymbol{C}[[\delta]]\to\boldsymbol{C}$ through φ). This is because, if necessary, we can take conjugation in $GL_2(F)$ such that a^+ is close to O.

Then we have $v(a^+-a^-)=v(a^+-b^+)=0$. Hence by the second formula of Lemma 3.4 we obtain

$$v(a^--b^-)-v(b^+-b^-)=v(\tau-1)=n.$$

Here, $v(a^--b^-)$ is equal to the distance between O and a branch point of a^- and b^-, and similar for $v(b^+-b^-)$. Therefore we have $|C_a\cap C_b|=n$. ∎

When $t_{j,k}$'s are as in illustrated in Fig. 3.7, $t_{j,0}$ and $t_{j,2}$ are conjugate by $t_{j,1}$, and so $\rho(t_{j,1})$ maps the fixed geodesic by $\rho(t_{j,0})$ in $\boldsymbol{H}^3$ to that of $\rho(t_{j,2})$. Hence the mutual position between two geodesics fixed by $\rho(t_{j,0})$ and $\rho(t_{j,1})$ are equal to the mutual position between two geodesics fixed by $\rho(t_{j,1})$ and $\rho(t_{j,2})$. Therefore complex distance between $\rho(t_{j,0})$ and $\rho(t_{j,1})$, $\rho(t_{j,1})$ and $\rho(t_{j,2})$ are equal to each other and so we have $|C_{t_{j,0}} \cap C_{t_{j,1}}| = |C_{t_{j,1}} \cap C_{t_{j,2}}|$. By the same argument we obtain $|C_{t_{j,0}} \cap C_{t_{j,1}}| = |C_{t_{j,1}} \cap C_{t_{j,2}}| = |C_{t_{j,2}} \cap C_{t_{j,3}}| = \cdots$.

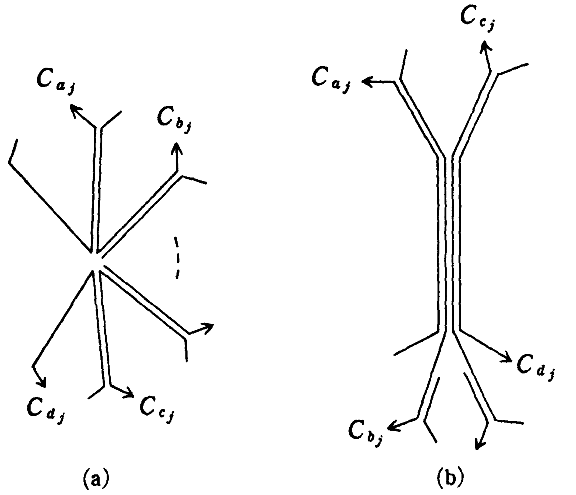


Fig. 3.8.

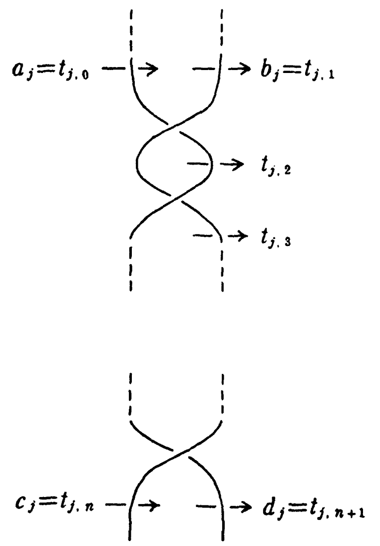


Fig. 3.7.

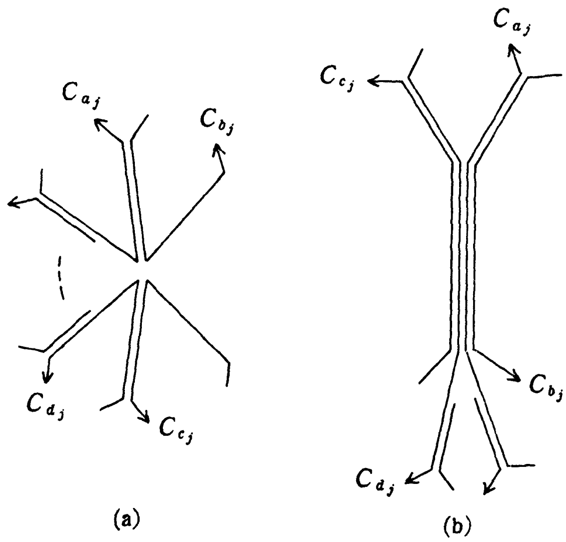


Fig. 3.9.

It follows that the arrangement of C_{a_j}, C_{b_j}, C_{c_j}, C_{d_j} is as shown in Fig. 3.8 (resp. Fig. 3.9) for $n_j > 0$ (resp. $n_j < 0$). In the case (a) of these figures (i.e. the case $|C_{a_j} \cap C_{b_j}| = \|m\|$), we call T_j being of *parasol* type, and (b) (i.e. the case $|C_{a_j} \cap C_{b_j}| > \|m\|$), being of *broom* type, respectively. We let a *center* of T_j be a center of the parasol or a midpoint of the broomstick.

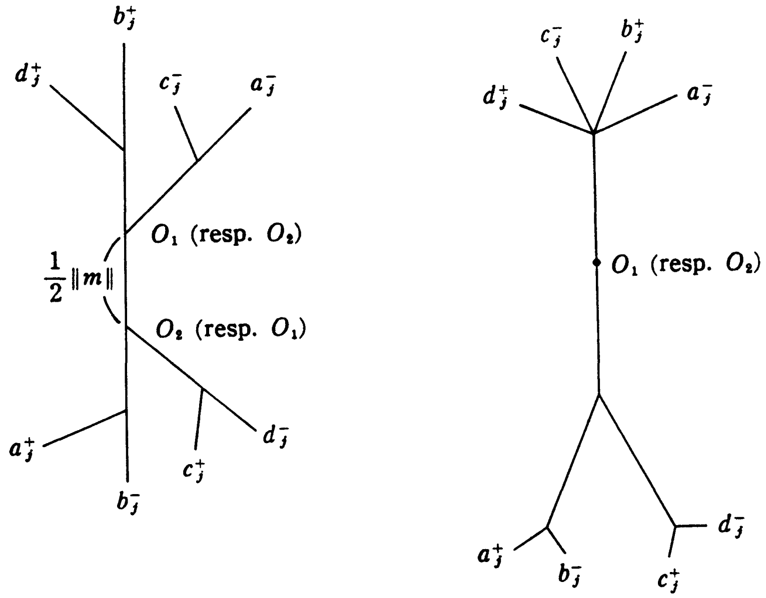


Fig. 3.10.

For some time, we assume Proposition 3.5 (i) below, and define some notations. When j odd (resp. even), the arrangement of axes is as shown in Fig. 3.10 where we denote by a_j^+, a_j^- the ends of C_{a_j} such that the action of a_j is a translation from a_j^- to a_j^+, as is the same with b_j^+, b_j^-, $\cdots$. We denote by δ a coordinate around the ideal point. Each end can be regarded as an element of $P^1(\boldsymbol{C}[[\delta]])$ depending on a choice of standard lattice $\mathcal{O}\oplus\mathcal{O}$ in § 2 i.e. we have an ambiguity of a choice of the canonical representation of Γ into $PGL_2(F)$ by taking conjugation in $PGL_2(F)$. Put $\varepsilon=\sqrt{-1}\mu^{-1}$ with an eigenvalue μ of $\rho(m)$, and we express each end by power series of ε instead of δ, that is, each end belongs to $P^1(\boldsymbol{C}[[\varepsilon^{2/n}]])$ where $v((\text{trace } \rho(m))^2)=-n$.

We take coordinates of $P^1(\boldsymbol{C}[[\varepsilon^{2/n}]])$ as follows.

$$\begin{array}{lccc} & \boldsymbol{C}[[\varepsilon^{2/n}]] & \longrightarrow & P^1(\boldsymbol{C}[[\varepsilon^{2/n}]]) \\ \text{the first coordinate} & f & \longmapsto & [f, 1] \\ \text{the second coordinate} & f & \longmapsto & [1, f] \end{array}$$

From now on, we express a_j^+, b_j^-, c_j^+, d_j^- by the first coordinate with origin O_1 (resp. O_2) and a_j^+, b_j^-, c_j^+, d_j^- by the second coordinate, such that in the link of origin: $P^1(\boldsymbol{C})$, O_2 (resp. O_1) indicates zero of the second coordinate. Since

$$a_j = c_{j-1}^{-1}, \qquad b_j = d_{j-2}, \qquad c_j = a_{j+1}^{-1}, \qquad d_j = b_{j+2}\,,$$

we obtain

$$a_j^+ = \varepsilon c_{j-1}^-, \quad b_j^+ = d_{j-2}^+, \quad c_j^+ = \varepsilon a_{j+1}^-, \quad d_j^+ = b_{j+2}^+,$$
$$a_j^- = \varepsilon^{-1} c_{j-1}^+, \quad b_j^- = d_{j-2}^-, \quad c_j^- = \varepsilon^{-1} a_{j+1}^+, \quad d_j^- = b_{j+2}^-,$$

remarking that the coordinate is multiplied by ε when origin slides by a length $\|m\|/2$.

By Fig. 3.10, if T_j is of parasol type, then

$$b_j^+ - a_j^- \in O(\varepsilon^0), \qquad \notin o(\varepsilon^0),$$
$$b_j^- - a_j^+ \in O(\varepsilon^2), \qquad \notin o(\varepsilon^2).$$

Hence we can define non-zero complex numbers A_j, B_j by

$$b_j^+ - a_j^- = A_j + o(\varepsilon^0),$$
$$b_j^- - a_j^+ = B_j \varepsilon^2 + o(\varepsilon^2).$$

When T_j is of broom type, we put $A_j = B_j = 0$.

Here is Proposition 3.5, whose proof is in the next section.

PROPOSITION 3.5. (i) *Let* O_1 (*resp.* O_2) *be the center of* T_1 (*resp.* T_2). *Then the center of* T_j *corresponds to* O_1 (*resp.* O_2) *for* j *odd* (*resp. even*), *and* $|O_1O_2| = \|m\|/2$. *Further each of* C_{a_j}, C_{b_j}, C_{c_j}, C_{d_j} *includes the segment* O_1O_2 *such that* O_2 (*resp.* O_1) *and each of* a_j^-, b_j^+, c_j^-, d_j^+ *are located in the opposite direction about* O_1 (*resp.* O_2).

(ii)

$$A_j = \begin{cases} 2\nu \cos \dfrac{k_j \pi}{n_j}, & \text{for } j \text{ odd} \\ 2\nu^{-1} \cos \dfrac{k_j \pi}{n_j}, & \text{for } j \text{ even} \end{cases}$$

$$B_j = \begin{cases} -2\nu^{-1} \cos \dfrac{k_j \pi}{n_j}, & \text{for } j \text{ odd} \\ -2\nu \cos \dfrac{k_j \pi}{n_j}, & \text{for } j \text{ even} \end{cases}$$

with some $k_j \in \boldsymbol{Z}/(n_j)$, $k_j \neq 0$ *and some non-zero complex constant* ν.

(iii) *Given* $\{k_j\}_{1 \leq j \leq N}$ *of* (ii). *If there exists* j *such that* $k_j \neq n_j/2$, *then there exists an ideal point realizing these* $\{k_j\}$.

§4. Proof of Proposition 3.5.

The whole of this section is devoted to the proof of Proposition 3.5. Proposition 3.5 is also valid in the case of two-bridge links. Their representation spaces have been defined in Remark 1.4.

Let $[n_1, n_2, \cdots, n_N]$ be a continued fraction expansion of β/α. We prove this proposition by induction on α, and so we can assume this proposition holds good for $[n_1, \cdots, n_{N'}]$ $(N'<N)$, which has smaller α. Moreover we can assume $N\geqq 2$ because the claim is valid when $N=1$ i.e. $\beta=1$ as discussed previously.

In the following Steps $1\sim 4$, we fix α, β and $[n_1, \cdots, n_N]$, and prove (i)(ii), constructing trees of T_j's in order, i.e. we determine mutual positions of axes C_{a_j}, C_{b_j}, C_{c_j}, C_{d_j} for $j=1, 2, \cdots, N$ in order.

Step 1. We consider the case that there are parasol types in succession from T_1 to T_l. We shall show these T_j's satisfy (i)(ii) in this step. This step is rather long. Firstly it will be proved that (i) and (ii) hold for T_1, secondly, for all T_j's by induction.

Since b_2a_2 belongs to vertex group and T_2 is of parasol type, $|C_{a_2}\cap C_{b_2}|=\|m\|$. Hence $|C_{a_1}\cap C_{c_1}|=\|m\|$ because $a_1=b_2$, $c_1=a_2^{-1}$. Since $|C_{a_1}\cap C_{c_1}|=\|m\|$, the arrangement of axes of T_1 is as shown in Fig. 4.1 where each length between two adjacent vertices is $\|m\|/2$.

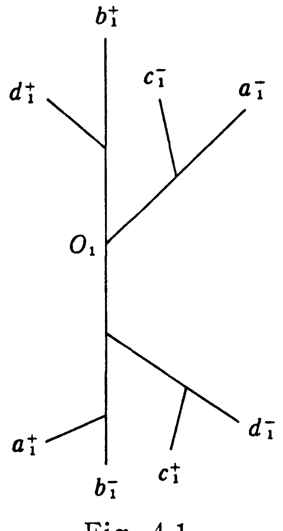


Fig. 4.1.

We can apply the next lemma for $j=1$.

LEMMA 4.1. *If T_j is of parasol type and C_{a_j} and C_{c_j} have a common part of length less than or equal to $2\|m\|$. Then the arrangement of axes is as shown in Fig.* 4.2(a) *with* $|OX_1|=|OX_2|=|OX_3|$.

PROOF. Since T_j is of parasol type, we obtain Fig. 4.2(b) and (c) with $t_{j,k}$'s defined in Fig. 3.7.

Moreover we have

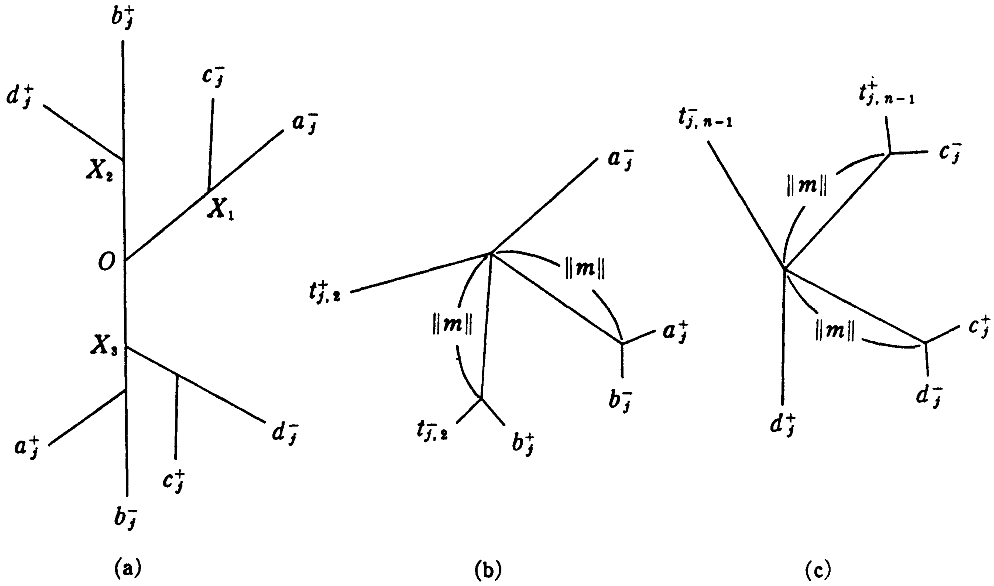


Fig. 4.2.

$$|C_{a_j}\cap C_{t_{j,n-1}}| = |C_{b_j}\cap C_{c_j}| = |C_{t_{j,2}}\cap C_{d_j}|,$$

because the complex distances between $\rho(a_j)$ and $\rho(t_{j,n-1})$, between $\rho(b_j)$ and $\rho(c_j)$, and between $\rho(t_{j,2})$ and $\rho(d_j)$ are equal to each other, and a length of common part of two axes depends on a complex distance by Lemma 3.3. Hence the lemma is proved. ■

By Lemma 3.3 the complex distance between $\rho(a_1)$ and $\rho(c_1)$ has the form $1+O(\varepsilon^2)$. Hence the complex distance τ_1 between $\rho(a_1)$ and $\rho(b_1)$ satisfies

$$(\tau+1)\frac{\varepsilon^{-2}}{2}-2\left(\cos\frac{2k_1\pi}{n_1}+1\right)\in O(\varepsilon^2)$$

for some $k_1\in \boldsymbol{Z}/(n_1)$ by Lemma 6.1, which will be shown in §6 (note that $\xi^{-1}=\varepsilon^{-2}/2+O(\varepsilon^0)$ and $\tau_1+1\in o(\varepsilon^0)$).

We can express a complex distance in terms of ends by Lemma 3.4. Noting that we express a_1^-, b_1^+ by the first coordinate and a_1^+, b_1^- by the second coordinate which is the inverse of the first coordinate, we apply the first formula of Lemma 3.4 for $a^-=a_1^-$, $b^+=b_1^+$, $a^+=1/a_1^+$, $b^-=1/b_1^-$. Then we have

$$(1-a_1^+a_1^-)(1-b_1^+b_1^-)\frac{\tau_1+1}{2} = -(b_1^- - a_1^+)(b_1^+ - a_1^-).$$

We can put $b_1^+ - a_1^- = A_1+o(\varepsilon^0)$ with some complex number A_1. Since a translation by a length $\|m\|$ corresponds to a multiplication by ε^2 in the coordinate, we can put $b_1^- - a_1^+ = B_1\varepsilon^2+o(\varepsilon^2)$. Further we have a_1^+, $b_1^-\in o(\varepsilon^0)$ because we defined the second coordinate such that the downward path from O_1 indicates zero. Substituting these to the above formula, we have

$$\frac{\tau_1+1}{2} = -A_1B_1\varepsilon^2+o(\varepsilon^2).$$

Hence we obtain

$$A_1B_1 = -2\left(\cos\frac{2k_1\pi}{n_1}+1\right) = -4\cos^2\frac{k_1\pi}{n_1},$$

and

$$A_1 = 2\nu\cos\frac{k_1\pi}{n_1}, \qquad B_1 = -2\nu^{-1}\cos\frac{k_1\pi}{n_1},$$

for some ν. Furthermore $|O_1O_2| = \|m\|/2$, because the location of O_2 is determined in Fig. 4.1 (note that $a_2=c_1^{-1}$, $b_2=a_1$).

Now, about a_{j-1}, b_{j-1}, d_{j-2}, b_1 which are at the junction of T_{j-2} and T_{j-1}, we assume the arrangement of the axes is as shown in Fig. 4.3 where each length between two adjoining vertices is $\|m\|/2$.

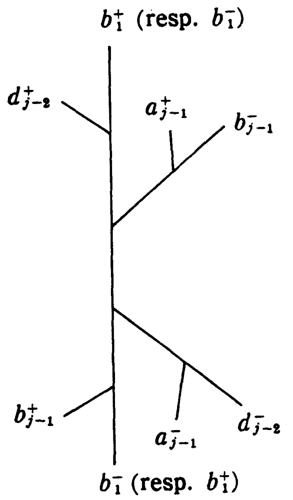


Fig. 4.3.

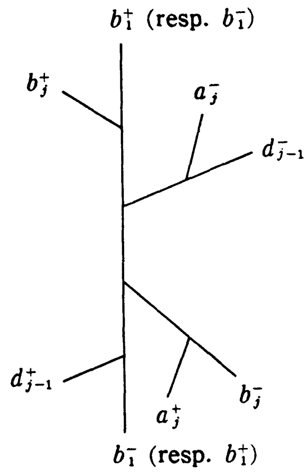


Fig. 4.4.

Since T_j is of parasol type, C_{a_j} and C_{b_j} have a common part of length $\|m\|$. By Lemma 4.1 on T_{j-1}, the midpoint of a common part is O_2, and similarly for $C_{d_{j-1}}$ and C_{b_1}. Moreover, since T_{j-1} is of parasol type, C_{a_j} and $C_{d_{j-1}}$ have a common part of length $\|m\|$, which has O_2 at an end, and similarly for C_{b_j} and C_{b_1} because $d_{j-1}a_j^{-1}=b_1^{-1}b_j$. Therefore, the arrangement of the axes is as shown in Fig. 4.4 where the elements of Γ are at the junction of T_{j-1} and T_j. As above, (i) is proved inductively.

Since $C_{a_{j+1}}$ and $C_{b_{j+1}}$ have an intersection, we obtain

$$A_jB_j = -4\cos^2\frac{k_j\pi}{n_j}$$

with some $k_j \in \boldsymbol{Z}/(n_j)$ in the similar way to the case of T_1.

To prove (ii), we need some lemmas which give relations between A_j's and B_j's. Our aim is to obtain Lemma 4.5. We omit the subscript j in Lemmas 4.2 and 4.3.

LEMMA 4.2. *Under the assumption of Lemma* 4.1 *and* $|OX_1| = (1/2)\|m\|$, *then*

$$c^- - a^- = d^+ - b^+ + o(\varepsilon)$$
$$b^- - a^+ = d^- - c^+ + o(\varepsilon^2).$$

PROOF. Let τ be the complex distance between $\rho(a)$ and $\rho(b)$, which is equal to the complex distance between $\rho(c)$ and $\rho(d)$.

Noting that the coordinate of a^-, b^+, c^-, d^+ is reciprocal to that of a^+, b^-, c^+, d^- respectively, we apply Lemma 3.4, to obtain

$$(1-a^+a^-)(1-b^+b^-)\frac{\tau+1}{2} = (b^- - a^+)(a^- - b^+)$$
$$(1-c^+c^-)(1-d^+d^-)\frac{\tau+1}{2} = (d^- - c^+)(c^- - d^+).$$

Further, we have

$$b^- - a^+, \qquad d^- - c^+ \in O(\varepsilon^2)$$
$$a^- - b^+, \qquad c^- - d^+ \in O(\varepsilon^0)$$
$$a^- - b^+ = c^- - d^+ + o(\varepsilon^0).$$

Comparing the leading coefficients, we have

$$b^- - a^+ = d^- - c^+ + o(\varepsilon^2).$$

Similarly, we can obtain

$$c^- - a^- = d^+ - b^+ + o(\varepsilon)$$

from the fact that the complex distances between $\rho(a)$ and $\rho(c)$ and between $\rho(b)$ and $\rho(d)$ are the same. ∎

LEMMA 4.3. *Under the assumption of Lemma* 4.2, *the next formula holds.*

$$A(c^+ - a^+) = B(c^- - a^-) + o(\varepsilon),$$

with complex numbers A, B *defined in* §3, *which satisfy*

$$b^+ - a^- = A + o(\varepsilon^0)$$
$$b^- - a^+ = B\varepsilon^2 + o(\varepsilon^2).$$

PROOF. Let σ be the complex distance of $\rho(a)$ and $\rho(t_{n-1})$, which is equal to the complex distance of $\rho(b)$ and $\rho(c)$, where t_{n-1} is as in Fig. 3.7 (we omit

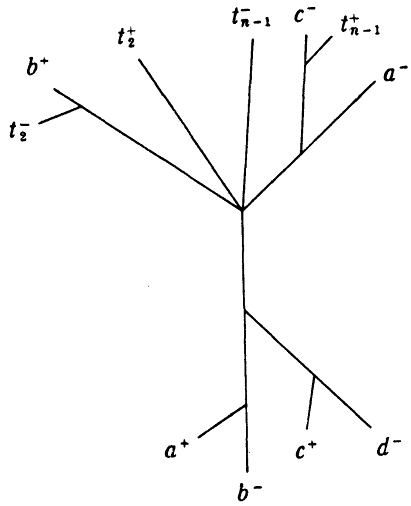


Fig. 4.5.

the subscript j now). The arrangement of axes is as shown in Fig. 4.5.

By Lemma 3.4,

$$(1-a^+a^-)(t_{n-1}^+-t_{n-1}^-)\frac{\sigma+1}{2}=(1-a^+t_{n-1}^-)(a^--t_{n-1}^+)$$

$$(1-b^+b^-)(1-c^+c^-)\frac{\sigma+1}{2}=(b^+-c^-)(c^+-b^-).$$

Hence

$$(*)\qquad (t_{n-1}^+-t_{n-1}^-)(b^+-c^-)(c^+-b^-)=a^--t_{n-1}^+.$$

Since c translates t_{n-1}^- to d^-, by Lemma 4.4 below we obtain

$$\varepsilon^{-2}(c^--t_{n-1}^-)(d^--c^+)=(1-t_{n-1}^-c^+)(1-d^-c^-).$$

Moreover, $c^-=t_{n-1}^++o(\varepsilon^0)$ and by Lemma 4.2 $d^--c^+=B\varepsilon^2+o(\varepsilon^2)$. Thus

$$B(t_{n-1}^+-t_{n-1}^-)=1+o(\varepsilon^0).$$

Furthermore, $b^+-c^-=A+o(\varepsilon^0)$, $c^+-b^-=c^+-a^-+o(\varepsilon)$ and $a^--t_{n-1}^+=a^--c^-+o(\varepsilon)$. Substituting these to (*), we obtain the required formula. ■

In the proof of Lemma 4.3 we use the next lemma.

LEMMA 4.4. *If meridian m translates an end e_1^+ to e_2^-, then*

$$-\varepsilon^{-2}(e_1^+-m^+)(e_2^+-m^-)=(e_1^+-m^-)(e_2^+-m^-).$$

PROOF. Matrix $\rho(m)$ is expressed as

$$\rho(m)=\frac{\mu}{m^+-m^-}\binom{m^+}{1}(1-m^-)+\frac{\mu^{-1}}{m^+-m^-}\binom{m^-}{1}(1-m^+).$$

Since $\rho(m)$ translates $\begin{bmatrix} e_1^+ \\ 1 \end{bmatrix}$ to $\begin{bmatrix} e_2^+ \\ 1 \end{bmatrix}$,

$$\binom{e_2^+}{1} \quad \text{and} \quad \mu\cdot\frac{e_1^+-m^-}{m^+-m^-}\binom{m^+}{1}+\mu^{-1}\cdot\frac{e_1^+-m^+}{m^+-m^-}\binom{m^-}{1}$$

are linearly dependent. Hence

$$e_2^+\{\mu(e_1^+-m^-)+\mu^{-1}(e_1^+-m^+)\}=\mu(e_1^+-m^-)m^++\mu^{-1}(e_1^+-m^+)m^-.$$

By the definition of ε, $\varepsilon=\sqrt{-1}\mu^{-1}$, and so we obtain the required formula. ∎

Now we can obtain relations between A_j's and B_j's.

LEMMA 4.5. *If all of T_{j-1}, T_j, T_{j+1} are of parasol type, then*

$$(A_{j-1}-A_{j+1})A_j=(B_{j-1}-B_{j+1})B_j.$$

PROOF.

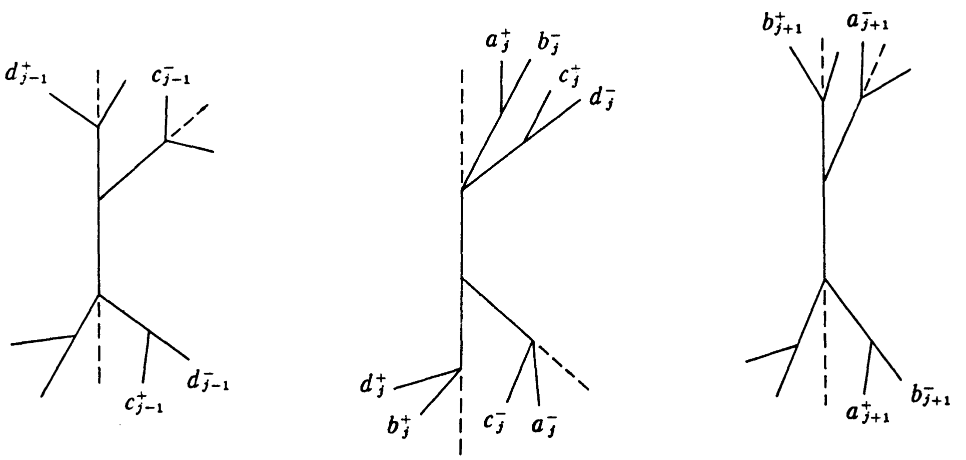


Fig. 4.6.

As is shown in Fig. 4.6, we have the arrangement of axes for the elements of T_{j-1}, T_j and T_{j+1}. Here $c_j^+-a_j^+$ is deformed as follows.

$$\begin{aligned} c_j^+-a_j^+ &= \varepsilon(a_{j+1}^--c_{j+1}^-) \\ &= \varepsilon((b_{j+1}^+-A_{j+1})-(d_{j-1}^+-A_{j-1}))+o(\varepsilon) \\ &= \varepsilon(A_{j-1}-A_{j+1})+o(\varepsilon) \end{aligned}$$

Similarly we obtain,

$$c_j^- - a_j^- = \varepsilon(B_{j-1} - B_{j+1}) + o(\varepsilon).$$

Moreover by Lemma 4.3 we have

$$A_j(c_j^+ - a_j^+) = B_j(c_j^- - a_j^-) + o(\varepsilon).$$

These give the required formula. ■

By similar arguments we obtain the following lemma.

LEMMA 4.6. *If both T_1 and T_2 are of parasol type, then*

$$A_1A_2 = B_1B_2.$$

By Lemmas 4.5 and 4.6, we have $A_jA_{j+1} = B_jB_{j+1}$ $(j=1, 2, \cdots)$. Further we have $A_jB_j = -4\cos^2(k_j\pi/n_j)$ before. Noting that $A_j \neq 0$, $B_j \neq 0$ when T_j is of parasol type, we can obtain (ii) inductively.

Step 2. We consider the case that some broom types continue from T_1 to T_l and T_{l+1} is of parasol type.

Axes C_{a_1} and C_{b_1} have a common part of finite length. If otherwise, a_1 and b_1 fixes a common end, and so does Γ, because a_1 and b_1 generate Γ. This is a contradiction to irreducibility of representation.

Hence the arrangement of axes is as shown in Fig. 4.7. It satisfies assumptions of the next lemma for $j=1$, i.e. for b_1, $d_0(=a_1)$, c_1, d_2 and $X^+ = Y^+$, $X^- = Y^-$.

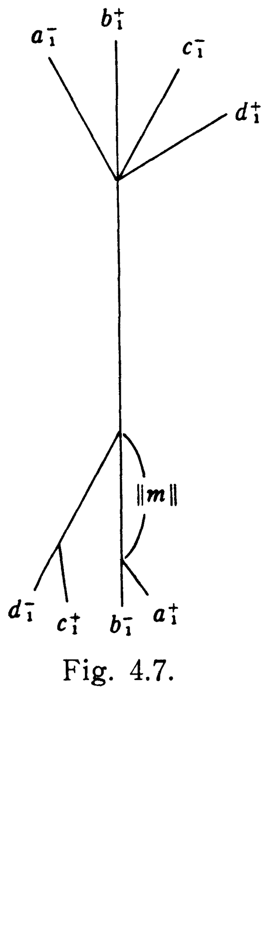


Fig. 4.7.

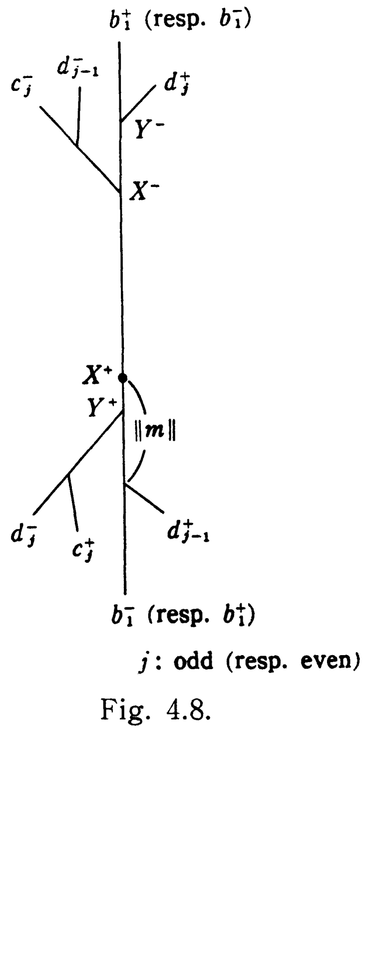


Fig. 4.8.

LEMMA 4.7. *If the arrangement of axes is as shown in Fig.* 4.8 *for elements* b_1, d_{j-1}, c_{j-1}, d_j *of* Γ *at the junction of* T_j *and* T_{j+1} (*see Fig.* 3.4) *and* $0 \leqq |X^+Y^+| < \|m\|$, $0 \leqq |X^-Y^-| < \|m\|$, *then* $|Y^+Y^-|/\|m\| = l$ *is a natural number. Furthermore* l *broom types continue from* T_j *to* T_{j+l-1} *and* T_{j+l} *is of parasol type.*

PROOF. We have $|Y^+Y^-| \geqq \|m\|$ because Y^+Y^- is a common part of C_{b_1} and C_{d_j}.

If $|Y^+Y^-| = \|m\|$, T_j is of parasol type and the lemma holds.

If $|Y^+Y^-| > \|m\|$, with $a_{j+1} = c_j^-$, $b_{j+1} = d_{j-1}$ we have the arrangement of axes as illustrated in Fig. 4.9.

Fig. 4.9 shows $|X_1^+Y_1^+| = |X_1^-Y_1^-| = 0$ for elements at the junction of T_{j+1} and T_{j+2}, which satisfy the assumption of Lemma 4.7 again. Repeating the same argument, we see that broom types continue and $|Y^+Y^-|$ become shorter by length $\|m\|$. Finally we have a parasol type T_{j+l}, where $l = |Y^+Y^-|/\|m\|$ is a natural number, because Y^+Y^- cannot have length $|Y^+Y^-| < \|m\|$ by the remark at the beginning of this proof. ■

Similarly we obtain the next lemma.

LEMMA 4.8. *If the arrangement of axes is as shown in Fig.* 4.10 *for the elements of* Γ *at the junction of* T_j *and* T_{j-1}, *and* $|X^+Y^+| < \|m\|$, $|X^-Y^-| < \|m\|$, *then* $|Y^+Y^-|/\|m\| = l$ *is a natural number. Furthermore* l *broom types continue*

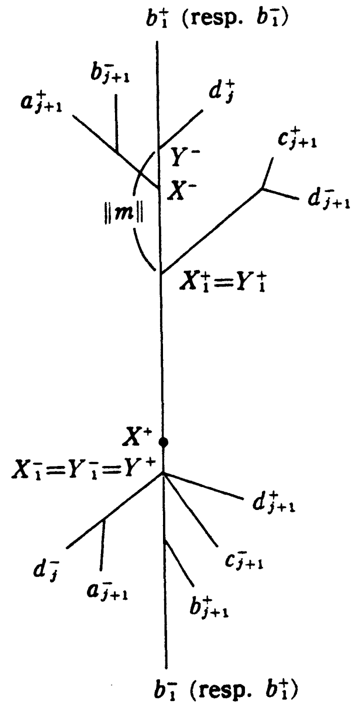


Fig. 4.9.

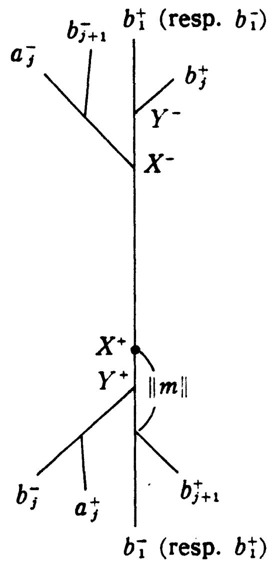


Fig. 4.10.

from T_j back to T_{j-l+1} and T_{j-l} is of parasol type.

If all of $T_1, \cdots, T_N$ are of broom type, Lemma 4.7 means the broomstick of T_1 is longest. On the other hand Lemma 4.8 shows that the broomstick of T_N is longest. This is a contradiction.

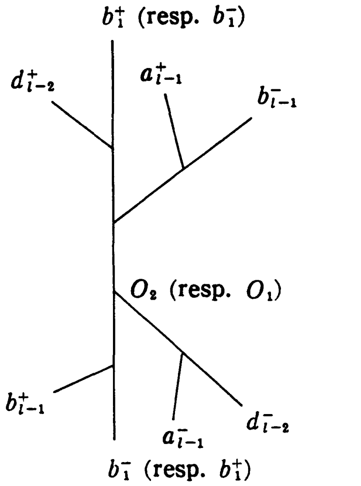
Therefore we can suppose there is a parasol type T_{l+1} after series of broom types.

By Lemma 4.7 the broomstick of T_1 is of length $l\|m\|$. Condition (i) is proved along the construction of the tree in the proof of Lemma 4.7. Furthermore, since n_j is even for broom type, put $k_j=n_j/2$, and (ii) holds good.

Step 3. We consider the case that there are some broom types in succession from T_l, after parasol type T_{l-1}. When these broom types continue to the end T_N, (i) is proved by Lemma 4.8, and (ii) holds good, putting $k_j=n_j/2$.

In the following, we consider the case that there is a parasol type again after these series of broom types.

Fig. 4.11 shows the arrangement of the axes at the junction of T_{l-2} and T_{l-1}. We discuss the location of the axes of c_{l-1} and d_{l-1} (see Fig. 4.12).



l: odd (resp. even)

Fig. 4.11.

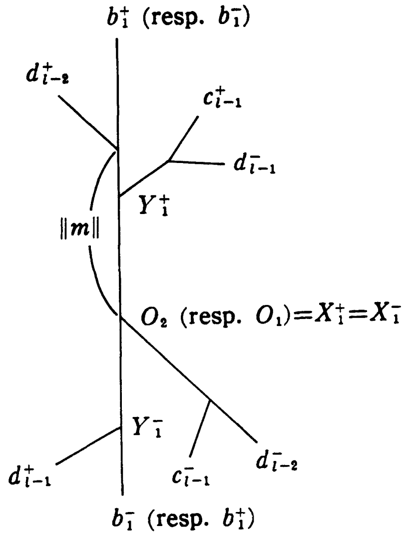


Fig. 4.12.

If $X_1^+Y_1^+$ and $X_1^-Y_1^-$ satisfy the assumption of Lemma 4.7 (both $|X_1^+Y_1^+|<\|m\|$ and $|X_1^-Y_1^-|<\|m\|$), broom types do not continue. Hence we must put Y_1^+ and Y_1^- in Fig. 4.12 such that the assumption is not satisfied. Now, the next claim holds.

CLAIM. $|X_1^+Y_1^+| = |X_1^-Y_1^-|$

PROOF. We denote by $T_1', \cdots, T_{l-1}'$ trees which has the same $[n_1, \cdots, n_{l-1}]$

and $\{k_j\}$ as the tree discussed now, and also express other corresponding notations with a prime. Such tree exists uniquely, because of the assumption of induction and Step 5 in this section which shows the uniqueness. Using Proposition 6.5, we take small perturbation from $T_1', \cdots, T_{l-1}'$ to obtain a tree we need.

Put formally $a_l'=(c_{l-1}')^{-1}$, $b_l'=d_{l-2}'$ and $\Delta a_1^+=a_1-a_1'$, $\Delta b_1^+=b_1-b_1'$, $\cdots$. We see the order of $\Delta b_l^+-\Delta a_l^-$ and $\varepsilon^{-2}(\Delta b_l^- - \Delta a_l^+)$ are equal to each other by applying Proposition 6.5, taking conjugation such that the corresponding axes lie on another as close as possible. It follows that the order of $c_{l-1}^+-d_{l-2}^+$ and $c_{l-1}^- - d_{l-2}^-$ are equal to each other. This completes the proof of the claim. ∎

After all we obtain $|X_1^+Y_1^+|=|X_1^-Y_1^-|=\|m\|$ in order that the assumption of Lemma 4.7 is not satisfied. Therefore Fig. 4.13 holds about the arrangement of the axes at the junction of T_{l-1} and T_l.

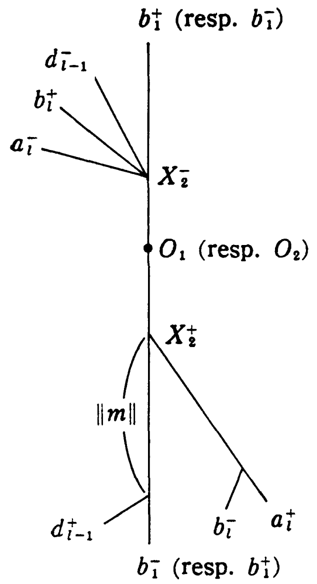


Fig. 4.13.

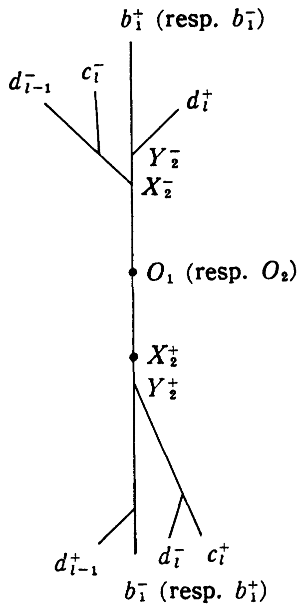


Fig. 4.14.

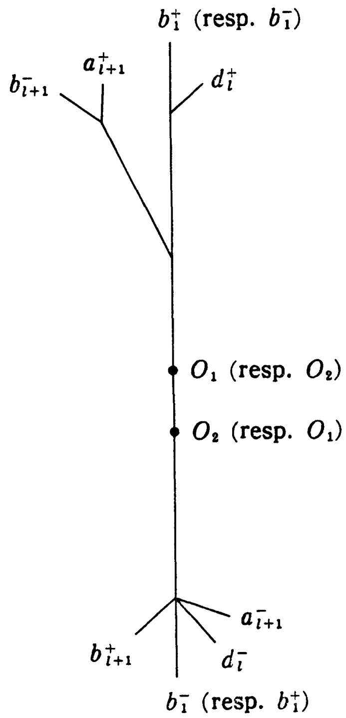


Fig. 4.15.

Repeat the process of obtaining Fig. 4.13 from Fig. 4.11, and we see $|X_2^+Y_2^+|=|X_2^-Y_2^-|=\|m\|$ in Fig. 4.14 and obtain Fig. 4.15 about the arrangement of axes at the junction of T_l and T_{l+1}. Repeating the same discussion, we see the broomsticks grow longer by $\|m\|$ and these T_j's satisfy (i).

Now, we come up to the middle in the line of broom types. When the number of broom types is odd, $X^+=Y^+$ and $X^-=Y^-$ for the middle broom type.

When the number of broom types is even, $|X^+Y^+|=|X^-Y^-|=\|m\|/2$ for the former of the two middle ones. As shown in the proof of Lemma 4.7, the rest broomsticks become shorter by $\|m\|$, and we find a parasol type at last. These construction of trees shows (i). For broom types (ii) holds automatically.

Step 4. We consider a parasol type T_l after series of broom types. If this is the first parasol type, there is no problem about values A_l, B_l. We shall show that (ii) holds on T_l using the same ν as the previous parasol type. For simplicity, we may let the previous ν be equal to 1, taking conjugation if necessary.

We proceed in the same way as in the proof of Claim in Step 3. We define $T'_1, \cdots, T'_{l-1}$, put formally $a'_l=(c'_{l-1})^{-1}$, $b'_l=d'_{l-2}$, and apply Proposition 6.5 with Remark 6.6, to obtain

$$b_l^+-a_l^- = -(b_l^- - a_l^+)\varepsilon^{-2}+o(\varepsilon^0).$$

This shows $A_l=-B_l$. We deal with the remainder of the proof as in Step 1, and obtain

$$A_l=\cos\frac{k_l\pi}{n_l}, \qquad B_l=-\cos\frac{k_l\pi}{n_l}$$

for some $k_l\in \boldsymbol{Z}/(n_l)$. Thus the claim of Step 4 is proved.

The last three steps are devoted to the proof of (iii).

Step 5. We show the uniqueness of the ideal point which has the same $\{k_j\}_{1\le j\le N}$.

We suppose there exist two ideal points which have the same $\{k_j\}_{1\le j\le N}$. Let $\Delta\tau_1$ be the difference of τ_1. If $\Delta\tau_1\notin o(\varepsilon^N)$, taking conjugate if necessary, we apply Proposition 6.5, to obtain $\Delta\tau_{N-l}\notin o(\varepsilon)$ where τ_j is the complex distance of $\rho(a_j)$ and $\rho(b_j)$ and l is the number of continued broom types from T_1; in particular $l=0$ if T_1 is of parasol type (note that when $l>0$, Δa_1^+, Δa_1^-, Δb_1^+, Δb_1^- are smaller by ε^l than the case $l=0$ corresponding to the same $\Delta\tau_1$). However, this contradicts the fact that τ_{N-l} is determined by k_{N-l} in $o(\varepsilon)$ order.

Hence $\Delta\tau_1\in o(\varepsilon^N)$. Moreover reversing up and down of the plumbing representation of the two-bridge knot, we repeat the above argument, to obtain $\Delta\tau_N\in o(\varepsilon^{2N})$. In a similar way, it follows that $\Delta\tau_1\in o(\varepsilon^{3N})$, $\Delta\tau_N\in o(\varepsilon^{4N}), \cdots$. Therefore $\Delta\tau_1=0$. Since τ_1 determines the conjugacy class of the ideal point, we have shown the claim of Step 5.

Step 6. We determine the value of $\phi_x(1,0)$ defined in §1 where x is an ideal point corresponding to $\{k_j\}_{1\le j\le N}$.

When there exists a parasol type T_j both for some odd j and for some even j, τ_1 is expressed by power series in ε. Because, suppose there is a term in the expansion of τ_1 which has an index of fraction. We put τ'_1 to be power series in ε which is obtained from τ_1 by eliminating terms with fraction indices.

Then Lemma 4.9 below shows τ_1' is expressed by power series in ε (note that ξ is expressed by power series in ε^2). We compare τ_j and τ_1' by Proposition 6.5, to obtain there exists τ_j which has a term of fraction index as the lowest term. This is a contradiction to the fact that τ_j has the lowest two terms with integer indices determined by k_j.

LEMMA 4.9. *τ_j is expressed as a polynomial in τ_1, ξ and ξ^{-1} where τ_j is the complex distance between $\rho(a_j)$ and $\rho(b_j)$.*

PROOF.

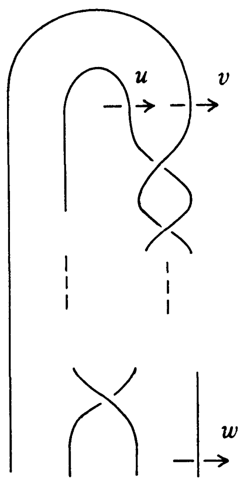


Fig. 4.16.

Any meridian w is obtained from u, v, u^{-1} or v^{-1} by taking conjugation some times by u, v, u^{-1} or v^{-1} where u, v are the standard two generators of Γ.

By a similar calculation as in § 1, τ_j is expressed as the trace of polynomial in P, Q, γ and s. When taking its trace, the surviving terms are of even degree with respect to P, Q. Thus γ, s are replaced by ξ, ξ^{-1}. Further, trace of product of even P, Q's is expressed as a polynomial in τ_1. The lemma is proved. ■

Therefore, in this case the representation space X is locally parametrized by ε. Hence $\phi_x(1, 0)=2$. We define these ideal points to be of *type* 2.

Next, we consider the case when there are parasol types T_j's only for odd j's or only for even j's. In this case there is no branch at O_2 (or at O_1) and each segment between two adjacent branch points has length $\|m\|=2|O_1O_2|$. We can see τ_1 is expressed by a power series of ε^2, in a similar argument as above. Hence $\phi_x(1, 0)=1$. We define these ideal points to be of *type* 1.

Step 7. Assuming there exists an ideal point x corresponding to each $\{k_j\}_{1\le j\le N}$, we calculate $\sum_x \phi_x(1, 0)$. Lemma 1.3 and Remark 1.4 show

$$\sum_x \phi_x(1, 0) = \begin{cases} \dfrac{\alpha-1}{2}, & \text{for } \alpha \text{ odd} \\ \dfrac{\alpha-2}{2}, & \text{for } \alpha \text{ even}. \end{cases}$$

If the result of the calculation is equal to this, we can conclude the existence of an ideal point corresponding to each $\{k_j\}_{1\leq j\leq N}$.

We denote two $\{k_j\}$'s are equivalent if they determine the same ideal point. The equivalence relation is generated by $\{k_j\}\sim\{-k_j\}$ and $\{k_j\}\sim\{(-1)^j k_j\}$. Firstly $\{k_j\}\sim\{-k_j\}$ derives from replacing ν by $\sqrt{-1}\nu$ in the statement of (ii). Secondly $\{k_j\}\sim\{(-1)^j k_j\}$, because ε has an ambiguity of sign as a function on the representation space X.

Now, given a continued fraction expansion $[n_j]$, we count the number of the ideal points which produce incompressible surfaces associated with $[n_j]$.

(a): the case that there exists odd i such that n_i odd and there exists even j such that n_j odd.

There exist ideal points only of type 2, because T_j is necessarily of parasol type if n_j odd. Thus the number of ideal points of type 2 is

$$\#(\{(k_1, \cdots, k_N) \mid k_j \in \boldsymbol{Z}/(n_j),\ k_j \neq 0\}/\sim) = \frac{1}{4}\prod_j(|n_j|-1).$$

(b): the case that there exists odd i such that n_i odd and n_j is even for each even j.

We obtain ideal points of type 1 when T_j is of broom type for each even j. The number of ideal points of type 1 is

$$\#(\{(k_1, \cdots, k_N) \mid k_i \in \boldsymbol{Z}/(n_i),\ k_i \neq 0, \text{ for odd } i,\ k_j = \frac{n_j}{2}, \text{ for even } j\}/\sim)$$
$$= \frac{1}{2}\prod_{i:\text{odd}}(|n_i|-1).$$

Since the rest is of type 2, the number is

$$\frac{1}{4}\prod_{i:\text{odd}}(|n_i|-1)\{\prod_{j:\text{even}}(|n_j|-1)-1\}.$$

By a similar calculation, we obtain the number of ideal points as follows.

(c): the case that n_i is even for each odd i and there exists even j such that n_j odd.

type 1 $\quad \dfrac{1}{2}\prod_{j:\text{even}}(|n_j|-1)$

type 2 $\quad \dfrac{1}{4}\prod_{j:\text{even}}(|n_j|-1)\{\prod_{i:\text{odd}}(|n_i|-1)-1\}$

(d): the case that n_i is even for each i.

type 1 $\quad \frac{1}{2}\prod_{i:\text{odd}}(|n_i|-1)+\frac{1}{2}\prod_{j:\text{even}}(|n_j|-1)-1$

type 2 $\quad \frac{1}{4}\{\prod_{i:\text{odd}}(|n_i|-1)-1\}\{\prod_{j:\text{even}}(|n_j|-1)-1\}$

Next, we calculate $\sum\phi_x(1, 0)$. Noting that $\phi_x(1, 0)=1$ (resp. 2) for type 1 (resp. type 2), we obtain $\sum\phi_x(1, 0)$ is equal to

$$\frac{1}{2}\prod_j(|n_j|-1) \qquad \text{in the case (a), (b) or (c)}$$

$$\frac{1}{2}\prod_j(|n_j|-1)-\frac{1}{2} \qquad \text{in the case (d)}$$

where the sum is taken over the ideal points which correspond to the fixed continued fraction expansion $[n_j]$. Since just one (resp. two) continued fraction expansion of β/α satisfies the condition of (d) for α odd (resp. α even), the sum about all ideal points is as follows.

$$\sum_x\phi_x(1, 0)=\begin{cases}\dfrac{1}{2}\sum_{[n_j]}\prod_j(|n_j|-1)-\dfrac{1}{2}, & \text{for } \alpha \text{ odd}\\ \dfrac{1}{2}\sum_{[n_j]}\prod_j(|n_j|-1)-1, & \text{for } \alpha \text{ even}.\end{cases}$$

This result meets our expectation, because $\sum_{[n_j]}\prod_j(|n_j|-1)=\alpha$ by Lemma 6.7. This completes the proof. ■

§ 5. Main results.

Summarizing the proof of Proposition 3.5, we obtain the following.

THEOREM 5.1. *There is a* 1 *to* 1 *correspondence between the ideal points for the two-bridge knot of type* (α, β) *and the elements of the set*:

$$\bigcup_{[n_j]}\left(\left\{(k_1, \cdots, k_N)\,|\,k_j\in \boldsymbol{Z}/(n_j),\ k_j\neq 0,\ \exists i \text{ such that } k_i\neq\frac{n_i}{2}\right\}/\sim\right)$$

where the union is taken over all continued fraction expansions of β/α *and the equivalence relation is generated by* $(k_j)\sim(-k_j)$ *and* $(k_j)\sim((-1)^j k_j)$.

We use type 1 or 2 as the same meaning in the previous section. Since the peripheral subgroup is abelian, the order of pole: $\phi_x(p, q)$ is of the form $|N_1p-N_2q|$ with some integers N_1, N_2. We obtain $\phi_x(N_{[n_j]}, 1)=0$ because $m^{N_{[n_j]}}l$ is the boundary slope of the incompressible surface and so $m^{N_{[n_j]}}l$ belongs to an edge group. Moreover considering the definition of types 1, 2, $\phi_x(p, q)$ must be as follows.

$$\phi_x(p,\ q)=\begin{cases} |p-N_{[n_j]}q|, & \text{type } 1 \\ 2|p-N_{[n_j]}q|, & \text{type } 2. \end{cases}$$

Since $\Phi(p,\ q)=\sum_x \phi_x(p,\ q)$ (see § 1) and the number of ideal points of each type is given in the previous section, we obtain the next formula.

PROPOSITION 5.2.

$$\Phi(p,\ q)=\sum_{[n_j]}|p-N_{[n_j]}q|\cdot\left[\frac{1}{2}\prod_j(|n_j|-1)\right]$$

COROLLARY 5.3. *Every two-bridge knot has property P.*

PROOF. We denote by $N_{\max}$ (resp. $N_{\min}$) the maximal (resp. minimal) $N_{[n_j]}$ for fixed α, β. Some calculation shows that $(N_{\max}+N_{\min})/2-1$ is equal to the number of subtractions which are necessary when $\{1, 0\}$ is obtained from $\{\alpha, \beta\}$ by Euclidean algorithm: the method of mutual subtraction. Therefore $N_{\max}-N_{\min}\geqq 6$. Since each $N_{[n_j]}$ is even, there exists some $N_{[n_j]}$ such that $|N_{[n_j]}|\geqq 4$. Hence by Proposition 5.2, $\Phi(1,\ q)>\Phi(1,\ 0)$ for any nonzero q.

The following argument is in [4]. Since the degree of $I_{1,q}$ is bigger than that of $I_{1,0}$, there exists x_0 in X such that $v(I_{1,q})>v(I_{1,0})$, with the valuation v at x_0. Taking a section s of $R\to X$ in a neighbourhood of x_0, we consider representations ρ parametrized by elements in the neighbourhood of x_0. In particular, we put $\rho_0=s(x_0)$. We can set

$$\rho(m)=\pm\begin{pmatrix}\mu & \eta \\ 0 & \mu^{-1}\end{pmatrix},\qquad \rho(ml^q)=\pm\begin{pmatrix}\lambda & \zeta \\ 0 & \lambda^{-1}\end{pmatrix},$$

taking conjugate if necessary. Then

$$I_{1,0}=(\mu+\mu^{-1})^2-4=\mu^{-2}(\mu^2-1)^2$$
$$I_{1,q}=(\lambda+\lambda^{-1})^2-4=\lambda^{-2}(\lambda^2-1)^2.$$

Since $v(\mu^2)=v(\lambda^2)=0$, with $v(I_{1,q})>v(I_{1,0})$ we obtain

$$v(\lambda^2-1)>v(\mu^2-1).$$

In particular $v(\lambda^2-1)>0$, and so $\lambda|_{x_0}=\pm 1$. Moreover, since $\rho(m)$ and $\rho(ml^q)$ commute, we have $\mu\zeta+\lambda^{-1}\eta=\lambda\eta+\mu^{-1}\zeta$. Hence

$$\mu^{-2}(\mu^2-1)^2\zeta^2=\lambda^{-2}(\lambda^2-1)^2\eta^2.$$

This means

$$2v(\mu^2-1)+v(\zeta^2)=2v(\lambda^2-1)+v(\eta^2).$$

With $v(\lambda^2-1)>v(\mu^2-1)$ we have $v(\zeta^2)>0$. Hence $\zeta|_{x_0}=0$. Therefore $\rho_0(ml^q)=\pm 1$ in $PSL_2\boldsymbol{C}$.

Next, we show ρ_0 is not an abelian representation. If ρ_0 is abelian, then

$\rho_0(l)=\pm 1$ because l is the identity element in the abelianization of Γ. With $\rho_0(ml^q)=\pm 1$, we have $\rho_0(m)=\pm 1$. Hence ρ_0 is the trivial representation. However, we cannot obtain any irreducible representations by infinitesimal deformation from the trivial representation. This is a contradiction.

We have found ρ_0 such that $\rho_0(ml^q)=\pm 1$ and $\rho(\Gamma)$ is not an abelian group. Hence $\Gamma/(ml^q)$ is a nontrivial group for any nonzero integer q. Therefore K has property P. ∎

Now, we study the correspondence between the ideal points and the incompressible surfaces. By Theorem 5.1 the ideal points are identified with $\{k_j\}$'s. We use the notations of incompressible surfaces defined in § 2.

THEOREM 5.4. *Give an ideal point $\{k_j\}$ associated with a continued fraction expansion $[n_j]$.*

If each n_i is even and $k_j=n_j/2$ either for each even j or for each odd j, then the incompressible surfaces which can be obtained from the ideal point by the method of § 2 are of the type $S_1(m_1, m_2, \cdots, m_{N-1})$ where $m_1, \cdots, m_{N-1}$ satisfy the following condition: there exist integers $M_1, M_2, \cdots, M_N$ such that

$$M_{j+1}=M_j+(-1)^j m_j$$

for $j=1, 2, \cdots, N-1$ and when T_j is of parasol type, $M_j=1$ (resp. 0) for j odd (resp. even).

Otherwise, the incompressible surfaces which can be obtained from the ideal point are of the type $S_2(m_1, m_2, \cdots, m_{N-1})$ where $m_1, \cdots, m_{N-1}$ satisfy the following condition: there exist $M_1, M_2, \cdots, M_N \in \{0, 1\}$ such that

$$M_{j+1}=\begin{cases} M_j, & \text{if } m_j=0, 2 \\ 1-M_j & \text{if } m_j=1 \end{cases}$$

for $j=1, 2, \cdots, N-1$ and when T_j is of parasol type, $M_j=1$ (resp. 0) for j odd (resp. even).

PROOF. When constructing an incompressible surface from Γ-tree T, we may replace T by $\hat{T}=\cup C_\gamma$ where the union is taken over γ which is conjugate to meridian. Because for any vertex of $T-\hat{T}$, its vertex group is a subgroup of a vertex group of some vertex in $\hat{T}$.

The first case of the theorem corresponds to the type 1 of (d) in the proof of Proposition 3.5. In this case $\hat{T}/\Gamma$ is a loop with one edge and one vertex. As in § 2 we have a map $\phi: (S^3-K)\to\hat{T}/\Gamma$ which induces an incompressible surface. Then we can take its infinite cyclic covering $\tilde{\phi}: \widetilde{S^3-K}\to\widetilde{\hat{T}/\Gamma}$. The value of M_j indicates the image of the component around T_j which has a path to the base point along $b_j a_j^{-1}$. If T_j is of parasol type, the vertex of the image is uniquely given by the construction of the tree in the proof of Proposition 3.5. It follows the condition of M_j's. Conversely if the condition of M_j's is satisfied, we can find ϕ concretely.

In the second case, $\hat{T}/\Gamma$ is an interval with one edge and two vertices. By similar argument, we obtain the condition of M_j's.

The theorem is proved. ■

In [**2**], by concrete calculation Burde remarks reducibility of representation space when $(\alpha, \beta)=(15, 11)$, $(21, 13)$, $(33, 23)$ which are the first three cases of the following proposition.

PROPOSITION 5.5. *If* $\beta^2\equiv 1 \mod \alpha$, $\beta \neq \pm 1$, *then the representation space consists of at least two components.*

PROOF. Under the assumption of the proposition, there exists an inversion on (S^3, K) which turns upside down of plumbing representation. This inversion induces an inversion on representation space which transfers an ideal point $\{k_j\}$ of $[n_j]$ to an ideal point $\{k_{N-j+1}\}$ of $[(-1)^{N+1}n_{N-j+1}]$. Hence there exists a component on which the action of the inversion is non-trivial.

On the other hand, when $\beta\neq\pm 1$, K is a hyperbolic knot, that is, S^3-K admits the complete hyperbolic structure. Since the inversion preserves an orientation of S^3-K, it fixes the representation which gives the hyperbolic structure. Moreover it also fixes representations in a neighbourhood of the representation because representations near a parabolic representation are decided by the images of peripheral subgroup and the inversion preserves them. Therefore there exists a component which is fixed by the inversion. The proposition is proved. ■

REMARK 5.6. In addition to the above case, there are two more cases that the representation space of two-bridge knot group is reducible. These are discussed in detail by R. Riley [**10**].

§6. Remaining lemmas.

In §4 we use some lemmas without proof. This section is devoted to the proof of those lemmas.

LEMMA 6.1. *Let a, b, c, $d(\in\Gamma)$ be as illustrated in Fig. 3.7. (We omit subscript j.) Then the complex distance σ_n between $\rho(a)$ and $\rho(c)$ is expressed by the complex distance τ between $\rho(a)$ and $\rho(b)$ as the following formula.*

$$\sigma_n-1=(\tau-1)\prod_{0<l<n}\left(\frac{\tau+1}{\xi}-2\left(1+\cos\frac{2l\pi}{n}\right)\right), \qquad \textit{for}\quad n=1, 2, \cdots$$

where $\xi=-2/((\mathrm{trace}\ \rho(a))^2-4)$ *as defined in* §1.

PROOF. Using $t_i = t_{i-1}t_{i-2}t_{i-1}^{-1}$ inductively, we obtain

$$t_n = \begin{cases} (t_1t_0)^{(n-1)/2}t_1(t_0^{-1}t_1^{-1})^{(n-1)/2}, & n: \text{odd} \\ (t_1t_0)^{(n-2)/2}t_1t_0t_1^{-1}(t_0^{-1}t_1^{-1})^{(n-2)/2}, & n: \text{even}. \end{cases}$$

Putting $\rho(t_i) = s(\gamma + P_i)$, we can express P_n by P_0 and P_1. P_n is calculated inductively as

$$P_n = s(\gamma + P_1)\cdot P'_{n-1}\cdot s(\gamma - P_1)$$

where P'_{n-1} is obtained from P_{n-1} by interchanging P_0 and P_1. Hence we have

$$P_n = f_nP_0 + g_nP_1 + \gamma h_n(P_0P_1 - P_1P_0)$$

where f_n, g_n and h_n are polynomials in τ and ξ^{-1}, characterized by the following recursive formulas

$$f_n = \left(1 - \frac{1}{\xi}\right)g_{n-1} + \left(4 - \frac{2}{\xi}\right)h_{n-1},$$

$$g_n = f_{n-1} + \frac{\tau}{\xi}g_{n-1} - \left(4 - \frac{2}{\xi}\right)\tau h_{n-1},$$

$$h_n = \frac{1}{2\xi}g_{n-1} - \left(1 - \frac{1}{\xi}\right)h_{n-1}.$$

Considering the matrix which represents the above formulas, we have

$$f_n + g_n + 2(1-\tau)h_n = f_{n-1} + g_{n-1} + 2(1-\tau)h_{n-1}$$

as the eigenvector with an eigenvalue 1. Since the value at $n=0$ is 1, we have $f_n + g_n + 2(1-\tau)h_n = 1$. With the above recursive formulas, we calculate, to obtain a recursive formula of σ_n:

$$\sigma_n = 2\left(\frac{\tau+1}{2\xi} - 1\right)\sigma_{n-1} - \sigma_{n-2} + (1+\tau)\left(2 - \frac{1}{\xi}\right), \qquad \sigma_0 = 1, \quad \sigma_1 = \tau$$

where $\sigma_n = -1/2 \operatorname{trace} P_0P_n = f_n + \tau g_n$ by definition.

Now, we define a family of polynomials $\varphi_n(t)$ by $\varphi_n(\cos\theta) = \cos n\theta$ for $n = 0, 1, 2, \cdots$. Some argument shows that $\varphi_n(t)$'s are characterized by the next recursive formula

$$\varphi_n(t) = 2t\varphi_{n-1}(t) - \varphi_{n-2}(t), \qquad \varphi_0(t) = 1, \qquad \varphi_1(t) = t$$

and moreover satisfy the following.

$$\varphi_n(t) - 1 = (t-1)\prod_{0<l<n} 2\left(t - \cos\frac{2l\pi}{n}\right).$$

Comparing the recursive formulas of the series $\{(\sigma_n - 1)/(\tau - 1)\}$ and $\{(\varphi_n(t) - 1)/(t-1)\}$, we obtain the formula of Lemma 6.1. ∎

Our next aim is to prove Proposition 6.5, which describe the way how small perturbations of a_1^+, a_1^-, b_1^+, b_1^- give rise to small perturbations of a_l^+, a_l^-, b_l^+, b_l^- (note that Γ is generated by a_1 and b_1). We express differentials of a_j^+, a_j^-, $\cdots$ by Δa_j^+, Δa_j^-, $\cdots$. Subscript j is often omitted in Lemmas 6.2, 6.3 and 6.4.

LEMMA 6.2. *If Δa^+, Δa^-, Δb^+, $\Delta b^- \in O(\varepsilon^q)$ and $\Delta a^+ - \Delta b^- \in O(\varepsilon^{q+2})$, then Δc^+, Δc^-, Δd^+, $O(\varepsilon^q)$, where q is a non-negative rational number.*

PROOF. In Fig. 3.7, c and d are expressed only by a and b. Taking small perturbation of its formula, we obtain this lemma. ∎

LEMMA 6.3. *Under the assumption of Lemma 6.2, we further assume a^+, b^-, $b^+ \in o(\varepsilon^0)$. Then $\Delta\tau \in O(\varepsilon^{q+2})$ and the next formula holds.*

$$\frac{\Delta\tau}{2} = -A(\Delta b^- - \Delta a^+) - B\varepsilon^2(\Delta b^+ - \Delta a^-) + A^2 B\varepsilon^2 \Delta a^+ + o(\varepsilon^{q+2})$$

PROOF. By Lemma 3.4 we have

$$(1-a^+a^-)(1-b^+b^-)\frac{\tau+1}{2} = -(b^+-a^-)(b^--a^+).$$

Taking small perturbation of this, we obtain the required formula. ∎

LEMMA 6.4. *Under the assumption of Lemma 6.2, we assume $AB = 2(1+\cos(2k\pi/n))$. Then the following formulas hold.*

When T is of parasol type,

$$\Delta c^+ - \Delta a^+ = \Delta d^- - \Delta b^- + o(\varepsilon^q) = -\frac{n}{8A}\varepsilon^{-2}\Delta\tau \operatorname{cosec}^2\frac{k\pi}{n} + o(\varepsilon^q)$$

$$\Delta c^- - \Delta a^- = \Delta d^+ - \Delta b^+ + o(\varepsilon^q) = -\frac{n}{8B}\varepsilon^{-2}\Delta\tau \operatorname{cosec}^2\frac{k\pi}{n} + o(\varepsilon^q).$$

When T is of broom type,

$$\Delta c^+ - \Delta a^+ = \Delta d^- - \Delta b^- + o(\varepsilon^q) = \frac{n}{2}(\Delta b^- - \Delta a^+)\varepsilon^{-2} + o(\varepsilon^q)$$

$$\Delta c^- - \Delta a^- = \Delta d^+ - \Delta b^+ + o(\varepsilon^q) = \frac{n}{2}(\Delta b^+ - \Delta a^-) + o(\varepsilon^q).$$

PROOF. Since the case $n<0$ is regarded as the "mirror image" of the case $n>0$, we can assume $n>0$. t_i's will be as shown in Fig. 3.7.

When T is of parasol type, we let σ be the complex distance between $\rho(b)$ and $\rho(c)$, which is equal to the complex distance between $\rho(a)$ and $\rho(t_{n-1})$, $\rho(t_2)$ and $\rho(d)$. By Lemma 3.4 we have,

$$(1-b^+b^-)(1-c^+c^-)\frac{\sigma+1}{2} = -(b^+-c^-)(b^--c^+)$$

$$(1-a^+a^-)(t_{n-1}^+-t_{n-1}^-)\frac{\sigma+1}{2}=-(1-a^+t_{n-1}^-)(a^--t_{n-1}^+)$$

$$(1-d^+d^-)(t_2^--t_2^+)\frac{\sigma+1}{2}=-(1-d^-t_2^+)(d^+-t_2^-)\,.$$

Moreover, we have

$$t_{n-1}^+-t_{n-1}^-=B^{-1}+o(\varepsilon^0)$$
$$t_2^+-t_2^-=B^{-1}+o(\varepsilon^0)$$

because the complex distances between $\rho(a)$ and $\rho(b)$, between $\rho(b)$ and $\rho(t_2)$, between $\rho(t_{n-1})$ and $\rho(c)$ are equal to each other. Therefore we obtain

$$\frac{\Delta\sigma}{2}=A(\Delta c^+-\Delta a^+)+o(\varepsilon^q)=A(\Delta d^--\Delta b^-)+o(\varepsilon^q)$$

$$\frac{\Delta\sigma}{2}=B(\Delta c^--\Delta a^-)+o(\varepsilon^q)=B(\Delta d^+-\Delta b^+)+o(\varepsilon^q)\,.$$

It remains to calculate the ratio of $\Delta\sigma$ and $\Delta\tau$. We take small perturbation of the formula of Lemma 6.1, to obtain

$$\Delta\sigma=\Delta\sigma_{n-1}=-2\xi^{-1}\Delta\tau\sum_j\prod_{\substack{0<l<n-1\\ l\neq j}}2\left(\cos\frac{2k\pi}{n}-\cos\frac{2l\pi}{n-1}\right)+o(\varepsilon^q)\,.$$

Further we have

$$\sum_j\prod_{\substack{0<l<n-1\\ l\neq j}}2\left(\cos\frac{2k\pi}{n}-\cos\frac{2l\pi}{n-1}\right)+o(\varepsilon^q)=\frac{n}{4}\operatorname{cosec}^2\frac{k\pi}{n}$$

by calculating $\varphi_n'(\cos(2k\pi/n))$ in two ways, where φ_n is in the proof of Lemma 6.1. Thus we obtain

$$\Delta\sigma=-\frac{n}{4}\varepsilon^{-2}\Delta\tau\operatorname{cosec}^2\frac{k\pi}{n}+o(\varepsilon^q)\,.$$

In this case the lemma is proved.

When T is of broom type, by Lemma 4.4 applied the relations of t_i's we have,

$$-\varepsilon^{-2}(1-t_i^+t_{i-1}^+)(t_{i-2}^+-t_{i-1}^-)=(t_i^+-t_{i-1}^-)(1-t_{i-2}^+t_{i-1}^+)$$
$$-\varepsilon^{-2}(t_i^--t_{i-1}^+)(1-t_{i-2}^-t_{i-1}^-)=(1-t_i^-t_{i-1}^-)(t_{i-2}^-t_{i-1}^+)\,.$$

Taking small perturbation of these, we obtain the following, inductively.

$$\Delta t_{i+2}^--\Delta t_i^-=\begin{cases}\Delta b^+-\Delta a^-+o(\varepsilon^q), & i:\text{ even}\\ (\Delta b^--\Delta a^+)\varepsilon^{-2}+o(\varepsilon^q), & i:\text{ odd}.\end{cases}$$

Hence we have,

$$\Delta c^+ - \Delta a^+ = \sum_{\substack{0<i<n \\ i:\mathrm{odd}}} (\Delta t_{i+2}^- - \Delta t_i^-) + o(\varepsilon^q) = \frac{n}{2}(\Delta b^- - \Delta a^+)\varepsilon^{-2} + o(\varepsilon^q),$$

$$\Delta c^- - \Delta a^- = \sum_{\substack{0\le i<n \\ i:\mathrm{even}}} (\Delta t_{i+2}^- - \Delta t_i^-) + o(\varepsilon^q) = \frac{n}{2}(\Delta b^+ - \Delta a^-)\varepsilon^{-2} + o(\varepsilon^q).$$

Since $\Delta d^+ - \Delta c^- = \Delta b^+ - \Delta a^- + o(\varepsilon^q)$ and $\Delta d^- - \Delta c^+$, $\Delta b^- - \Delta a^+ \in O(\varepsilon^{q+2})$, we obtain the required formula. ∎

The next Proposition shows when take small perturbation of τ_1 as power series in ε (note that τ_1 determine the whole tree), this perturbation affects $b_l^+ - a_l^-$ and $b_l^- - a_l^+$ such that $\Delta b_l^+ - \Delta a_l^-$ and $\Delta b_l^- - \Delta a_l^+$ have the same ratio as A_j and B_j $(j<l)$.

PROPOSITION 6.5. *If Δa_1^+, Δa_1^-, Δb_1^+, $\Delta b_1^- \in O(\varepsilon^q)$, $\Delta b_1^+ - \Delta a_1^-$, $(\Delta b_1^- - \Delta a_1^+)\varepsilon^{-2} \in O(\varepsilon^q)$, $\notin o(\varepsilon^q)$ and*

$$\Delta b_1^+ - \Delta a_1^- = -\nu^2(\Delta b_1^- - \Delta a_1^+)\varepsilon^{-2} + o(\varepsilon^q)$$

$$A_j = \begin{cases} -\nu^2 B_j, & j:\ odd \\ -\nu^{-2} B_j, & j:\ even \end{cases}$$

for some nonzero complex number ν and $j=1, 2, \cdots, l-1$, then $\Delta b_l^+ - \Delta a_l^-$, $(\Delta b_l^- - \Delta a_l^+)\varepsilon^{-2} \in O(\varepsilon^{q-l+1})$, $\notin o(\varepsilon^{q-l+1})$ and

$$\Delta b_l^+ - \Delta a_l^- = \begin{cases} -\nu^2(\Delta b_l^- - \Delta a_l^+)\varepsilon^{-2} + o(\varepsilon^{q-l+1}), & l:\ odd \\ -\nu^{-2}(\Delta b_l^- - \Delta a_l^+)\varepsilon^{-2} + o(\varepsilon^{q-l+1}), & l:\ even. \end{cases}$$

PROOF. We proceed by induction on l. For simplicity we can put $\nu=1$.

By Lemma 6.2 we obtain Δa_j^+, Δa_j^-, $\cdots$, Δd_j^+, $\Delta d_j^- \in O(\varepsilon^{q-j+1})$ inductively. Hence some argument shows

$$\Delta b_l^+ - \Delta a_l^- = -(\Delta c_{l-1}^+ - \Delta a_{l-1}^+)\varepsilon^{-1} + o(\varepsilon^{q-l+1})$$

$$(\Delta b_l^- - \Delta a_l^+)\varepsilon^{-2} = -(\Delta c_{l-1}^- - \Delta a_{l-1}^-)\varepsilon^{-1} + o(\varepsilon^{q-l+1})$$

by checking the relation in Γ.

Therefore the next claim is sufficient to complete the proof.

CLAIM.

$$\Delta c_{l-1}^+ - \Delta a_{l-1}^+ = \frac{n_{l-1}}{2}\operatorname{cosec}^2\frac{k_{l-1}\pi}{n_{l-1}}(\Delta b_{l-1}^- - \Delta a_{l-1}^+)\varepsilon^{-2} + o(\varepsilon^{q-l+1})$$

$$\Delta c_{l-1}^- - \Delta a_{l-1}^- = \frac{n_{l-1}}{2}\operatorname{cosec}^2\frac{k_{l-1}\pi}{n_{l-1}}(\Delta b_{l-1}^+ - \Delta a_{l-1}^-) + o(\varepsilon^{q-l+1})$$

When T_{l-1} is of parasol type, by Lemma 6.3 with $A_{l-1} = -B_{l-1}$, $\Delta a_{l-1}^+ \in o(\varepsilon^{q-l})$ we have

$$\begin{aligned}\varepsilon^{-2}\varDelta\tau_{l-1} &= -4A_{l-1}(\varDelta b_{l-1}^{-}-\varDelta a_{l-1}^{+})\varepsilon^{-2}+o(\varepsilon^{q-l+2}) \\ &= -4A_{l-1}(\varDelta b_{l-1}^{+}-\varDelta a_{l-1}^{-})+o(\varepsilon^{q-l+2})\end{aligned}$$

noting that $\varDelta b_{l-1}^{+}-\varDelta a_{l-1}^{-}=-(\varDelta b_{l-1}^{-}-\varDelta a_{l-1}^{+})\varepsilon^{-2}+o(\varepsilon^{q-l+2})$ which is the assumption of induction. With the formulas of Lemma 6.4 we obtain the required formulas.

When T_{l-1} is of broom type, just the formulas of Lemma 6.4 mean the required formulas with $k_{l-1}=n_{l-1}/2$. ■

REMARK 6.6. In the statement of Proposition 6.5, the assumption $\varDelta b_{1}^{+}-\varDelta a_{1}^{-}=-\nu^{2}(\varDelta b_{1}^{-}-\varDelta a_{1}^{+})\varepsilon^{-2}+o(\varepsilon^{q})$ is not necessary if there exist a broom type among $T_1, T_2, \cdots, T_{l-1}$.

At the end we prove the next lemma, which counts the number of ideal points.

LEMMA 6.7. *For any co-prime integers α, β such that $0<\beta<\alpha$, we have*

$$\sum_{[n_j]}\prod_{j}(|n_j|-1)=\alpha$$

where the sum is taken over all continued fraction expansion $[n_j]$'s of β/α.

PROOF. We put $m_{\alpha,\beta}=\sum\prod_j(|n_j|-1)$ where the sum is taken over continued fraction expansion $[n_j]$'s which have the form

$$\frac{\alpha}{\beta}=\cfrac{1}{n_1+\cfrac{1}{n_2+\ddots+\cfrac{1}{n_N}}}, \qquad \text{with} \quad |n_j|\geqq 2.$$

By definition, the left side of the required formula is $m_{\alpha,\beta}+m_{\alpha,\alpha-\beta}$. We shall show that this is equal to α.

We can assume $2\beta<\alpha$, replacing β by $\alpha-\beta$ if necessary. Let n, r be as in

$$\alpha=n\beta+r \qquad \text{with} \quad n\geqq 2, \qquad 0\leqq r<\beta.$$

We obtain

$$\begin{aligned}m_{\alpha,\beta} &= (n-1)m_{\beta,r}+nm_{\beta,\beta-r} \\ m_{\alpha-\beta,\beta} &= (n-2)m_{\beta,r}+(n-1)m_{\beta,\beta-r} \\ m_{\alpha,\alpha-\beta} &= m_{\alpha-\beta,\alpha-2\beta}\end{aligned}$$

because β/α, $\beta/(\alpha-\beta)$ and $(\alpha-\beta)/\alpha$ have continued fractions as follows,

$$\frac{\beta}{\alpha}=\frac{1}{n+(r/\beta)}=\frac{1}{n+1-(\beta-r)/\beta}$$

$$\frac{\beta}{\alpha-\beta}=\frac{1}{n-1+(r/\beta)}=\frac{1}{n-(\beta-r)/\beta}$$

$$\frac{\alpha-\beta}{\alpha}=\frac{1}{2-(\alpha-2\beta)/(\alpha-\beta)}.$$

Therefore we have,

$$m_{\alpha,\beta}+m_{\alpha,\alpha-\beta}=(m_{\alpha-\beta,\beta}+m_{\alpha-\beta,\alpha-2\beta})+(m_{\beta,r}+m_{\beta,\beta-r}).$$

We proceed by induction on α, to obtain $m_{\alpha,\beta}+m_{\alpha,\alpha-\beta}=\alpha$. ■

References

[1] M. Bestvina, Degenerations of the hyperbolic space, Duke Math. J., **56** (1988), 143-161.
[2] G. Burde, $SU(2)$-representation spaces for two-bridge knot groups, Math. Ann., **288** (1990), 103-119.
[3] G. Burde and H. Zieschang, "Knots," Studies in Mathematics 5, de Gruyter, 1985.
[4] M. Culler, C. M. Gordon, J. Luecke and P. B. Shalen, Dehn surgery on knots, Ann. of Math., **125** (1987), 237-300.
[5] M. Culler and J. W. Morgan, Group actions on $\boldsymbol{R}$-trees, Proc. London Math. Soc. (3), **55** (1987), 571-604.
[6] M. Culler and P. B. Shalen, Varieties of group representations and splittings of 3-manifolds, Ann. of Math., **117** (1983), 109-146.
[7] A. Hatcher and W. Thurston, Incompressible surface in 2-bridge knot complements, Invent. Math., **79** (1985), 225-246.
[8] D. Mumford, An analytic construction of degenerating curves over complete local rings, Compositio Math., **24** (1972), 129-174.
[9] R. Riley, Nonabelian representations of 2-bridge knot groups, Quart. J. Math. Oxford Ser. (2), **35** (1984), 191-208.
[10] R. Riley, Algebra for Heckoid groups, Trans. Amer. Math. Soc., **334** (1992), 389-409.
[11] J. P. Serre (with H. Bass), "Trees", Springer-Verlag, New York, 1980.
[12] M. Takahashi, Two-bridge knots have property P, Mem. Amer. Math. Soc., **29-239** (1981).
[13] M. Tretkoff, A topological approach to the theory of groups acting on trees, J. Pure Appl. Algebra, **16** (1980), 323-333.

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