J. Math. Soc. Japan
Vol. 45, No. 2, 1993

Elliptic differential inequalities with applications to harmonic maps

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(Received Aug. 13, 1991)
(Revised April 10, 1992)

Introduction.

Harmonic maps $\phi:(M, g)\rightarrow(N, h)$ between Riemannian manifolds are the smooth critical points of the *energy functional*

$$E(\phi)=\int_{M}e(\phi)dV_{M}\,,$$

where $e(\phi)=(1/2)|d\phi|^{2}$ is the *energy density* of ϕ. Or, equivalently, the C^{2} solutions of the elliptic system

$$\text{(0.1)}\qquad \mathrm{Trace}_{g}\nabla d\phi=0\,.$$

The left-hand side of (0.1) is the *tension field* of ϕ, denoted $\tau(\phi)$; it is a vector field along ϕ: we refer to the surveys [**5**], [**6**] for complete definitions and background.

Since the pioneering work of Eells and Sampson ([**7**] (1964)), harmonic maps have attracted the interest of both geometers and analysts: during the early stages of the theory, research was focused on maps between compact manifolds. Indeed, in a compact setting a harmonic map provides a strong candidate for a "best map" in a prescribed homotopy class; and a natural generalization of the concept of closed geodesic.

More recently, harmonic maps of non-compact domains have become object of growing interest: as a significant example, we quote the discovery of a new family of harmonic maps $\phi:\boldsymbol{R}^{2}\rightarrow\boldsymbol{H}^{2}$ of rank two almost everywhere; that was obtained by Choi and Treibergs [**4**], using a version of Ruh-Vilms' Theorem for constant mean curvature hypersurfaces of Minkowski 3-space. It is natural to view the study of harmonic maps of non-compact domains as a generalization of the theory of harmonic functions $f: M\rightarrow\boldsymbol{R}$ on complete Riemannian manifolds [**18**]; however, we point out two key differences:

a) a single equation — i.e., $\Delta f=0$ — is replaced by a system — i.e., (0.1).

b) the curvature of the range plays a role, making system (0.1) non-linear.

Nevertheless, Liouville's type Theorems for harmonic maps have been obtained ([**3**], [**16**], [**17**]); more generally, one expects relations between the growth at ∞ of solutions — or of their energy density — and the geometry of the manifolds.

In this paper we undertake this type of study under the hypothesis that the domain M is a complete, m-dimensional Riemannian manifold such that Ricci $(M) \geqq -AG(r)$, where A is a positive constant and r denotes distance from a fixed point $q \in M$ (the choice of q plays no role in what follows); $G(r)$ is a positive, non-decreasing function such that $G(0)=1$ and $G(r) \to +\infty$ as $r \to +\infty$. In the sequel, we shall always abbreviate this by simply writing that M satisfies Ricci $(M) \geqq -AG(r)$.

We prove our results by studying the relations between the growth of G at $+\infty$ and the existence of C^2 solutions on M of differential inequalities of the type

$$\Delta u \geqq b(x)\varphi(u), \qquad x \in M \tag{0.2}$$

(the sign convention is $\Delta = \operatorname{div}(\nabla u)$). Osserman [**12**] and Redheffer ([**14**], [**15**]) studied (0.2) in the case $M = \boldsymbol{R}^m$, while Calabi [**1**] analysed the case Ricci $(M) \geqq 0$: their use of the maximum principle has inspired our work.

In Section 2 we obtain a priori estimates for the energy density of bounded harmonic maps $\phi: M \to N$, where N has non-positive sectional curvature ((2.12) below). As a by-pass product of our analysis we also obtain refinements (Theorem 2.17 and Corollary 2.24) of results of [**2**] and [**9**] on the image diameter of maps with bounded tension field; in the case of isometric immersions this also complements work of Karp [**10**].

In Section 3 we illustrate further applications and extensions to rotationally symmetric manifolds (i.e., models in the sense of [**8**]); in particular, we extend work of Tachikawa ([**16**], [**17**]), proving non-existence results for certain harmonic maps into Hadamard manifolds or models (see (3.26), (3.37-41) below).

Most of the technicalities of this paper rely on the analysis (Section 1) of an O.D.E. which arise from the study of rotationally symmetric solutions of (0.2): reading Sections 2 and 3 requires some familiarity with notation and facts of Section 1.

We also remark that the methods of this paper can be applied to study other elliptic equations of geometric interest; in particular, they yield some non-existence results for the non-compact Yamabe problem, as we shall illustrate in a forthcoming paper. Finally, we mention here that the works [**10**], [**11**], [**13**] deal — by different methods — with problems related to this paper.

1. Analysis of the O.D.E..

In this section we establish some qualitative properties of solutions of (1.1) below: the key technical result is Proposition 1.11.

$$\alpha''(t)+(m-1)[\tilde{g}'(t)/\tilde{g}(t)]\alpha'(t) = f(\alpha(t)) \tag{1.1}$$
$$\alpha'(0) = 0\,, \qquad \alpha(0) = \alpha_0\,, \qquad t \geqq 0$$

where $m\geqq 2$, $f\in \mathrm{Lip}_{\mathrm{loc}}(\boldsymbol{R})$, f is non-decreasing and nonnegative; $\tilde{g}\in C^1([0, +\infty))$, $\tilde{g}>0$ on $(0, +\infty)$, $\tilde{g}(0)=0$ and $\tilde{g}'(0)>0$. Unless otherwise specified, in the sequel we shall tacitly assume that the above assumptions on f, $\tilde{g}$ and m hold (but we note that the assumption f non-decreasing is unnecessary in Lemma 1.2 below).

LEMMA 1.2. *The Cauchy problem* (1.1) *has a unique solution* α *which is defined on a maximal interval* $[0, T)$. *Moreover, if* $f(\alpha_0)>0$, *then* $\alpha'>0$ *on* $(0, T)$; *and if* $T<+\infty$, *then* $\alpha(t)\to+\infty$ *as* $t\to T^-$.

PROOF. First we write (1.1) in integral form

$$\alpha(t) = \alpha_0+\int_0^t [\tilde{g}(s)]^{1-m}\left\{\int_0^s [\tilde{g}(u)]^{m-1} f(\alpha(u))du\right\}ds\,. \tag{1.3}$$

Existence and uniqueness for small t is standard: it can be obtained by applying the Picard iteration procedure. To see that $\alpha'(t)>0$ for $t>0$, we write (1.1) as

$$(\tilde{g}^{m-1}\alpha')' = \tilde{g}^{m-1}f(\alpha)\,.$$

Integrating over $[0, t]$ and using $\alpha'(0)=0$ we find

$$[\tilde{g}(t)]^{m-1}\alpha'(t) = \int_0^t [\tilde{g}(s)]^{m-1}f(\alpha(s))ds$$

from which the assertion follows immediately. Finally, let $T<+\infty$ and suppose that $\alpha(t)\to c<+\infty$ as $t\to T^-$. Then $[\tilde{g}(s)]^{m-1}f(\alpha(s))\in L^1([0, T])$; therefore differentiating (1.3) we find that $\alpha'(t)$ converges to a finite limit as $t\to T^-$, a fact which contradicts the maximality of $[0, T)$. //

LEMMA 1.4. *Let* α *be a solution of* (1.1) *on* $[0, +\infty)$ *such that* $\alpha(t)\to+\infty$ *as* $t\to+\infty$. *Suppose that* $\tilde{g}'\geqq 0$ *and*

$$([f(t)]^{\eta}/t) \longrightarrow +\infty \quad \text{as } t\to+\infty, \quad \text{for some } \eta>0\,. \tag{1.5}$$

Then for any $c>0$ *there exists* $\tau_0\in(0, +\infty)$ *such that*

$$\{1-(m-1)(2/c)^{1/2}[\tilde{g}'(t)/\tilde{g}(t)][f(\alpha(t))]^{(\eta-1)/2}\}f(\alpha(t)) < \alpha''(t) \tag{1.6}$$

for all $t \geqq \tau_0$.

PROOF. Because $\alpha' \geqq 0$ and $\tilde{g}' \geqq 0$, (1.1) implies $f(\alpha)\alpha' \geqq \alpha''\alpha'$. Integrating this inequality over $[0, t]$ and using $\alpha'(0)=0$, $\alpha'>0$ on $(0, +\infty)$ we obtain

$$2\int_{\alpha_0}^{\alpha(t)} f(s)ds \geqq [\alpha'(t)]^2. \tag{1.7}$$

Given $c>0$, (1.5) guarantees the existence of t_0 such that

$$[f(t)]^{\eta} > c(t-\alpha_0) \qquad \text{for all } t \geqq t_0.$$

We choose $\tau_0 \geqq t_0$ in such a way that $\alpha(t) \geqq t_0$ for all $t \geqq \tau_0$; it follows that

$$[f(\alpha(t))]^{\eta} > c[\alpha(t)-\alpha_0] \qquad \text{for all } t \geqq \tau_0. \tag{1.8}$$

From now on let $t \geqq \tau_0$. From (1.7) and f non-decreasing we get

$$[\alpha'(t)]^2 \leqq 2f(\alpha(t))[\alpha(t)-\alpha_0].$$

Thus, applying (1.8) and elevating to 1/2, we obtain

$$\alpha'(t) < (2/c)^{1/2}[f(\alpha(t))]^{(\eta+1)/2}. \tag{1.9}$$

Now multiplying (1.9) by $(m-1)[\tilde{g}'/\tilde{g}]$ and using (1.1) gives (1.6). //

(1.10) In order to measure the rate of growth of $[\tilde{g}'(t)/\tilde{g}(t)]$ as $t \to +\infty$ it is convenient to introduce two classes $\mathcal{F}$, $\mathcal{G}$ of C^1 functions F defined in a neighbourhood of $+\infty$: namely, we say that $F \in \mathcal{G}$ if $[F]^{-1} \notin L^1(+\infty)$, $F'(t) \geqq 0$ (t large) and $F(t) \to +\infty$ as $t \to +\infty$. And $F \in \mathcal{F}$ if $F \in \mathcal{G}$ and furthermore

$$\lim_{t \to +\infty} F'(t)[F'(t)]^{-\varepsilon} \in \boldsymbol{R} \qquad \text{for any } \varepsilon > 0.$$

Examples of $F(t) \in \mathcal{F}$ are: t, $t \log t$, $t[\log t][\log(\log t)]$, $\cdots$.

NOTATION. $[\tilde{g}'/\tilde{g}] = \mathcal{O}(k)$ means that $[\tilde{g}'(t)/\tilde{g}(t)]/k(t) \to 0$ as $t \to +\infty$.

PROPOSITION 1.11. *Let* α *be a solution of* (1.1) *such that* $f(\alpha_0)>0$. *Assume that*

i) $([f(t)]^{\eta}/t) \longrightarrow +\infty$ *as* $t \to +\infty$, *for some* $0<\eta<1$;

ii) *there exists* $0<\gamma<(1-\eta)/(1+\eta)$ *and a nonnegative function* $D(t)$ *such that*

$$[\tilde{g}'(t)/\tilde{g}(t)] \leqq D(t); \text{ and } D(t) = \mathcal{O}(F^{\gamma}) \qquad \text{for some } F \in \mathcal{F} \text{ as in (1.10).}$$

Then α *is defined on a maximal interval* $[0, T)$ *with* $T<+\infty$.

PROOF. For technical reasons (the application of Lemma 1.4) we begin with proving the Proposition under the additional hypothesis that $\tilde{g}' \geqq 0$. We define

$$h(t) = [\tilde{g}(t)]^{1-m}\int_0^t [\tilde{g}(s)]^{m-1} ds\,, \qquad k(t) = [\tilde{g}'(t)/\tilde{g}(t)]\left\{\int_0^t h(s)ds\right\}^{-\delta/\eta}$$

where $\delta=(1-\eta)/2$; for a moment, let us suppose that

iii) $h(t)\notin L^1(+\infty)$; iv) $k(t)\in L^\infty(+\infty)$ and v) $\left\{\int_0^t f(s)ds\right\}^{-1/2}\in L^1(+\infty)$.

We show that iii), iv) and v) together imply the Proposition: by contradiction, let $T=+\infty$; $\alpha'\geqq 0$ and f non-decreasing force $f(\alpha(t))\geqq f(\alpha_0)$ for all $t\geqq 0$. Therefore from (1.3) we have

$$\alpha(t) \geqq \alpha_0+f(\alpha_0)\int_0^t h(s)ds\,. \tag{1.12}$$

Now (1.12) together with iii) imply that $\alpha(t)\to+\infty$ as $t\to+\infty$, so that the hypotheses of Lemma 1.4 are satisfied. Moreover, for t large, $f(\alpha(t))\geqq[\alpha(t)]^{1/\eta}$ by i). It follows that

$$[\tilde{g}'/\tilde{g}][f(\alpha)]^{-\delta} \leqq [\tilde{g}'/\tilde{g}][\alpha]^{-\delta/\eta} \leqq c_1 k$$

for some $c_1>0$. Applying iv) and (1.6) with a sufficiently large $c>0$ we obtain the existence of $B>0$ such that

$$Bf(\alpha) < \alpha'' \qquad \text{for all } t\geqq\tau_0,\ \tau_0 \text{ large.} \tag{1.13}$$

Multiplying both members of (1.13) by α' and integrating over $[\tau_0, t]$ gives

$$(2B)\int_{\alpha(\tau_0)}^{\alpha(t)} f(s)ds+[\alpha'(\tau_0)]^2 < [\alpha'(t)]^2\,. \tag{1.14}$$

Because $\alpha'>0$ on $[\tau_0, t]$, (1.14) gives

$$\alpha'(t)\left\{(2B)\int_{\alpha(\tau_0)}^{\alpha(t)} f(s)ds+[\alpha'(\tau_0)]^2\right\}^{-1/2} > 1\,. \tag{1.15}$$

Integrating (1.15) over $[\tau_0, \tau]$ we obtain

$$\int_{\alpha(\tau_0)}^{\alpha(\tau)}\left\{(2B)\int_{\alpha(\tau_0)}^{u} f(s)ds+[\alpha'(\tau_0)]^2\right\}^{-1/2} du > \tau-\tau_0\,. \tag{1.16}$$

Letting $\tau\to+\infty$ we see that (1.16) contradicts v): so (if $\tilde{g}'\geqq 0$) the proof is complete provided that we show that iii), iv) and v) hold.

Proof of iii). If $\tilde{g}(t)$ is bounded the conclusion is obvious. So we assume that $\tilde{g}(t)$ tends to $+\infty$ as t goes to $+\infty$: since $[F]^{-1}\notin L^1(+\infty)$, also $[F]^{-\gamma}\notin L^1(+\infty)$ for γ as in (1.11) ii): therefore it is enough to show that

$$[F]^\gamma h(t) \longrightarrow +\infty \qquad \text{as } t\to+\infty \tag{1.17}$$

Now, using the explicit expression of $h(t)$, (1.17) follows easily from de l'Hôpital's rule, (1.10) and (1.11) ii).

Proof of iv). Using (1.17) we deduce that

$$k(t) \leqq c_2+[\tilde{g}'(t)/\tilde{g}(t)]\left\{\int_0^t [F(s)]^{-\gamma} ds\right\}^{-\delta/\eta} \qquad \text{for some } c_2>0\,.$$

Because of (1.11) ii) it suffices to show that $[F(t)]^{\eta\gamma/\delta}/\left\{\int_0^t [F(s)]^{-\gamma} ds\right\}$ converges to a finite limit as $t\to+\infty$: but this follows easily from de l'Hôpital's rule and the fact that $F'F^{-\varepsilon}$ converges for all $\varepsilon>0$ because $F\in\mathcal{F}$.

Proof of v). Clearly $\int_0^t f(s)ds\to+\infty$ as $t\to+\infty$, because $f(\alpha_0)>0$ and f is non-decreasing. Since $0<\eta<1$ we can choose $\sigma>0$ such that $[2\sigma+1-(1/\eta)]<0$. Now we apply de l'Hôpital's rule and (1.11) i) to obtain

$$\lim_{t\to+\infty}\left\{t^{2\sigma+2}/\int_0^t f(s)ds\right\} = \lim_{t\to+\infty}(2\sigma+2)\{t^{1/\eta}/f(t)\}\,t^{2\sigma+1-(1/\eta)} = 0$$

from which v) follows.

Finally, we show that the assumption $\tilde{g}'\geqq 0$ is unnecessary. Indeed, we can consider

$$\alpha''(t)+(m-1)D(t)\alpha'(t) = f(\alpha(t))$$

$$\alpha'(0) = 0\,, \qquad \alpha(0) = \alpha_0\,, \qquad f(\alpha_0) > 0$$

where $D(t)$ is a suitable function as in (1.11) ii). The previous argument (with $\exp\left[\int_1^t D(s)ds\right]$ in place of $\tilde{g}(t)$) tells us that the unique solution of this Cauchy problem is defined on a maximal interval $[0, T_1)$ with $T_1<+\infty$. Now standard comparison arguments (using $[\tilde{g}'(t)/\tilde{g}(t)]\leqq D(t)$) imply that the solution of the original problem (1.1) blows up in finite time $T\leqq T_1$. //

A modification of the arguments of Proposition 1.11 gives

LEMMA 1.18. *Let α be a solution of* (1.1) *which is defined on* $[0, +\infty)$, *with $f(\alpha_0)>0$. Suppose that there exists a nonnegative function $D(t)$ such that $[\tilde{g}'(t)/\tilde{g}(t)]\leqq D(t)$; and $D(t)=\mathcal{O}(F)$ for some $F\in\mathcal{G}$ as in* (1.10). *Then $\alpha(t)\to+\infty$ as $t\to+\infty$.*

REMARKS 1.19. a) Hypothesis (1.1) ii) is quite sharp, as the following example shows:

$$\alpha''(t)+\{[(t^2+3)^\delta-2]/2t\}\,\alpha'(t) = [\alpha(t)]^\delta\,, \qquad \delta>1$$

$$\alpha'(0) = 0\,, \qquad \alpha(0) = 3$$

admits the global solution $\alpha(t)=t^2+3$ $\left(\text{here } \tilde{g}(t)=\exp\left(\int_1^t \{[(s^2+3)^\delta-2]/2s\}\,ds\right)\right)$.

b) If $\tilde{g}'(t)\geqq 0$, then the natural choice for the function $D(t)$ is $D(t)=$

$[\tilde{g}'(t)/\tilde{g}(t)]$; in general, the function $D(t)$ serves a technical purpose of comparison, based on the fact that $\tilde{g}'(t)<0$ — and, all the more reason, $\tilde{g}'(t)\to-\infty$ as $t\to+\infty$ — is a condition which contributes to a faster growth of solutions and so to their blowing up in finite time.

The following is a standard fact:

LEMMA 1.20. *Suppose that the function* f *in* (1.1) *satisfies*

$$f(s)\leqq a_1 s+a_2 \qquad \textit{for some } a_1,\, a_2>0\,. \tag{1.21}$$

Then any solution $\alpha(t)$ *of* (1.1) *is defined for all* $t\geqq 0$.

(1.22) For our purposes it will be useful to consider a variant of (1.1): namely, let $a\in C^1([0,+\infty))$ be a positive function such that $a^{1/2}\notin L^1(+\infty)$. We consider

$$\beta''(r)+(m-1)[g'(r)/g(r)]\beta'(r)=a(r)f(\beta(r)) \tag{1.23}$$

and set

$$h(r)=\int_0^r [a(s)]^{1/2}ds\,. \tag{1.24}$$

Then the change of variable $h(r)=t$, $t\in[0,+\infty)$, defines a bijection between solutions of

$$\alpha''(t)+(m-1)[\tilde{g}'(t)/\tilde{g}(t)]\alpha'(t)=f(\alpha(t)) \tag{1.25}$$

where

$$\alpha(t)=\beta(h^{-1}(t))\,,\qquad \tilde{g}(t)=g(r)[a(r)]^{1/(2m-2)} \quad\text{and} \tag{1.26}$$

$$[\tilde{g}'(t)/\tilde{g}(t)]=[a(r)]^{-1/2}\{[g'(r)/g(r)]+[1/2(m-1)][a'(r)/a(r)]\}\,. \tag{1.27}$$

The proof of these facts is a straightforward computation and therefore we omit it. We observe that (1.25) is of type (1.1); thus we can apply (modulo the change of variable $t=h(r)$) the results of this section to (1.23).

REMARK 1.28. The methods of this section apply to the more general Cauchy problem

$$|\alpha'(t)|^{-p}\{\alpha''(t)+(m-1)[\tilde{g}'(t)/\tilde{g}(t)]\alpha'(t)\}=f(\alpha(t)) \tag{1.29}$$

$$\alpha'(0)=0\,,\qquad \alpha(0)=\alpha_0\,,\qquad t\geqq 0\,.$$

In particular, Proposition 1.11 holds in this case provided that $p<1$, $0<\eta<1/(1-p)$, $0<\gamma<[1-(1-p)\eta]/[1+(1-p)^2\eta]$ and $[\tilde{g}'/\tilde{g}]=O(F^{\gamma(1-p)})$.

Equation (1.29) would permit us to study inequalities of the type

$$\Delta u \geqq b(x)\varphi(u)|\nabla u|^p, \qquad x \in M \tag{1.30}$$

which arise in the study of the operator $\operatorname{div}(|\nabla u|^{-p}\nabla u)$. However, we shall not pursue this generalization in this paper, because the case $p=0$ suffices for the geometric applications of the next sections.

2. Estimates for harmonic maps.

Differential equations of type (1.1) arise in geometry from problems involving Δr, where r is the distance function from a fixed point $q \in M$. We will use the following estimate which can be derived from [8]: suppose that $\operatorname{Ricci}(M) \geqq -AG(r)$, as in the introduction; then, at each $x \notin C_q$ (the cut locus of q), we have

$$\Delta r \leqq (m-1)[g'(r)/g(r)] \tag{2.1}$$

where $m = \dim M$ and, setting $\Omega = (A/(m-1))^{1/2}$,

$$g(r) = [\sinh(\Omega r)] \exp \int_0^r \Omega \coth(\Omega s)\{[1+(G(s)-1)\tanh^2(\Omega s)]^{1/2}-1\}\, ds.$$

We observe that

$$[g'(r)/g(r)] \approx \Omega[G(r)]^{1/2} \qquad \text{as } r \to +\infty\text{; and} \tag{2.2}$$

also recall that if $\operatorname{Ricci}(M) \geqq 0$, then (2.1) holds with $g(r)=r$; and if $\operatorname{Ricci}(M) \geqq -A$, $A>0$, we can take $g(r)=\sinh(\Omega r)$.

LEMMA 2.3. [**15**] *Let M be a complete Riemannian manifold and u a C^2 solution on M of the differential inequality*

$$\Delta u \geqq b(x)\varphi(u), \qquad x \in M, \tag{2.4}$$

where $b(x) \geqq 0$, $b \not\equiv 0$ and $\varphi \geqq 0$. Fix $q \in M$ and let r be the distance from q: If there exists a C^2 function v such that, for some $R>0$,

$$\Delta v < b \qquad \text{on } M/B_R(q) \quad \text{and} \tag{2.5}$$

$$v(x) \longrightarrow +\infty \qquad \text{as } r(x) \to +\infty, \tag{2.6}$$

then either $\sup_M\{u\} = +\infty$ or $\sup_M\{u\} \in Z(\varphi) = \{t \in \boldsymbol{R} : \varphi(t)=0\}$.

PROOF. This was proved by Redheffer in case $M = \boldsymbol{R}^m$ ([**15**], Theorem 1): in this general case the proof is essentially the same and therefore omitted. //

In the notation of the introduction and Section 1, we have

LEMMA 2.7. *Assume* $\operatorname{Ricci}(M) \geqq -AG(r)$. *Let $a(r)$ be a function as in* (1.22) *and suppose that*

$$(2.8)\qquad [\tilde{g}'(t)/\tilde{g}(t)] = \mathcal{O}(F(t))\,,$$

where $t=h(r)$, *as in* (1.24); $[\tilde{g}'/\tilde{g}]$ *is defined by* (1.27) *with* g *as in* (2.1); *and* $F\in\mathcal{G}$ *as in* (1.10). *Consider inequality* (2.4) *and furthermore suppose that* $b(x)\geqq a(r(x))$ *on* $M/B_R(q)$, *for some* $R>0$. *If* u *is a* C^2 *solution of* (2.4) *then either* $\sup_M\{u\}=+\infty$ *or* $\sup_M\{u\}\in Z(\varphi)=\{t\in\boldsymbol{R}\,;\,\varphi(t)=0\}$. *Moreover, in the special case* $a\equiv 1\equiv b$, *the conclusion holds with* (2.8) *replaced by*

$$(2.9)\qquad [G(t)]^{1/2} = \mathcal{O}(F(t))\,.$$

PROOF. We proceed to the construction of a function v as in Lemma 2.3. Let β be the unique solution of

$$(2.10)\qquad \beta''(r)+(m-1)[g'(r)/g(r)]\beta'(r) = (1/2)a(r)$$

determined by $\beta(0)=0$, $\beta'(0)=0$. Equation (2.10) is of type (1.23), with $f\equiv 1/2$: so we can transform it into (1.25) (via (1.24)) and apply Lemma 1.20 to conclude that β is defined for all $r\geqq 0$. Moreover, (2.8) enables us to apply Lemma 1.18 and deduce that $\beta(r)\to+\infty$ as $r\to+\infty$. Next, set $v(x)=\beta(r(x))$; we compute using Gauss Lemma, (2.1) and (2.10) to get

$$\Delta v = \beta''(r)+\beta'(r)\Delta r \leqq \beta''(r)+(m-1)[g'(r)/g(r)]\beta'(r) = (1/2)a(r) < a(r) \leqq b$$

outside some $B_R(q)$. Thus we can apply Lemma 2.3 to conclude. If furthermore $a(r)\equiv 1$, then $t=h(r)=r$; so $\tilde{g}=g$ and (2.2) tells us that in this case (2.9) is equivalent to (2.8). //

(2.11) Let N be a complete Riemannian manifold such that $\mathrm{Riem}\,N\leqq K$, for some nonpositive constant K. We study harmonic maps $\psi: M\to N$ under the assumption $\mathrm{Ricci}\,(M)\geqq -AG(r)$. We say that any such ψ is bounded if its image is relatively compact in N. We have

THEOREM 2.12. *Let* $\psi: M\to N$ *be a harmonic map between manifolds as in* (2.11). *Suppose that*

$$e(\psi)(x) \geqq [\varepsilon+r(x)]^{-2d} \qquad \textit{outside } B_R(q)\,,$$

for some R, $\varepsilon>0$ *and* $d\leqq(1/2)$. *If*

$$(2.13)\qquad [t]^{d/(1-d)}[G(t^{1/(1-d)})]^{1/2} = \mathcal{O}(F(t))$$

for some $F\in\mathcal{G}$ *as in* (1.10), *then* ψ *is unbounded.*

PROOF. Let ρ be the distance in N from $\psi(q)$. We prove the theorem for $K<0$ (the case $K=0$ is similar). Without loss of generality we can assume $K=-1$: setting $h=(\cosh\rho)/2$ and $u=h\circ\psi$, we compute (see [**5**])

$$(2.14)\qquad \Delta u = \textstyle\sum_i \mathrm{Hess}\,(h)(\psi_* e_i,\, \psi_* e_i)+dh(\tau(\psi))$$

where $\{e_i\}$, $1\leqq i\leqq m$, is a local orthonormal frame in TM. But $\tau(\psi)=0$, because ψ is harmonic: thus, applying the Hessian comparison theorem [8] to (2.14), we obtain

$$\Delta u \geqq (\cosh \rho)e(\psi) \geqq e(\psi)\,. \tag{2.15}$$

Now we show that we can apply Lemma 2.7 with $\varphi\equiv 1$, $b=e(\psi)$ and $a(r)=[\varepsilon+r]^{-2d}$: indeed, (1.24) is explicitly integrable and gives

$$t = h(r) = (1/(1-d))[(\varepsilon+r)^{1-d}-\varepsilon^{1-d}]\,;$$

from this, together with (1.27) and (2.2) it is not difficult to see that there exists $c_1>0$ such that

$$0 \leqq [\tilde{g}'(t)/\tilde{g}(t)] \leqq c_1[t]^{d/(1-d)}[G(t^{1/(1-d)})]^{1/2} \qquad \text{for } t \text{ large}\,; \tag{2.16}$$

this latter is $\mathcal{O}(F(t))$ by hypothesis (2.13); thus (2.8) holds and we can apply Lemma 2.7 (with $\varphi\equiv 1$) to conclude that u — and so ψ — is unbounded. //

THEOREM 2.17. *Assume* $\operatorname{Ricci}(M)\geqq -AG(r)$, *with* $[G(r)]^{1/2}=\mathcal{O}(F(r))$, *for some* $F\in\mathcal{G}$ *as in* (1.10). *Let N be a Riemannian manifold such that* $\operatorname{Riem} N\leqq K$, $K\in\boldsymbol{R}$; *and let $B_R(\tilde{q})$ be a geodesic ball centered at $\tilde{q}\in N$ and inside the cut locus of $\tilde{q}$ ($R<\pi/2(K)^{1/2}$ if $K>0$). If $\psi: M\to N$ is a smooth map with $|\tau(\psi)|\leqq\tau_0$, $\tau_0\in[0, +\infty)$, and $\psi(M)\subset B_R(\tilde{q})$, then setting $\chi=\inf_M\{e(\psi)\}$*

$$R \geqq (K)^{-1/2}\tan^{-1}\{2(K)^{1/2}\chi/\tau_0\} \qquad \textit{when } K>0\,; \tag{2.18}$$

$$R \geqq 2\chi/\tau_0 \qquad \textit{when } K=0\,; \tag{2.19}$$

$$R \geqq (-K)^{-1/2}\tanh^{-1}\{2(-K)^{1/2}\chi/\tau_0\} \qquad \textit{when } K<0\,. \tag{2.20}$$

PROOF. Again, we only prove the theorem in the case $K=-1$ (the other cases are similar). Proceeding as in the proof of (2.12) we obtain (2.14) and deduce that

$$\Delta u \geqq u\{2e(\psi)+\tanh(\rho\circ\psi)\langle\nabla\rho, \tau(\psi)\rangle\} \tag{2.21}$$

Since $u\geqq(1/2)$ and $-\tanh(R)\tau_0\leqq\tanh(\rho\circ\psi)\langle\nabla\rho, \tau(\psi)\rangle$(using $|\nabla\rho|=1$), we have

$$\Delta u \geqq \chi-(1/2)\tanh(R)\tau_0\,. \tag{2.22}$$

Now, suppose that $\chi-(1/2)\tanh(R)\tau_0=C>0$: then we apply Lemma 2.7 with $a\equiv 1\equiv b$ and $\varphi\equiv C$ to conclude that u is unbounded — contradiction. Thus

$$\chi-(1/2)\tanh(R)\tau_0 \leqq 0$$

and (2.20) follows readily. //

REMARK 2.23. Theorem 2.17 was proved — with different methods — in [2] in the special case $G(r)=[1+\{r\log(r+2)\}^2]$ (compare with (1.10)). Similarly, Corollary 3.2, 3.5 and Theorems 3.3, 3.4 of [2] still hold if assumption Ricci $(M)\geqq -A[1+\{r\log(r+2)\}^2]$ is replaced by Ricci $(M)\geqq -AG(r)$ as in our Theorem 2.17. If ϕ is an isometric immersion, then $\tau(\phi)=mH$, where $m=\dim M$ and H is the mean curvature vector; the boundedness of $|H|$, together with Gauss equations, ensure that in this case the assumption Ricci $(M)\geqq -AG(r)$ can be substituted by the corresponding assumption on the scalar curvature: in particular (compare with Theorem 3.3 of [2]), we obtain

COROLLARY 2.24. *Let M be a complete, non-compact immersed submanifold of $\boldsymbol{R}^n$ with parallel mean curvature H and scalar curvature bounded below by $-AG(r)$, with $[G(r)]^{1/2}=\mathcal{O}(F(r))$, for some $F\in\mathcal{G}$ as in* (1.10). *If the image of the Gauss map $\gamma: M\to G_m(\boldsymbol{R}^n)$ lies in a geodesic ball $B_R(\tilde{q})$ with $R<\pi/(2\sqrt{B})$ (where $B=1$ if $n-m=1$ and $B=2$ otherwise), then M is minimal.*

REMARK 2.25. Let M be the 2-dimensional plane with metric $dr^2+k^2(r)d\theta^2$ and assume that $k(r)=\exp[r^2(\log r)]$ for $r\gg 1$. Since Ricci $(M)=-k''/k$, we see that Ricci $(M)\geqq -Ar^2(\log r)^2$ for $r\gg 1$ and some $A>0$. So we can apply Theorem 2.17 with $F(r)=r(\log r)(\log\log r)$; on the other hand, M has no sub-quadratic exponential growth. Indeed

$$\lim_{r\to+\infty}\{\log(\mathrm{Vol}\,B_r)\}/r^2=\lim_{r\to+\infty}\log r=+\infty .$$

Thus Theorem 2.17 extends Theorem 3.1 (and related Corollaries) of [10].

3. Applications to models and Hadamard manifolds.

(3.1) We begin with some differential geometric preliminaries: a *model* (see [8]) is a complete Riemannian manifold

$$M^m(g)=(S^{m-1}\times[0,+\infty),\ g^2(r)d\theta^2+dr^2),\qquad m\geqq 2, \tag{3.2}$$

where $d\theta^2$ is the standard metric of S^{m-1} and $g(r)$ is a smooth function, odd at the origin and such that

$$g(0)=0,\qquad g'(0)=1\ \text{ and }\ g(r)>0\qquad \text{for all } r>0. \tag{3.3}$$

The point of $M^m(g)$ corresponding to $r=0$ is called pole and denoted by p. If $g(r)=r$, $\sinh r$, $\sin r$ ($r\in[0,\pi/2)$), we have $M^m(g)=\boldsymbol{R}^m$, $\boldsymbol{H}^m$, S^m_+ respectively. (Of course, S^m_+ is not a model.)

(3.4) Let (N, ds^2) be a complete, n-dimensional Riemannian manifold; and let $B_R(q)$ be a geodesic ball inside the cut locus of $q\in N$: following [8] we say that $B_R(q)$ *dominates* an n-dimensional model $M^n(\tilde{k})$ if $z\in B_R(q)$, $y\in M^n(\tilde{k})$ and $\rho(z)=\tilde{\rho}(y)$ (ρ, $\tilde{\rho}$ distances from q and the pole of $M^n(\tilde{k})$ respectively) imply

$$K_{\mathrm{rad}}(z) \leqq K_{\mathrm{rad}}(y)\,, \tag{3.5}$$

where K_{rad} is the radial curvature. Under these hypotheses, the hessian comparison theorem and Proposition 2.20 of [8] give

$$\mathrm{Hess}\,(\rho)_z > \mathrm{Hess}\,(\tilde{\rho})_y = \tilde{k}'(\tilde{\rho}(y))\tilde{k}(\tilde{\rho}(y))d\theta^2 \tag{3.6}$$

where $d\theta^2$ is the standard metric of S^{n-1} and the symbol $>$ is explained in [8], p. 19.

LEMMA 3.7. *Let M, N be Riemannian manifolds, $p \in M$, $q \in N$,* $\dim N = n$, *and let ds^2 be the metric on N. Let ρ be the distance function from q and $B_R(q)$ a ball which dominates $M^n(\tilde{k})$ as in* (3.4). *Let $\lambda^2(z)$ be the minimum eigenvalue of $ds^2 - d\rho^2$ at $z \in B_R(q)$ and $\phi: M \to N$ a smooth map such that $\phi(p) = q$. Define $u = \rho \circ \phi$, $U = \{x \in M : u(x) \neq 0\}$, and $\xi = \pi \circ \phi$ on U, where $\pi: B_R(q) = [0, R) \times S^{n-1} \to S^{n-1}$ denotes projection on the second factor. Then on $U \cap \phi^{-1}(B_R(q))$*

$$\langle \nabla \rho, \tau(\phi) \rangle \leqq \Delta u - 2\tilde{k}'(u)\tilde{k}(u)e(\xi)[(\lambda^2 \circ \phi)/(\tilde{k}(u))^2]\,. \tag{3.8}$$

PROOF. A standard computation (see [5]) gives

$$\langle \nabla \rho, \tau(\phi) \rangle = \Delta u - \textstyle\sum_i \mathrm{Hess}\,(\rho)_\phi(d\phi(e_i), d\phi(e_i)) \tag{3.9}$$

where $\{e_i\}$ is a local orthonormal frame on M. Now we apply (3.6) to (3.9) to get

$$\langle \nabla \rho, \tau(\phi) \rangle \leqq \Delta u - \tilde{k}'(u)\tilde{k}(u) \textstyle\sum_i d\theta^2(d\tilde{\phi}(e_i), d\tilde{\phi}(e_i)) \tag{3.10}$$

where $d\tilde{\phi}(e_i)$ are defined as follows: Let θ^A, $A = 1, \cdots, n-1$, be a local orthonormal coframe for S^{n-1}; using polar coordinates (ρ, θ) we can express ds^2 on $B_R(q)$ in the form

$$ds^2 = d\rho^2 + [h^2_{AB}(\rho, \theta)]\theta^A \theta^B \tag{3.11}$$

(the sum over repeated indexes is understood). Because we perform the computations at a point $z = \phi(x) \neq q$, we can assume that we have diagonalized (3.11) by means of an orthogonal transformation of the θ^A's, so that at z

$$ds^2 = d\rho^2 + [h^2{}_A(\rho, \theta)][\theta^A]^2\,.$$

Let $\{E_A\}$ be the frame field dual to $\{\theta^A\}$. Then

$$d\phi(e_i) = B_i^0[\partial/\partial\rho] + B_i^A[h_A(\rho, \theta)]^{-1}E_A$$

with $|d\phi(e_i)|^2 = (B_i^0)^2 + \sum_A (B_i^A)^2$. It follows that we can define, at y,

$$d\tilde{\phi}(e_i) = B_i^0[\partial/\partial\tilde{\rho}] + B_i^A[\tilde{k}(\tilde{\rho}(y))]^{-1}E_A\,.$$

From this we deduce that

$$(3.12)\qquad \sum_i d\theta^2(d\tilde{\psi}(e_i),\ d\tilde{\psi}(e_i)) = \{\sum_{A,i}(B_i^A)^2\}/\tilde{k}^2(u)\,.$$

Now we observe that, on U, $2e(\xi)=\sum_{A,i}\{B_i^A/h_A\}^2$ and therefore

$$(3.13)\qquad 2e(\xi) \leqq \{\sum_{A,i}(B_i^A)^2\}/\lambda^2\,.$$

Now (3.8) follows from (3.10), (3.12) and (3.13). //

REMARK 3.14. In many instances $[\lambda^2/\tilde{k}^2]\geqq 1$ and so (3.8) yields the more manageable inequality

$$(3.15)\qquad \langle\nabla\rho,\ \tau(\psi)\rangle \leqq \Delta u - 2\tilde{k}'(u)\tilde{k}(u)e(\xi)\,.$$

For instance, a model N dominates itself with $[\lambda^2/\tilde{k}^2]\equiv 1$. Or else, let the sectional curvature on $B_R(q)$ be bounded above by $K\in\boldsymbol{R}$. We have the three cases $K<0$, $K=0$ and $K>0$ or, to simplify notation, $K=-1$, $K=0$ and $K=1$. Using Rauch comparison theorem we obtain $\lambda^2\geqq\sinh^2\rho$, $\lambda^2\geqq\rho^2$ and $\lambda^2\geqq\sin^2\rho$ respectively. Thus, choosing $\boldsymbol{H}^n$, $\boldsymbol{R}^n$ and S_+^n respectively as dominated "models", we find $[\lambda^2/\tilde{k}^2]\geqq 1$. Lemma 3.7 provides a key ingredient in the proof of the next results and can be applied to manifolds M, N in considerable generality: however, in order to limit technical assumptions on the cut locus of points, we shall only state and prove the next theorems for especially interesting choices of M and N, leaving to the interested reader the details of further possible extensions to the other cases covered by Lemma 3.7 (see also (3.40), as an example).

(3.16) Recall that a Hadamard manifold N is a complete, simply connected Riemannian manifold with non-positive sectional curvature; in particular, the cut locus of any point is empty. In case the sectional curvature is bounded above by $-B^2$, $B>0$, then N dominates $M_{\tilde{k}}^n$ with $\tilde{k}(r)=\sinh(Br)$.

THEOREM 3.17. *Let N be a Hadamard manifold. Suppose that $M^m(g)$ is a model such that*

$$(3.18)\qquad [g(r)]^{-1}\notin L^1(+\infty)\quad\text{and}\quad g'(r)=\mathcal{O}(F(h(r)))$$

for some $F\in\mathcal{G}$ as in (1.10) *and $h(r)=\int_0^r[1+g(s)]^{-1}ds$. Then there are no bounded harmonic maps $\psi\colon M^m(g)\to N$ such that $e(\xi)>0$ on $U=\{x\in M^m(g)\colon\psi(x)\neq\psi(p)\}$ and*

$$(3.19)\qquad e(\xi)\geqq[c/g^2]\quad\text{on } U\cap\{M^m(g)\setminus B_{R_0}(p)\}\quad\text{for some } c,\ R_0>0$$

(ξ is defined as in Lemma 3.7).

PROOF. By contradiction, suppose that there exists a harmonic map $\psi\colon M^m(g)\to N$ whose image is contained in $B_R(q)$, for $q=\psi(p)$ and some $R>0$.

By Remark (3.14) and (3.16) $B_R(q)$ dominates a model $M^n(\tilde{k})$ that, without loss of generality, we can assume to be either $\boldsymbol{H}^n$ or $\boldsymbol{R}^n$: in particular, we have

$$\tilde{k}'(\rho) > 0 . \tag{3.20}$$

Let $u=\rho\circ\psi$; because ψ is harmonic, $\tau(\psi)=0$. Thus, using Lemma 3.7 in the form (3.15), we obtain

$$\Delta u \geqq 2\tilde{k}'(u)\tilde{k}(u)e(\xi) \tag{3.21}$$

on the open, dense set U. By a version of Lemma 2.3 (with $\varphi(u)=2\tilde{k}'(u)\tilde{k}(u)$, $b(x)=e(\xi)(x)$) it suffices to construct a C^2 function v such that

$$\Delta v < e(\xi) \qquad \text{on } U\cap\{M^m(g)\setminus B_{R_0}(p)\} ; \quad \text{and} \tag{3.22}$$

$$v(x) \longrightarrow +\infty \qquad \text{as } r(x)\to+\infty . \tag{3.23}$$

For this purpose we consider

$$\beta''(r)+(m-1)[g'(r)/g(r)]\beta'(r) = c/[1+g(r)]^2 \tag{3.24}$$

$$\beta(0)=0 , \qquad \beta'(0)=0 .$$

This is of type (1.23) with $f\equiv 1$ and $a(r)=c/[1+g(r)]^2$; using (1.24) we transform (3.24) in an equation of type (1.25): then, using (1.27), (3.18) and computing it is not difficult to deduce that $[\tilde{g}'(t)/\tilde{g}(t)]\leqq D(t)$, for some function $D(t)$ which satisfies the hypotheses of Lemma 1.18. Thus Lemmas 1.2, 1.18 and 1.20 (with $f\equiv 1$) enable us to conclude that (3.24) has a solution β which is defined for all $r>0$ and tends to $+\infty$ as $r\to+\infty$. We set $v=\beta\circ r$: clearly (3.23) holds. Moreover, on $U\cap\{M^m(g)\setminus B_{R_0}(p)\}$,

$$\Delta v = \beta''(r)+(m-1)[g'(r)/g(r)]\beta'(r) = c/[1+g(r)]^2 < c/[g(r)]^2 \leqq e(\xi) \tag{3.25}$$

as required by (3.22).

THEOREM 3.26. *Let N be a Hadamard manifold with sectional curvature bounded above by a negative constant. Let $M^m(g)$ be a model such that*

$$[g(r)]^{-1} \notin L^1(+\infty) \quad \textit{and} \quad g'(r) = \mathcal{O}(F^{\gamma}(h(r))) \tag{3.27}$$

for some $F\in\mathcal{F}$ as in (2.10), $0<\gamma<1$ *and* $h(r)=\int_0^r[1+g(s)]^{-1}ds$. *Then there are no harmonic maps $\psi: M^m(g)\to N$ such that, on $U=\{x\in M^m(g): \psi(x)\neq\psi(p)\}$,*

$$e(\xi) \geqq [c/g^2] \qquad \textit{for some } c>0 . \tag{3.28}$$

(*ξ is defined as in Lemma* 3.7).

PROOF. We consider

$$\beta''(r)+(m-1)[g'(r)/g(r)]\beta'(r)=\{c/[1+g(r)]^2\}\tilde{k}'(\beta)\tilde{k}(\beta). \tag{3.29}$$

with $\tilde{k}$ as in (3.16). As usual, we transform (3.29) into an equation of type (1.25): applying Proposition 1.11 (with η so small as to have $\gamma<(1-\eta)/(1+\eta)$) we obtain a solution β of (3.29) corresponding to initial conditions

$$\beta(0)=\beta_0>0\,,\qquad \beta'(0)=0\,. \tag{3.30}$$

Such a β is defined on a maximal finite interval $[0, R)$ and satisfies

$$\beta(r)\longrightarrow+\infty\qquad \text{as } r\rightarrow R^-\,. \tag{3.31}$$

Moreover, given arbitrary ε, $\delta>0$, we can assume that $R>\delta$ and

$$\beta(r)<\varepsilon\qquad \text{for all } r\in[0, \delta] \tag{3.32}$$

provided that β_0 is sufficiently small; indeed, $\tilde{k}'\tilde{k}$ is locally Lipschitz on $[0, +\infty)$ and so, if β_0 is small, the solution determined by (3.30) approximates the trivial solution $\beta\equiv0$ on compact sets.

Next, for $x\in M^m(g)$ we define $v(x)=\beta(r(x))$; so, using (3.29), we have

$$\begin{aligned}\Delta v&=\beta''(r)+(m-1)[g'(r)/g(r)]\beta'(r)\\ &=\{c/[1+g(r)]^2\}\tilde{k}'(v)\tilde{k}(v)<\{c/[g(r)]^2\}\tilde{k}'(v)\tilde{k}(v)\,.\end{aligned} \tag{3.33}$$

Moreover,

$$v(x)<\varepsilon\quad \text{on } B_\delta(p)\,;\quad \text{and}\quad v(x)\longrightarrow+\infty\quad \text{as } x\rightarrow\partial B_R(p) \tag{3.34}$$

by (3.31) and (3.32). Now we assume that there exists a nonconstant $\psi: M^m(g)\rightarrow N$ as in the statement of the theorem. Let ρ be the distance in N from $q=\psi(p)$ and set $u=\rho\circ\psi$. Since ψ is nonconstant, there exist δ, $\varepsilon>0$ and $y\in B_\delta(p)$ such that $u(y)>\varepsilon$. Let v be constructed as above with respect to this latter choice of ε, δ. Since ψ is nonconstant, U is dense in $M^m(g)$; thus the open set $U\cap B_R(p)$ is not empty. On it we consider the function $w=u-v$: if $z\in\partial\{U\cap B_R(p)\}$, then either $r(z)=R$ or $\psi(z)=q$; hence w is nonpositive near $\partial\{U\cap B_R(p)\}$. On the other hand at y we have $w(y)=u(y)-v(y)>\varepsilon-\varepsilon=0$. It follows that w attains a positive maximum at some interior point $\tilde{y}\in U\cap B_R(p)$. Using Lemma 3.7 and (3.33), at $\tilde{y}$ we have

$$0\geqq\Delta w\geqq\tilde{k}'(u)\tilde{k}(u)e(\xi)-\tilde{k}'(v)\tilde{k}(v)c/g^2\geqq[c/g^2]\{\tilde{k}'(u)\tilde{k}(u)-\tilde{k}'(v)\tilde{k}(v)\}\,.$$

Now $u(\tilde{y})>v(\tilde{y})$ together with $\tilde{k}'\tilde{k}$ increasing give the desired contradiction. //

APPLICATION 3.36. In the case of rotationally symmetric maps between models $e(\xi)=(m-1)/g^2$ (see [**13**]). We also observe that the condition $g'(r)=\mathcal{O}(F^\gamma(h(r)))$ in (3.27) (and similarly in (3.18)) can be relaxed: indeed, in order to be able to apply Proposition 1.11, as required in the proof of Theorem 3.26, it suffices that $g'(r_i)=\mathcal{O}(F^{\gamma}(h(r_i)))$ as $i\to+\infty$, for each sequence r_i such that $g'(r_i)\to+\infty$ as $i\to+\infty$ (see also Remark 1.19 b)). As a special case, blowing up of solutions occurs if $g'(r)$ is bounded from above; these facts together lead us to

COROLLARY 3.37. *Let $M^m(g)$ be a model such that $[g]^{-1}\notin L^1(+\infty)$ and g' is bounded above by some positive constant. Then any rotationally symmetric harmonic map $\phi: M^m(g)\to \boldsymbol{H}^m$ is constant.*

Similarly,

COROLLARY 3.38. *Let N be a Hadamard manifold with sectional curvature bounded above by a negative constant and $M^m(g)$ a model such that $[g]^{-1}\notin L^1(+\infty)$ and g' is bounded above by some positive constant. Then there are no nonconstant harmonic maps $\phi: M^m(g)\to N$ such that, on $U=\{x\in M^m(g): \phi(x)\neq\phi(p)\}$,*

$$e(\xi)\geqq[c/g^2]\quad \textit{for some } c>0\,. \tag{3.39}$$

(ξ is defined as in Lemma 3.7).

APPLICATION 3.40. Instead of a Hadamard manifold N we can take a model, say $N=N^n(k)$, and consider maps $\phi: M^m(g)\to N^n(k)$ which send pole into pole: then Theorem 3.17 (resp., Theorem 3.26 and its Corollaries 3.37 and 3.38) holds true — with the same proofs — if $k'>0$ (resp., $k'>0$, $(kk')'>0$ and kk' verifies (1.11) i)), the remaining assumptions being unchanged. That is of interest because the sectional curvature of these models $N^n(k)$ is not necessarily nonpositive.

REMARK 3.41. Assumption (3.19) (or, equivalently, (3.28) or (3.39)) gives an extension of condition (0.2) in Theorem 0.1 (resp., Theorem 1) of [**16**] (resp., [**17**]). In particular, Theorem 3.26, Corollaries 3.37, 3.38 and (3.40) extend the main theorems of [**16**], [**17**].

Acknowledgement. We wish to thank the referee for pointing out to us a mistake in a first version of Theorem 3.26.

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