J. Math. Soc. Japan
Vol. 44, No. 4, 1992

Fourth order quasilinear evolution equations of hyperbolic type

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(Received Dec. 26, 1990)
(Revised Oct. 7, 1991)

1. Introduction.

Some vibratory phenomena of beams may be described by the fourth order quasilinear evolution equation

$$(1.1)\qquad (\partial_t^2+A_2(t, u))\cdot(\partial_t^2+A_1(t, u))u+G(t, u, \partial_t u, \partial_t^2 u, \partial_t^3 u)=0 \qquad (t>0),$$

where $A_i(t, u)$, $i=1, 2$, are (unbounded) self-adjoint positive definite operators in a Hilbert space H, and G is a lower order nonlinear perturbation.

In such a generality, (1.1) is not so easy to be dealt with. One could imagine that it is possible to reduce it to a first order equation in a 4-ple of Hilbert space, and then apply known theories (see e.g. [**K**]). However in this case those methods seem to be too hard to be handled.

In [**P**], [**AP**] a very special semilinear case was studied by an *ad hoc* method, which provided global existence and boundedness of the solutions of the Cauchy problem.

As a preparation for the study of (1.1) here we confine ourselves to study the local well-posedness of the Cauchy problem for the equation

$$(1.2)\qquad (\partial_t^2+\gamma_2(u)A)\cdot(\partial_t^2+\gamma_1(u)A)u=0 \qquad (t>0),$$

where A is an (unbounded) self-adjoint positive definite operator in H, and γ_i $(i=1, 2)$ are real functionals on $D(A)$, the domain of A.

THEOREM 1.1 (Main result). *Let A be an (unbounded) self-adjoint positive definite operator in a Hilbert space H, with inner product $(\cdot, \cdot)$.*

For $i=1, 2$ let $m_i: [0, +\infty[\to[0, +\infty[$ satisfy:

i) *m_i is thrice continuously differentiable $(i=1, 2)$;*

ii) $m_i(r)\geqq\nu>0 \qquad (r\geqq 0;\ i=1, 2)$;

iii) $|m_2(r)-m_1(r)|\geqq\delta>0 \qquad (r\geqq 0)$.

Then the Cauchy problem for the equation

(1.3) $$(\partial_t^2+m_2((Au, u))A)\cdot(\partial_t^2+m_1((Au, u))A)u = 0 \qquad (t>0),$$

is locally well-posed in the phase space

$$D(A^{s/2})\times D(A^{(s-1)/2})\times D(A^{(s-2)/2})\times D(A^{(s-3)/2}),$$

for any $s\geqq 5/2$.

Moreover the life span of the solution depends only on the norm of the initial data in the phase space $D(A^{5/4})\times D(A^{3/4})\times D(A^{1/4})\times D(A^{-1/4})$.

By inspection of the proof, it is easy to extend Theorem 1.1 to cover the case of (1.2).

(1.3) is the abstract version (of the principal part) of the Timoshenko-Kirchhoff equation [**A**], which describes the nonlinear transversal vibrations of a simply supported beam of length L

(1.4) $$\begin{cases} (\partial_t^2-\gamma_2(u(\cdot, t))\partial_x^2)(\partial_t^2-\gamma_1(u(\cdot, t))\partial_x^2)u \\ \quad +\alpha(\partial_t^2-\gamma_0(u(\cdot, t))\partial_x^2)u = 0 & (0<x<L,\ t>0), \\ u(x, t) = \partial_x^2 u(x, t) = 0 & \text{for } x=0,\ L\ (t\geqq 0). \end{cases}$$

In this case α is a positive constant, and

$$\gamma_0(v) := m\left(\int_0^L |\partial_x v|^2 dx\right),$$

$$\gamma_1(v) := c_1+\gamma_0(v),$$

$$\gamma_2(v) := c_2+\gamma_0(v),$$

where m: $[0, +\infty[\to[0, +\infty[$ is a continuously differentiable function, and the c_i's are constants satisfying $0<c_1<c_2$. [1]

We note that in this case it is

$$\gamma_2(v)-\gamma_1(v) = \text{constant}.$$

By exploiting this condition, a quick proof of the well-posedness of the Cauchy problem for (1.4) was given in [**A**]. Tucsnak [**Tu**] also proved, by a different method, that the concrete equation (1.4) may be solved in the particular case when γ_2=const. (the Hirschhorn-Reiss model [**HR**]). However, both methods seem hard to be extended to the general case of (1.1).

Here we provide two different proofs of Theorem 1.1: the second one leans upon Kato's method, and it is our hope that that proof may be extended to the general equation (1.1).

The paper is organized as follows. In section 2 one linearizes (1.3), thus obtaining the equation

(1.5) $$(\partial_t^2+a_2(t)A)\cdot(\partial_t^2+a_1(t)A)u = f(t) \qquad (t>0).$$

We apply to (1.5) the estimates in [**APP2**] (for the convenience of the reader, the estimates for this simple case are here completely derived in the Appendix). Then in section 3 we prove the result for (1.3) by a contraction argument (with two norms), in the spirit of [**AG**].

2. The linear case.

In this section we will study the linear version of (1.3). This is a preliminary step to the nonlinear case.

Let A be a (unbounded) self adjoint positive definite operator on a Hilbert space H. For any $s \geqq 0$ we consider the Hilbert space

$$Y_s := D(A^{s/2})$$

endowed with the norm

$$\|u\|_s := |A^{s/2}u|_H .$$

For $s<0$ we set

$$Y_s := (D(A^{-s/2}))' ,$$

the (anti)dual space of Y_{-s}, and we endow it with the (anti)dual norm. Let $T>0$ be fixed. We denote I: $[0, T]$. By analogy with [**K**] we set for any $s \in \mathbf{R}$, $k \in \mathbf{N}$

$$C_{s,k}(I; Y) := \bigcap_{j=0}^{k} C^j(I; Y_{s-j}) .$$

Let us denote, for $u \in C_{s,k}(I; Y)$

$$E_{s,k}(u, t) := \sum_{j=0}^{k} \|\partial_t^j u(t)\|_{s-j}^2 \qquad (t \in I) .$$

In the space $C_{s,k}(I; Y)$ we will consider the following norm:

$$|||u|||_{s,k,T}^2 := \sup_{t \in I} E_{s,k}(u, t) .$$

We will make also the following (strict) hyperbolicity assumptions: let a_1, a_2 be two real functions such that

(R) $$a_i \in C^1(I) \qquad (i=1, 2) ;$$

(H) $$a_i(t) \geqq \nu > 0 \qquad (i=1, 2,\ \forall t \in I) ;$$

(SH) $$|a_2(t) - a_1(t)| \geqq \delta > 0 \qquad (\forall t \in I) ;$$

Finally, for the known term we need the regularity assumption

(2.1) $$f \in L^1(I; Y_{s-3}) .$$

Then we have the following result:

THEOREM 2.1. *Let a_1 and a_2 satisfy assumptions (R), (H), (SH) and let* (2.1) *be satisfied. Then, for any $s \in \boldsymbol{R}$, the Cauchy problem*

$$\left\{\begin{array}{ll} (\partial_t^2+a_2(t)A)\cdot(\partial_t^2+a_1(t)A)u = f(t) & (t\in I), \\ (\partial_t^j u)(0) = u_j \in Y_{s-j} & (j=0, \cdots, 3), \end{array}\right. \tag{2.2}$$

has a unique solution $u \in C_{s,3}(I; Y)$. For $t\in I$ we have the estimate

$$E_{s,3}(u, t) \leqq C(T; a_1, a_2)\left(E_{s,3}^{1/2}(u, 0)+\int_0^t \|f(\tau)\|_{s-3}d\tau\right)^2. \tag{2.3}$$

Moreover there exists a constant C_0 depending only on $a_i(0)$ and $\partial_t a_i(0)$ ($i=1, 2$) such that if $\partial_t a_1$ and $\partial_t a_2$ vary in an equicontinuous set of functions, then

$$C(T; a_1, a_2) = C_0+o(1) \quad \textit{as } T\to 0^+. \tag{2.4}$$

1ST PROOF of THEOREM 2.1. (by Fourier series).

For simplicity, we assume that there exists a sequence (e_n) of eigenvectors of A, which form an orthogonal base for H (in the general case, proof may be given by spectral decomposition). Let (λ_n^2) denote the sequence of the relative eigenvalues. We look for the solution in the form of Fourier development:

$$u(t) = \sum_{n=1}^{\infty} y_n(t)e_n \qquad (t\in I).$$

Then u is a solution of the problem (2.2) if and only if, for each n, y_n solves the following Cauchy problem for an ordinary differential equation:

$$\left\{\begin{array}{ll} (\partial_t^2+a_2(t)\lambda_n^2)(\partial_t^2+a_1(t)\lambda_n^2)y_n = f_n(t) & (t\in I), \\ (\partial_t^j y_n)(0) = y_{j,n} & (j=0, \cdots, 3). \end{array}\right. \tag{2.5$_n$}$$

Here $(y_{j,n})$ ($j=0, \cdots, 3$) and $(f_n(t))$ are the coefficients of the Fourier developments for u_j ($j=0, \cdots, 3$) and $f(t)$ respectively.

It is easily seen that for every n there exists a unique solution $y_n \in C^3(I; \boldsymbol{R})$. Now we want to estimate the energy

$$e(y_n, t) := \sum_{j=0}^{3} \lambda_n^{2(3-j)} |\partial_t^j y_n(t)|^2, \tag{2.6}$$

in terms of the initial data and of the known term f.

We have the following proposition (for its proof, which follows the line of [**APP2**], see the Appendix).

PROPOSITION 2.2. *For each n let y_n be the solution of* (2.5)$_n$. *Let the assumptions (R), (H) and (SH) be satisfied. Then the following estimate holds:*

$$e(y_n, t) \leqq C(T; a_1, a_2)\left(e^{1/2}(y_n, 0)+\int_0^t |f_n(\tau)|d\tau\right)^2, \tag{2.7}$$

where $C(T; a_1, a_2)$ verifies the last statement of Theorem 2.1.

Now we have for $t \in I$

$$E_{s,3}(u, t) = \sum_{n=0}^{\infty} \lambda_n^{2(s-3)} e(y_n, t),$$

and

$$\sum_{n=0}^{\infty} \lambda_n^{2(s-3)} \left(\int_0^t |f_n(\tau)| d\tau \right)^2 = \left(\int_0^t \|f(\tau)\|_{s-3} \mathrm{d}\tau \right)^2.$$

Therefore from (2.7) we have

$$E_{s,3}(u, t) \leqq C(T; a_1, a_2) \sum_{n=0}^{\infty} \lambda_n^{2(s-3)} \left(e^{1/2}(y_n, 0) + \int_0^t |f_n(\tau)| \mathrm{d}\tau \right)^2$$

$$\leqq C(T; a_1, a_2) \left(E_{s,3}^{1/2}(u, 0) + \int_0^t \|f(\tau)\|_{s-3} \mathrm{d}\tau \right)^2.$$

(the last inequality follows from Minkowski's one). □

2ND PROOF of THEOREM 2.1 (via Kato's theory).

An alternative proof of theorem 2.1 may also be given by means of Kato's theory [**K**], after reduction to a first order problem. For this, we reduce the problem (2.2) to a first order one. First we set

$$\begin{cases} v_1(t) := (\partial_t^2 + a_1(t)A)u(t), \\ v_2(t) := (\partial_t^2 + a_2(t)A)u(t). \end{cases} \tag{2.8}$$

If we set $v := (v_1, v_2)$, then problem (2.2) is equivalent to find $v \in C^1(I; Y_{s-2} \times Y_{s-2})$ such that

$$\begin{cases} \partial_t^2 v + \mathfrak{A}(t)v + \mathfrak{B}(t)v + \mathfrak{C}(t)\partial_t v + \partial_t(\mathfrak{C}(t)v) = \mathfrak{F}(t) \qquad (t \in I), \\ \qquad v(0) = v_1, \qquad \partial_t v(0) = v_1, \end{cases}$$

where

$$\mathfrak{A}(t) := \begin{pmatrix} a_2(t) & 0 \\ 0 & a_1(t) \end{pmatrix} A,$$

$$\mathfrak{B}(t) := \begin{pmatrix} 0 & 0 \\ -\mathrm{d}^2(t) & \mathrm{d}^2(t) \end{pmatrix}, \qquad \mathfrak{C}(t) := \begin{pmatrix} 0 & 0 \\ \mathrm{d}(t) & -\mathrm{d}(t) \end{pmatrix},$$

$$\mathrm{d} := \frac{\partial_t a_2 - \partial_t a_1}{a_2 - a_1}.$$

Also we set

$$\mathfrak{F}(t) := \begin{pmatrix} f(t) \\ f(t) \end{pmatrix},$$

while the initial conditions v_0 and v_1 for v and $\partial_t v$ are determined in an obvious way.

Now we perform a second transformation. Let

$$\begin{cases} w_1 := v \\ w_2 := \partial_t v + \mathfrak{C}(t)v \,. \end{cases}$$

If we set $w := (w_1, w_2)$, then the problems (2.8) and (2.2) are equivalent to find $w \in C^0(I; Y_{s-2} \times Y_{s-2} \times Y_{s-3} \times Y_{s-3})$ such that

$$\begin{cases} \partial_t w + \mathcal{A}(t)w + \mathcal{B}(t)w = \mathcal{F}(t) \qquad (t \in I)\,, \\ \qquad w(0) = w_0 \,. \end{cases} \tag{2.9}$$

Here

$$\mathcal{A}(t) := \begin{pmatrix} 0 & -I \\ \mathfrak{A}(t) & 0 \end{pmatrix}, \qquad \mathcal{B}(t) := \begin{pmatrix} \mathfrak{C}(t) & 0 \\ 0 & \mathfrak{C}(t) \end{pmatrix}, \qquad \mathcal{F}(t) := \begin{pmatrix} 0 \\ \mathfrak{F}(t) \end{pmatrix}.$$

Now problem (2.9) satisfies the assumptions of Theorem 3.3 of [**K**], hence the thesis follows. □

Remark. The last check is somewhat cumbersome. As far as one is concerned merely with problem (1.4) (or (1.2)) the first proof is clearly more convenient. On the other hand, Kato's approach seems to be the unique one which might allow to deal with the more general equation (1.1).

3. Proof of the main result.

This section is devoted to prove Theorem 1.1.

The line of the proof follows the one of [**AG**], and is made in two steps. The first one consists in linearizing equation (1.3). Then we use the estimates of section 2 and a contraction argument to complete the proof.

Step 1. In the following, s will be a fixed real number $\geqq 5/2$. If $v \in C_{s,3}(I, Y)$ it is easily seen that $m_i(\langle Av, v\rangle) \in C^3(I)$ $(i=1, 2)$.

Now we apply Theorem 2.1 to problem (2.2) with the choice

$$a_i(t) := m_i(\langle Av, v\rangle) \qquad (i=1, 2) \tag{3.1}$$

and $f \equiv 0$, i.e.

$$\begin{cases} (\partial_t^2 + m_2(\langle Av, v\rangle)A)\cdot(\partial_t^2 + m_1(\langle Av, v\rangle)A)u = 0 \qquad (t \in I)\,, \\ \qquad (\partial_t^j u)(0) = u_j \qquad (j=0, \cdots, 3)\,, \end{cases} \tag{3.2}$$

to get that problem (3.2) has a unique solution $u \in C_{s,3}(I; Y)$.

Step 2. Let us fix for the moment $s=5/2$. We introduce the following subset of $C_{5/2,3}(I; Y)$:

$$\Xi_{T,R} := \{v \in C_{5/2,3}(I; Y); |||v|||_{5/2,3,T} \leqq R, (\partial_t^j v)(0) = u_j \ (j=0, \cdots, 3)\}.$$

Here R is any real value such that $R^2 > C_0 E_{5/2,3}(0)$ where C_0 is the constant provided in (2.4) of Theorem 2.1 for the choice (3.1) (C_0 depends only on the initial data u_j ($j=0, 1$)), and $E_{5/2,3}(0) := \sum_{j=0}^{3} \|u_j\|_{5/2-j}^2$. Let S be the resolvent map defined, for each $v \in \Xi_{T,R}$, by $S(v) = u$ where u is the solution of the problem (3.2).

We note that for a_i given by (3.1), $\partial_t a_i$ vary in an equicontinuous class as v varies in $\Xi_{T,R}$ ($i=1, 2$). Then it is easily seen from (2.4) that we can find $T > 0$ so small that

$$C(T; a_1, a_2) \leqq C := \frac{R^2}{E_{5/2,3}(0)},$$

so that from the estimates (2.3) (set there $s=5/2$).

$$S \text{ maps } \Xi_{T,R} \text{ into itself.}$$

Now we claim that there exists $T' \in (0, T]$ such that S has a fixed point in $\Xi_{T',R}$. In order to use Banach's fixed point theorem, we show that there exists $T' \in (0, T)$ such that S is a contraction map in $\Xi_{T',R}$ with respect to the weaker norm of $C_{3/2,3}(I'; Y)$ (see (2.1)), where $I' := [0, T']$.

Let $v_1, v_2 \in \Xi_{T',R}$. Set

$$w := S(v_1) - S(v_2);$$

$$\mu_{ij} := m_i(\langle Av_j, v_j \rangle) \qquad (i=1, 2).$$

Then w solves the problem

$$\begin{cases} (\partial_t^2 + \mu_{21}(t)A)(\partial_t^2 + \mu_{11}(t)A)w = \tilde{f}(t) & (t \in I'), \\ (\partial_t^j w)(0) = 0 \qquad (j=0, \cdots, 3). \end{cases}$$

Here

$$\tilde{f}(t) := \partial_t^2(\mu_{12} - \mu_{11})AS(v_2) + 2\partial_t(\mu_{12} - \mu_{11})A\partial_t S(v_2)$$
$$+ (\mu_{12} - \mu_{11} + \mu_{22} - \mu_{21})A\partial_t^2 S(v_2) + (\mu_{22}\mu_{12} - \mu_{21}\mu_{11})A^2 S(v_2).$$

It is easy to show that $\tilde{f} \in L^1(I'; Y_{-3/2})$.

According to estimate (2.3) (set there $s=3/2$) we have that for $T' \leqq T$

$$E_{3/2,3}(w, t) \leqq C\left(\int_0^t \|\tilde{f}(\tau)\|_{-3/2} d\tau\right)^2 \qquad (t \in I'). \tag{3.3}$$

Now we evaluate $\|\tilde{f}(t)\|_{-3/2}$ in terms of the norm of the difference $(v_1 - v_2)$ in $C_{3/2,3}(I', Y)$.

We note that $A^{-\alpha} \in B(H)$, for $\alpha \geqq 0$. If we set $K := \|A^{-1}\|$, it is

$$\|A^{-\alpha}\|_{B(H)} \leqq K^\alpha \qquad \text{for } 0 \leqq \alpha \leqq 1,$$

and we have the crude inequality:

$$|\langle Av, w\rangle| \leqq \|A^{1-(i+j)/2}\|_{B(H)}\|v\|_i\|w\|_j \leqq K^{(i+j)/2-1}\|v\|_i\|w\|_j, \tag{3.4}$$

valid for each $v\in Y_i$, $w\in Y_j$, whenever $2\leqq i+j\leqq 4$.

From (3.4) we deduce that the following estimates hold for $v_1, v_2\in \Xi_{T',R}$:

$$|\langle Av_1, v_1\rangle-\langle Av_2, v_2\rangle| = |\langle A(v_1+v_2), v_1-v_2\rangle| \leqq 2KR|||v_1-v_2|||_{3/2,3,T'};$$

$$\begin{aligned}
&|\langle Av_1, \partial_t v_1\rangle-\langle Av_2, \partial_t v_2\rangle| \\
&\quad = |\langle A\partial_t(v_1-v_2), v_1+v_2\rangle+\langle A(v_1-v_2), \partial_t(v_1+v_2)\rangle|/2 \\
&\quad \leqq K^{1/2}R[\|\partial_t(v_1-v_2)\|_{1/2}+\|v_1-v_2\|_{3/2}] \\
&\quad \leqq \sqrt{2}K^{1/2}R|||v_1-v_2|||_{3/2,3,T'};
\end{aligned}$$

$$\begin{aligned}
&|\langle Av_1, \partial_t^2 v_1\rangle+\langle A\partial_t v_1, \partial_t v_1\rangle-\langle Av_2, \partial_t^2 v_2\rangle-\langle A\partial_t v_2, \partial_t v_2\rangle| \\
&\quad = |\langle A\partial_t^2(v_1-v_2), v_1+v_2\rangle+2\langle A\partial_t(v_1-v_2), \partial_t(v_1+v_2)\rangle \\
&\qquad +\langle A(v_1-v_2), \partial_t^2(v_1+v_2)\rangle|/2 \\
&\quad \leqq R\|\partial_t^2(v_1-v_2)\|_{-1/2}+2R\|\partial_t(v_1-v_2)\|_{1/2}+R\|v_1-v_2\|_{3/2} \\
&\quad \leqq \sqrt{6}R|||v_1-v_2|||_{3/2,3,T'}.
\end{aligned}$$

Now we observe that, for any $v\in\Xi_{T',R}$ the following estimate holds:

$$|\langle A\partial_t^h v, \partial_t^k v\rangle| \leqq K^{(3-h-k)/2}R^2 \qquad \text{for } 0\leqq h+k\leqq 2.$$

Let us set

$$M_k := \max_{\substack{i=1,2\\ |\xi|\leqq KR^2}}\left|\frac{d^k}{d\xi^k}m_i(\xi)\right| \qquad (k=0,\cdots,3).$$

To evaluate the norm of the term $\tilde{f}$, it suffices to apply the following estimates, valid for $t\in I'$:

$$|\mu_{i2}-\mu_{i1}| \leqq 2M_1KR|||v_1-v_2|||_{3/2,3,T'} \qquad (i=1,2);$$

$$\begin{aligned}
|\partial_t(\mu_{12}-\mu_{11})| &= 2|\mu'_{12}\langle Av_2, \partial_t v_2\rangle-\mu'_{11}\langle Av_1, \partial_t v_1\rangle| \\
&\leqq 2|\mu'_{12}(\langle Av_2, \partial_t v_2\rangle-\langle Av_1, \partial_t v_1\rangle)+(\mu'_{12}-\mu'_{11})\langle Av_1, \partial_t v_1\rangle| \\
&\leqq C_1|||v_1-v_2|||_{3/2,3,T'},
\end{aligned}$$

where we have set

$$\mu'_{ij} := \left.\frac{dm_i}{d\xi}\right|_{\xi=\langle Av_j, v_j\rangle} \qquad (i, j=1,2),$$

$$C_1 := 2\sqrt{2}K^{1/2}M_1R + 4K^2M_2R_3\,.$$

$$\begin{aligned}|\partial_t^2(\mu_{12}-\mu_{11})| &= |4[\mu_{12}''\langle Av_2,\, \partial_t v_2\rangle^2 - \mu_{11}''\langle Av_1,\, \partial_t v_1\rangle^2] + 2\mu_{12}'(\langle Av_2,\, \partial_t^2 v_2\rangle \\ &\quad + \langle A\partial_t v_2,\, \partial_t v_2\rangle) - 2\mu_{11}'(\langle Av_1,\, \partial_t^2 v_1\rangle + \langle A\partial_t v_1,\, \partial_t v_1\rangle)| \\ &\leqq |4\mu_{12}''(\langle Av_2,\, \partial_t v_2\rangle + \langle Av_1,\, \partial_t v_1\rangle)(\langle Av_2,\, \partial_t v_2\rangle - \langle Av_1,\, \partial_t v_1\rangle) \\ &\quad + 4(\mu_{12}'' - \mu_{11}'')\langle Av_1,\, \partial_t v_1\rangle^2 + 2\mu_{12}'(\langle Av_2,\, \partial_t^2 v_2\rangle + \langle A\partial_t v_2,\, \partial_t v_2\rangle - \langle Av_1,\, \partial_t^2 v_1\rangle \\ &\quad - \langle A\partial_t v_1,\, \partial_t v_1\rangle) + 2(\mu_{12}' - \mu_{11}')(\langle Av_1,\, \partial_t^2 v_1\rangle + \langle A\partial_t v_1,\, \partial_t v_1\rangle)| \\ &\leqq C_2|||v_1 - v_2|||_{3/2,3,T'}\end{aligned}$$

where

$$\mu_{ij}'' := \frac{d^2 m_i}{d\xi^2}\bigg|_{\xi=\langle Av_j,\, v_j\rangle} \qquad (i,\, j=1,\, 2)\,.$$

$$C_2 := 2\sqrt{6}M_1R + 8(\sqrt{2}+1)K^{3/2}M_2R^3 + 8K^3M_3R^5\,.$$

By performing analogous calculations we have in sum

$$\|\tilde{f}(t)\|_{-3/2} \leqq C_3|||v_1 - v_2|||_{3/2,3,T'} \qquad (t\in I')\,. \tag{3.5}$$

Here C_3 is depending only on constants K, R and M_i, $i=0, \cdots, 3$. By using (3.5) in the estimate (3.3) we get

$$|||S(v_1) - S(v_2)|||_{3/2,3,T'} \leqq CC_3T'|||v_1 - v_2|||_{3/2,3,T'}$$

for every $v_1, v_2 \in \Xi_{T',R}$. So we can choose $T' \leqq T$ such that S is a contraction map from $\Xi_{T',R}$ into itself with respect to the norm of $C_{3/2,3}$. As a matter of fact, $\Xi_{T',R}$ is not a complete metric space under this norm. So we have to consider the larger set

$$\begin{aligned}\Theta_{T',R} := \{u \in \bigcap_{j=0}^{3} C^j(I'\,;\, Y_{5/2-j}-\text{weak}) \cap C^3(I'\,;\, Y_{-3/2})\,; \\ |||u|||_{5/2,3,T'} \leqq R, \qquad (\partial_t^j u)(0) = u_j \quad (j=0, \cdots, 3)\}\,.\end{aligned}$$

Then map S still maps $\Theta_{T,R}$ into $\Xi_{T,R}$ (hence a fortiori into itself). Moreover S is a contraction map in $\Theta_{T,R}$ endowed with the norm of $C_{3/2,3}(I'\,;Y)$. Now, under this norm, $\Theta_{T',R}$ is a complete metric space, so by the classical contraction mapping principle, there exists a unique fixed point $u \in \Theta_{T',R}$ which solves problem (1.3). Finally a standard boot-strap argument (regularity and uniqueness arguments for the linear problem (2.2)) provide that $u \in \Xi_{T',R}$.

Now let s any number $>5/2$ and consider initial data in the phase space $Y_s \times Y_{s-1} \times Y_{s-2} \times Y_{s-3}$. From the above case $s=5/2$, we know that the Cauchy problem for the equation (1.3) admits a unique solution in the class $\Xi_{T',R}$. From Theorem 2.1 this solution belongs to the class $C_{s,3}(I, Y)$, hence the conclusion follows. □

Appendix.

PROOF OF PROPOSITION 2.2. In spite of the fact that the problem is finite dimensional, it is convenient to follow the idea of [**APP2**]. We transform the problem $(2.5)_n$ in an equivalent one (cf. the first step in the second proof of Theorem 2.1). For simplicity let us eliminate the subscript n everywhere and use the following notations:

$$\mathrm{d} := \frac{\partial_t a_2 - \partial_t a_1}{a_2 - a_1};$$

$$M_0 := \max_{\substack{t\in I \\ i=1,2}} |a_i(t)|; \qquad M_1 := \max_{\substack{t\in I \\ i=1,2}} |\partial_t a_i(t)|.$$

Let us consider the two functions

$$\begin{cases} v_1(t) := (\partial_t^2 + a_1(t)\lambda^2)y(t), \\ v_2(t) := (\partial_t^2 + a_2(t)\lambda^2)y(t). \end{cases}$$

We have

$$y = \frac{v_1 - v_2}{\lambda^2(a_1 - a_2)}. \tag{A.1}$$

The vector-valued function

$$v := (v_1, v_2)$$

solves the following second order problem:

$$\begin{cases} \partial_t^2 v + A(t)v + B(t)v + C(t)\partial_t v + \partial_t(C(t)v) = F(t) \qquad (t\in I), \\ v(0) = v_0; \quad v'(0) = v_1, \end{cases}$$

where

$$A(t) := \lambda^2\begin{pmatrix} a_2(t) & 0 \\ 0 & a_1(t) \end{pmatrix},$$

$$B(t) := \begin{pmatrix} 0 & 0 \\ -\mathrm{d}^2(t) & \mathrm{d}^2(t) \end{pmatrix}, \qquad C(t) := \begin{pmatrix} 0 & 0 \\ \mathrm{d}(t) & -\mathrm{d}(t) \end{pmatrix}, \qquad F(t) := \begin{pmatrix} f(t) \\ f(t) \end{pmatrix}.$$

Let us define the perturbed energy

$$\tilde{e}(v, t) := \langle A(t)v(t), v(t)\rangle + |v'(t) + C(t)v(t)|^2.$$

For this quantity it is possible, by standard energy methods, to obtain the inequality:

$$\tilde{e}(v, t) \leq \left(\tilde{e}^{1/2}(v, 0) + \int_0^t |f(\tau)|\mathrm{d}\tau\right)^2 e^{At}$$

where

$$\Lambda:=\frac{4\sqrt{2}M_1}{\delta}\left(1+\sqrt{\frac{M_0}{\nu}}\right)+\frac{M_1}{\nu}.$$

In order to get an estimate for the energy of the unknown y, let us write again the energy (2.6):

$$e(y,t)=\sum_{j=0}^{3}\lambda^{2(3-j)}|\partial_t^j y(t)|^2.$$

Now, thanks to the identities (A.1) and

$$\partial_t^2 y=\frac{a_2v_1-a_1v_2}{a_2-a_1},$$

it is easy to show that there exists a constant $c>0$ which depends only on M_0, M_1, δ, ν and λ_1 such that

$$c^{-1}e(y,0)\leqq\tilde{e}(v,0)\leqq ce(y,0).$$

We get

$$e(y,t)\leqq C\left(e^{1/2}(y,0)+\int_0^t|f(\tau)|\mathrm{d}\tau\right)^2e^{\Lambda t},$$

where the constant C depends only on M_0, M_1, ν and δ in a continuous way. Moreover, let $M_{0,0}=\max\{a_1(0),a_2(0)\}$ and $M_{0,1}=\max\{\partial_1 a_1(0),\partial_t a_2(0)\}$. If $\partial_t a_1$ and $\partial_t a_2$ lie in a equicontinuous subset of $C^0(I)$, then

$$\begin{aligned}M_1&=M_{0,1}+o(1)\\ M_0&=M_{0,0}+O(T)\end{aligned}\qquad\text{as } T\to 0^+$$

and (2.4) follows. □

Note.

[1] For simplicity the beam is supposed here to be not precompressed. (1.3) may be compared with [**HR**], and with other nonlinear models quoted in [**A**]. The term m is a nonlinear correction (to the well-known Timoshenko beam equation [**T**]) which takes into account the fact that the length of the beam, and then the axial tension, varies during the evolution. For the elastic string with fixed ends, such a correction is firstly due to G. KIRCHHOFF [**Ki**], cf. [**Ka**] [**N**] [**NM**] and the references quoted there.

ACKNOWLEDGEMENT. We wish to thank A. SESTIERI for calling our attention upon the Timoshenko beam equation. The research was supported by a contract with C.N.R.. The first and third authors are member of G.N.A.F.A. (C.N.R.).

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