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Applications of spreading models to regular methods of summabilities and growth rate of Cesàro means

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§ 0. Introduction.

Applications of Brunel-Sucheston's spreading model are presented. One application is to show an alternative theorem concerning weakly null sequences of Banach spaces, another application is to show an alternative theorem of summabilities of bounded sequences in Banach spaces and the other one is to estimate, from above or below, the growth rate of Cesàro means.

1. One application of Brunel-Sucheston's spreading model is to show that each weakly null sequence of Banach spaces has a subsequence which is either "uniformly completely Cesàro summable" or "uniformly completely non Cesàro summable".

THEOREM I. *For every weakly null sequence* $\{x_n\}_n$ *of a Banach space* X, *one can extract a subsequence* $\{x'_n\}_n$ *of* $\{x_n\}_n$ *such that either*

$$\lim_{k\to\infty}\left(\sup_{n_1<\cdots<n_k,\ |a_i|\leq 1}\left\|\frac{1}{k}\sum_{i=1}^{k} a_i x'_{n_i}\right\|\right)=0 \tag{1}$$

or

$$\inf_{k}\left(\inf_{n_1<\cdots<n_k,\ |\theta_i|=1}\left\|\frac{1}{k}\sum_{i=1}^{k}\theta_i x'_{n_i}\right\|\right)>0\,. \tag{2}$$

2. A real or complex infinite matrix $(a_{n,m})_{n,m}$ defines *a regular method of summability*, if (and only if) the following conditions hold:

$$\sup_{n}\left(\sum_{m=1}^{\infty}|a_{n,m}|\right)<\infty\,, \tag{1}$$

$$\lim_{n\to\infty}\left(\sum_{m=1}^{\infty}a_{n,m}\right)=1 \tag{2}$$

and

$$\lim_{n\to\infty}a_{n,m}=0 \qquad (m\in \boldsymbol{N})\,. \tag{3}$$

Let Λ denote the set of all regular methods of summability and put

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$$\Lambda_0 := \left\{(a_{n,m})_{n,m} \in \Lambda :\ \lim_{n\to\infty}\left(\sup_m |a_{n,m}|\right) = 0\right\},$$

$$\Lambda_+ := \left\{(a_{n,m})_{n,m} \in \Lambda :\ \overline{\lim_{n\to\infty}}\left(\sup_m |a_{n,m}|\right) > 0\right\}.$$

A "typical" element of Λ_0 is that of Cesàro's:

$$C := (c_{n,m})_{n,m} \text{ with } c_{n,m} := \frac{1}{n}(1 \leqq m \leqq n) \text{ and } c_{n,m} := 0\ (1 \leqq n < m).$$

On the other hand, "the most trivial" element of Λ_+ is the identity summability:

$$I := (\delta_{n,m})_{n,m} \text{ with } \delta_{n,m} := 1\ (n=m) \text{ and } \delta_{n,m} := 0\ (n \neq m).$$

For a regular method of summability $R=(a_{n,m})_{n,m}$, a bounded sequence $\{x_n\}_n$ in a Banach space X is called R-summable to an element $x_0 \in X$ if

$$\lim_{n\to\infty}\left\| \sum_{m=1}^{\infty} a_{n,m} x_m - x_0 \right\| = 0 .$$

Now we introduce stronger notions of summability and non summability as follows. A bounded sequence $\{x_n\}_n$ in a Banach space X is said to be *completely R-summable* to an element $x_0 \in X$ if each subsequence of $\{x_n\}_n$ is R-summable to x_0. For example, the canonical basis $\{e_n\}_n$ of ℓ_2 is not norm convergent, but it is completely C-summable to zero. On the other hand, a bounded sequence $\{x_n\}_n$ in a Banach space X is said to be *completely non R-summable* if each subsequence of $\{x_n\}_n$ is non R-summable. For example, the canonical basis $\{e_n\}_n$ of ℓ_1 is completely non C-summable.

With respect to this definition, we have the following:

THEOREM II. *Let $\{x_n\}_n$ be a bounded sequence with no norm convergent subsequence in a Banach space X. Then $\{x_n\}_n$ has a subsequence $\{x'_n\}_n$ which satisfies one of the following conditions:*

(1) *$\{x'_n\}_n$ is completely R-summable for every $R \in \Lambda_0$ and completely non R-summable for every $R \in \Lambda_+$.*

(2) *$\{x'_n\}_n$ is completely non R-summable for every $R \in \Lambda_+$, and for each $R \in \Lambda_0$ there is a subsequence $\{x''_n\}_n$ of $\{x'_n\}_n$ which is completely non R-summable.*

This theorem will be proved in a more precise formulation in § 2 (see Theorem 4).

3. In [2], Banach and Saks proved that $L_p[0, 1]$ $(1<p<\infty)$ has the so-called Banach-Saks property by actually showing the following:

Each weakly null sequence $\{x_n\}_n$ in $L_p[0, 1]$ has a subsequence $\{x'_n\}_n$ which satisfies

$$\left\| \sum_{i=1}^{k} x'_i \right\|_p = \begin{cases} O(k^{1/p}) & (1<p\leqq 2), \\ O(k^{1/2}) & (2<p<\infty). \end{cases}$$

We shall show a natural generalization of this result. Recall that a Banach space X is said to be of *type* p for some $1<p\leqq 2$, of *cotype* q for some $2\leqq q<\infty$, if there exists a constant $M\geqq 1$ so that for every finite set of vectors $\{x_i\}_{i=1}^k \subset X$, we have

$$\int_0^1 \left\| \sum_{i=1}^{k} r_i(t) x_i \right\| dt \leqq M \Big(\sum_{i=1}^{k} \|x_i\|^p \Big)^{1/p},$$

respectively,

$$\frac{1}{M} \Big(\sum_{i=1}^{k} \|x_i\|^q \Big)^{1/q} \leqq \int_0^1 \left\| \sum_{i=1}^{k} r_i(t) x_i \right\| dt,$$

where $\{r_n\}_n$ denotes the sequence of the Rademacher functions, i.e., $r_n(t)=\operatorname{sign}(\sin 2^{n-1}\pi t)$ $(n\in\boldsymbol{N})$.

Then we have the following:

THEOREM III.

(1) *Let X be a Banach space of type p for some $1<p\leqq 2$. Then from each weakly null sequence $\{x_n\}_n$ in X one can extract a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ such that*

$$\sup_k \Big(\sup_{n_1<\cdots<n_k,\, |a_i|\leq 1} \left\| \frac{1}{k^{1/p}} \sum_{i=1}^{k} a_i x'_{n_i} \right\| \Big) < \infty .$$

(2) *Let X be a Banach space of cotype q for some $2\leqq q<\infty$. Then from each weakly null sequence $\{x_n\}_n$ with $\inf_n \|x_n\|>0$ in X one can extract a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ such that*

$$\inf_k \Big(\inf_{n_1<\cdots<n_k,\, |\theta_i|=1} \left\| \frac{1}{k^{1/q}} \sum_{i=1}^{k} \theta_i x'_{n_i} \right\| \Big) > 0 .$$

This theorem will be proved in a slightly more precise formulation in §3 (see Theorems 5 and 6).

Throughout this paper, X denotes a (real or complex) Banach space with the dual space X^*, $\boldsymbol{N}$ denotes the set of all positive integers and S_0 denotes the vector space of all (real or complex) finite scalar sequences with the canonical unit basis $\{e_n\}_n$. Let us agree to write an element in S_0 in the form $(a_i)_{i=1}^k$ or $\sum_{i=1}^k a_i e_i$ for the sake of convenience. We may also use $\{e_n\}_n$ to denote spreading models, since there seems to be no difficulty to understand what $\{e_n\}_n$ means. Recall that a Banach space X has *the Banach-Saks property* if every bounded sequence $\{x_n\}_n$ of X has a subsequence whose Cesàro means converge strongly, and a Banach space X has *the weak Banach-Saks property* if every *weakly null sequence* $\{x_n\}_n$ (i.e., $w\text{-}\lim_{n\to\infty} x_n=0$) of X has a subsequence whose

Cesàro means converge strongly.

§ 1. Spreading models and an alternative theorem.

We shall first state some fundamental properties about Brunel-Sucheston's spreading model which are needed in the proofs of theorems and lemmas in this paper. Since Brunel-Sucheston's spreading model is investigated in detail by Beauzamy and Lapresté [4], for proofs, we refer to it.

Let $\{x_n\}_n$ be a bounded sequence with no norm Cauchy subsequence in a Banach space X. Suppose that the limit

$$\lim_{\substack{m\to\infty \\ m\leq n_1<\cdots<n_k}} \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\|$$

exists for each $(a_i)_{i=1}^k \in S_0$. We shall call such a sequence $\{x_n\}_n$ *a Brunel-Sucheston sequence*. Then we can define the nonnegative function Ψ on S_0 by

$$\Psi((a_i)_{i=1}^k) := \lim_{\substack{m\to\infty \\ m\leq n_1<\cdots<n_k}} \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\| .$$

Clearly Ψ defines a seminorm on S_0. Furthermore, since $\{x_n\}_n$ is assumed to have no norm Cauchy subsequence, Ψ indeed defines a norm on S_0 (see Brunel and Sucheston [6]). Hence we shall write $\|\sum_{i=1}^k a_i e_i\|$ in place of $\Psi((a_i)_{i=1}^k)$ for each $(a_i)_{i=1}^k \in S_0$. Let E be the completion of $[S_0, \|\cdot\|]$. Then $\{x_n\}_n$ and $[E, \{e_n\}_n]$ have the following properties (1), (2) and (3) which are referred as *Spreading Model Property* in the sequel:

(1) (a) The norm $\|\cdot\|$ for E is invariant under spreading in the sense that

$$\left\| \sum_{i=1}^{k} a_i e_i \right\| = \left\| \sum_{i=1}^{k} a_i e_{n_i} \right\|$$

holds for each k, $n_i \in \boldsymbol{N}$ $(i=1, 2, \cdots, k)$ with $n_1<n_2<\cdots<n_k$ and $(a_i)_{i=1}^k \in S_0$.

(b) $\{e_{2n-1}-e_{2n}\}_n$ is a monotone unconditional basic sequence, i.e.,

$$\left\| \sum_{i\in A_1} a_i(e_{2i-1}-e_{2i}) \right\| \leqq \left\| \sum_{i\in A_2} a_i(e_{2i-1}-e_{2i}) \right\|$$

for each finite subsets A_1, A_2 of $\boldsymbol{N}$ with $A_1 \subset A_2$ and $(a_i)_i \in S_0$.

Moreover if, in addition, $\{x_n\}_n$ is a weakly null sequence, then $\{e_n\}_n$ is a monotone unconditional Schauder basis for E, hence (b) can be improved as follows:

(b′) $$\left\| \sum_{i\in A_1} a_i e_i \right\| \leqq \left\| \sum_{i\in A_2} a_i e_i \right\|$$

for each finite subsets A_1, A_2 of $\boldsymbol{N}$ with $A_1 \subset A_2$ and $(a_i)_i \in S_0$.

(2) $$\lim_{\substack{m\to\infty \\ m\leq n_1<\cdots<n_k}} \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\| = \left\| \sum_{i=1}^{k} a_i e_i \right\|$$

for every vector $(a_i)_{i=1}^k \in S_0$.

(3) For any $\varepsilon>0$ and $k\in \boldsymbol{N}$ there exists an $L(\varepsilon, k)\in \boldsymbol{N}$ so that for every $(a_i)_{i=1}^k\in S_0$ and $n_i\in \boldsymbol{N}$ $(i=1, 2, \cdots, k)$ with $L(\varepsilon, k)\leqq n_1<n_2<\cdots<n_k$,

$$(1-\varepsilon)\left\| \sum_{i=1}^{k} a_i e_i \right\| \leqq \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\| \leqq (1+\varepsilon)\left\| \sum_{i=1}^{k} a_i e_i \right\| .$$

We shall call $[E, \{e_n\}_n]$ *the spreading model* of a Brunel-Sucheston sequence $\{x_n\}_n$. We pay much attention to the fact that each subsequence of $\{x_n\}_n$ is a Brunel-Sucheston sequence and $[E, \{e_n\}_n]$ is (isometrically isomorphic to) the spreading model of each subsequence of $\{x_n\}_n$.

It is easy to see that there is no Brunel-Sucheston sequence in any finite dimensional Banach space. On the other hand, Brunel-Sucheston sequences indeed exist in any infinite dimensional Banach space. This result is due to Brunel and Sucheston [**6**]. By using the classical Ramsey's theorem, they actually proved the following:

THEOREM (Brunel-Sucheston [**6**]). *In any Banach space, every bounded sequence with no norm Cauchy subsequence has a subsequence which is a Brunel-Sucheston sequence.*

Now we shall prove the following Theorems 1 and 2 which are main results in this section and also necessary for our later applications. Theorem 2 will be formulated free from spreading models. We have first the following lemma.

LEMMA 1. *Let $\{x_n\}_n$ be a weakly null Brunel-Sucheston sequence in a Banach space X and $[E, \{e_n\}_n]$ be its spreading model. We put*

$$\rho(k):=\left\| \frac{1}{k} \sum_{i=1}^{k} e_i \right\| \qquad (k\in \boldsymbol{N}).$$

Then $\rho:=\lim_{k\to\infty}\rho(k)$ exists and is equal to $\inf_k \rho(k)$.

PROOF. We set $\nu:=\inf_k \rho(k)$, then for all $\eta>0$ there exists an $m\in \boldsymbol{N}$ such that $\rho(m)\leqq \nu+\eta$. Take any $k\in \boldsymbol{N}$ and decompose as $k=pm+q$ for some $p, q\in \boldsymbol{N}\cup\{0\}$ with $0\leqq q\leqq m-1$. Note that

$$\begin{aligned}\rho(k) &= \left\| \frac{1}{k} \sum_{i=1}^{k} e_i \right\| \leqq \frac{1}{k} \sum_{j=0}^{p-1} \left\| \sum_{i=1}^{m} e_{i+jm} \right\| + \frac{1}{k} \sum_{i=pm+1}^{k} \|e_i\| \\ &= \frac{1}{k} p \left\| \sum_{i=1}^{m} e_i \right\| + \frac{q}{k} \|e_1\| = \frac{pm}{k} \rho(m) + \frac{q}{k} \|e_1\| \\ &\leqq \frac{pm}{k}(\nu+\eta) + \frac{q}{k} \|e_1\| .\end{aligned}$$

Letting $k\to\infty$, we get

$$\overline{\lim_{k\to\infty}}\,\rho(k) \leqq \nu+\eta \qquad \text{for all } \eta>0,$$

hence we have

$$\overline{\lim_{k\to\infty}}\,\rho(k) \leqq \nu = \inf_k \rho(k) \leqq \underline{\lim_{k\to\infty}}\,\rho(k).$$

Therefore $\rho=\lim_{k\to\infty}\rho(k)$ exists and is equal to $\inf_k\rho(k)$.

The following well known facts are repeatedly used in the sequel.

LEMMA 2 (Singer [**18**, p. 499]). *For a monotone unconditional basic sequence* $\{x_n\}_n$ *in a Banach space* X, *we have*

$$\left\|\sum_{i=1}^{k} a_i x_i\right\| \leqq 4\left\|\sum_{i=1}^{k} b_i x_i\right\|$$

for all $(a_i)_{i=1}^k$, $(b_i)_{i=1}^k \in S_0$ *with* $|a_i|\leqq|b_i|$ $(i=1, 2, \cdots, k)$. *The constant* 4 *can be replaced by* 2 *if the scalars are real.*

LEMMA 3 (Bessaga-Pelczyński [**5**]). *Let* $\{x_n\}_n$ *be a weakly null sequence with* $\overline{\lim}_{n\to\infty}\|x_n\|>0$ *in a Banach space* X. *Then for every* $\varepsilon>0$ *one can choose a subsequence of* $\{x_n\}_n$ *which is a basic sequence with basis constant* $1+\varepsilon$.

We shall now state the following proposition concerning the case $\rho>0$, of which (1) follows from Beauzamy [**3**] together with Lemma 2, and (2) follows from (3) of Spreading Model Property.

PROPOSITION 1. *Let* $\{x_n\}_n$ *be a Brunel-Sucheston sequence in a Banach space* X *and* $[E, \{e_n\}_n]$ *be its spreading model. Assume that*

$$\rho = \lim_{k\to\infty}\left\|\frac{1}{k}\sum_{i=1}^{k} e_i\right\| = \inf_k\left\|\frac{1}{k}\sum_{i=1}^{k} e_i\right\| > 0.$$

Then the following statements hold.

(1) E *is isomorphic to* ℓ_1. *In fact we have the following inequality:*

$$\frac{\rho}{4}\sum_{i=1}^{k}|a_i| \leqq \left\|\sum_{i=1}^{k} a_i e_i\right\|$$

for all $(a_i)_{i=1}^k\in S_0$.

(2) X *contains finite dimensional* ℓ_1-*spaces uniformly. More precisely, for any strictly increasing sequence* $\{j(n)\}_n$ *of* $\boldsymbol{N}$ *one can choose a subsequence* $\{x'_n\}_n$ *of* $\{x_n\}_n$ *so that*

$$\frac{\rho}{8}\sum_{i=1}^{j(k)}|a_i| \leqq \left\|\sum_{i=1}^{j(k)} a_i x'_{n_i}\right\|$$

for all k, $n_i\in\boldsymbol{N}$ $(i=1, 2, \cdots, j(k))$ *with* $k\leqq n_1<n_2<\cdots<n_{j(k)}$ *and* $(a_i)_{i=1}^{j(k)}\in S_0$.

The following result is the key to our applications of the Brunel-Sucheston's spreading model.

THEOREM 1. *Let $\{x_n\}_n$ be a Brunel-Sucheston sequence in a Banach space X and $[E, \{e_n\}_n]$ be its spreading model. Then for any $\varepsilon>0$ and integer $t\geqq 2$ one can select a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ with the following property:*

For every k, $n_i\in \boldsymbol{N}$ $(i=1, 2, \cdots, k)$ with $n_1<n_2<\cdots<n_k$ and $(a_i)_{i=1}^k\in S_0$, we have

$$(1.1)\qquad \begin{cases} (1-\varepsilon)\left\|\sum_{i=1}^k a_ie_i\right\|-(2\log_t k)\sup_{1\leq i\leq k}|a_i|\sup_n\|x'_n\| \\ \leqq\left\|\sum_{i=1}^k a_ix'_{n_i}\right\|\leqq(1+\varepsilon)\left\|\sum_{i=1}^k a_ie_i\right\|+(3\log_t k)\sup_{1\leq i\leq k}|a_i|\sup_n\|x'_n\|\,. \end{cases}$$

PROOF. Let $0<\varepsilon<1$ and an integer $t\geqq 2$ be given. By the property (3) of Spreading Model Property, for any $k\in\boldsymbol{N}$ there exists an $L(k)\in\boldsymbol{N}$ such that for all $n_i\in\boldsymbol{N}$ $(i=1, 2, \cdots, k)$ with $L(k)\leqq n_1<n_2<\cdots<n_k$ and $(a_i)_{i=1}^k\in S_0$,

$$(1-\varepsilon)\left\|\sum_{i=1}^k a_ie_i\right\|\leqq\left\|\sum_{i=1}^k a_ix_{n_i}\right\|\leqq(1+\varepsilon)\left\|\sum_{i=1}^k a_ie_i\right\|\,.$$

We may assume that $\{L(k)\}_k$ is strictly increasing, and we set

$$x'_n:=x_{L(t^n)}\qquad(n\in\boldsymbol{N})\,.$$

We shall show that this subsequence $\{x'_n\}_n$ meets the requirement. Let k, $m\in\boldsymbol{N}$ with $t^m\leqq k<t^{m+1}$, $n_i\in\boldsymbol{N}$ $(i=1, 2, \cdots, k)$ with $n_1<n_2<\cdots<n_k$ and $(a_i)_{i=1}^k\in S_0$ be given. We observe the following decomposition:

$$\sum_{i=1}^k a_ix'_{n_i}=\sum_{i=1}^m a_ix'_{n_i}+\sum_{i=m+1}^k a_ix'_{n_i}\,.$$

Since $x'_{n_{m+1}}=x_{L(t^{n_{m+1}})}$ and $L(k)<L(t^{m+1})\leqq L(t^{n_{m+1}})$, it follows that

$$\begin{aligned}\left\|\sum_{i=m+1}^k a_ix'_{n_i}\right\| &\leqq(1+\varepsilon)\left\|\sum_{i=m+1}^k a_ie_i\right\| \\ &\leqq(1+\varepsilon)\left(\left\|\sum_{i=1}^k a_ie_i\right\|+\left\|\sum_{i=1}^m a_ie_i\right\|\right) \\ &\leqq(1+\varepsilon)\left\|\sum_{i=1}^k a_ie_i\right\|+2m\sup_{1\leq i\leq m}|a_i|\sup_{1\leq i\leq m}\|e_i\| \\ &\leqq(1+\varepsilon)\left\|\sum_{i=1}^k a_ie_i\right\|+(2\log_t k)\sup_{1\leq i\leq k}|a_i|\sup_n\|x'_n\|\end{aligned}$$

and

$$\left\|\sum_{i=m+1}^{k} a_i x'_{n_i}\right\| \geqq (1-\varepsilon)\left(\left\|\sum_{i=1}^{k} a_i e_i\right\| - \left\|\sum_{i=1}^{m} a_i e_i\right\|\right)$$

$$\geqq (1-\varepsilon)\left\|\sum_{i=1}^{k} a_i e_i\right\| - (\log_t k) \sup_{1\leqq i\leqq k} |a_i| \sup_n \|x'_n\| .$$

Moreover the following estimate is easily verified:

$$\left\|\sum_{i=1}^{m} a_i x'_{n_i}\right\| \leqq (\log_t k) \sup_{1\leqq i\leqq k} |a_i| \sup_n \|x'_n\| .$$

Hence we get (1.1).

This completes the proof of Theorem 1.

By using Theorem 1, we can show an "alternative" theorem concerning weakly null sequences which is a generalization of Rosenthal's result [**16**].

THEOREM 2. *For every weakly null sequence $\{x_n\}_n$ in X one can extract a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ such that either*

(1) $$\lim_{k\to\infty}\left(\sup_{n_1<\cdots<n_k,\, |a_i|\leqq 1}\left\|\frac{1}{k}\sum_{i=1}^{k} a_i x'_{n_i}\right\|\right) = 0$$

or

(2) $$\inf_k\left(\inf_{n_1<\cdots<n_k,\, |\theta_i|=1}\left\|\frac{1}{k}\sum_{i=1}^{k} \theta_i x'_{n_i}\right\|\right) > 0 .$$

If the case (2) *happens, then, in addition to* (2), *we have the following: for any strictly increasing sequence $\{j(n)\}_n$ of $\boldsymbol{N}$, one can choose further a subsequence $\{x''_n\}_n$ of $\{x'_n\}_n$ so that*

(3) $$\inf_k\left(\inf_{k\leqq n_1<\cdots<n_{j(k)},\, (a_i)_i\neq 0}\frac{1}{\sum_{i=1}^{j(k)}|a_i|}\left\|\sum_{i=1}^{j(k)} a_i x''_{n_i}\right\|\right) > 0 .$$

PROOF. If $\{x_n\}_n$ has a norm convergent subsequence whose limit is necessarily zero, then one can choose a subsequence which satisfies (1). So we may suppose that $\{x_n\}_n$ has no norm convergent subsequence. By virtue of Brunel-Sucheston's theorem, Theorem 1, Lemma 2 and Spreading Model Property (1) (b′), there is a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ which is a Brunel-Sucheston sequence with its spreading model $[E, \{e_n\}_n]$ and satisfies the following estimates:

(1.2) $$\frac{1}{5}\left\|\sum_{i=1}^{k} e_i\right\| - (2\log_2 k)\sup_n \|x'_n\| \leqq \left\|\sum_{i=1}^{k} \theta_i x'_{n_i}\right\|$$

and

(1.3) $$\left\|\sum_{i=1}^{k} a_i x'_{n_i}\right\| \leqq 5\left\|\sum_{i=1}^{k} e_i\right\| + (3\log_2 k)\sup_n \|x'_n\|$$

for every k, $n_i\in\boldsymbol{N}$ $(i=1, 2, \cdots, k)$ with $n_1<n_2<\cdots<n_k$ and $(a_i)_{i=1}^k$, $(\theta_i)_{i=1}^k$ with

$|a_i|\leqq 1$, $|\theta_i|=1$ $(i=1, 2, \cdots, k)$.

We investigate two cases $\rho=0$ and $\rho>0$, where ρ is as in Lemma 1, i.e.,

$$\rho=\lim_{k\to\infty}\left\|\frac{1}{k}\sum_{i=1}^{k}e_i\right\|=\inf_k\left\|\frac{1}{k}\sum_{i=1}^{k}e_i\right\|.$$

In the case of $\rho=0$, one can obtain from (1.3),

$$\varlimsup_{k\to\infty}\Big(\sup_{n_1<\cdots<n_k,\,|a_i|\leq 1}\Big\|\frac{1}{k}\sum_{i=1}^{k}a_ix'_{n_i}\Big\|\Big)$$

$$\leqq\varlimsup_{k\to\infty}\Big(\frac{5}{k}\Big\|\sum_{i=1}^{k}e_i\Big\|+3\frac{\log_2 k}{k}\sup_n\|x'_n\|\Big)=0.$$

For the case $\rho>0$, one can get from the inequality (1.2) that

$$\varliminf_{k\to\infty}\Big(\inf_{n_1<\cdots<n_k,\,|\theta_i|=1}\Big\|\frac{1}{k}\sum_{i=1}^{k}\theta_ix'_{n_i}\Big\|\Big)$$

$$\geqq\varliminf_{k\to\infty}\Big(\frac{1}{5k}\Big\|\sum_{i=1}^{k}e_i\Big\|-2\frac{\log_2 k}{k}\sup_n\|x'_n\|\Big)=\frac{1}{5}\rho>0.$$

Hence there is an $m\in\boldsymbol{N}$ such that

$$\inf_{k\geq m}\Big(\inf_{n_1<\cdots<n_k,\,|\theta_i|=1}\Big\|\frac{1}{k}\sum_{i=1}^{k}\theta_ix'_{n_i}\Big\|\Big)\geqq\frac{1}{6}\rho.$$

We may assume that

$$\frac{2}{3}\Big\|\sum_{i=1}^{m}a_ie_i\Big\|\leqq\Big\|\sum_{i=1}^{m}a_ix'_{n_i}\Big\|$$

for all $n_i\in\boldsymbol{N}$ $(i=1, 2, \cdots, m)$ with $n_1<n_2<\cdots<n_m$ and $(a_i)_{i=1}^m\in S_0$. Therefore we have

$$\inf_{k\leq m}\Big(\inf_{n_1<\cdots<n_k,\,|\theta_i|=1}\Big\|\frac{1}{k}\sum_{i=1}^{k}\theta_ix'_{n_i}\Big\|\Big)\geqq\frac{2}{3}\inf_{k\leq m}\Big(\inf_{n_1<\cdots<n_k,\,|\theta_i|=1}\Big\|\frac{1}{k}\sum_{i=1}^{k}\theta_ie_i\Big\|\Big)$$

$$\geqq\frac{1}{6}\inf_{k\leq m}\Big\|\frac{1}{k}\sum_{i=1}^{k}e_i\Big\|\geqq\frac{1}{6}\rho,$$

since $\{e_n\}_n$ is monotone unconditional, hence we get

$$\inf_k\Big(\inf_{n_1<\cdots<n_k,\,|\theta_i|=1}\Big\|\frac{1}{k}\sum_{i=1}^{k}\theta_ix'_{n_i}\Big\|\Big)\geqq\frac{1}{6}\rho>0.$$

Finally, it is easy to see that the condition (3) is a direct consequence of Proposition 1. This completes the proof of the theorem.

As a direct consequence, we get the following fact which is a slight generalization of Partington's result [**14**].

COROLLARY 1. *Let X be a Banach space with the weak Banach-Saks pro-*

perty. Then every weakly null sequence $\{x_n\}_n$ *in* X *has a subsequence* $\{x'_n\}_n$ *which satisfies the condition* (1) *in the above theorem.*

§ 2. Regular methods of summability.

In this section, we shall first prove that the subsequences which are chosen with respect to the condition (1) or (2) in Theorem 2 are closely related to "complete R-summability" and "complete non R-summability" (see Definition 1). Secondly we show an alternative theorem concerning bounded sequences of Banach spaces with respect to regular methods of summability.

Recall that a real or complex infinite matrix $(a_{n,m})_{n,m}$ is called a regular method of summability, if given a sequence of scalars $\{x_n\}_n$ converging to x_0, then the sequence

$$y_n := \sum_{m=1}^{\infty} a_{n,m} x_m \qquad (n \in \boldsymbol{N})$$

also converges to x_0. It is well known that $(a_{n,m})_{n,m}$ is a regular method of summability if and only if

$$\sup_n \Big(\sum_{m=1}^{\infty} |a_{n,m}| \Big) < \infty, \tag{1}$$

$$\lim_{n\to\infty} \Big(\sum_{m=1}^{\infty} a_{n,m} \Big) = 1 \tag{2}$$

and

$$\lim_{n\to\infty} a_{n,m} = 0 \qquad (m \in \boldsymbol{N}). \tag{3}$$

For a proof, see DeVito [7, p. 96].

By the conditions (1), (2) and (3), it is easy to see the following fact:

Let $(a_{n,m})_{n,m}$ *be a regular method of summability and* $\{x_n\}_n$ *be a sequence in a Banach space* X *which converges strongly* (*resp. weakly*) *to an element* $x_0 \in X$. *Then the sequence*

$$y_n := \sum_{m=1}^{\infty} a_{n,m} x_m \qquad (n \in \boldsymbol{N})$$

also converges strongly (*resp. weakly*) *to* x_0.

Let Λ denote the set of all regular methods of summability and put

$$\Lambda_0 := \Big\{ (a_{n,m})_{n,m} \in \Lambda : \lim_{n\to\infty} \Big(\sup_m |a_{n,m}| \Big) = 0 \Big\},$$

$$\Lambda_+ := \Big\{ (a_{n,m})_{n,m} \in \Lambda : \overline{\lim_{n\to\infty}} \Big(\sup_m |a_{n,m}| \Big) > 0 \Big\},$$

$$I := (\delta_{n,m})_{n,m} \text{ and } C := (c_{n,m})_{n,m},$$

where $\delta_{n,m}$ denotes the Kronecker delta, and we set $c_{n,m} := 1/n$ $(1 \leqq m \leqq n)$ and $c_{n,m} := 0$ $(1 \leqq n < m)$. It is easy to see that Λ is the disjoint union of Λ_0 and Λ_+, C is a "typical" element in Λ_0 and I is a "trivial" element in Λ_+. In order to state the main results of this section, we need more definitions.

DEFINITION 1. Let $\{x_n\}_n$ be a bounded sequence in X, x_0 be an element of X and $R = (a_{n,m})_{n,m}$ be an element of Λ.

(1) We say that $\{x_n\}_n$ is R-summable to x_0 if

$$\lim_{n\to\infty} \left\| \sum_{m=1}^{\infty} a_{n,m} x_m - x_0 \right\| = 0 .$$

(2) We say that $\{x_n\}_n$ is completely R-summable to x_0 if each subsequence of $\{x_n\}_n$ is R-summable to x_0.

(3) We say that $\{x_n\}_n$ is completely non R-summable if no subsequence of $\{x_n\}_n$ is R-summable.

DEFINITION 2. Let $\{x_n\}_n$ be a bounded sequence in X, x_0 be an element of X and $R = (a_{n,m})_{n,m}$ be an element of Λ.

(1) We say that $\{x_n\}_n$ is R-summable to x_0 with respect to the weak topology, denoted by w-R-summable to x_0, if

$$x_0 = w\text{-}\lim_{n\to\infty} \left(\sum_{m=1}^{\infty} a_{n,m} x_m \right) .$$

(2) We say that $\{x_n\}_n$ is completely R-summable to x_0 with respect to the weak topology, denoted by completely w-R-summable to x_0 if each subsequence of $\{x_n\}_n$ is w-R-summable to x_0.

(3) We say that $\{x_n\}_n$ is completely non R-summable with respect to the weak topology, denoted by completely non w-R-summable, if no subsequence of $\{x_n\}_n$ is w-R-summable.

In order to clarify the above definitions, we give examples.

EXAMPLE 1 (cf. Theorem 4). Let $\{e_n\}_n$ denote the unit vector basis of ℓ_1.

(1) We put for $n, m \in \boldsymbol{N}$,

$$x_n := \begin{cases} e_m & (n = 2^m), \\ 0 & (n \neq 2^m). \end{cases}$$

Then $\{x_n\}_n$ is C-summable to zero and has a subsequence $\{x_{2^n-1}\}_n = \{0\}_n$ which is completely C-summable to zero. However, $\{x_n\}_n$ also has a subsequence $\{x_{2^n}\}_n = \{e_n\}_n$ which is completely non C-summable (in fact, $\{e_n\}_n$ is completely non w-C-summable).

(2) We put for $n, m \in \boldsymbol{N}$,

$$x_n := \begin{cases} e_m & (n=2m-1), \\ -e_m & (n=2m). \end{cases}$$

Then $\{x_n\}_n$ is C-summable to zero. Since every subsequence of $\{x_n\}_n$ has further a subsequence which is equivalent to $\{e_n\}_n$, no subsequence of $\{x_n\}_n$ is completely C-summable.

Now we shall prove lemmas.

LEMMA 4.

(1) *Let $\{x_n\}_n$ be a bounded sequence in X, x_0 be an element of X and $R=(a_{n,m})_{n,m}$ be an element of Λ. Assume that $\{x_n\}_n$ is completely w-R-summable to x_0. Then $\{x_n\}_n$ converges weakly to x_0.*

(2) *Let $\{x_n\}_n$ be a bounded sequence in X, x_0 be an element of X and $R=(a_{n,m})_{n,m}$ be an element of Λ_+. Assume that $\{x_n\}_n$ is completely R-summable to x_0. Then $\{x_n\}_n$ converges strongly to x_0.*

PROOF. PROOF of (1). Suppose the conclusion is false, then one can choose an element $x^*\in X^*$ and a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ such that $\lim_{n\to\infty}x^*(x'_n)$ exists, but $x^*(x_0)\neq\lim_{n\to\infty}x^*(x'_n)$. Note that $x_0=w\text{-}\lim_{n\to\infty}(\sum_{m=1}^{\infty}a_{n,m}x'_m)$, hence one has

$$x^*(x_0)=\lim_{n\to\infty}x^*\Big(\sum_{m=1}^{\infty}a_{n,m}x'_m\Big)=\lim_{n\to\infty}\Big\{\sum_{m=1}^{\infty}a_{n,m}x^*(x'_m)\Big\}=\lim_{n\to\infty}x^*(x'_n)$$

by the regularity of $R=(a_{n,m})_{n,m}$, which is a contradiction.

PROOF of (2). Let $\{x_n\}_n$ be a bounded sequence in X which is completely R-summable to x_0 for some $R=(a_{n,m})_{n,m}\in\Lambda_+$. By Lemma 4 (1), $x_0=w\text{-}\lim_{n\to\infty}x_n$ and we may assume without loss of generality that $x_0=0$, hence $\{x_n\}_n$ is a weakly null sequence. Suppose that $\lim_{n\to\infty}\|x_n\|=0$ does not hold, then there exists a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ which is a basic sequence with basis constant 2 and $L:=\inf_n\|x'_n\|>0$ (cf. Lemma 3). Therefore we have

$$\sup_m|a_{n,m}|\leqq\frac{4}{L}\Big\|\sum_{m=1}^{\infty}a_{n,m}x'_m\Big\| \qquad (n\in N),$$

but this contradicts our assumptions

$$\lim_{n\to\infty}\Big\|\sum_{m=1}^{\infty}a_{n,m}x'_m\Big\|=0 \quad\text{and}\quad \overline{\lim_{n\to\infty}}\Big(\sup_m|a_{n,m}|\Big)>0\,.$$

This completes the proof of the lemma.

LEMMA 5.

(1) *Let $\{e_n\}_n$ be the unit vector basis of ℓ_1. Then $\{e_n\}_n$ is completely non w-R-summable for every $R\in\Lambda$.*

(2) *Let* $\{x_n\}_n$ *be a weakly Cauchy sequence in* X *with no weak limit. Then* $\{x_n\}_n$ *is completely non* w-R-*summable for every* $R \in \Lambda$.

(3) *Let* $\{x_n\}_n$ *be a weakly null basic sequence in* X *with* $L := \inf_n \|x_n\| > 0$. *Then* $\{x_n\}_n$ *is completely non* R-*summable for every* $R \in \Lambda_+$.

PROOF. PROOF of (1). Let $R = (a_{n,m})_{n,m}$ be an element of Λ. Suppose that for an element $x_0 = (b_n)_n \in \ell_1$, we have

$$x_0 = w\text{-}\lim_{n\to\infty}\Big(\sum_{m=1}^{\infty} a_{n,m} e_m\Big).$$

Since weak convergence in ℓ_1 implies coordinatewise convergence, by using the fact $\lim_{n\to\infty} a_{n,m} = 0$ ($m \in \boldsymbol{N}$), one can see that $b_n = 0$ ($n \in \boldsymbol{N}$). But for the element $x^* \in \ell_\infty = \ell_1^*$ whose coordinates are all 1, we have

$$\sum_{n=1}^{\infty} b_n = x^*(x_0) = \lim_{n\to\infty}\Big\{\sum_{m=1}^{\infty} a_{n,m} x^*(e_m)\Big\} = \lim_{n\to\infty}\Big(\sum_{m=1}^{\infty} a_{n,m}\Big) = 1\,,$$

which is a contradiction. Since every subsequence of $\{e_n\}_n$ is equivalent to $\{e_n\}_n$ itself, the conclusion follows.

PROOF of (2). Let $R = (a_{n,m})_{n,m}$ be an element of Λ. Suppose that for a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ and an element $x_0 \in X$, we have

$$x_0 = w\text{-}\lim_{n\to\infty}\Big(\sum_{m=1}^{\infty} a_{n,m} x'_m\Big).$$

Take any $x^* \in X^*$. Since $\lim_{n\to\infty} x^*(x'_n)$ exists and R is a regular method of summability, we have

$$x^*(x_0) = \lim_{n\to\infty}\Big\{\sum_{m=1}^{\infty} a_{n,m} x^*(x'_m)\Big\} = \lim_{n\to\infty} x^*(x'_n) = \lim_{n\to\infty} x^*(x_n)\,.$$

This means that $x_0 = w\text{-}\lim_{n\to\infty} x_n$, which is a contradiction. Hence $\{x_n\}_n$ is completely non w-R-summable for every $R \in \Lambda$.

PROOF of (3). Let $R = (a_{n,m})_{n,m}$ be an element of Λ_+ and M be a basis constant of $\{x_n\}_n$. Since we have

$$\max_{1\le i\le k} |c_i| \le \frac{2M}{L}\Big\|\sum_{i=1}^{k} c_i x_i\Big\|$$

for all $(c_i)_{i=1}^k \in S_0$ and $\overline{\lim}_{n\to\infty}(\sup_m |a_{n,m}|) > 0$, no subsequence of $\{x_n\}_n$ is R-summable (to zero), hence $\{x_n\}_n$ is completely non R-summable.

Thus we complete the proof of Lemma 5.

Now we will prove Theorem 3 and 4 which are main results in this section.

THEOREM 3.

(1) *Let* $\{x_n\}_n$ *be a bounded sequence in* X *which satisfies the condition* (1) *of Theorem* 2, *i.e.*,

$$\lim_{k\to\infty}\Big(\sup_{n_1<\cdots<n_k,\,|a_i|\le 1}\Big\|\frac{1}{k}\sum_{i=1}^{k}a_i x_{n_i}\Big\|\Big)=0\,.$$

Then $\{x_n\}_n$ *is completely R-summable* (*to zero*) *for every* $R\in\Lambda_0$.

(2) *Let* $\{x_n\}_n$ *be a weakly null sequence in X which satisfies the condition* (2) *of Theorem* 2, *i.e.*,

$$\inf_{k}\Big(\inf_{n_1<\cdots<n_k,\,|\theta_i|=1}\Big\|\frac{1}{k}\sum_{i=1}^{k}\theta_i x_{n_i}\Big\|\Big)>0\,.$$

Then $\{x_n\}_n$ *has a subsequence* $\{x'_n\}_n$ *which is completely non R-summable for each* $R\in\Lambda_+$, *and for each* $R\in\Lambda_0$ *there is a subsequence* $\{x''_n\}_n$ *of* $\{x'_n\}_n$ *which is completely non R-summable.*

PROOF. PROOF of (1). Let $\{x_n\}_n$ be a bounded sequence in X which satisfies the condition (1) of Theorem 2 and $R=(a_{n,m})_{n,m}$ be an element of Λ_0. We put for any subsequence $\{x'_n\}_n$ of $\{x_n\}_n$,

$$y_n:=\sum_{m=1}^{\infty}a_{n,m}x'_m \qquad (n\in\boldsymbol{N})$$

and

$$M:=\max\Big\{\sup_n\|x_n\|,\ \sup_n\Big(\sum_{m=1}^{\infty}|a_{n,m}|\Big)\Big\}<\infty\,.$$

By our assumption laid on $\{x_n\}_n$, for any $\varepsilon>0$ there exists a $k\in\boldsymbol{N}$ such that

$$\sup_{n_1<\cdots<n_k,\,|a_i|\le 1}\Big\|\frac{1}{k}\sum_{i=1}^{k}a_i x_{n_i}\Big\|<\varepsilon\,.$$

Fix any $n\in\boldsymbol{N}$ and let $h(m)$ $(m\in\boldsymbol{N})$ be a permutation of $\boldsymbol{N}$ such that

$$|a_{n,h(1)}|\geqq|a_{n,h(2)}|\geqq|a_{n,h(3)}|\geqq\cdots\,.$$

Let $x^*\in X^*$ with $\|x^*\|=1$. We choose $|\theta_m|=1$ $(m\geqq k+1)$ such that

$$|x^*(x'_{h(m)})|=\theta_m x^*(x'_{h(m)})\,.$$

Then we have the following estimates:

$$\begin{aligned}|x^*(y_n)|&\leqq\sum_{m=1}^{\infty}|a_{n,m}||x^*(x'_m)|=\sum_{m=1}^{\infty}|a_{n,h(m)}||x^*(x'_{h(m)})|\\&=\sum_{m=1}^{k}|a_{n,h(m)}||x^*(x'_{h(m)})|+\sum_{j=1}^{\infty}\sum_{m=jk+1}^{(j+1)k}|a_{n,h(m)}||x^*(x'_{h(m)})|\\&\leqq Mk|a_{n,h(1)}|+\sum_{j=1}^{\infty}\Big(k|a_{n,h(jk)}|\Big\|\frac{1}{k}\sum_{m=jk+1}^{(j+1)k}\theta_m x'_{h(m)}\Big\|\Big)\end{aligned}$$

$$\leqq Mk|a_{n,h(1)}|+\varepsilon\sum_{j=1}^{\infty}k|a_{n,h(jk)}| \leqq Mk|a_{n,h(1)}|+\varepsilon\sum_{m=1}^{\infty}|a_{n,h(m)}|$$

$$\leqq Mk\Big(\sup_{m}|a_{n,m}|\Big)+M\varepsilon .$$

Therefore we get

$$\|y_n\| \leqq Mk\Big(\sup_{m}|a_{n,m}|\Big)+M\varepsilon \qquad (n\in \boldsymbol{N}),$$

hence we have

$$\overline{\lim_{n\to\infty}}\|y_n\| \leqq M\varepsilon$$

by virtue of $\lim_{n\to\infty}(\sup_m|a_{n,m}|)=0$. Since $\varepsilon>0$ is arbitrary, we get

$$\lim_{n\to\infty}\|y_n\|=0,$$

hence $\{x_n\}_n$ is completely R-summable.

PROOF of (2). By our assumption, Brunel-Sucheston's theorem, Theorems 1, 2 and Lemma 3, $\{x_n\}_n$ has a subsequence which is still denoted by $\{x_n\}_n$ such that $\{x_n\}_n$ is a Brunel-Sucheston sequence with its spreading model $[E, \{e_n\}_n]$ with $\rho>0$ and is a basic sequence with basis constant 2, where

$$\rho=\lim_{k\to\infty}\left\|\frac{1}{k}\sum_{i=1}^{k}e_i\right\|=\inf_{k}\left\|\frac{1}{k}\sum_{i=1}^{k}e_i\right\|.$$

Then, by Lemma 5 (3), $\{x_n\}_n$ is completely non R-summable for each $R\in\Lambda_+$. Let $R=(a_{n,m})_{n,m}$ be an element of Λ_0 (or Λ). Since

$$\lim_{n\to\infty}\Big(\sum_{m=1}^{\infty}a_{n,m}\Big)=1 \quad\text{and}\quad \lim_{n\to\infty}a_{n,m}=0 \qquad (m\in\boldsymbol{N}),$$

one can choose suitable subsequences $\{j(k)\}_k$, $\{h(k)\}_k$ of $\boldsymbol{N}$ so as to satisfy

$$\sum_{m=k}^{j(k)}|a_{h(k),m}| \geqq \frac{1}{2} \qquad (k\in\boldsymbol{N}).$$

Then, by Proposition 1 (2), there is a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ such that

$$\frac{\rho}{8}\sum_{i=1}^{j(k)}|a_i| \leqq \left\|\sum_{i=1}^{j(k)}a_i x'_{m_i}\right\|$$

for all k, $m_i\in\boldsymbol{N}$ $(i=1, 2, \cdots, j(k))$ with $k\leqq m_1<m_2<\cdots<m_{j(k)}$ and $(a_i)_{i=1}^{j(k)}\in S_0$. Let $\{x''_n\}_n$ be any subsequence of $\{x'_n\}_n$ and put

$$y_n:=\sum_{m=1}^{\infty}a_{n,m}x''_m \qquad (n\in\boldsymbol{N}).$$

Then we have

$$0<\frac{\rho}{16}\leqq\frac{\rho}{8}\sum_{m=k}^{j(k)}|a_{h(k),m}|\leqq\left\|\sum_{m=k}^{j(k)}a_{h(k),m}x''_m\right\|$$

$$\leqq 4\left\|\sum_{m=1}^{\infty}a_{h(k),m}x''_m\right\|=4\|y_{h(k)}\| \qquad (k\in\boldsymbol{N}).$$

Hence $\{y_n\}_n$ is non R-summable, therefore $\{x'_n\}_n$ is completely non R-summable.
This completes the proof of Theorem 3.

THEOREM 4. *Let $\{x_n\}_n$ be a bounded sequence with no norm convergent subsequence in a Banach space X. Then $\{x_n\}_n$ has a subsequence $\{x'_n\}_n$ which satisfies one of the following three cases:*

(1) *$\{x'_n\}_n$ is completely R-summable for every $R\in\Lambda_0$ and completely non R-summable for every $R\in\Lambda_+$.*

(2) *$\{x'_n\}_n$ converges weakly and is completely non R-summable for every $R\in\Lambda_+$, and for each $R\in\Lambda_0$ there is a subsequence $\{x''_n\}_n$ of $\{x'_n\}_n$ which is completely non R-summable. Moreover, $\{x'_n\}_n$ has no subsequence which is completely non R-summable for every $R\in\Lambda_0$.*

(3) *$\{x'_n\}_n$ is completely non w-R-summable for every $R\in\Lambda$.*

PROOF. We rely upon Rosenthal's ℓ_1-theorem ([**15**], Dor [8] for complex scalars), so we know that $\{x_n\}_n$ has a subsequence $\{x'_n\}_n$ which is either equivalent to the unit vector basis of ℓ_1 or a weakly Cauchy sequence. If $\{x'_n\}_n$ is either equivalent to the unit vector basis of ℓ_1 or a weakly Cauchy sequence with no weak limit, then by Lemma 5 (1) and (2), $\{x'_n\}_n$ satisfies the case (3) of Theorem 4. Hence we need only to consider the case where $\{x'_n\}_n$ is a weakly convergent sequence. We may assume that $\{x'_n\}_n$ is a weakly null sequence. Moreover, by Lemma 3 and Theorem 2, we may suppose that $\{x'_n\}_n$ is a basic sequence with $\inf_n\|x'_n\|>0$ and satisfies one of the conditions stated in Theorem 2. If $\{x'_n\}_n$ satisfies the condition (1) of Theorem 2, then, by Theorem 3 (1) and Lemma 5 (3), $\{x'_n\}_n$ itself satisfies the case (1) of Theorem 4. On the other hand, suppose that $\{x'_n\}_n$ satisfies the condition (2) of Theorem 2. Then, by virtue of Theorem 3 (2), a subsequence of $\{x'_n\}_n$, which we still denote by $\{x'_n\}_n$, satisfies the first part of the case (2) of Theorem 4. Let $\{x''_n\}_n$ be any subsequence of $\{x'_n\}_n$. Since $\{x''_n\}_n$ converges weakly to zero, by Mazur's theorem [**13**], there is a subsequence $\{j_k\}_k$ of $\boldsymbol{N}$ and a sequence $\{\alpha_i\}_i$ of nonnegative real numbers such that

$$\sum_{i=j_k}^{j_{k+1}-1}\alpha_i=1 \quad (k\in\boldsymbol{N}) \quad \text{and} \quad \lim_{k\to\infty}\left\|\sum_{i=j_k}^{j_{k+1}-1}\alpha_i x''_i\right\|=0.$$

Then, $R=(a_{n,m})_{n,m}$ defined by

$$a_{n,m}=\begin{cases}\alpha_m & (j_n\leqq m<j_{n+1},\ n\in \boldsymbol{N}),\\ 0, & \text{otherwise},\end{cases}$$

is a regular method of summability and $\{x''_n\}_n$ is R-summable (to zero). Hence, by the first part of the case (2), R belongs to Λ_0. This means that $\{x'_n\}_n$ satisfies the second part of the case (2).

Therefore we complete the proof of Theorem 4.

Before stating some corollaries, we give examples of sequences appeared in three cases of Theorem 4.

EXAMPLE 2.

(1) The canonical unit vector basis $\{e_n\}_n$ of ℓ_p $(1<p<\infty)$ or c_0 satisfies the case (1) of Theorem 4.

(2) The Schauder basis $\{e_n\}_n$ of Baernstein's space [**1**] satisfies the case (2) of Theorem 4. More generally, let $\{x_n\}_n$ be a weakly null basic sequence with $\inf_n\|x_n\|>0$ which has no C-summable subsequence. Then $\{x_n\}_n$ itself satisfies the case (2) of Theorem 4.

(3) By Lemma 5, a bounded sequence which is equivalent to the unit vector basis of ℓ_1 or a weakly Cauchy sequence with no weak limit satisfies the case (3) of Theorem 4.

The first corollary of Theorem 4 is the following:

COROLLARY 2 (Erdös-Magidor [**9**]). *Let $\{x_n\}_n$ be a bounded sequence in X and R be a regular method of summability. Then there is a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ such that either*

(1) *$\{x'_n\}_n$ is completely R-summable*

or

(2) *$\{x'_n\}_n$ is completely non R-summable.*

Since, by virtue of Lemma 4 (2), a bounded sequence which is completely R-summable for some $R\in\Lambda_+$ is norm convergent, it is completely R-summable for every $R\in\Lambda_+$ (or Λ). For Λ_0, we have the following:

COROLLARY 3. *Let $\{x_n\}_n$ be a bounded sequence in X. Suppose that $\{x_n\}_n$ is completely R-summable for some $R\in\Lambda_0$. Then there is a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ which is completely R-summable for every $R\in\Lambda_0$.*

COROLLARY 4. *A Banach space X has the Banach-Saks property if* (*and only if*) *there is an $R\in\Lambda$ such that every bounded sequence in X has a subsequence which is R-summable.*

By combining Theorem 4 and Lemma 4 (1), we have the following:

COROLLARY 5 (Singer [17]). *A Banach space X is reflexive if (and only if) every bounded sequence in X is w-R-summable for some $R\in\Lambda$.*

COROLLARY 6. *Let X be a Banach space with the weak Banach-Saks property. Then for every weakly convergent sequence $\{x_n\}_n$ in X, one can choose a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ which is completely R-summable for every $R\in\Lambda_0$.*

§ 3. Growth rate of Cesàro means.

In [2], Banach and Saks proved that $L_p[0, 1]$ $(1<p<\infty)$ has the weak Banach-Saks property by actually showing the following:

Each weakly null sequence $\{x_n\}_n$ in $L_p[0, 1]$ has a subsequence $\{x'_n\}_n$ which satisfies

$$\left\|\sum_{i=1}^{k} x'_i\right\|_p=\begin{cases} O(k^{1/p}) & (1<p\leqq 2), \\ O(k^{1/2}) & (2<p<\infty). \end{cases}$$

Note that this result implies that $L_p[0, 1]$ $(1<p<\infty)$ has the Banach-Saks property, since $L_p[0, 1]$ $(1<p<\infty)$ is reflexive. As we stated in the introduction, a Banach space X is called of *type p* for some $1<p\leqq 2$, if there exists a constant $M\geqq 1$ so that for every finite set of vectors $\{x_i\}_{i=1}^k\subset X$, we have

$$\int_0^1\left\|\sum_{i=1}^{k} r_i(t)x_i\right\|dt\leqq M\Big(\sum_{i=1}^{k}\|x_i\|^p\Big)^{1/p},$$

where $\{r_n\}_n$ denotes the sequence of the Rademacher functions, i.e., $r_n(t)=\operatorname{sign}(\sin 2^{n-1}\pi t)$ $(n\in\boldsymbol{N})$. Any constant M satisfying the above inequality is called a type p constant of X. We note the following equalities:

$$\operatorname*{Average}_{\theta_i=\pm 1}\left\|\sum_{i=1}^{k}\theta_i x_i\right\|=\frac{1}{2^k}\sum_{\theta_i=\pm 1}\left\|\sum_{i=1}^{k}\theta_i x_i\right\|=\int_0^1\left\|\sum_{i=1}^{k} r_i(t)x_i\right\|dt\,.$$

It is well known that $L_p[0, 1]$ $(1<p<\infty)$ is of type $\min(2, p)$ (see Lindenstrauss and Tzafriri [11, p. 73]).

The notion of type (and cotype) was first introduced by Hoffmann-Jørgensen [10] and was studied extensively by Maurey and Pisier (in particular [12]). We shall see that the notion of type (and cotype) is closely related to the rates of convergence of Cesàro means. First we generalize the result of Banach and Saks with regard to the rates of convergence of Cesàro means.

THEOREM 5. *Let X be a Banach space of type p $(1<p\leqq 2)$ and M be a type p constant of X. Then for each weakly null sequence $\{x_n\}_n$ with $\sup_n\|x_n\|\leqq 1$ in X, one can extract a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ so that*

$$\left\|\sum_{i=1}^{k} a_i x'_{n_i}\right\|\leqq 78Mk^{1/p}$$

for every k, $n_i \in \boldsymbol{N}$ $(i=1, 2, \cdots, k)$ *with* $n_1 < n_2 < \cdots n_k$ *and* $(a_i)_{i=1}^k$ *with* $|a_i| \leqq 1$ $(i=1, 2, \cdots, k)$.

PROOF. Let $\{x_n\}_n$ be a weakly null sequence with $\sup_n \|x_n\| \leqq 1$ in X. If $\{x_n\}_n$ has a norm convergent subsequence whose limit is necessarily zero, then one can easily choose a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ which meets the requirement. For example, let $\{x'_n\}_n$ be chosen so that for all $m \geqq n \geqq 1$,

$$\|x'_m\| \leqq n^{1/p} - (n-1)^{1/p} .$$

So we may assume that $\{x_n\}_n$ has no norm convergent subsequence. By virtue of Brunel-Sucheston's theorem and Theorem 1, $\{x_n\}_n$ has a subsequence $\{x'_n\}_n$ which is a Brunel-Sucheston sequence with its spreading model $[E, \{e_n\}_n]$ and satisfies the following:

$$\frac{1}{5}\left\|\sum_{i=1}^k e_i\right\| - 2\log_3 k \leqq \inf_{|\theta_i|=1}\left\|\sum_{i=1}^k \theta_i x'_{n_i}\right\| \tag{3.1}$$

and

$$\sup_{|a_i|\leqq 1}\left\|\sum_{i=1}^k a_i x'_{n_i}\right\| \leqq 5\left\|\sum_{i=1}^k e_i\right\| + 3\log_3 k \tag{3.2}$$

for each k, $n_i \in \boldsymbol{N}$ $(i=1, 2, \cdots, k)$ with $n_1 < n_2 < \cdots < n_k$.

We show this subsequence $\{x'_n\}_n$ has the desired property. By using the inequality (3.1) we have

$$\begin{aligned}\frac{1}{5}\left\|\sum_{i=1}^k e_i\right\| - 2\log_3 k &\leqq \underset{\theta_i=\pm 1}{\text{Average}}\left\|\sum_{i=1}^k \theta_i x'_i\right\| \\ &\leqq M\Big(\sum_{i=1}^k \|x'_i\|^p\Big)^{1/p} \leqq Mk^{1/p} ,\end{aligned}$$

hence by the inequality (3.2) and the fact

$$\log_3 k \leqq k^{1/p} \qquad (k \in \boldsymbol{N}),$$

we obtain

$$\left\|\sum_{i=1}^k a_i x'_{n_i}\right\| \leqq \{5(5Mk^{1/p} + 10k^{1/p}) + 3k^{1/p}\} \leqq 78Mk^{1/p}$$

for all k, $n_i \in \boldsymbol{N}$ $(i=1, 2, \cdots, k)$ with $n_1 < n_2 < \cdots < n_k$ and $(a_i)_{i=1}^k$ with $|a_i| \leqq 1$ $(i=1, 2, \cdots, k)$.

This completes the proof of the theorem.

The following result which is a direct consequence of Theorem 5 is also derived from the works of Rosenthal [**16**] and Maurey and Pisier [**12**].

COROLLARY 7. *Let X be a Banach space of type p for some $1 < p \leqq 2$. Then X has the weak Banach-Saks property.*

Recall that a Banach space X is said to be of *cotype* q $(2\leqq q<\infty)$, if there exists a constant $M\geqq 1$ such that

$$\frac{1}{M}\left(\sum_{i=1}^{k}\|x_i\|^q\right)^{1/q}\leqq\int_0^1\left\|\sum_{i=1}^{k}r_i(t)x_i\right\|dt$$

for every finite set of vectors $\{x_i\}_{i=1}^{k}\subset X$, where $\{r_n\}_n$ is the sequence of the Rademacher functions. Any constant M satisfying the above inequality is called a cotype q constant of X. For instance, $L_p[0, 1]$ $(1<p<\infty)$ is of cotype $\max(2, p)$ ([**11**, p. 73]).

We now describe a cotype version of Theorem 5.

THEOREM 6. *Let X be a Banach space of cotype q $(2\leqq q<\infty)$ and M be a cotype q constant of X. Then each weakly null sequence $\{x_n\}_n$ with $\inf_n\|x_n\|\geqq 1$ in X admits of a subsequence $\{x'_n\}_n$ which satisfies*

$$\frac{1}{50M}k^{1/q}\leqq\left\|\sum_{i=1}^{k}\theta_i x'_{n_i}\right\|$$

for all k, $n_i\in\boldsymbol{N}$ $(i=1, 2, \cdots, k)$ with $n_1<n_2<\cdots<n_k$ and $(\theta_i)_{i=1}^{k}$ with $|\theta_i|=1$ $(i=1, 2, \cdots, k)$.

PROOF. Let $\{x_n\}_n$ be a weakly null sequence with $\inf_n\|x_n\|\geqq 1$ in X. Since $\{x_n\}_n$ has no norm convergent subsequence, by Brunel-Sucheston's theorem and Theorem 1, one can select a subsequence $\{x'_n\}_n$ of $\{x_n\}_n$ which is a Brunel-Sucheston sequence with its spreading model $[E, \{e_n\}_n]$ and satisfies the following:

$$\frac{1}{5}\left\|\sum_{i=1}^{k}e_i\right\|-(2\log_t k)\sup_n\|x'_n\|\leqq\inf_{|\theta_i|=1}\left\|\sum_{i=1}^{k}\theta_i x'_{n_i}\right\| \tag{3.3}$$

and

$$\sup_{|a_i|\leqq 1}\left\|\sum_{i=1}^{k}a_i x'_{n_i}\right\|\leqq 5\left\|\sum_{i=1}^{k}e_i\right\|+(3\log_t k)\sup_n\|x'_n\| \tag{3.4}$$

for all k, $n_i\in\boldsymbol{N}$ $(i=1, 2, \cdots, k)$ with $n_1<n_2<\cdots<n_k$, where an integer $t\geqq 2$ is chosen so large that

$$\sup_k\left\{\frac{1}{k^{1/q}}(\log_t k)\sup_n\|x'_n\|\right\}\leqq\frac{1}{106M}.$$

We now see that the subsequence $\{x'_n\}_n$ meets our requirement. Note that by the inequality (3.4) we have

$$\frac{1}{M}\left(\sum_{i=1}^{k}\|x'_{n_i}\|^q\right)^{1/q}\leqq\underset{\theta_i=\pm 1}{\text{Average}}\left\|\sum_{i=1}^{k}\theta_i x'_{n_i}\right\|\leqq 5\left\|\sum_{i=1}^{k}e_i\right\|+(3\log_t k)\sup_n\|x'_n\|$$

for all k, $n_i\in\boldsymbol{N}$ $(i=1, 2, \cdots, k)$ with $n_1<n_2<\cdots<n_k$, hence we obtain

$$\frac{1}{M}k^{1/q} \leqq 5\left\|\sum_{i=1}^{k} e_i\right\| + (3\log_t k)\sup_n \|x'_n\| .$$

Therefore we get by the inequality (3.3),

$$\begin{aligned}\inf_{|\theta_i|=1}\left\|\sum_{i=1}^{k}\theta_i x'_{n_i}\right\| &\geqq \frac{1}{5}\left\|\sum_{i=1}^{k} e_i\right\| - (2\log_t k)\sup_n \|x'_n\| \\ &\geqq \frac{1}{5}\left\{\frac{1}{5M}k^{1/q} - \frac{3}{5}(\log_t k)\sup_n \|x'_n\|\right\} - (2\log_t k)\sup_n \|x'_n\| \\ &= \frac{1}{25M}k^{1/q} - \frac{53}{25}(\log_t k)\sup_n \|x'_n\| \geqq \frac{1}{50M}k^{1/q}\end{aligned}$$

for all k, $n_i \in \boldsymbol{N}$ ($i=1, 2, \cdots, k$) with $n_1 < n_2 < \cdots < n_k$.

Thus the proof of the theorem is completed.

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