

## The birational action of $\mathfrak{S}_5$ on $P^2(\mathbb{C})$ and the icosahedron

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### § 1. Introduction.

The symmetric group  $\mathfrak{S}_5$  on five letters 1, 2, 3, 4, 5 is generated by permutations  $s_i = (i, i+1)$  ( $i=1, 2, 3, 4$ ). As is known, the 2-dimensional complex projective space  $P^2$  admits a birational action of  $\mathfrak{S}_5$  in the following manner:

$$\begin{aligned} s_1: (\xi_1: \xi_2: \xi_3) &\longrightarrow (\xi_1^{-1}: \xi_2^{-1}: \xi_3^{-1}), \\ s_2: (\xi_1: \xi_2: \xi_3) &\longrightarrow (\xi_1: \xi_1 - \xi_2: \xi_1 - \xi_3), \\ s_3: (\xi_1: \xi_2: \xi_3) &\longrightarrow (\xi_2: \xi_1: \xi_3), \\ s_4: (\xi_1: \xi_2: \xi_3) &\longrightarrow (\xi_1: \xi_3: \xi_2). \end{aligned}$$

Here  $\xi = (\xi_1: \xi_2: \xi_3)$  means a homogeneous coordinate of  $P^2$ . Putting  $S = \{\xi \in P^2; \xi_1 \xi_2 \xi_3 (\xi_2 - \xi_3)(\xi_3 - \xi_1)(\xi_1 - \xi_2) = 0\}$ , we find that each  $s_i$  defines an automorphism of  $P^2 - S$ . Moreover, it is known that  $\mathfrak{S}_5$  coincides with the group of birational actions  $\varphi$  of  $P^2$  such that  $\varphi|_{(P^2 - S)}$  are automorphisms.

Suggested by a result of [S], N. Takayama showed that there are mutually disjoint twenty simply connected domains  $D_i$  ( $i=1, \dots, 20$ ) of  $P^2 - S$  such that their union is open dense in  $P^2$  and that they are transitive by the  $\mathfrak{S}_5$ -action. On the other hand, it is well-known that the alternative group  $\mathfrak{U}_5$  of the fifth degree is the symmetry group of the icosahedron which has twenty faces.

The purpose of the present paper is to give a description of the fundamental group  $\pi_1(P^2 - S)$  in terms of combinatorial properties of the icosahedron. In particular, we shall introduce a group  $B(\gamma)$  consisting of certain equivalence classes of sequences of the twenty simply connected domains and show that  $\pi_1(P^2 - S) \cong B(\gamma)$ . The precise statement is given in Theorem 8.11.

We are going to explain the contents of this paper briefly. In § 2, we shall define twenty simply connected domains of  $P^2 - S$  and study their properties. In § 3, we shall construct the blowing up space  $Z$  of  $P^2$  so that the proper transform  $\tilde{S}$  of  $S$  is the union of ten lines whose intersecting points are of

normal crossing type and in § 4, we shall determine the boundary components of the simply connected domains in the space  $Z$ . In § 5, we shall name the twenty faces of the icosahedron in a natural way and in § 6, we shall construct a non-orientable surface  $\mathcal{K}$  from the icosahedron by using the results of § 5. The surface  $\mathcal{K}$  is topologically isomorphic to the connected sum of three Klein bottles and has a triangular decomposition with twenty triangles. Then we shall prove the existence of a natural bijective map between the set of twenty domains of  $P^2 - S$  given above and that of twenty triangles of  $\mathcal{K}$ . In this way, the set of the twenty triangles of  $\mathcal{K}$  admits an  $\mathfrak{S}_5$ -action and in this sense,  $\mathcal{K}$  is regarded as a non-orientable analogue of the icosahedron. In § 7, we shall define a group  $B(\gamma)$  consisting of certain equivalence classes of sequences of twenty letters which are in a one to one correspondence with the twenty domains of  $P^2 - S$  introduced above, where  $\gamma$  is a letter corresponding to one of the simply connected domains. In § 8, after a detailed study of the structure of the group  $B(\gamma)$ , we shall prove that  $B(\gamma) \cong \pi_1(P^2 - S)$ .

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## § 2. Twenty simply connected open subsets of $P^2 - S$ .

We begin with introducing polynomials  $\varphi_i(\xi)$  ( $1 \leq i \leq 5$ ) of  $\operatorname{Re} \xi_j$ ,  $\operatorname{Im} \xi_j$  ( $j = 1, 2, 3$ ) defined by

$$\varphi_1(\xi) = \operatorname{Im} \xi_2 \bar{\xi}_3 (\bar{\xi}_1 - \bar{\xi}_2)(\xi_1 - \xi_3),$$

$$\varphi_2(\xi) = \operatorname{Im} (\bar{\xi}_1 - \bar{\xi}_2)(\xi_1 - \xi_3),$$

$$\varphi_3(\xi) = \operatorname{Im} \bar{\xi}_2 \xi_3,$$

$$\varphi_4(\xi) = \operatorname{Im} \bar{\xi}_1 \xi_3,$$

$$\varphi_5(\xi) = \operatorname{Im} \bar{\xi}_1 \xi_2.$$

In spite that  $\varphi_i(\xi)$  ( $1 \leq i \leq 5$ ) are not functions on  $P^2$ , it is possible to define open subsets  $U_i^\varepsilon$  ( $1 \leq i \leq 5$ ,  $\varepsilon = \pm$ ) of  $P^2$  by

$$U_i^\varepsilon = \{\xi \in P^2; \varepsilon \varphi_i(\xi) > 0\}.$$

LEMMA 2.1. Put  ${}^0U_i^\varepsilon = U_i^\varepsilon \cap (P^2 - S)$ . Then

$$s_i({}^0U_j^\varepsilon) = \begin{cases} {}^0U_{i+1}^\varepsilon & \text{if } j = i \\ {}^0U_i^\varepsilon & \text{if } j = i+1 \\ {}^0U_j^{-\varepsilon} & \text{otherwise.} \end{cases}$$

This lemma follows from direct computation.

LEMMA 2.2.

- (i)  $U_3^+ \cap U_4^- \cap U_5^+ \subset U_1^- \cap U_2^+$ .  
(ii)  $U_3^+ \cap U_4^- \cap U_5^+$  is simply connected.

PROOF. (i) From the equations

$$\begin{aligned}\varphi_1(\xi) &= -|\xi_1|^2 \varphi_3(\xi) + |\xi_2|^2 \varphi_4(\xi) - |\xi_3|^2 \varphi_5(\xi), \\ \varphi_2(\xi) &= \varphi_3(\xi) - \varphi_4(\xi) + \varphi_5(\xi),\end{aligned}$$

we find that  $\varphi_1(\xi) < 0$ ,  $\varphi_2(\xi) > 0$  on the set  $U_3^+ \cap U_4^- \cap U_5^+$  and (i) follows.

(ii) is proved by direct computation (cf. [S]).

Combining Lemmas 2.1, 2.2, we have the following:

$$\begin{aligned}A_{12} &= U_3^- \cap U_4^+ \cap U_5^- \subset U_1^+ \cap U_2^-, & B_{12} &= U_3^+ \cap U_4^- \cap U_5^+ \subset U_1^- \cap U_2^+, \\ A_{13} &= U_2^+ \cap U_4^+ \cap U_5^- \subset U_1^+ \cap U_3^+, & B_{13} &= U_2^- \cap U_4^- \cap U_5^+ \subset U_1^- \cap U_3^-, \\ A_{14} &= U_2^+ \cap U_3^- \cap U_5^- \subset U_1^+ \cap U_4^-, & B_{14} &= U_2^- \cap U_3^+ \cap U_5^+ \subset U_1^- \cap U_4^+, \\ A_{15} &= U_2^+ \cap U_3^- \cap U_4^+ \subset U_1^+ \cap U_5^+, & B_{15} &= U_2^- \cap U_3^+ \cap U_4^- \subset U_1^- \cap U_5^-, \\ A_{23} &= U_1^+ \cap U_4^- \cap U_5^+ \subset U_2^+ \cap U_3^-, & B_{23} &= U_1^- \cap U_4^+ \cap U_5^- \subset U_2^- \cap U_3^+, \\ A_{24} &= U_1^+ \cap U_3^+ \cap U_5^+ \subset U_2^+ \cap U_4^+, & B_{24} &= U_1^- \cap U_3^- \cap U_5^- \subset U_2^- \cap U_4^-, \\ A_{25} &= U_1^+ \cap U_3^+ \cap U_4^- \subset U_2^+ \cap U_5^-, & B_{25} &= U_1^- \cap U_3^- \cap U_4^+ \subset U_2^- \cap U_5^+, \\ A_{34} &= U_1^+ \cap U_2^- \cap U_5^+ \subset U_3^- \cap U_4^+, & B_{34} &= U_1^- \cap U_2^+ \cap U_5^- \subset U_3^+ \cap U_4^-, \\ A_{35} &= U_1^+ \cap U_2^- \cap U_4^- \subset U_3^- \cap U_5^-, & B_{35} &= U_1^- \cap U_2^+ \cap U_4^+ \subset U_3^+ \cap U_5^+, \\ A_{45} &= U_1^+ \cap U_2^- \cap U_3^+ \subset U_4^+ \cap U_5^-, & B_{45} &= U_1^- \cap U_2^+ \cap U_3^- \subset U_4^- \cap U_5^+.\end{aligned}$$

We frequently write  $A_{ij} = A_{ji}$ ,  $B_{ij} = B_{ji}$  for simplicity.

THEOREM 2.3.

- (i) The action of  $\mathfrak{S}_5$  on the open sets  ${}^0U_i^\varepsilon$  ( $1 \leq i \leq 5$ ,  $\varepsilon = \pm$ ) induces that on the set  $\mathcal{D} = \{A_{ij}, B_{ij}; 1 \leq i < j \leq 5\}$ . The concrete description of the action of  $\mathfrak{S}_5$  is given in Figure I. In particular, the  $\mathfrak{S}_5$ -action on  $\mathcal{D}$  is transitive.  
(ii)  $(\cup_{i < j} A_{ij}) \cup (\cup_{i < j} B_{ij})$  is open dense in  $\mathbf{P}^2$ .

REMARK. The main part of this theorem was first formulated and shown by N. Takayama whose proof is very complicated. We give here a simpler proof based on Lemma 2.2.

PROOF. (i) First we determine the isotropy subgroup  $\text{Iso}(A_{12})$  of  $A_{12}$  in  $\mathfrak{S}_5$ . It follows from Lemma 2.1 that  $\text{Iso}(A_{12})$  is contained in  $\langle s_1, s_3, s_4 \rangle$ . From this, it is easy to show that  $\text{Iso}(A_{12}) = \langle s_1 s_3, s_1 s_4 \rangle$ . Therefore  $\mathcal{D} = \mathfrak{S}_5 \cdot A_{12}$ .

The statement (ii) is a consequence of (i) and Lemma 2.2.

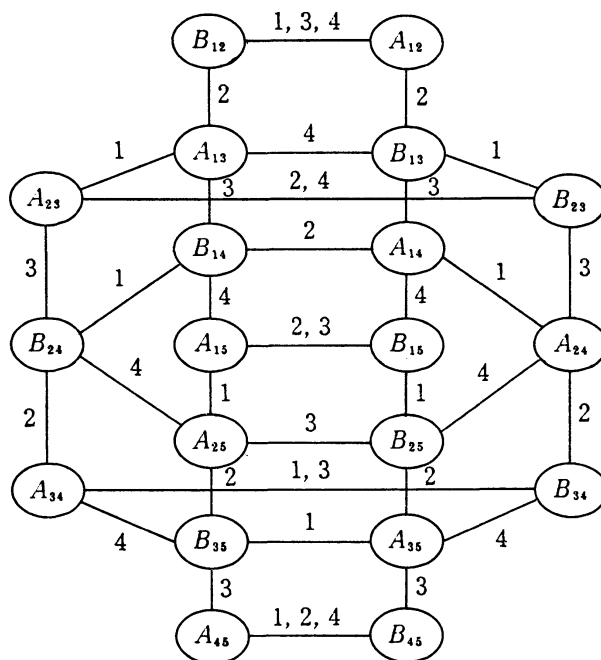


Figure I.

### § 3. The blowing up space of $P^2$ .

Following [ST], we shall construct the blowing up space of  $P^2$  at four points  $(1:0:0)$ ,  $(0:1:0)$ ,  $(0:0:1)$ ,  $(1:1:1)$ . First we introduce the space  $Z$  defined by

$$Z = \{((\xi_1 : \xi_2 : \xi_3), (\eta_1 : \eta_2 : \eta_3), (\zeta_1 : \zeta_2 : \zeta_3)) \in P^2 \times P^2 \times P^2;$$

$$\xi_1 \eta_1 = \xi_2 \eta_2 = \xi_3 \eta_3, \quad \xi_1 \zeta_1 + \xi_2 \zeta_2 + \xi_3 \zeta_3 = 0, \quad \zeta_1 + \zeta_2 + \zeta_3 = 0\}$$

and the projection  $\pi$  of  $Z$  to  $P^2$  by  $\pi(\xi, \eta, \zeta) = \xi$ . Moreover we define ten lines  $L(ij)$  ( $1 \leq i < j \leq 5$ ) of  $Z$  by

$$L(12): \xi_1 = \xi_2 = \xi_3, \quad \eta_1 = \eta_2 = \eta_3,$$

$$L(13): \xi_2 = \xi_3 = \eta_1 = 0,$$

$$L(14): \xi_1 = \xi_3 = \eta_2 = 0,$$

$$L(15): \xi_1 = \xi_2 = \eta_3 = 0,$$

$$L(23): \xi_1 = \eta_2 = \eta_3 = 0,$$

$$L(24): \xi_2 = \eta_1 = \eta_3 = 0,$$

$$L(25): \xi_3 = \eta_1 = \eta_2 = 0,$$

$$L(34): \xi_1 = \xi_2, \quad \eta_1 = \eta_2,$$

$$L(35): \xi_1 = \xi_3, \quad \eta_1 = \eta_3,$$

$$L(45): \xi_2 = \xi_3, \quad \eta_2 = \eta_3.$$

We frequently write  $L(ij) = L(ji)$  ( $i > j$ ) for simplicity. From the definition, we find that

$$\pi(L(12)) = \{(1:1:1)\},$$

$$\pi(L(13)) = \{(1:0:0)\},$$

$$\pi(L(14)) = \{(0:1:0)\},$$

$$\pi(L(15)) = \{(0:0:1)\},$$

$$\pi(L(23)) = \{(\xi_1: \xi_2: \xi_3); \xi_1 = 0\},$$

$$\pi(L(24)) = \{(\xi_1: \xi_2: \xi_3); \xi_2 = 0\},$$

$$\pi(L(25)) = \{(\xi_1: \xi_2: \xi_3); \xi_3 = 0\},$$

$$\pi(L(34)) = \{(\xi_1: \xi_2: \xi_3); \xi_1 = \xi_2\},$$

$$\pi(L(35)) = \{(\xi_1: \xi_2: \xi_3); \xi_1 = \xi_3\},$$

$$\pi(L(45)) = \{(\xi_1: \xi_2: \xi_3); \xi_2 = \xi_3\}.$$

We find that  $\pi^{-1}(\mathbf{P}^2 - S)$  is biholomorphic to  $\mathbf{P}^2 - S$  and the complement of  $\pi^{-1}(\mathbf{P}^2 - S)$  in  $Z$  is  $\tilde{S} = \pi^{-1}(S)$  which is the union of the lines  $L(ij)$  ( $i < j$ ) which intersect at normal crossing points. To explain more precisely, we state the following which is easy to prove.

LEMMA 3.1. *If  $L(ij) \cap L(i'j') \neq \emptyset$ , then  $i, j, i', j'$  are mutually different and the intersection  $L(ij) \cap L(i'j')$  consists of a unique point at which  $L(ij)$  and  $L(i'j')$  intersect normal crossingly.*

As a result, we find that  $Z$  is a blowing up space of  $\mathbf{P}^2$  at the four points  $(1:0:0)$ ,  $(0:1:0)$ ,  $(0:0:1)$ ,  $(1:1:1)$ .

There are holomorphic involutive automorphisms  $t_i$  ( $i=1, 2, 3, 4$ ) of  $Z$  such that  $\pi \circ t_i = s_i \circ \pi$ . In particular,  $(i, i+1) \rightarrow t_i$  implies an isomorphism of  $\mathfrak{S}_5$  with the group generated by  $t_1, t_2, t_3, t_4$ . In this way,  $\mathfrak{S}_5$  acts on the space  $Z$ . If  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 \end{pmatrix}$  is a permutation of  $\mathfrak{S}_5$ , the corresponding automorphism of  $Z$  maps the line  $L(ij)$  to  $L(\sigma_i \sigma_j)$ .

We put  $\tilde{U}_i^\varepsilon = \pi^{-1}(U_i^\varepsilon)$  ( $\varepsilon = \pm$ ),  $\tilde{U}_i^0 = \overline{\tilde{U}_i^+} \cap \overline{\tilde{U}_i^-}$  ( $i=1, \dots, 5$ ),  $\tilde{A}_{ij} = \pi^{-1}(A_{ij})$ ,  $\tilde{B}_{ij} = \pi^{-1}(B_{ij})$  ( $i < j$ ) and  $\tilde{\mathcal{D}} = \{\tilde{A}_{ij}, \tilde{B}_{ij}\}$ . It follows from the definition that  $\tilde{A}_{ij} \cong A_{ij}$ ,  $\tilde{B}_{ij} \cong B_{ij}$ . Moreover, we denote by  $Z_R$  the real locus of  $Z$ , that is,

$$Z_R = \{((\xi_1 : \xi_2 : \xi_3), (\eta_1 : \eta_2 : \eta_3), (\zeta_1 : \zeta_2 : \zeta_3)) \in Z; \xi_1, \xi_2, \xi_3 \in \mathbf{R}\}.$$

REMARK. In Terada [T2], the blowing up space  $Z$  is constructed in an alternative way.

#### § 4. The boundary components of $\tilde{A}_{ij}, \tilde{B}_{ij}$ .

In the sequel, for a closed algebraic set  $Y$  of  $Z$  and its locally closed subset  $V$  which contains an inner point of  $Y$ ,  $V^\circ$  denotes the interior of  $V$  in  $Y$ .

The purpose of this section is to study the boundary components of the sets  $\tilde{A}_{ij}, \tilde{B}_{ij}$ .

PROPOSITION 4.1.

- (i) The six subsets  $\partial\tilde{A}_{23} \cap \partial\tilde{B}_{45}, \partial\tilde{B}_{23} \cap \partial\tilde{A}_{45}, \partial\tilde{A}_{24} \cap \partial\tilde{B}_{35}, \partial\tilde{B}_{24} \cap \partial\tilde{A}_{35}, \partial\tilde{A}_{25} \cap \partial\tilde{B}_{34}, \partial\tilde{B}_{25} \cap \partial\tilde{A}_{34}$  of  $Z$  are contained in  $\tilde{U}_1^0$  and their union coincides with  $\tilde{U}_1^0$ . Moreover, their interiors are simply connected and mutually disjoint.
- (ii) The six subsets  $\partial\tilde{A}_{13} \cap \partial\tilde{A}_{45}, \partial\tilde{B}_{13} \cap \partial\tilde{B}_{45}, \partial\tilde{A}_{14} \cap \partial\tilde{A}_{35}, \partial\tilde{B}_{14} \cap \partial\tilde{B}_{35}, \partial\tilde{A}_{15} \cap \partial\tilde{A}_{34}, \partial\tilde{B}_{15} \cap \partial\tilde{B}_{34}$  of  $Z$  are contained in  $\tilde{U}_2^0$  and their union coincides with  $\tilde{U}_2^0$ . Moreover, their interiors are simply connected and mutually disjoint.
- (iii) The six subsets  $\partial\tilde{A}_{12} \cap \partial\tilde{A}_{45}, \partial\tilde{B}_{12} \cap \partial\tilde{B}_{45}, \partial\tilde{A}_{14} \cap \partial\tilde{A}_{25}, \partial\tilde{B}_{14} \cap \partial\tilde{B}_{25}, \partial\tilde{A}_{15} \cap \partial\tilde{A}_{24}, \partial\tilde{B}_{15} \cap \partial\tilde{B}_{24}$  of  $Z$  are contained in  $\tilde{U}_3^0$  and their union coincides with  $\tilde{U}_3^0$ . Moreover, their interiors are simply connected and mutually disjoint.
- (iv) The six subsets  $\partial\tilde{A}_{12} \cap \partial\tilde{A}_{35}, \partial\tilde{B}_{12} \cap \partial\tilde{B}_{35}, \partial\tilde{A}_{13} \cap \partial\tilde{A}_{25}, \partial\tilde{B}_{13} \cap \partial\tilde{B}_{25}, \partial\tilde{A}_{15} \cap \partial\tilde{A}_{23}, \partial\tilde{B}_{15} \cap \partial\tilde{B}_{23}$  of  $Z$  are contained in  $\tilde{U}_4^0$  and their union coincides with  $\tilde{U}_4^0$ . Moreover, their interiors are simply connected and mutually disjoint.
- (v) The six subsets  $\partial\tilde{A}_{12} \cap \partial\tilde{A}_{34}, \partial\tilde{B}_{12} \cap \partial\tilde{B}_{34}, \partial\tilde{A}_{13} \cap \partial\tilde{A}_{24}, \partial\tilde{B}_{13} \cap \partial\tilde{B}_{24}, \partial\tilde{A}_{14} \cap \partial\tilde{A}_{23}, \partial\tilde{B}_{14} \cap \partial\tilde{B}_{23}$  of  $Z$  are contained in  $\tilde{U}_5^0$  and their union coincides with  $\tilde{U}_5^0$ . Moreover, their interiors are simply connected and mutually disjoint.

PROOF. From the definitions of  $\tilde{A}_{ij}, \tilde{B}_{ij}$ , we find that each  $\tilde{U}_i^0$  is contained in the union of  $\bigcup_{i < j} (\partial\tilde{A}_{ij} \cup \partial\tilde{B}_{ij})$ . Noting the  $\mathfrak{S}_5$ -action on  $Z$ , the proposition is a consequence of Lemma 4.2 below.

LEMMA 4.2. Take two open sets  $C_1, C_2$  of  $\mathcal{D}$ . Assume that  $C_1 \neq C_2$ . If  $\partial C_1 \cap \partial C_2$  has 3 dimensions over  $\mathbf{R}$  and contained in  $U_5^0$ , then  $\{C_1, C_2\}$  equals one of the following six sets  $\{B_{14}, B_{23}\}, \{A_{14}, A_{23}\}, \{A_{13}, A_{24}\}, \{B_{13}, B_{24}\}, \{B_{12}, B_{34}\}, \{A_{12}, A_{34}\}$ . Moreover,  $(\partial C_1 \cap \partial C_2)^\circ$  is simply connected and disjoint with  $S$ .

PROOF. We take  $\xi \in U_5^0$  and assume that  $\xi_2 \neq 0$  for a moment. Then, from

the assumption, we have

$$\begin{aligned}\xi_1 \bar{\xi}_2 &= \bar{\xi}_1 \xi_2, & \varphi_1(\xi) &= (\xi_1 \bar{\xi}_2 - |\xi_1|^2) \varphi_3(\xi), \\ \varphi_2(\xi) &= \frac{\xi_2 - \bar{\xi}_1}{\xi_2} \varphi_3(\xi), & \varphi_4(\xi) &= \frac{\xi_1}{\xi_2} \varphi_3(\xi).\end{aligned}$$

From these equations, we find the following:

- (1) If  $\xi_1/\xi_2 > 1$ ,  $\varphi_3(\xi) > 0$ , then  $\varphi_1(\xi) < 0$ ,  $\varphi_2(\xi) < 0$ ,  $\varphi_4(\xi) > 0$ .
- (2) If  $\xi_1/\xi_2 > 1$ ,  $\varphi_3(\xi) < 0$ , then  $\varphi_1(\xi) > 0$ ,  $\varphi_2(\xi) > 0$ ,  $\varphi_4(\xi) < 0$ .
- (3) If  $0 < \xi_1/\xi_2 < 1$ ,  $\varphi_3(\xi) > 0$ , then  $\varphi_1(\xi) > 0$ ,  $\varphi_2(\xi) > 0$ ,  $\varphi_4(\xi) > 0$ .
- (4) If  $0 < \xi_1/\xi_2 < 1$ ,  $\varphi_3(\xi) < 0$ , then  $\varphi_1(\xi) < 0$ ,  $\varphi_2(\xi) < 0$ ,  $\varphi_4(\xi) < 0$ .
- (5) If  $\xi_1/\xi_2 < 0$ ,  $\varphi_3(\xi) > 0$ , then  $\varphi_1(\xi) < 0$ ,  $\varphi_2(\xi) > 0$ ,  $\varphi_4(\xi) < 0$ .
- (6) If  $\xi_1/\xi_2 < 0$ ,  $\varphi_3(\xi) < 0$ , then  $\varphi_1(\xi) > 0$ ,  $\varphi_2(\xi) < 0$ ,  $\varphi_4(\xi) > 0$ .

Moreover, comparing the above inequalities with the definition of  $A_{ij}$ ,  $B_{ij}$  we have the following:

In the case (1),  $\{C_1, C_2\} = \{B_{14}, B_{23}\}$ .

In the case (2),  $\{C_1, C_2\} = \{A_{14}, A_{23}\}$ .

In the case (3),  $\{C_1, C_2\} = \{A_{13}, A_{24}\}$ .

In the case (4),  $\{C_1, C_2\} = \{B_{13}, B_{24}\}$ .

In the case (5),  $\{C_1, C_2\} = \{B_{12}, B_{34}\}$ .

In the case (6),  $\{C_1, C_2\} = \{A_{12}, A_{34}\}$ .

Now we consider the case  $C_1=B_{14}$ ,  $C_2=B_{23}$ , that is, the case (1). Then, from the argument above, we find that the set

$$M = \{\xi \in \mathbf{P}^2; |\xi_2|^2 < \xi_1 \bar{\xi}_2 = \bar{\xi}_1 \xi_2 < |\xi_1|^2, \operatorname{Im} \bar{\xi}_2 \xi_3 > 0\}$$

is an open subset of  $(\partial B_{14} \cap \partial B_{23})^\circ$ . Since it is easy to show that  $(\partial B_{14} \cap \partial B_{23})^\circ$  and  $\xi_2=0$  are disjoint, we find that  $M=(\partial B_{14} \cap \partial B_{23})^\circ$ . On the other hand, putting  $x=\xi_1/\xi_2$ ,  $y=\xi_3/\xi_2$ , we find that  $M$  is isomorphic to the set  $\{(x, y) \in \mathbf{C}^2; 1 < x = \bar{x}, \operatorname{Im} y > 0\}$  which is clearly simply connected. Hence  $M=(\partial B_{14} \cap \partial B_{23})^\circ$  is simply connected. It is clear from the definition that  $M \cap S = \emptyset$ . By an argument similar to the above case, we can prove that  $(\partial A_{14} \cap \partial A_{23})^\circ$ ,  $(\partial A_{13} \cap \partial A_{24})^\circ$ ,  $(\partial B_{13} \cap \partial B_{24})^\circ$ ,  $(\partial B_{12} \cap \partial B_{34})^\circ$ ,  $(\partial A_{12} \cap \partial A_{34})^\circ$  are simply connected and are disjoint with  $S$ .

From Proposition 4.1, we have the following: Let  $j_1, \dots, j_5$  be numbers such that  $\{j_1, \dots, j_5\} = \{1, 2, 3, 4, 5\}$ .

(i) The case where  $1 \in \{j_1, j_2\}$ .

(i.1) The union of  $(\partial\tilde{A}_{j_1j_2} \cap \partial\tilde{A}_{j_3j_4})^\circ$ ,  $(\partial\tilde{A}_{j_1j_2} \cap \partial\tilde{A}_{j_3j_5})^\circ$ ,  $(\partial\tilde{A}_{j_1j_2} \cap \partial\tilde{A}_{j_4j_5})^\circ$  is open dense in  $\partial\tilde{A}_{j_1j_2}$  and

$$\partial(\partial\tilde{A}_{j_1j_2}) = \partial\tilde{A}_{j_1j_2} - \{(\partial\tilde{A}_{j_1j_2} \cap \partial\tilde{A}_{j_3j_4})^\circ \cup (\partial\tilde{A}_{j_1j_2} \cap \partial\tilde{A}_{j_3j_5})^\circ \cup (\partial\tilde{A}_{j_1j_2} \cap \partial\tilde{A}_{j_4j_5})^\circ\}.$$

(i.2) The union of  $(\partial\tilde{B}_{j_1j_2} \cap \partial\tilde{B}_{j_3j_4})^\circ$ ,  $(\partial\tilde{B}_{j_1j_2} \cap \partial\tilde{B}_{j_3j_5})^\circ$ ,  $(\partial\tilde{B}_{j_1j_2} \cap \partial\tilde{B}_{j_4j_5})^\circ$  is open dense in  $\partial\tilde{B}_{j_1j_2}$  and

$$\partial(\partial\tilde{B}_{j_1j_2}) = \partial\tilde{B}_{j_1j_2} - \{(\partial\tilde{B}_{j_1j_2} \cap \partial\tilde{B}_{j_3j_4})^\circ \cup (\partial\tilde{B}_{j_1j_2} \cap \partial\tilde{B}_{j_3j_5})^\circ \cup (\partial\tilde{B}_{j_1j_2} \cap \partial\tilde{B}_{j_4j_5})^\circ\}.$$

(ii) The case where  $j_5=1$ .

(ii.1) The union of  $(\partial\tilde{A}_{j_1j_2} \cap \partial\tilde{B}_{j_3j_4})^\circ$ ,  $(\partial\tilde{A}_{j_1j_2} \cap \partial\tilde{B}_{j_3j_5})^\circ$ ,  $(\partial\tilde{A}_{j_1j_2} \cap \partial\tilde{B}_{j_4j_5})^\circ$  is open dense in  $\partial\tilde{A}_{j_1j_2}$  and

$$\partial(\partial\tilde{A}_{j_1j_2}) = \partial\tilde{A}_{j_1j_2} - \{(\partial\tilde{A}_{j_1j_2} \cap \partial\tilde{B}_{j_3j_4})^\circ \cup (\partial\tilde{A}_{j_1j_2} \cap \partial\tilde{B}_{j_3j_5})^\circ \cup (\partial\tilde{A}_{j_1j_2} \cap \partial\tilde{B}_{j_4j_5})^\circ\}.$$

(ii.2) The union of  $(\partial\tilde{B}_{j_1j_2} \cap \partial\tilde{A}_{j_3j_4})^\circ$ ,  $(\partial\tilde{B}_{j_1j_2} \cap \partial\tilde{A}_{j_3j_5})^\circ$ ,  $(\partial\tilde{B}_{j_1j_2} \cap \partial\tilde{A}_{j_4j_5})^\circ$  is open dense in  $\partial\tilde{B}_{j_1j_2}$  and

$$\partial(\partial\tilde{B}_{j_1j_2}) = \partial\tilde{B}_{j_1j_2} - \{(\partial\tilde{B}_{j_1j_2} \cap \partial\tilde{A}_{j_3j_4})^\circ \cup (\partial\tilde{B}_{j_1j_2} \cap \partial\tilde{A}_{j_3j_5})^\circ \cup (\partial\tilde{B}_{j_1j_2} \cap \partial\tilde{A}_{j_4j_5})^\circ\}.$$

In the rest of this section, we study the subsets  $\partial(\partial\tilde{A}_{ij})$ ,  $\partial(\partial\tilde{B}_{ij})$  which have two dimensions.

PROPOSITION 4.3.

$$\bigcup_{i < j} (\tilde{U}_i^\circ \cap \tilde{U}_j^\circ) = \bigcup_{i < j} \{\partial(\partial\tilde{A}_{ij}) \cup \partial(\partial\tilde{B}_{ij})\} = Z_R \cup \{\bigcup_{i < j} L(ij)\}.$$

PROOF. It follows from Proposition 4.1 and the definitions of  $\partial(\partial\tilde{A}_{ij})$ ,  $\partial(\partial\tilde{B}_{ij})$  that

$$\bigcup_{i < j} (\tilde{U}_i^\circ \cap \tilde{U}_j^\circ) = \bigcup_{i < j} \{\partial(\partial\tilde{A}_{ij}) \cup \partial(\partial\tilde{B}_{ij})\}.$$

On the other hand, it is clear from the definition that  $Z_R$  and  $(\bigcup_{i < j} L(ij))$  are contained in  $\bigcup_{i < j} \{\partial(\partial\tilde{A}_{ij}) \cup \partial(\partial\tilde{B}_{ij})\}$ . The converse inclusion relation follows from the following lemma.

LEMMA 4.4. If  $\{j_1, j_2, j_3, j_4, j_5\} = \{1, 2, 3, 4, 5\}$ , then  $\tilde{U}_{j_1}^\circ \cap \tilde{U}_{j_2}^\circ \subset Z_R \cup (\bigcup_{i < j} L(ij))$ .

PROOF. Noting the  $\mathfrak{S}_5$ -action, it suffices to prove the lemma in the case where  $j_1=4$ ,  $j_2=5$ . Then it follows from the definition that  $\tilde{U}_4^\circ \cap \tilde{U}_5^\circ$  is contained in the set

$$\{(\xi, \eta, \zeta) \in Z; \xi_1\bar{\xi}_2 = \bar{\xi}_1\xi_2, \xi_1\bar{\xi}_3 = \bar{\xi}_1\xi_3, \eta_1\bar{\eta}_2 = \bar{\eta}_1\eta_2, \eta_1\bar{\eta}_3 = \bar{\eta}_1\eta_3\}.$$



Therefore the lemma follows from the definition of  $Z$  and that of ten lines.

Next we determine the connected components of  $Z_R - \tilde{S}$ . For this purpose, we introduce twelve subsets of  $Z_R$  defined by

$$\begin{aligned} D(12345) &= \{(\xi, \eta, \zeta) \in Z_R; 0 < \xi_1 < \xi_2 < \xi_3\}, \\ D(12354) &= \{(\xi, \eta, \zeta) \in Z_R; 0 < \xi_1 < \xi_3 < \xi_2\}, \\ D(12435) &= \{(\xi, \eta, \zeta) \in Z_R; 0 < \xi_2 < \xi_1 < \xi_3\}, \\ D(12453) &= \{(\xi, \eta, \zeta) \in Z_R; 0 < \xi_2 < \xi_3 < \xi_1\}, \\ D(12534) &= \{(\xi, \eta, \zeta) \in Z_R; 0 < \xi_3 < \xi_1 < \xi_2\}, \\ D(12543) &= \{(\xi, \eta, \zeta) \in Z_R; 0 < \xi_3 < \xi_2 < \xi_1\}, \\ D(13245) &= \{(\xi, \eta, \xi) \in Z_R; \xi_1 < 0 < \xi_2 < \xi_3\}, \\ D(13254) &= \{(\xi, \eta, \zeta) \in Z_R; \xi_1 < 0 < \xi_3 < \xi_2\}, \\ D(13425) &= \{(\xi, \eta, \zeta) \in Z_R; \xi_3 < 0 < \xi_2 < \xi_1\}, \\ D(13524) &= \{(\xi, \eta, \zeta) \in Z_R; \xi_2 < 0 < \xi_3 < \xi_1\}, \\ D(14235) &= \{(\xi, \eta, \zeta) \in Z_R; \xi_2 < 0 < \xi_1 < \xi_3\}, \\ D(14325) &= \{(\xi, \eta, \zeta) \in Z_R; \xi_3 < 0 < \xi_1 < \xi_2\}. \end{aligned}$$

PROPOSITION 4.5. (i) *The twelve sets  $D(12345)$ ,  $D(12354)$ ,  $D(12435)$ ,  $D(12453)$ ,  $D(12534)$ ,  $D(12543)$ ,  $D(13245)$ ,  $D(13425)$ ,  $D(13524)$ ,  $D(13254)$ ,  $D(14235)$ ,  $D(14325)$  are simply connected, mutually disjoint. Their union is open dense in  $Z_R$  and its complement equals  $Z_R \cap (\bigcup_{i < j} L(ij))$ .*

(ii) *The action of  $\mathfrak{S}_5$  on the set*

$$\mathcal{E} = \{D(12345), D(12354), D(12435), D(12453), D(12534), D(12543), \\ D(13245), D(13425), D(13524), D(13254), D(14235), D(14325)\}.$$

*is transitive. The isotropy subgroup of  $D(12345)$  is generated by  $w_1 = s_1 s_2 s_3 s_4$  and  $w_2 = s_1 s_2 s_1 s_4$  and is isomorphic to the dihedral group of order 10.*

The proof of this proposition is straightforward and so we omit it.

Last we remark on some properties of the set  $D(12345)$ . The set  $D(12345)$  is simply connected and is surrounded by the lines  $L(12)_R$ ,  $L(23)_R$ ,  $L(34)_R$ ,  $L(45)_R$ ,  $L(15)_R$ , where  $L(ij)_R = L(ij) \cap Z_R$  which is isomorphic to  $\mathbf{P}^1(\mathbf{R})$ , the 1-dimensional real projective space. More precisely, we have

$$s(12345, 12) = \overline{D(12345)} \cap \overline{D(12543)} \subset L(12)_R,$$

$$s(12345, 23) = \overline{D(12345)} \cap \overline{D(13245)} \subset L(23)_R,$$

$$s(12345, 34) = \overline{D(12345)} \cap \overline{D(12435)} \subset L(34)_R,$$

$$s(12345, 45) = \overline{D(12345)} \cap \overline{D(12354)} \subset L(45)_R,$$

$$s(12345, 15) = \overline{D(12345)} \cap \overline{D(14325)} \subset L(15)_R,$$

and each  $s(12345, ij)$  is a segment of the real line  $L(ij)_R$ . Moreover, there are five vertices of  $D(12345)$  which are defined by

$$s(12345, 12) \cap s(12345, 34) = L(12) \cap L(34),$$

$$s(12345, 34) \cap s(12345, 15) = L(34) \cap L(15),$$

$$s(12345, 15) \cap s(12345, 23) = L(15) \cap L(23),$$

$$s(12345, 23) \cap s(12345, 45) = L(23) \cap L(45),$$

$$s(12345, 45) \cap s(12345, 12) = L(45) \cap L(12).$$

REMARK. The non-orientable surface  $Z_R$  with the set of faces  $\mathcal{E}$  is regarded as a non-orientable analogue of a dodecahedron admitting an  $\mathfrak{S}_5$ -action. Later, we shall construct a non-orientable surface  $\mathcal{K}$  with a triangular decomposition with twenty triangles which is regarded as a non-orientable analogue of an icosahedron (cf. § 6).

## § 5. Some elementary properties of the icosahedron.

The purpose of this section is to name the twenty faces of an icosahedron in a natural way.

We begin with giving a 2-1 map from the totality  $\mathcal{F}$  of the twenty faces of an icosahedron to the set  $\mathcal{P} = \{(i, j); 1 \leq i < j \leq 5\}$ .

Let  $f$  be a map of  $\mathcal{F}$  to  $\mathcal{P}$  with the following condition:

(\*) Take a face  $\Delta$  of the icosahedron. Since  $\Delta$  is an triangle, there are three faces  $\Delta_1, \Delta_2, \Delta_3 \in \mathcal{F}$  which have common sides with  $\Delta$ . Then there is a permutation  $\sigma \in \mathfrak{S}_5$  such that

$$f(\Delta) = (\sigma(1), \sigma(2)),$$

$$f(\Delta_1) = (\sigma(4), \sigma(5)),$$

$$f(\Delta_2) = (\sigma(5), \sigma(3)),$$

$$f(\Delta_3) = (\sigma(3), \sigma(4)).$$

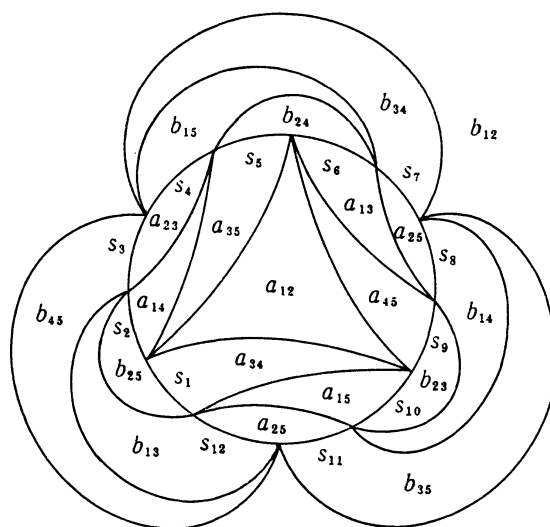


Figure II.

It is easy to show that such a map  $f$  of  $\mathcal{F}$  to  $\mathcal{P}$  is unique up to  $\mathfrak{S}_5$ -action. A concrete correspondence is given in Figure II. In this way, we attach a pair of the letters 1, 2, 3, 4, 5 to each face of the icosahedron. Then it is possible to attach one of the five letters 1, 2, 3, 4, 5 to each of thirty sides of the icosahedron in the following manner. Let  $s$  be a side of the icosahedron. Then there are two faces both of which have  $s$  as the common side. If  $(i_1, i_2), (i_3, i_4)$  are the pairs of letters which are attached to the faces in question. Then, it follows from the definition that there is a letter  $i_5$  such that  $\{i_1, i_2, i_3, i_4, i_5\}$  coincides with  $\{1, 2, 3, 4, 5\}$ . Then we attach  $i_5$  to the side  $s$ . Since there are thirty sides on the icosahedron, there are six sides attached with the same letter.

We are going to decompose the twenty faces into two sets each of which has ten faces by cutting off along twelve sides of the icosahedron with an additional condition which will be explained soon. We first take six sides which are attached with the same letter, say 1. Next we choose three letters from the remaining letters 2, 3, 4, 5, say, 3, 4, 5. Then there is a sequence of sides  $s_1, \dots, s_{12}$  such that 1, 3, 1, 4, 1, 5, 1, 3, 1, 4, 1, 5 is the sequence of letters which are attached to  $s_1, s_2, \dots, s_{12}$ , respectively. From the definition, we find that  $s_1, s_2, \dots, s_{12}$  decompose the icosahedron into two parts, say,  $I_1, I_2$ . Let  $\mathcal{F}_i$  be the set of faces belonging to  $I_i$  ( $i=1, 2$ ). Then it is easy to see that both  $\mathcal{F}_i$  contain ten faces and that the map of  $\mathcal{F}$  to  $\mathcal{P}$  defined above induces a bijection of  $\mathcal{F}_i$  to  $\mathcal{P}$  ( $i=1, 2$ ). We note here that the decomposition of  $\mathcal{F}$  to  $\mathcal{F}_1, \mathcal{F}_2$  does not depend on the permutation of 3, 4, 5 in the choice of the sequence  $s_1, s_2, \dots, s_{12}$ . As a result, we name all the faces of the icosahedron so that

$$\mathcal{F}_1 = \{a_{ij}; 1 \leq i < j \leq 5\},$$

$$\mathcal{F}_2 = \{b_{ij}; 1 \leq i < j \leq 5\},$$

where  $(i, j)$  is the pairs attached to both of  $a_{ij}$ ,  $b_{ij}$ . For later purpose, we specify the twelve sides  $s_i$  ( $1 \leq i \leq 12$ ) in the following manner. We first take as  $s_1$  the side of the face  $a_{34}$  attached with the letter 1 and next take as  $s_2$  the side of  $a_{14}$  attached with the letter 3 and so on. Then the twelve sides in question are uniquely determined (cf. Figure II).

### § 6. The connected sum of three Klein bottles.

As usual, we consider the two dimensional sphere  $S^2$  on which all the vertices of the icosahedron in question lie on. Projecting the thirty sides of the icosahedron to the sphere from the center of  $S^2$ , we obtain thirty segments and twenty faces on  $S^2$ . There is a one to one correspondence between the set of faces of the icosahedron and that of faces on the sphere obtained as above. For this reason, we use the same name for the face on  $S^2$  with that of the corresponding face of the icosahedron. We now concentrate our attention to the twelve segments  $s'_i$  ( $1 \leq i \leq 12$ ) which are obtained by the projection of the sides  $s_i$  ( $1 \leq i \leq 12$ ) given in § 5. Let  $P_i$  (resp.  $Q_i$ ) be the vertex of  $S^2$  which is the end of the segments  $s'_{2i-1}$  and  $s'_{2i}$  (resp.  $s'_{2i}$  and  $s'_{2i+1}$ ) ( $1 \leq i \leq 6$ ), where we put  $s'_{13} = s'_1$ . Now we cut off  $S^2$  along the six segments  $s'_{2i}$  ( $1 \leq i \leq 6$ ). Then we obtain new segments  $t_{2i}$ ,  $t'_{2i}$  whose ends are  $P_i$  and  $Q_i$ . Under the specification of the sides  $s_i$  given in the last of the previous section, we designate  $t_2, t_4, t_6, t_8, t_{10}, t_{12}$  the sides of  $a_{14}, a_{23}, a_{13}, a_{25}, a_{15}, a_{24}$  respectively and designate  $t'_2, t'_4, t'_6, t'_8, t'_{10}, t'_{12}$  the sides of  $b_{25}, b_{15}, b_{24}, b_{34}, b_{23}, b_{13}$ , respectively. In this way, we obtain a surface  $\mathcal{Q}$  with six holes.

We are going to construct a non-orientable surface  $\mathcal{K}$  from  $\mathcal{Q}$  identifying the three pairs of the six holes as follows:

(\*\*) We identify  $t_{2i}$  with  $t_{2i+6}$ ,  $t'_{2i}$  with  $t'_{2i+6}$ ,  $P_i$  with  $P_{i+3}$  and  $Q_i$  with  $Q_{i+3}$ , respectively.

In this way, we obtain a new surface  $\mathcal{K}$  which has twenty faces which come from the icosahedron. Hereafter, we write  $a'_{ij}$  and  $b'_{ij}$  as faces of  $\mathcal{K}$  corresponding to the faces  $a_{ij}$  and  $b_{ij}$  of the icosahedron, respectively and put  $\mathcal{F}' = \{a'_{ij}, b'_{ij}; 1 \leq i < j \leq 5\}$ .

**LEMMA 6.1.** *The surface  $\mathcal{K}$  is isomorphic to the connected sum of three Klein bottles. In particular,  $\mathcal{K}$  is non-orientable.*

This lemma follows from the construction of  $\mathcal{K}$ . The following explains the reason why we introduce the surface  $\mathcal{K}$ .

PROPOSITION 6.2. *The map of the set  $\tilde{\mathcal{D}}$  to  $\mathcal{F}'$  defined by  $\tilde{A}_{ij} \rightarrow a'_{ij}$ ,  $\tilde{B}_{ij} \rightarrow b'_{ij}$  preserves adjacent relation, that is, if,  $C_1, C_2 \in \tilde{\mathcal{D}}$  are adjacent (i.e. they have a common codimension one boundary) and if  $c_i$  is the corresponding element of  $\mathcal{F}'$  ( $i=1, 2$ ), then  $c_1$  and  $c_2$  have the common side and vice versa.*

COROLLARY. *The surface  $\mathcal{K}$  admits a  $\mathfrak{S}_5$ -action induced from that of  $\mathbf{P}^2$ .*

In spite that the icosahedron only permits the  $\mathfrak{A}_5$ -action, the surface  $\mathcal{K}$  admits an  $\mathfrak{S}_5$ -action in the above sense. We have defined the  $\mathfrak{S}_5$ -action on  $\mathcal{K}$  in a combinatorial way. Therefore we are lead to the following problem.

PROBLEM 6.3. *Is there a smooth  $\mathfrak{S}_5$ -action on the surface  $\mathcal{K}$  so that the induced action on the set  $\mathcal{F}'$  coincides with that given above?*

As explained in §4, the space  $Z_R$  with the set  $\mathcal{E}$  of its faces is regarded as a non-orientable analogue of a dodecahedron. Therefore, it is natural to ask whether there is a duality between  $Z_R$  and the surface  $\mathcal{K}$  something like the duality between the dodecahedron and the icosahedron or not?

## §7. An equivalence relation on the set of words.

The purpose of this section is to introduce a group  $B(\gamma)$  consisting of equivalence classes of certain sequences of letters under an equivalence relation (see below). In the next section, we shall prove that  $B(\gamma)$  is isomorphic to the fundamental group  $\pi_1(\mathbf{P}^2 - S)$ .

Let  $\alpha_{ij}, \beta_{ij}$  ( $1 \leq i < j \leq 5$ ) be letters which are in a one to one correspondence with the simply connected domains  $\tilde{A}_{ij}, \tilde{B}_{ij}$  in such a way that  $\tilde{A}_{ij} \rightarrow \alpha_{ij}$ ,  $\tilde{B}_{ij} \rightarrow \beta_{ij}$ . We denote by  $\delta$  the map of  $\tilde{\mathcal{D}}$  to  $\mathcal{L} = \{\alpha_{ij}, \beta_{ij}\}$  thus defined. Then the action of  $\mathfrak{S}_5$  on  $\tilde{\mathcal{D}}$  induces that on  $\mathcal{L}$ .

DEFINITION 7.1. Take  $\gamma, \gamma' \in \mathcal{L}$ . Then  $\gamma$  and  $\gamma'$  are adjacent in the weak sense if  $\gamma = \gamma'$  or  $\delta^{-1}(\gamma)$  and  $\delta^{-1}(\gamma')$  have a common codimension one boundary.

DEFINITION 7.2. (i) A word  $\sigma$  is a sequence of a finite number of letters  $\gamma_1 \gamma_2 \cdots \gamma_n$  ( $n > 0$ ) such that  $\gamma_i \in \mathcal{L}$  ( $i=1, \dots, n$ ) and that  $\gamma_i$  and  $\gamma_{i+1}$  are adjacent in the weak sense ( $i=1, \dots, n-1$ ).

(ii) For any word  $\sigma = \gamma_1 \gamma_2 \cdots \gamma_n$  and  $w \in \mathfrak{S}_5$ , we define  $w\sigma = (w\gamma_1)(w\gamma_2) \cdots (w\gamma_n)$ . In this way,  $\mathfrak{S}_5$  acts on the set of words.

(iii) Let  $\sigma = \gamma_1 \gamma_2 \cdots \gamma_m, \tau = \gamma'_1 \gamma'_2 \cdots \gamma'_n$  be two words. If  $\gamma_m$  and  $\gamma'_1$  are adjacent in the weak sense, then  $\sigma\tau = \gamma_1 \gamma_2 \cdots \gamma_m \gamma'_1 \gamma'_2 \cdots \gamma'_n$  is called the product of  $\sigma$  and  $\tau$ . On the other hand,  $\sigma^{-1} = \gamma_m \cdots \gamma_2 \gamma_1$  is called the inverse of  $\sigma$ . By definition, both  $\sigma\tau$  and  $\sigma^{-1}$  are words.

DEFINITION 7.3. (i) Let  $\sigma = \gamma_1 \gamma_2 \cdots \gamma_n$  be a word. If  $\gamma_i = \gamma_j$  for  $i < j < i+3$ ,

then  $\tau = \gamma_1 \gamma_2 \cdots \gamma_i \gamma_{i+1} \cdots \gamma_n$  is also a word. Then we write  $\sigma \sim_T \tau$  or  $\tau \sim_T \sigma$ .

(ii) Let  $\sigma, \tau$  be words. Then  $\sigma \sim_N \tau$  if there are words  $\rho, \rho'$  and  $w \in \mathfrak{S}_5$  such that both the products  $\rho \sigma \rho', \rho \tau \rho'$  are well-defined and that

$$\{w(\rho \sigma \rho'), w(\rho \tau \rho')\} = \{\alpha_{12} \alpha_{35} \alpha_{14} \alpha_{25} \beta_{34} \beta_{12}, \alpha_{12} \alpha_{34} \beta_{25} \beta_{14} \beta_{35} \beta_{12}\}.$$

(iii) Let  $\sigma, \tau$  be words. Then  $\sigma$  and  $\tau$  are equivalent and write  $\sigma \sim \tau$  if there are finitely many words  $\sigma_1 = \sigma, \sigma_2, \dots, \sigma_m = \tau$  such that  $\sigma_i \sim_T \sigma_{i+1}$  or  $\sigma_i \sim_N \sigma_{i+1}$  ( $i=1, \dots, m-1$ ). By definition, " $\sim$ " actually defines an equivalence relation on the set of words.

The following lemma is a consequence of Definition 7.3 and Proposition 4.1.

LEMMA 7.4. *Let  $\sigma, \tau$  be words. If  $\sigma \sim \tau$ , then  $w\sigma \sim w\tau$  for any  $w \in \mathfrak{S}_5$ .*

DEFINITION 7.5. (i) Let  $\sigma$  be a word. Then  $\langle \sigma \rangle$  denotes the equivalence classes of  $\sigma$  and is called an e-word (=an abbreviation of the equivalence class of the word  $\sigma$ ).

(ii) If  $\sigma, \tau$  are words such that the product  $\sigma\tau$  is well-defined, then  $\langle \sigma\tau \rangle$  depends only on  $\langle \sigma \rangle, \langle \tau \rangle$ . So we write  $\langle \sigma\tau \rangle = \langle \sigma \rangle \langle \tau \rangle$  and call it the product of  $\langle \sigma \rangle$  and  $\langle \tau \rangle$ . Similarly,  $\langle \sigma^{-1} \rangle$  depend only on  $\langle \sigma \rangle$ . So we write  $\langle \sigma \rangle^{-1} = \langle \sigma^{-1} \rangle$  and we call it the inverse of  $\langle \sigma \rangle$ .

It follows from Lemma 7.4 that  $\mathfrak{S}_5$  acts on the set of e-words.

DEFINITION 7.6. For any  $\gamma, \gamma' \in \mathcal{L}$ , we define

$$B(\gamma, \gamma') = \{\langle \sigma \rangle; \sigma = \gamma_1 \gamma_2 \cdots \gamma_m \text{ is a word such that } \gamma_1 = \gamma, \gamma_m = \gamma'\}$$

and put  $B(\gamma) = B(\gamma, \gamma)$  for any  $\gamma \in \mathcal{L}$ .

It follows from the definition that  $B(\gamma)$  has a group structure and  $\langle \gamma \rangle$  is the unit element of  $B(\gamma)$ .

### §8. The fundamental group $\pi_1(\mathbf{P}^2 - S)$ and the group $B(\gamma)$ .

The purpose of this section is to prove that the group  $B(\gamma)$  defined in the previous section is isomorphic to  $\pi_1(\mathbf{P}^2 - S) \cong \pi_1(Z - \tilde{S})$ .

In the sequel, if  $c: [0, 1] \rightarrow Z - \tilde{S}$  is a path, we denote by  $[c]$  its homotopy class.

Let  $c: [0, 1] \rightarrow Z - \tilde{S}$  be a path with the conditions  $(D, D': i), (D, D': ii):$

$(D, D': i)$   $c(0) \in D, c(1) \in D'$ .

$(D, D': ii)$  There is a sequence of a finite number of domains  $E_1 = D,$

$E_2, \dots, E_m = D'$  and a decomposition  $t_0 = 0 < t_1 < \dots < t_m = 1$  of the interval  $[0, 1]$  such that  $E_j \in \mathfrak{D}$ , that  $\dim_{\mathbf{R}}(\partial E_{j-1} \cap \partial E_j) = 3$ , that  $c((t_{j-1}, t_j)) \subset E_j$  ( $j = 1, \dots, m$ ) and that  $c(t_j) \in (\partial E_j \cap \partial E_{j+1})^\circ$  ( $j = 1, \dots, m-1$ ).

Here  $D, D' \in \mathfrak{D}$ . Then we put  $\zeta(c) = \delta(D_1)\delta(D_2) \cdots \delta(D_m)$  which is a word and therefore  $\langle \zeta(c) \rangle$  is an e-word contained in  $B(\delta(D), \delta(D'))$ . In the sequel, we frequently write  $(D : i)$  and  $(D : ii)$  instead of  $(D, D : i)$ ,  $(D, D : ii)$  for simplicity.

PROPOSITION 8.1. *There are closed paths  $c_1, \dots, c_6$  with the conditions  $(\tilde{A}_{12} : i)$ ,  $(\tilde{A}_{12} : ii)$  and satisfy the following :*

(i)  $c_1$  (resp.  $c_2, c_3, c_4, c_5, c_6$ ) makes a round of the line  $L(24)$  (resp.  $L(45)$ ,  $L(25)$ ,  $L(35)$ ,  $L(23)$ ,  $L(34)$ ).

(ii)  $[c_1 c_2 c_3 c_4 c_5 c_6] = 1$ .

(iii)  $\zeta(c_1) = \alpha_{12}\alpha_{35}\alpha_{14}\alpha_{23}\beta_{45}\beta_{12}\beta_{34}\alpha_{25}\alpha_{14}\alpha_{35}\alpha_{12}$ ,

$$\zeta(c_2) = \alpha_{12}\alpha_{35}\alpha_{14}\alpha_{25}\beta_{34}\beta_{15}\beta_{24}\alpha_{35}\alpha_{12},$$

$$\zeta(c_3) = \alpha_{12}\alpha_{35}\beta_{24}\beta_{15}\beta_{34}\beta_{12}\beta_{45}\alpha_{23}\alpha_{15}\alpha_{34}\alpha_{12},$$

$$\zeta(c_4) = \alpha_{12}\alpha_{34}\alpha_{15}\alpha_{23}\beta_{45}\beta_{13}\beta_{25}\alpha_{34}\alpha_{12},$$

$$\zeta(c_5) = \alpha_{12}\alpha_{34}\beta_{25}\beta_{13}\beta_{24}\alpha_{35}\alpha_{12},$$

$$\zeta(c_6) = \alpha_{12}\alpha_{35}\beta_{24}\beta_{13}\beta_{45}\alpha_{23}\alpha_{14}\alpha_{35}\alpha_{12}.$$

PROOF. We consider the section  $L : \xi_1 + 2\xi_2 - i\xi_3 = 0$  of  $\mathbf{P}^2$ . Then  $L = L' \cup \{(i : 0 : 1)\}$ , where  $L' = \{(x : 1 : -i(x+2)) ; x \in \mathbf{C}\}$ . For simplicity, we put  $x = x_1 + ix_2$  ( $x_1, x_2 \in \mathbf{R}$ ) and define polynomials  $f_j(x) = f_j(x_1, x_2)$  ( $j = 1, \dots, 5$ ) by

$$f_1(x) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + x_1^2 + 4x_1 x_2 + x_2^2 - 2x_1 + 4x_2,$$

$$f_2(x) = x_1^2 + x_2^2 + x_1 - x_2 - 2,$$

$$f_3(x) = -x_1 - 2,$$

$$f_4(x) = -x_1^2 - x_2^2 - 2x_1,$$

$$f_5(x) = -x_2.$$

Then, by direct calculation, we find that  $\pm \varphi_j((x : 1 : -i(x+2))) > 0$  if and only if  $\pm f_j(x) > 0$ . In the sequel, we identify the  $(x_1, x_2)$ -space with  $L'$  by  $x = x_1 + ix_2 \rightarrow (x : 1 : -i(x+2))$ . Then we find that

$$L' \cap \pi(L(34)) = \{(x_1, x_2) = (1, 0)\},$$

$$L' \cap \pi(L(35)) = \{(x_1, x_2) = (-1, -1)\},$$

$$L' \cap \pi(L(45)) = \{(x_1, x_2) = (-2, 1)\},$$

$$L' \cap \pi(L(23)) = \{(x_1, x_2) = (0, 0)\},$$

$$L' \cap \pi(L(24)) = \emptyset,$$

$$L' \cap \pi(L(25)) = \{(x_1, x_2) = (-2, 0)\}.$$

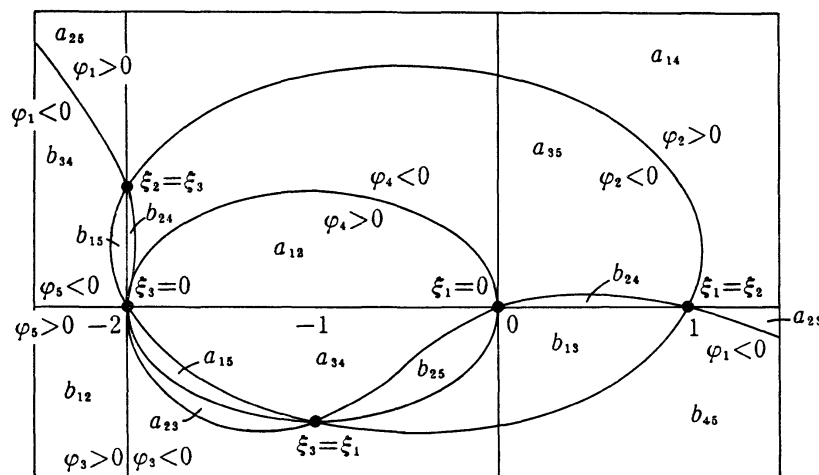


Figure III.

It is easy to construct closed paths  $c_j$  ( $j=1, \dots, 6$ ) with the conditions in the proposition by looking at the Figure III. For example, we explain how to construct a path  $c_5$  in  $L'$  which starts at a point  $p$  in the intersection  $\tilde{A}_{12} \cap L'$  and passes through the domains  $\tilde{A}_{12}, \tilde{A}_{34}, \tilde{B}_{25}, \tilde{B}_{13}, \tilde{B}_{24}, \tilde{A}_{35}, \tilde{A}_{12}$  in this order and ends at the point  $p$  and which satisfies (i), (ii), (iii). We can construct paths  $c_2, c_3, c_4, c_6$  by a similar way. To construct a path  $c_1$ , we need a modification. The point  $q = L \cap \pi(L(24))$  is outside of  $L'$  and so is regarded as the infinite point of  $L \cong \mathbf{P}^1$ . For this reason, we construct a path  $c_1$  which makes a round of  $\infty \in \mathbf{P}^1$ , identifying  $L$  with  $\mathbf{P}^1$ . The rest of the argument is similar to that for the case  $c_5$ .

LEMMA 8.2. *We define*

$$\begin{aligned} \mu_{12} &= \alpha_{13}\alpha_{25}\alpha_{14}\alpha_{23}\alpha_{15}\alpha_{24}\alpha_{13}, & \mu_{13} &= \alpha_{12}\alpha_{34}\alpha_{15}\alpha_{23}\alpha_{14}\alpha_{35}\alpha_{12}, \\ \mu_{14} &= \alpha_{12}\alpha_{34}\alpha_{15}\alpha_{24}\alpha_{13}\alpha_{45}\alpha_{12}, & \mu_{15} &= \alpha_{12}\alpha_{35}\alpha_{14}\alpha_{25}\alpha_{13}\alpha_{45}\alpha_{12}, \\ \mu_{23} &= \alpha_{12}\alpha_{34}\beta_{25}\beta_{13}\beta_{24}\alpha_{35}\alpha_{12}, & \mu_{24} &= \alpha_{12}\alpha_{34}\beta_{25}\beta_{14}\beta_{23}\alpha_{45}\alpha_{12}, \\ \mu_{25} &= \alpha_{12}\alpha_{35}\beta_{24}\beta_{15}\beta_{23}\alpha_{45}\alpha_{12}, & \mu_{34} &= \alpha_{13}\alpha_{24}\beta_{35}\beta_{14}\beta_{23}\alpha_{45}\alpha_{13}, \\ \mu_{35} &= \alpha_{13}\alpha_{25}\beta_{34}\beta_{15}\beta_{23}\alpha_{45}\alpha_{13}, & \mu_{45} &= \beta_{25}\alpha_{34}\alpha_{15}\alpha_{24}\beta_{35}\beta_{14}\beta_{25}. \end{aligned}$$

Then there is a closed path  $h_{ij}$  which makes a round of the line  $L(ij)$  such



that  $\zeta(h_{ij}) = \mu_{ij}$  ( $i < j$ ).

PROOF. The path  $c_5$ , for example, makes a round of the line  $L(23)$  in the positive direction and

$$\zeta(c_5) = \alpha_{12}\alpha_{34}\beta_{25}\beta_{13}\beta_{24}\alpha_{35}\alpha_{12}.$$

Considering the  $\mathfrak{S}_5$ -action on the space  $Z$  and the set of words, we imply the lemma.

LEMMA 8.3. Put  $\gamma_j = \zeta(c_j)$  ( $j=1, \dots, 6$ ). Then

$$\begin{aligned} \langle \alpha_{12}\alpha_{45} \cdot \mu_{12} \cdot \alpha_{45}\alpha_{12} \rangle &= \langle \gamma_1\gamma_5\gamma_3 \rangle, & \langle \mu_{13} \rangle &= \langle \gamma_4\gamma_5\gamma_6 \rangle, \\ \langle \mu_{14} \rangle &= \langle \gamma_3\gamma_4\gamma_5 \rangle, & \langle \mu_{15} \rangle &= \langle \gamma_2\gamma_3\gamma_4 \rangle, \\ \langle \mu_{23} \rangle &= \langle \gamma_5 \rangle, & \langle \mu_{24} \rangle &= \langle \gamma_1 \rangle^{-1}, \\ \langle \mu_{25} \rangle &= \langle \gamma_3 \rangle, & \langle \alpha_{12}\alpha_{45} \cdot \mu_{34} \cdot \alpha_{45}\alpha_{12} \rangle &= \langle \gamma_6 \rangle, \\ \langle \alpha_{12}\alpha_{45} \cdot \mu_{35} \cdot \alpha_{45}\alpha_{12} \rangle &= \langle \gamma_4 \rangle^{-1}, & \langle \alpha_{12}\alpha_{34} \cdot \mu_{45} \cdot \alpha_{34}\alpha_{12} \rangle &= \langle \gamma_2 \rangle^{-1}. \end{aligned}$$

This lemma is an easy consequence of the definition of  $\mu_{ij}$  and Definition 7.3.

On the other hand, we have the following.

LEMMA 8.4. There is a closed path  $c: [0, 1] \rightarrow Z - \tilde{S}$  with the conditions  $(\tilde{A}_{23}: i)$ ,  $(\tilde{A}_{23}: ii)$  such that  $[c]=1$  and that

$$\zeta(c) = \alpha_{23}\alpha_{15}\alpha_{24}\beta_{35}\beta_{14}\beta_{23}\beta_{15}\beta_{24}\alpha_{35}\alpha_{14}\alpha_{23}.$$

PROOF. We prove this lemma by constructing a path  $c$  in an explicit way. We consider  $\mathbf{P}^2$  instead of  $Z$ . We first put

$$\begin{aligned} \xi_j &= u_j + iv_j \quad (j=1, 2, 3), \\ \frac{\xi_2}{\xi_1} &= p_2 + iq_2, \quad \frac{\xi_3}{\xi_1} = p_3 + iq_3, \end{aligned}$$

and define

$$\begin{aligned} f_1 &= \frac{1}{u_1^2} \{v_1^2 q_2 q_3 (q_2 - q_3) + 2v_1 q_2 q_3 (u_2 - u_3) \\ &\quad + u_1^2 q_2 q_3 (q_2 - q_3) + u_1 (q_2 u_3 - q_3 u_2) - (q_2 u_3^2 - q_3 u_2^2)\}, \\ f_2 &= \frac{1}{u_1} \{(u_1 - u_3)q_2 + (u_2 - u_1)q_3\}, \\ f_3 &= \frac{1}{u_1} (u_2 q_3 - u_3 q_2), \quad f_4 = q_3, \quad f_5 = q_2. \end{aligned}$$

Then, by direct computation, we find that

$$|\xi_1|^4 f_1 = \varphi_1(\xi),$$

$$|\xi_1|^2 f_j = \varphi_j(\xi) \quad (j=2, 3, 4, 5).$$

Now we assume that  $0 < u_3 < u_2 < u_1$  and  $0 < v_1$  and fix them for a moment. Moreover, we define a closed path  $\tilde{c}_\varepsilon(t) = (\varepsilon \cos(2\pi t - \pi/4), \varepsilon \sin(2\pi t - \pi/4))$  ( $t \in [0, 1]$ ) on the  $(p_2, p_3)$ -space for  $\varepsilon > 0$ . From the definition,  $\tilde{c}_\varepsilon$  is regarded as a closed path on  $Z$  which we denote by  $c_\varepsilon$ . Then, it is possible to show that for some  $(u_1, u_2, u_3, v_1)$  with the condition above and for some  $0 < \varepsilon \ll 1$ , we have

$$[c_\varepsilon] = 1.$$

$$\zeta(c_\varepsilon) = \alpha_{23}\alpha_{15}\alpha_{24}\beta_{35}\beta_{14}\beta_{23}\beta_{15}\beta_{24}\alpha_{35}\alpha_{14}\alpha_{23}.$$

In fact, we take  $(u_1, u_2, u_3, v_1) = (4, 3, 1, 3)$ . Then

$$f_1 = \frac{1}{16} \{25q_2q_3(q_2 - q_3) + 12q_2q_3 + 3q_2 - 3q_3\},$$

$$f_2 = \frac{1}{4} (3q_2 - q_3),$$

$$f_3 = \frac{1}{4} (3q_3 - q_2).$$

Therefore from Figure IV, we can easily see the claim above.

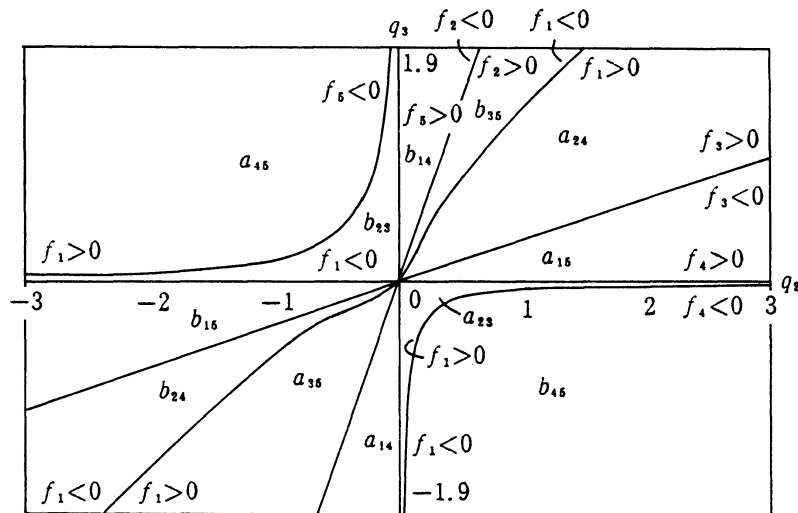


Figure IV.

This lemma has an easy consequence.

PROPOSITION 8.5. *For any  $E \in \mathcal{E}$  (for the definition of  $\mathcal{E}$ , see Proposition 4.5.), there is a subset  $\{D_j; j=1, \dots, 10\}$  of  $\mathfrak{D}$  and a closed path  $c_F: [0, 1] \rightarrow Z - \tilde{S}$  with the conditions  $(D_1: i)$ ,  $(D_1: ii)$  such that  $\dim_{\mathbf{R}}(\partial D_j \cap \partial D_{j+1}) = 3$  ( $j=1, 2, \dots, 10$ ), where we put  $D_{11} = D_1$  and that*

$$[c_F] = 1,$$

$$\zeta(c_F) = \delta(D_1)\delta(D_2) \cdots \delta(D_{10})\delta(D_{11}).$$

We here give a concrete correspondence between the twelve connected components of  $Z_{\mathbf{R}} - \bigcup_{i < j} L(ij)$  and words given in Proposition 8.5:

$$\begin{aligned} D(12345) &\longrightarrow \alpha_{24}\beta_{35}\beta_{14}\beta_{25}\beta_{13}\beta_{24}\alpha_{35}\alpha_{14}\alpha_{25}\alpha_{13}\alpha_{24}, \\ D(12354) &\longrightarrow \alpha_{25}\beta_{34}\beta_{15}\beta_{24}\beta_{13}\beta_{25}\alpha_{34}\alpha_{15}\alpha_{24}\alpha_{13}\alpha_{25}, \\ D(12435) &\longrightarrow \alpha_{23}\beta_{45}\beta_{13}\beta_{25}\beta_{14}\beta_{23}\alpha_{45}\alpha_{13}\alpha_{25}\alpha_{14}\alpha_{23}, \\ D(12453) &\longrightarrow \alpha_{25}\beta_{34}\beta_{15}\beta_{23}\beta_{14}\beta_{25}\alpha_{34}\alpha_{15}\alpha_{23}\alpha_{14}\alpha_{25}, \\ D(12534) &\longrightarrow \alpha_{23}\beta_{45}\beta_{13}\beta_{24}\beta_{15}\beta_{23}\alpha_{45}\alpha_{13}\alpha_{24}\alpha_{15}\alpha_{23}, \\ D(12543) &\longrightarrow \alpha_{24}\beta_{35}\beta_{14}\beta_{23}\beta_{15}\beta_{24}\alpha_{35}\alpha_{14}\alpha_{23}\alpha_{15}\alpha_{24}, \\ D(13245) &\longrightarrow \alpha_{34}\beta_{25}\beta_{14}\beta_{35}\beta_{12}\beta_{34}\alpha_{25}\alpha_{14}\alpha_{35}\alpha_{12}\alpha_{34}, \\ D(13254) &\longrightarrow \alpha_{35}\beta_{24}\beta_{15}\beta_{34}\beta_{12}\beta_{35}\alpha_{24}\alpha_{15}\alpha_{34}\alpha_{12}\alpha_{35}, \\ D(13425) &\longrightarrow \alpha_{23}\beta_{45}\beta_{12}\beta_{35}\beta_{14}\beta_{23}\alpha_{45}\alpha_{12}\alpha_{35}\alpha_{14}\alpha_{23}, \\ D(13524) &\longrightarrow \alpha_{23}\beta_{45}\beta_{12}\beta_{34}\beta_{15}\beta_{23}\alpha_{45}\alpha_{12}\alpha_{34}\alpha_{15}\alpha_{23}, \\ D(14235) &\longrightarrow \alpha_{34}\beta_{25}\beta_{13}\beta_{45}\beta_{12}\beta_{34}\alpha_{25}\alpha_{13}\alpha_{45}\alpha_{12}\alpha_{34}, \\ D(14325) &\longrightarrow \alpha_{24}\beta_{35}\beta_{12}\beta_{45}\beta_{13}\beta_{24}\alpha_{35}\alpha_{12}\alpha_{45}\alpha_{13}\alpha_{24}. \end{aligned}$$

We are going to explain the reason why we introduced the relation  $\sim_N$  on the set of words (cf. Definition 7.3, (ii)). We first consider the domains  $\tilde{A}_{12}$  and  $\tilde{B}_{12}$ . Among the twelve words above, the words containing the letters  $\alpha_{12}$ ,  $\beta_{12}$  are those corresponding to the sets  $D(13245)$ ,  $D(13425)$ ,  $D(13524)$ ,  $D(13254)$ ,  $D(14235)$ ,  $D(14325)$ . Now we take our attention to the word corresponding to  $D(13245)$ . Then it follows from Proposition 8.5 that there are two paths  $\phi_1$ ,  $\phi_2: [0, 1] \rightarrow Z - \tilde{S}$  with the conditions  $(\tilde{A}_{12}, \tilde{B}_{12}: i)$ ,  $(\tilde{A}_{12}, \tilde{B}_{12}: ii)$  such that

$$\zeta(\phi_1) = \alpha_{12}\alpha_{35}\alpha_{14}\alpha_{25}\beta_{34}\beta_{12},$$

$$\zeta(\phi_2) = \alpha_{12}\alpha_{34}\beta_{25}\beta_{14}\beta_{35}\beta_{12},$$

and that

$$[\phi_1] = [\phi_2].$$

For the remaining cases  $D(13425)$ ,  $D(13524)$ ,  $D(13254)$ ,  $D(14235)$ ,  $D(14325)$ , there

are pairs of paths on  $Z-\tilde{S}$  with the conditions similar to the above. Moreover, taking a pair  $\{\tilde{A}_{ij}, \tilde{B}_{ij}\}$  instead of  $\{\tilde{A}_{12}, \tilde{B}_{12}\}$ , we can obtain similar results. Therefore we have the following proposition which is a consequence of Definition 7.3 and the argument above.

**PROPOSITION 8.6.** *For any  $\langle \gamma \rangle \in B(\alpha_{12})$ , there is a closed path  $c: [0, 1] \rightarrow Z-\tilde{S}$  with the conditions  $(\tilde{A}_{12}: i)$ ,  $(\tilde{A}_{12}: ii)$  such that  $\zeta(c) = \gamma$ . We put  $\tau(\gamma) = [c]$ . Then  $\tau$  is a group homomorphism of  $B(\alpha_{12})$  to  $\pi_1(Z-\tilde{S}, p_0)$  for  $p_0 \in \tilde{A}_{12}$ .*

What we want to prove next is that  $\tau$  is an isomorphism. For this purpose, we are going to obtain the relations among  $\langle \gamma_1 \rangle$ ,  $\langle \gamma_2 \rangle$ ,  $\langle \gamma_3 \rangle$ ,  $\langle \gamma_4 \rangle$ ,  $\langle \gamma_5 \rangle$ ,  $\langle \gamma_6 \rangle$ . Before doing this, we define

$$CC(\langle \gamma \rangle) = \{ \langle \gamma' \rangle \langle \gamma \rangle \langle \gamma' \rangle^{-1}; \langle \gamma' \rangle \in B(\alpha_{12}) \}$$

for any  $\langle \gamma \rangle \in B(\alpha_{12})$ . As we showed in Lemma 8.3, to each line  $L(ij)$ , there associates an e-word. Let  $\gamma_{ij}$  be a word which is associated with  $L(ij)$  there. We note here that both  $CC(\langle \gamma_{ij} \rangle)$  and  $CC(\langle \gamma_{ij} \rangle^{-1})$  are associated with  $L(ij)$ . But it is enough to choose one of them in the subsequent discussion. We put  $CC(ij) = CC(\langle \gamma_{ij} \rangle)$  for simplicity. A concrete correspondence between the ten lines and the conjugacy classes in  $B(\alpha_{12})$  is given as follows (cf. Lemma 8.3):

$$\begin{aligned} CC(12) &= CC(\langle \gamma_1 \gamma_5 \gamma_3 \rangle), & CC(13) &= CC(\langle \gamma_4 \gamma_5 \gamma_6 \rangle), \\ CC(14) &= CC(\langle \gamma_3 \gamma_4 \gamma_5 \rangle), & CC(15) &= CC(\langle \gamma_2 \gamma_3 \gamma_4 \rangle), \\ CC(23) &= CC(\langle \gamma_5 \rangle), & CC(24) &= CC(\langle \gamma_1^{-1} \rangle), \\ CC(25) &= CC(\langle \gamma_3 \rangle), & CC(34) &= CC(\langle \gamma_6 \rangle), \\ CC(35) &= CC(\langle \gamma_4^{-1} \rangle), & CC(45) &= CC(\langle \gamma_2^{-1} \rangle). \end{aligned}$$

We now recall the following well-known result:

**LEMMA 8.7.** *Let  $D = \{x \in \mathbb{C}; |x| < 1\}$  be the unit disk and let  $(x, y)$  be a coordinate of  $X = D \times D$ . Put  $X' = \{(x, y) \in X; xy \neq 0\}$ . Then  $\pi_1(X')$  is abelian.*

Noting this lemma and Lemma 3.1, we can expect that if  $L(ij)$  and  $L(i'j')$  intersect at a point, then  $\langle \gamma_{ij} \gamma_{i'j'} \rangle = \langle \gamma_{i'j'} \gamma_{ij} \rangle$  for some  $\langle \gamma_{ij} \rangle \in CC(ij)$ ,  $\langle \gamma_{i'j'} \rangle \in CC(i'j')$ . In fact, we have the following.

**PROPOSITION 8.8.**

$$\langle \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \gamma_6 \rangle = 1, \quad \langle \gamma_1 \gamma_4 \rangle = \langle \gamma_4 \gamma_1 \rangle, \quad \langle \gamma_2 \gamma_5 \rangle = \langle \gamma_5 \gamma_2 \rangle,$$

$$\begin{aligned}\langle \gamma_3 \gamma_6 \rangle &= \langle \gamma_6 \gamma_3 \rangle, & \langle \gamma_1 \gamma_2 \gamma_3 \rangle &= \langle \gamma_2 \gamma_3 \gamma_1 \rangle = \langle \gamma_3 \gamma_1 \gamma_2 \rangle, \\ \langle \gamma_3 \gamma_4 \gamma_5 \rangle &= \langle \gamma_4 \gamma_5 \gamma_3 \rangle = \langle \gamma_5 \gamma_3 \gamma_4 \rangle, & \langle \gamma_5 \gamma_6 \gamma_1 \rangle &= \langle \gamma_6 \gamma_1 \gamma_5 \rangle = \langle \gamma_1 \gamma_5 \gamma_6 \rangle.\end{aligned}$$

PROOF. First we pay attention to the lines  $L(12)$ ,  $L(34)$ . They intersect at a point in a normal crossing. On the other hand,  $CC(\langle \gamma_1 \gamma_5 \gamma_3 \rangle) \in CC(12)$  and  $CC(\langle \gamma_6 \rangle) \in CC(34)$ . Then one can show the relation  $\langle \gamma_1 \gamma_5 \gamma_3 \gamma_6 \rangle = \langle \gamma_6 \gamma_1 \gamma_5 \gamma_3 \rangle$  by using the equivalence relation “ $\sim$ ”. By a similar way, we obtain 15 relations among the generators  $\langle \gamma_1 \rangle$ ,  $\langle \gamma_2 \rangle$ ,  $\langle \gamma_3 \rangle$ ,  $\langle \gamma_4 \rangle$ ,  $\langle \gamma_5 \rangle$ ,  $\langle \gamma_6 \rangle$  corresponding to 15 normal crossing points of the lines  $L(ij)$  (cf. Lemma 3.1). It is easy to show that the 15 relations thus obtained combined with the relation  $\langle \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \gamma_6 \rangle = 1$  are equivalent to the 10 relations in the proposition.

We review the structure of the fundamental group  $\pi_1(\mathbf{P}^2 - S, p_0)$  which was studied in Terada [T2].

PROPOSITION 8.9. *The fundamental group  $\pi_1(\mathbf{P}^2 - S, p_0)$  is generated by  $\rho(ij)$  ( $i < j$ ,  $i, j = 0, 1, 2, 3$ ) and the defining relations are*

$$\begin{aligned}\rho(01)\rho(02)\rho(12)\rho(03)\rho(13)\rho(23) &= 1, \\ \rho(01)\rho(02)\rho(12) &= \rho(02)\rho(12)\rho(01) = \rho(12)\rho(01)\rho(02), \\ \rho(01)\rho(03)\rho(13) &= \rho(03)\rho(13)\rho(01) = \rho(13)\rho(01)\rho(03), \\ \rho(02)\rho(03)\rho(23) &= \rho(03)\rho(23)\rho(02) = \rho(23)\rho(02)\rho(03), \\ \rho(12)\rho(13)\rho(23) &= \rho(13)\rho(23)\rho(12) = \rho(23)\rho(12)\rho(13).\end{aligned}$$

Instead of [T2], we used the notation in [YY, Proposition 2.1]. The defining relations given in [YY] is slightly different but it is easy to prove that the relations above are equivalent to those in [YY, Proposition 2.1].

We are now in a position to state the main theorem of this paper.

THEOREM 8.10. *The group  $B(\alpha_{12})$  is generated by  $\langle \gamma_1 \rangle$ ,  $\langle \gamma_2 \rangle$ ,  $\langle \gamma_3 \rangle$ ,  $\langle \gamma_4 \rangle$ ,  $\langle \gamma_5 \rangle$ ,  $\langle \gamma_6 \rangle$  and  $\tau$  is an isomorphism of  $B(\alpha_{12})$  to  $\pi_1(Z - \hat{S}, p_0)$  for  $p_0 \in \tilde{A}_{12}$ .*

PROOF. Since  $\tau$  is surjective, it suffices to show that  $\text{Ker } \tau = 1$ . By direct computation, we find that, by the correspondence

$$\begin{aligned}\langle \gamma_1 \rangle &\longrightarrow \rho(01), & \langle \gamma_3^{-1} \gamma_2 \gamma_3 \rangle &\longrightarrow \rho(13), & \langle \gamma_3 \rangle &\longrightarrow \rho(03), \\ \langle \gamma_4 \rangle &\longrightarrow \rho(23), & \langle \gamma_5 \rangle &\longrightarrow \rho(02), & \langle \gamma_6 \rangle &\longrightarrow \rho(12),\end{aligned}$$

the relations given in Proposition 8.8 are equivalent to those given in Proposi-

tion 8.9. This means that  $\text{Ker } \tau = 1$ .

REMARK. It is clear from the definition that for any  $\gamma, \gamma' \in \mathcal{L}$ , there is a group isomorphism between  $B(\gamma), B(\gamma')$ .

Last in this section, we mention some statements on the relations between  $\pi_1(Z - \tilde{S})$  and  $\pi_1(\mathcal{K})$  which seem interesting. We recall the relation " $\sim_T$ " (cf. Definition 7.3). Noting that " $\sim_T$ " is also an equivalence relation on the set of words, we introduce the set  $A(\gamma)$  for  $\gamma \in \mathcal{L}$  by

$$A(\gamma) = \{ \langle \sigma \rangle_T; \sigma = \gamma_1 \gamma_2 \cdots \gamma_m \text{ is a word such that } \gamma_1 = \gamma_m = \gamma \},$$

where  $\langle \sigma \rangle_T = \{ \sigma'; \text{ there are finitely many words } \sigma_1, \dots, \sigma_m$

such that  $\sigma = \sigma_1, \sigma' = \sigma_m$  and that  $\sigma_i \sim_T \sigma_{i+1} \ (i=1, \dots, m-1)\}$ ,

and in particular, we denote by  $A(\alpha_{12})'$  the subgroup of  $A(\alpha_{12})$  generated by  $\langle \gamma_1 \rangle_T, \langle \gamma_2 \rangle_T, \langle \gamma_3 \rangle_T, \langle \gamma_4 \rangle_T, \langle \gamma_5 \rangle_T, \langle \gamma_6 \rangle_T$ .

On the other hand, we recall the set of triangles of the surface  $\mathcal{K}$ . In particular, we take a point  $q_0$  of  $a'_{12}$ . Then, to each word  $\gamma$  with  $\langle \gamma \rangle_T \in A(\alpha_{12})$ , there associates a closed path  $g: [0, 1] \rightarrow \mathcal{K}$  such that  $g(0) = g(1) = q_0$  and satisfying the condition similar to  $(\tilde{A}_{12}: \text{ii})$ . Let  $[g]_{\mathcal{K}}$  denote the homotopy class in  $\pi_1(\mathcal{K}, q_0)$  of the path  $g$ . Then it seems true that the correspondence  $\langle \gamma \rangle_T \rightarrow [g]_{\mathcal{K}}$  defines a group homomorphism  $\tau'$  of  $A(\alpha_{12})$  to  $\pi_1(\mathcal{K}, q_0)$ . Moreover, it is known (cf. [M]) that

$$\pi_1(\mathcal{K}) = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 x_6^2 = 1 \rangle.$$

PROBLEM 8.11. *Are the following statements true or false?*

(i)  $A(\alpha_{12})$  is generated by  $\langle \gamma_1 \rangle_T, \langle \gamma_2 \rangle_T, \langle \gamma_3 \rangle_T, \langle \gamma_4 \rangle_T, \langle \gamma_5 \rangle_T, \langle \gamma_6 \rangle_T$ , in other words,  $A(\alpha_{12})' = A(\alpha_{12})$ .

(ii)  $A(\alpha_{12}) \cong \pi_1(Z - (\mathbf{Z}_R \cup_{i < j} L(ij)))$ .

(iii) The generators  $x_1, x_2, x_3, x_4, x_5, x_6$  of  $\pi_1(\mathcal{K}, q_0)$  given above are so taken that  $\tau'(\langle \gamma_j \rangle_T) = x_j^2 \ (j=1, \dots, 6)$ .

## References

- [H] S. Hitotumatu, Solve regular polytopes, Tokai Daigaku Shuppankai (in Japanese), 1983.
- [M] Y. Matsumoto, Introduction to Topology, Iwanami Shoten (in Japanese).
- [S] J. Sekiguchi, A global representation of solutions to the zonal spherical system for  $SL(3, \mathbf{R})/SO(3)$ , preprint.
- [ST] J. Sekiguchi and N. Takayama, A global representation of solutions to the system of differential equations for Appell hypergeometric function  $F_1(a, b, b', c; x, y)$ , in preparation.

- [T1] T. Terada, Problème de Riemann et fonctions automorphes provenant des fonctions hypergéométriques de plusieurs variables, J. Math. Kyoto Univ., **13** (1973), 557-578.
- [T2] T. Terada, Quelques propriétés géométrique de domaine de  $F_1$  et le groupe de tresses colorées, Publ. Res. Inst. Math. Sci., **17** (1981), 95-111.
- [YY] T. Yamazaki and M. Yoshida, On Hirzebruch's examples of surfaces with  $c_1^2=3c_2$ , Math. Ann., **266** (1984), 421-431.

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