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Decomposition of non-decreasing slowly varying functions and the domain of attraction of Gaussian distributions

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1. Introduction.

The class of non-decreasing slowly varying functions is a significant subclass of the class of slowly varying functions. Such functions appear as the truncated variances of distributions in the domain of attraction $\boldsymbol{D}(2)$ of Gaussian distributions and the slowly varying function parts of their normalizing constants are also asymptotically equal to non-decreasing ones. Let us call a function $g(x)$ a component of a non-negative non-decreasing function $f(x)$, if both $g(x)$ and $f(x)-g(x)$ are non-negative non-decreasing. A non-zero component of a non-decreasing slowly varying function is not necessarily slowly varying. But, there are slowly varying functions of which all non-zero components are slowly varying. In this paper, we will give a simple criterion for a function to have this property. We will also consider some properties of components of non-decreasing slowly varying functions. We will then apply the results to a topic of the domain of attraction of Gaussian distributions. Our purpose is to study a conjecture of Tucker's. The conjecture says that every non-trivial factor of a distribution in $\boldsymbol{D}(2)$ will also belong to $\boldsymbol{D}(2)$. In general, this conjecture is not true, as a counter-example in [4] shows. This problem is connected with the general results of decomposition of non-decreasing slowly varying functions. Using the results, we supply a general method to construct counter-examples. Main consequence is the existence of distributions, none of which belongs to $\boldsymbol{D}(2)$ but the convolution of which belongs to it. We also give a sufficient condition for all non-trivial factors to be in $\boldsymbol{D}(2)$.

2. Preliminaries.

First we introduce some notations. The totality of all probability measures on real numbers $\boldsymbol{R}^1$ is denoted by $\boldsymbol{P}(\boldsymbol{R}^1)$. We call a delta distribution *trivial distribution.* For $\mu, \nu \in \boldsymbol{P}(\boldsymbol{R}^1)$, $\mu * \nu$ denotes the convolution of μ and ν. A distribution μ_1 is called a *factor* of a distribution μ, if $\mu=\mu_1*\nu$ with some $\nu\in \boldsymbol{P}(\boldsymbol{R}^1)$. A distribution μ_1 is called a *non-trivial factor* of μ if μ_1 is a factor of μ and μ_1 is not a trivial distribution. Two functions $f_1(x)$ and $f_2(x)$ are said to be *asymptotically equal* if $\lim_{x\to\infty} f_1(x)/f_2(x)=1$. The set of all positive integers is denoted by $\boldsymbol{N}$.

All the facts in this section are proved in [1], [2], [3] and [7].

DEFINITION 2.1. A function $f(x)$ is said to be slowly varying (at infinity) if it is real-valued, positive and measurable on $[A, \infty)$ for some $A>0$ and if

$$\lim_{x\to\infty} f(kx)/f(x)=1 \qquad \text{for each } k>0\,. \tag{2.1}$$

REMARK. If $f(x)$ is non-decreasing, (2.1) is equivalent to the following ([7] p. 37 Lemma 1.15):

$$\lim_{x\to\infty} f(2x)/f(x)=1\,. \tag{2.2}$$

Slowly varying functions have the following representation ([7] p. 2 Theorem 1.2).

PROPOSITION 2.2. *A function $f(x)$ defined on $[A, \infty)$, $A>0$, is slowly varying, if and only if there exists a positive number $B\geqq A$ such that for all $x\geqq B$ we have*

$$f(x)=c(x)\exp\left(\int_B^x \varepsilon(t)t^{-1}dt\right), \tag{2.3}$$

where $c(x)$ is a bounded positive measurable function on $[B, \infty)$ such that $\lim_{x\to\infty} c(x) =c$ $(0<c<\infty)$, *and $\varepsilon(t)$ is a continuous function on $[B, \infty)$ such that* $\lim_{t\to\infty}\varepsilon(t)=0$. *We call this $\varepsilon(t)$ an ε-function of $f(x)$.*

DEFINITION 2.3. For $\nu\in\boldsymbol{P}(\boldsymbol{R}^1)$, we define *the truncated variance* of ν by

$$V(R)=\int_{|x|<R} |x|^2\nu(dx)\,. \tag{2.4}$$

The truncated variance of the distribution of a random variable X is denoted by $V_X(R)$.

DEFINITION 2.4. Let $X, X_1, X_2, \cdots, X_n, \cdots$ be $\boldsymbol{R}^1$-valued i. i. d. (independent and identically distributed) random variables with distribution ν. If, for suitably chosen constants $B_n>0$ and $A_n\in\boldsymbol{R}^1$, the distribution of

$$S_n=B_n^{-1}\sum_{k=1}^n X_k-A_n \tag{2.5}$$

converges to a distribution μ as $n\to\infty$, then we say that ν is *attracted* to μ. We call the totality of distributions attracted to μ *the domain of attraction* of μ.

We say that ν belongs to *the domain of partial attraction* of a distribution μ if there is an increasing sequence m_n of positive integers such that, for some constants $A_n \in \boldsymbol{R}^1$ and $B_n > 0$, the distribution of

$$S_n = B_n^{-1} \sum_{k=1}^{m_n} X_k - A_n \tag{2.6}$$

converges to μ as $n \to \infty$.

It is well-known that a distribution has a non-empty domain of attraction if and only if it is stable. We denote by $\boldsymbol{D}(2)$ the domain of attraction of Gaussian distributions (stable with index 2) on $\boldsymbol{R}^1$. The following facts on $\boldsymbol{D}(2)$ are known.

PROPOSITION 2.5. *A non-trivial distribution ν on $\boldsymbol{R}^1$ belongs to $\boldsymbol{D}(2)$ if and only if its truncated variance is slowly varying, which is equivalent to that*

$$\lim_{x \to \infty} x^2 \nu(\boldsymbol{R}^1 \setminus [-x, x]) / V(x) = 0 . \tag{2.7}$$

If $\nu \in \boldsymbol{D}(2)$, then $E|X|^{2-\varepsilon} < \infty$ for every $\varepsilon > 0$ and the normalizing constants B_n and A_n in (2.5) *are of the form*

$$\begin{cases} B_n = n^{1/2} h(n) , \\ A_n = n^{1/2} (h(n))^{-1} EX + c(n) , \end{cases} \tag{2.8}$$

where $h(x)$ is a slowly varying function and $\lim_{n \to \infty} c(n) = c$ ($c \in \boldsymbol{R}$).

The slowly varying $h(x)$ appearing in (2.8) is called *the slowly varying function part* of the normalizing constant B_n.

PROPOSITION 2.6. *Let $h(x)$ be a slowly varying function. Then there exists a distribution in $\boldsymbol{D}(2)$ with normalizing constants* (2.8), *if and only if $h(x)$ is asymptotically equal to a non-decreasing slowly varying function.* ([**9**]; *see Appendix for the proof.*)

A characterization of the domain of partial attraction of Gaussian distributions is known.

PROPOSITION 2.7. *A non-trivial distribution ν on $\boldsymbol{R}^1$ belongs to the domain of partial attraction of Gaussian distributions on $\boldsymbol{R}^1$ if and only if*

$$\liminf_{x \to \infty} x^2 \nu(\boldsymbol{R}^1 \setminus [-x, x]) / V(x) = 0 . \tag{2.9}$$

3. Decomposition of non-decreasing slowly varying functions.

In this section, a general result on decomposition of a non-decreasing slowly varying function into the sum of non-decreasing functions is given. Here is our main theorem.

THEOREM 3.1. *Let $f(x)$ be a non-negative non-decreasing function on $[A, \infty)$, not identically zero.*

(1) *If $f(x)$ satisfies*

$$\limsup_{x\to\infty}(f(2x)-f(x))<\infty \tag{3.1}$$

and $f(x)$ is represented as the sum of two non-negative non-decreasing functions $f_1(x)$ and $f_2(x)$, then each of $f_1(x)$ and $f_2(x)$ is slowly varying or identically zero.

(2) *If $f(x)$ satisfies*

$$\limsup_{x\to\infty}(f(2x)-f(x))=\infty\,, \tag{3.2}$$

then we can construct two non-negative non-decreasing functions $f_1(x)$ and $f_2(x)$ which are not slowly varying and not identically zero and satisfy

$$f(x)=f_1(x)+f_2(x)\,.$$

PROOF. (1) Since

$$f(2x)-f(x)=\sum_{j=1}^{2}(f_j(2x)-f_j(x))\geqq f_j(2x)-f_j(x)$$

for each j, $\sup_{x\geqq A}(f_j(2x)-f_j(x))<\infty$. Hence, if $f_j(x)$ is not identically zero, $\lim_{x\to\infty}(f_j(2x)-f_j(x))/f_j(x)=0$. This implies that $f_j(x)$ is a slowly varying function.

(2) Let x_0 be a point such that $f(x_0)>0$. It follows from (3.2) that there exists $x_1>x_0$ such that $f(2x_1)-f(x_1)\geqq 2^{-1}f(x_0)$. For $x_0\leqq x\leqq x_1$, we define $f_1(x)$ and $f_2(x)$ as

$$f_1(x)=2^{-1}f(x_0),\qquad f_2(x)=f(x)-2^{-1}f(x_0)\,.$$

There exists x_2 such that $x_2>2x_1$ and

$$f(2x_2)-f(x_2)\geqq f_2(x_1)\,.$$

For $x_1<x\leqq x_2$, define $f_1(x)$ and $f_2(x)$ as

$$f_1(x)=f(x)-f_2(x_1),\qquad f_2(x)=f_2(x_1)\,.$$

Similarly, choose x_3 such that $x_3>2x_2$ and $f(2x_3)-f(x_3)\geqq f_1(x_2)$. Define, for $x_2<x\leqq x_3$,

$$f_1(x)=f_1(x_2),\qquad f_2(x)=f(x)-f_1(x_2)\,.$$

Repeating this procedure, we define $f_1(x)$ and $f_2(x)$ in the following way. Given x_{2k-1}, choose x_{2k} such that $x_{2k}>2x_{2k-1}$ and

$$f(2x_{2k})-f(x_{2k})\geqq f_2(x_{2k-1})\,.$$

For $x_{2k-1}<x\leqq x_{2k}$, $f_1(x)$ and $f_2(x)$ are defined as

$$f_1(x)=f(x)-f_2(x_{2k-1}), \qquad f_2(x)=f_2(x_{2k-1}).$$

Next choose x_{2k+1} such that $x_{2k+1}>2x_{2k}$ and

$$f(2x_{2k+1})-f(x_{2k+1})\geqq f_1(x_{2k}).$$

For $x_{2k}<x\leqq x_{2k+1}$, $f_1(x)$ and $f_2(x)$ are defined as

$$f_1(x)=f_1(x_{2k}), \qquad f_2(x)=f(x)-f_1(x_{2k}).$$

Then obviously $f(x)=f_1(x)+f_2(x)$ and each $f_j(x)$ is non-negative. In order to prove non-decrease of $f_1(x)$ and $f_2(x)$, it is enough to think about a neighbourhood of x_n ($n\in \boldsymbol{N}$). For sufficiently small $\varepsilon>0$,

$$f_1(x_{2k}+\varepsilon)=f_1(x_{2k}), \qquad f_2(x_{2k-1}+\varepsilon)=f_2(x_{2k-1}). \tag{3.3}$$

Since $f(x_{2k-1})\leqq f(x_{2k-1}+\varepsilon)=f_1(x_{2k-1}+\varepsilon)+f_2(x_{2k-1}+\varepsilon)$, (3.3) implies $f_1(x_{2k-1})\leqq f_1(x_{2k-1}+\varepsilon)$. Similarly, we have $f_2(x_{2k})\leqq f_2(x_{2k}+\varepsilon)$. Thus $f_1(x)$ and $f_2(x)$ are non-decreasing. Since

$$\begin{aligned} f_1(x_{2k}) &\leqq f(2x_{2k+1})-f(x_{2k+1}) \\ &= f_1(2x_{2k+1})-f_1(x_{2k+1})+f_2(2x_{2k+1})-f_2(x_{2k+1}) \\ &= f_1(2x_{2k+1})-f_1(x_{2k+1}), \end{aligned}$$

we have

$$f_1(2x_{2k+1})/f_1(x_{2k+1})\geqq 1+f_1(x_{2k})/f_1(x_{2k+1})=2.$$

Hence $\limsup_{x\to\infty}f_1(2x)/f_1(x)\geqq 2$. Similarly, $\limsup_{x\to\infty}f_2(2x)/f_2(x)\geqq 2$. This implies that neither $f_1(x)$ nor $f_2(x)$ is slowly varying. □

DEFINITION 3.2. We say that a non-negative non-decreasing function $f(x)$ is *dominatedly non-decreasing* (*undominatedly non-decreasing*) if $\limsup_{x\to\infty}(f(2x)-f(x))<\infty$ $(=\infty)$.

REMARK. The class of dominatedly non-decreasing functions is a proper subset of the class of non-decreasing slowly varying functions. Any component of a dominatedly non-decreasing function is also dominatedly non-decreasing.

REMARK. The dominated non-decrease of $f(x)$ is equivalent to the dominated variation of $\exp(f(x))$ ([7] p. 99 Definition A.4). Using this fact and some known results ([7] p. 99 Lemma A.4, p. 93 Theorem A.1), we can give a representation of a dominatedly non-decreasing function $f(x)$ as follows:

$$f(x)=c(x)+\int_A^x \varepsilon(t)t^{-1}dt, \tag{3.4}$$

where $c(x)$ and $\varepsilon(t)$ are bounded measurable and $\varepsilon(t)$ is non-negative on $[A,\infty)$.

We can give a stronger assertion on decomposition of an undominatedly non-decreasing function as follows.

PROPOSITION 3.3. *If $f(x)$ is an undominatedly non-decreasing function, then, for each $n\in\boldsymbol{N}$, $f(x)$ can be represented as $f(x)=\sum_{j=1}^{n}f_j(x)$, where each $f_j(x)$ is unbounded non-negative non-decreasing and the sum of an arbitrary proper subset of the set $\{f_j(x): j=1, \cdots, n\}$ is not slowly varying. Moreover $f(x)$ has a representation $f(x)=\sum_{j=1}^{\infty}f_j(x)$ with the same properties.*

PROOF. Let x_0 be a point such that $f(x_0)>0$. Define $f_j(x)=n^{-1}f(x_0)$ for each j. For $x>x_0$, define $f_j(x)$ inductively as follows: for given x_{nk+m} $(0\leqq m \leqq n-1)$, choose x_{nk+m+1} such that $x_{nk+m+1}>2x_{nk+m}$ and

$$f(2x_{nk+m+1})-f(x_{nk+m+1})\geqq \sum_{\substack{j=1\\ j\neq m}}^{n} f_j(x_{nk+m}).$$

For $x_{nk+m}<x\leqq x_{nk+m+1}$, each $f_j(x)$ is defined as

$$f_m(x)=f(x)-\sum_{\substack{j=1\\ j\neq m}}^{n} f_j(x_{nk+m}), \qquad f_j(x)=f_j(x_{nk+m}) \qquad (j\neq m).$$

It is easy to see that each $f_j(x)$ is non-decreasing. Let S be a proper subset of $\{1, 2, \cdots, n\}$ and $g(x)$ be $\sum_{j\in S}f_j(x)$. Take m such that $m+1$ (mod n)$\in S$ and $m\notin S$. Then, we have

$$\begin{aligned} g(2x_{nk+m+1})-g(x_{nk+m+1}) &\geqq f_{m+1}(2x_{nk+m+1})-f_{m+1}(x_{nk+m+1}) \\ &= f(2x_{nk+m+1})-\sum_{\substack{j=1\\ j\neq m+1}}^{n} f_j(x_{nk+m+1})-f_{m+1}(x_{nk+m+1}) \\ &= f(2x_{nk+m+1})-f(x_{nk+m+1})\geqq \sum_{\substack{j=1\\ j\neq m}}^{n} f_j(x_{nk+m}) \end{aligned}$$

and

$$g(x_{nk+m+1})\leqq \sum_{\substack{j=1\\ j\neq m}}^{n} f_j(x_{nk+m+1})=\sum_{\substack{j=1\\ j\neq m}}^{n} f_j(x_{nk+m}).$$

Thus $g(2x_{nk+m+1})/g(x_{nk+m+1})\geqq 2$. Hence $\limsup_{x\to\infty}g(2x)/g(x)\geqq 2$. This implies that $g(x)$ is not slowly varying.

Next, in order to get decomposition into infinite sum, define $f_j(x)$ $(j\in\boldsymbol{N})$ as follows. First, define a sequence of positive integers $n(k)$ $(k\in\boldsymbol{N})$ as follows: $n(k)=1$ if $k=i(i+1)/2$ for some i. For $j\geqq 2$, $n(k)=j$ if $k=j-1+(j+i-1)(j+i-2)/2$ for some i. That is, the sequence $n(1), n(2), \cdots$ is

$$1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \cdots.$$

Define $f_j(x_0)=2^{-j}f(x_0)$ $(j\in\boldsymbol{N})$. For $x>x_0$, define $f_j(x)$ inductively as follows: for given x_{k-1}, choose x_k such that $x_k>2x_{k-1}$ and

$$f(2x_k)-f(x_k)\geqq\sum_{j\neq n(k)}f_j(x_{k-1}).$$

For $x_{k-1}<x\leqq x_k$, each $f_j(x)$ is defined as

$$f_{n(k)}(x)=f(x)-\sum_{j\neq n(k)}f_j(x_{k-1}),\qquad f_j(x)=f_j(x_{k-1})\qquad (j\neq n(k)).$$

It is easy to see that each $f_j(x)$ is non-decreasing. Let S be a proper subset of $\boldsymbol{N}$ and $g(x)$ be $\sum_{j\in S}f_j(x)$. Take k such that $n(k+1)\in S$ and $n(k)\notin S$. Then, we have

$$\begin{aligned}g(2x_k)-g(x_k)&\geqq f_{n(k+1)}(2x_k)-f_{n(k+1)}(x_k)\\&=f(2x_k)-\sum_{j\neq n(k+1)}f_j(x_k)-f_{n(k+1)}(x_k)\\&=f(2x_k)-f(x_k)\geqq\sum_{j\neq n(k)}f_j(x_{k-1})\end{aligned}$$

and

$$g(x_k)\leqq\sum_{j\neq n(k)}f_j(x_k)=\sum_{j\neq n(k)}f_j(x_{k-1}).$$

Thus $g(2x_k)/g(x_k)\geqq 2$. Hence $\limsup_{x\to\infty}g(2x)/g(x)\geqq 2$. This implies that $g(x)$ is not slowly varying. □

EXAMPLES.

1. The function $f(x)=\log x$ is dominatedly non-decreasing.
2. The function $f(x)=(\log x)^2$ is undominatedly non-decreasing.
3. If $f(x)$ and $g(x)$ are dominatedly non-decreasing, then, for any $a, b>0$ and $0\leqq\alpha,\ \beta\leqq 1$, $a(f(x))^{\alpha}+b(g(x))^{\beta}$ is dominatedly non-decreasing.
4. If $f(x)$ is dominatedly non-decreasing and $g(x)$ is non-decreasing slowly varying, then $f(g(x))$ is dominatedly non-decreasing.

Now we show some facts concerning the dominated non-decrease.

PROPOSITION 3.4. *For any unbounded non-decreasing slowly varying function $f(x)$, there exists an undominatedly non-decreasing function $\tilde{f}(x)$ asymptotically equal to $f(x)$.*

PROOF. Given $f(x)$, we can define $\tilde{f}(x)$ as follows:

$$\tilde{f}(x)=f(x)+r^{n(x)},\tag{3.5}$$

where $r>1$ is a constant and $n(x)\in\boldsymbol{N}$ such that

$$r^{n(x)}\leqq(f(x))^{1/2}<r^{n(x)+1}.$$

Obviously, $\tilde{f}(x)$ is a non-decreasing function and

$$1\leqq\tilde{f}(x)/f(x)=1+r^{n(x)}/f(x)\leqq 1+(f(x))^{-1/2}.$$

Hence $\tilde{f}(x)$ and $f(x)$ are asymptotically equal. And

$$\tilde{f}(2x)-\tilde{f}(x)=f(2x)-f(x)+r^{n(x)}(r^{n(2x)-n(x)}-1).$$

Since $\lim_{x\to\infty} r^{n(x)}=\infty$ and $\limsup_{x\to\infty} r^{n(2x)-n(x)}-1=r-1$, $\tilde{f}(x)$ is undominatedly non-decreasing. Thus it is proved that $\tilde{f}(x)$ satisfies all conditions. □

The above proposition shows that the dominated non-decrease is not guaranteed by asymptotic growth order. However, we have some conditions concerning asymptotic equality to a dominatedly non-decreasing function.

PROPOSITION 3.5. *Let $f(x)$ be a non-decreasing slowly varying function on $[A, \infty)$.*

(1) *If $\limsup_{x\to\infty} f(x)/\log x=\infty$, then $f(x)$ is undominatedly non-decreasing.*

(2) *If $\limsup_{x\to\infty} f(x)/\log x<\infty$ and an ε-function of $f(x)$ is non-negative and satisfies $\limsup_{x\to\infty}\int_x^{2x}\varepsilon(t)t^{-1}dt(\log x)<\infty$, then there exists a dominatedly non-decreasing function $\tilde{f}(x)$ asymptotically equal to $f(x)$.*

PROOF. (1) Let us show that if $f(x)$ is dominatedly non-decreasing, then $\limsup_{x\to\infty} f(2x)/\log x<\infty$. Choose c $(0<c<\infty)$ such that $f(2x)-f(x)<c$ for all x. Summing up the inequalities $f(2^k)-f(2^{k-1})<c$ for $k=1, \cdots, n$, we have $f(2^n)<nc+f(1)$. Hence $\limsup_{n\to\infty} f(2^n)/\log 2^n \leqq \limsup_{n\to\infty}(nc+f(1))/(n\log 2)=c/\log 2<\infty$. For x such that $2^n\leqq x\leqq 2^{n+1}$, $f(x)/\log x\leqq f(2^{n+1})/\log 2^n=\{(n+1)/n\}\cdot\{f(2^{n+1})/\log 2^{n+1}\}$. Therefore $\limsup_{x\to\infty} f(x)/\log x\leqq c/\log 2<\infty$.

(2) By the representation theorem of slowly varying function (Proposition 2.2), $f(x)$ is written as follows.

$$f(x)=c(x)\exp\left(\int_B^x \varepsilon(t)t^{-1}\right)dt\,, \qquad \lim_{x\to\infty} c(x)=c\,.$$

By the assumption, there exist positive constants c_1 and c_2 such that $f(x)/\log x<c_1$ and $\int_x^{2x}\varepsilon(t)t^{-1}dt(\log x)<c_2$. Now we set

$$\tilde{f}(x)=c\exp\left(\int_B^x \varepsilon(t)t^{-1}dt\right).$$

Then

$$\begin{aligned}\tilde{f}(2x)-\tilde{f}(x)&=\tilde{f}(x)(\tilde{f}(2x)/\tilde{f}(x)-1)\\&=(\tilde{f}(x)/f(x))f(x)\Big(\exp\left(\int_x^{2x}\varepsilon(t)t^{-1}dt\right)-1\Big)\\&\leqq c_1(\log x)\Big(\exp\left(\int_x^{2x}\varepsilon(t)t^{-1}dt\right)-1\Big)(1+o(1))\\&\leqq c_1c_2(1+o(1))\,.\end{aligned}$$

Thus $\tilde{f}(2x)-\tilde{f}(x)$ is bounded. □

If $\varepsilon(t)=\alpha/\log t$, then $f(x)=c(x)(\log x)^{\alpha}$ ($\lim_{x\to\infty}c(x)=c$ $(0<c<\infty)$) and this $\varepsilon(t)$ satisfies the condition $\limsup_{x\to\infty}\int_x^{2x}\varepsilon(t)t^{-1}dt(\log x)<\infty$ in (2). This $f(x)$ is undominatedly non-decreasing, if $\alpha>1$, by (1). However the proof of the following proposition shows that $f(x)$ can be dominatedly non-decreasing when we assume $\varepsilon(t)=\alpha/\log t$ with $\alpha>1$ only on a sequence of sets.

PROPOSITION 3.6. *Fix a and b such that $0\leqq a<b<\infty$ or $a=b=0$. Consider the class of non-decreasing slowly varying functions satisfying* $\liminf_{x\to\infty}f(x)/\log x=a$ *and* $\limsup_{x\to\infty}f(x)/\log x=b$. *We can construct two functions $f_1(x)$ and $f_2(x)$ in this class such that $f_1(x)$ is dominately non-decreasing, while any non-decreasing function $\tilde{f}(x)$ asymptotically equal to $f_2(x)$ is undominatedly non-decreasing.*

PROOF. Consider the case $0<a<b$. We make $f_1(x)$ and $f_2(x)$ oscillate between $a\log x$ and $b\log x$. We will choose a sequence $\{x_k : k\in \boldsymbol{N}\}$ in a suitable manner and make $f_j(x)$ flat in $[x_{2k-1}, x_{2k}]$ and increasing in $[x_{2k}, x_{2k+1}]$. Thus we make $a\log x\leqq f_j(x)\leqq b\log x$ and $f_j(x_{2k-1})=b\log x_{2k-1}=f_j(x_{2k})=a\log x_{2k}$. More precisely, choice of $\{x_k\}$ and definition of $f_j(x)$ are as follows. We can assume $b=1$ without loss of generality. Fix $x_1>1$. For $x_1\leqq x\leqq x_2=x_1^{1/a}$, let $f_j(x)=f_j(x_1)=\log x_1$. For $x_2<x\leqq x_3$, let $f_j(x)=f_j(x_2)\exp\left(\int_{x_2}^{x}\varepsilon_j(t)t^{-1}dt\right)$, where $\varepsilon_1(t)=2(\log t)^{-1}$ and $\varepsilon_2(t)=r(\log^2 t)^{-1}$ with $r=2^{-1}\log a^{-1}$ (>0). Here x_3 is determined by

$$f_j(x_2)\exp\left(\int_{x_2}^{x_3}\varepsilon_j(t)t^{-1}dt\right)=\log x_3\,.$$

For $x>x_3$, $f_j(x)$ is defined inductively as follows. We assume that $f_j(x)$ is defined on $[x_1, x_{2k-1}]$. x_{2k} is defined to satisfy

$$f_j(x_{2k-1})=a\log x_{2k}\,.$$

For $x_{2k-1}<x\leqq x_{2k}$, $f_j(x)$ is defined as

$$f_j(x)=f_j(x_{2k-1})\,.$$

Define x_{2k+1} to satisfy

$$f_j(x_{2k})\exp\left(\int_{x_{2k}}^{x_{2k+1}}\varepsilon_j(t)t^{-1}dt\right)=\log x_{2k+1}\,.$$

For $x_{2k}<x\leqq x_{2k+1}$, $f_j(x)$ is defined as

$$f_j(x)=f_j(x_{2k})\exp\left(\int_{x_{2k}}^{x}\varepsilon_j(t)t^{-1}dt\right).$$

Let us prove that $f_1(x)$ and $f_2(x)$ have the desired properties. Since $\int_x^{2x}\varepsilon_1(t)t^{-1}dt(\log x)$ is bounded, $f_1(x)$ is dominatedly non-decreasing by Proposition

3.5 (2). We consider the behavior of $f_2(x)$ at $x_{2k}\log x_{2k}$. Since $\int_x^{x\log x}(\log^2 t)^{-1}t^{-1}dt$ is non-decreasing and converges to 1 as x goes to infinity, we have

$$\lim_{k\to\infty}\exp\left(\int_{x_{2k}}^{x_{2k}\log x_{2k}}(r/\log^2 t)t^{-1}dt\right)=e^r=a^{-1/2}$$

and

$$a\log x_{2k}\exp\left(\int_{x_{2k}}^{x_{2k}\log x_{2k}}(r/\log^2 t)t^{-1}dt\right)<\log(x_{2k}\log x_{2k}).$$

This means that $x_{2k}\log x_{2k}<x_{2k+1}$. Therefore

$$f_2(x_{2k}\log x_{2k})=f_2(x_{2k})\exp\left(\int_{x_{2k}}^{x_{2k}\log x_{2k}}(r/\log^2 t)t^{-1}dt\right).$$

Thus we get

$$\lim_{k\to\infty}f_2(x_{2k}\log x_{2k})/f_2(x_{2k})=a^{-1/2}. \tag{3.6}$$

Assume that $\tilde{f}(x)$ is a non-decreasing function and $\tilde{f}(2x)-\tilde{f}(x)<c<\infty$. Define $n_k\in\boldsymbol{N}$ as $2^{n_k-1}x_{2k}<x_{2k}\log x_{2k}\leqq 2^{n_k}x_{2k}$. Then,

$$\tilde{f}(x_{2k}\log x_{2k})-\tilde{f}(x_{2k})\leqq n_k c<(\log^2 x_{2k}/\log 2+1)c.$$

Hence

$$\tilde{f}(x_{2k}\log x_{2k})/\tilde{f}(x_{2k})<1+(a\log x_{2k})^{-1}(\log^2 x_{2k}/\log 2+1)c.$$

Thus we have

$$\lim_{k\to\infty}\tilde{f}(x_{2k}\log x_{2k})/\tilde{f}(x_{2k})=1. \tag{3.7}$$

(3.6) and (3.7) imply that $\tilde{f}(x)$ is not asymptotically equal to $f_2(x)$. Thus we conclude that if $\tilde{f}(x)$ is non-decreasing and asymptotically equal to $f_2(x)$, then $\tilde{f}(x)$ is undominatedly non-decreasing.

The case $0=a<b$ can be proved by using $(\log x)^{1/2}$ instead of $a\log x$. In the case $a=b=0$, $(\log x)^{1/2}$ and $2^{-1}(\log x)^{1/2}$ can be used. □

The following proposition gives properties of components in decomposition of a non-decreasing slowly varying function.

PROPOSITION 3.7. *Let $f(x)$ be a non-decreasing slowly varying function. Suppose that $f(x)=f_1(x)+f_2(x)$, where $f_1(x)$ and $f_2(x)$ are non-negative non-decreasing functions.*

(1) *If $f_j(x)$ is not identically zero, then*

$$\liminf_{x\to\infty}f_j(kx)/f_j(x)=1 \qquad \textit{for every } k>1.$$

(2) *If $f_1(x)$ is not slowly varying, then*

$$\liminf_{x\to\infty} f_1(x)/f_2(x) = 0 .$$

Especially, if none of $f_1(x)$ *and* $f_2(x)$ *is slowly varying, then*

$$\liminf_{x\to\infty} f_1(x)/f_2(x) = 0 \quad \textit{and} \quad \limsup_{x\to\infty} f_1(x)/f_2(x) = \infty .$$

PROOF. (1) Suppose that $\liminf_{x\to\infty} f_j(kx)/f_j(x) > 1$. There exist $\varepsilon > 0$ and x_0 such that $f_j(kx)/f_j(x) \geqq 1+\varepsilon$ for all $x \geqq x_0$. Hence $f_j(k^n x_0)/f_j(x_0) \geqq (1+\varepsilon)^n$ for all $n \in \boldsymbol{N}$, that is,

$$f_j(k^n x_0)/(1+\varepsilon)^n \geqq f_j(x_0) > 0 .$$

select sufficiently small δ such that $k^\delta < 1+\varepsilon$. Then

$$\lim_{n\to\infty} f_j(k^n x_0)/(1+\varepsilon)^n \leqq \lim_{n\to\infty} \{f(k^n x_0)/(k^n)^\delta\}\{k^\delta/(1+\varepsilon)\}^n = 0 .$$

This is a contradiction.

(2) Since

$$\begin{aligned} f(2x)/f(x)-1 &= (f_1(2x)/f_1(x)-1)/(1+f_2(x)/f_1(x)) \\ &\quad +(f_2(2x)/f_2(x)-1)/(1+f_1(x)/f_2(x)) \end{aligned}$$

and the left-hand side goes to zero, we have

$$\lim_{x\to\infty} (f_1(2x)/f_1(x)-1)/(1+f_2(x)/f_1(x)) = 0$$

and

$$\lim_{x\to\infty} (f_2(2x)/f_2(x)-1)/(1+f_1(x)/f_2(x)) = 0 .$$

Hence the assertions follow immediately. □

We close this section with a theorem on another kind of decomposition.

THEOREM 3.8. *Let* $f(x)$ *be an undominatedly non-decreasing slowly varying function and* r *be a constant such that* $0 \leqq r < \infty$. *Then there exist unbounded non-negative non-decreasing functions* $f_1(x)$ *and* $f_2(x)$ *satisfying* $f(x) = f_1(x)+f_2(x)$ *such that* $f_1(x)$ *is slowly varying,* $f_2(x)$ *is not slowly varying and*

$$\limsup_{x\to\infty} f_2(x)/f_1(x) = r . \tag{3.8}$$

PROOF. By Theorem 3.1, $f(x)$ is represented as $f(x) = \tilde{f}_1(x)+\tilde{f}_2(x)$, where $\tilde{f}_1(x)$ and $\tilde{f}_2(x)$ are unbounded non-negative non-decreasing and not slowly varying. Assume $r>0$. Define u by $r = u/(1-u)$. Then, $0<u<1$. Set $f_1(x) = \tilde{f}_1(x)+(1-u)\tilde{f}_2(x)$ and $f_2(x) = u\tilde{f}_2(x)$. Then, $f_1(x)$ and $f_2(x)$ are unbounded non-decreasing functions, $f_2(x)$ is not slowly varying and $f(x) = f_1(x)+f_2(x)$. Now we have only to prove that $f_1(x)$ and $f_2(x)$ satisfy (3.8) and $f_1(x)$ is slowly varying. We have

$$f_2(x)/f_1(x)=\{u\tilde{f}_2(x)/\tilde{f}_1(x)\}/\{1+(1-u)\tilde{f}_2(x)/\tilde{f}_1(x)\},$$

and hence $f_2(x)/f_1(x)\leqq u/(1-u)$. By Proposition 3.7 (2), $\limsup_{x\to\infty}\tilde{f}_2(x)/\tilde{f}_1(x)=\infty$. Thus we get (3.8). Again, by Proposition 3.7 (2), it follows from (3.8) that $f_1(x)$ is slowly varying. The proof in the case $r>0$ is finished. If $r=0$, then define $f_1(x)=f(x)-(\tilde{f}_2(x))^{1/2}$ and $f_2(x)=(\tilde{f}_2(x))^{1/2}$ for large x. □

4. Decomposition problem of distributions in $\boldsymbol{D}(2)$.

In this section, we apply the results in the preceding section to a topic on $\boldsymbol{D}(2)$ in probability theory. We are interested in properties of factors of distributions in $\boldsymbol{D}(2)$. According to [4], H. Tucker made the following conjecture: if μ is a distribution in $\boldsymbol{D}(2)$, then any non-trivial factor of μ belongs to $\boldsymbol{D}(2)$. Hahn and Klass [4] give a counter-example to this conjecture. They show the existence of two non-trivial distributions μ_1 and μ_2 such that μ_1 and $\mu_1*\mu_2$ belong to $\boldsymbol{D}(2)$ and μ_2 does not belong to $\boldsymbol{D}(2)$. In their example, $\lim_{R\to\infty}V_2(R)/V_1(R)=0$ holds, where $V_j(R)$ is the truncated variance of μ_j $(j=1, 2)$. This implies that, if X_k, Y_k $(k\in\boldsymbol{N})$ are independent, X_k has distribution μ_1, and Y_k has distribution μ_2, then, with some normalizing constant B_n, the distribution of $B_n^{-1}\sum_{k=1}^n(X_k-EX_1)$ converges to Gaussian distribution with mean 0 and the distribution of $B_n^{-1}\sum_{k=1}^n(Y_k-EY_1)$ converges to the delta distribution concentrated at 0. We deal with Tucker's conjecture from a more general point of view in connection with the results in the preceding section. First, we will give a sufficient condition for a distribution to have the property that all non-trivial factors of it belong to $\boldsymbol{D}(2)$. The Lévy-Cramér theorem says that any non-trivial factor of Gaussian distribution is Gaussian. Obviously, any non-trivial factor of a distribution with finite variance has finite variance, and hence belongs to the domain of normal attraction of Gaussian distributions. We extend the above fact: if μ has a dominatedly non-decreasing truncated variance, then every non-trivial factor of μ belongs to $\boldsymbol{D}(2)$. Second, we will construct another kind of counter-examples to the conjecture: we give μ_1 and μ_2 such that neither μ_1 nor μ_2 belongs to $\boldsymbol{D}(2)$ but the convolution $\mu_1*\mu_2$ belongs to $\boldsymbol{D}(2)$.

We prepare two propositions.

PROPOSITION 4.1. *Let $V(R)=V_X(R)$. The following are equivalent:*

$$\limsup_{R\to\infty}(V(2R)-V(R))<\infty. \tag{4.1}$$

$$\limsup_{R\to\infty}R^2P(|X|>R)<\infty. \tag{4.2}$$

PROOF. (4.2) implies (4.1) because

$$V(2R)-V(R)=EX^2 1(R\leqq|X|<2R)\leqq 4R^2P(R\leqq|X|<2R)\leqq 4R^2P(|X|\geqq R).$$

Conversely, assume (4.1). Then, since $V(2R)-V(R)\geqq R^2P(R\leqq|X|<2R)$, there exists a positive constant c such that $R^2P(R\leqq|X|<2R)<c$ for every $R>0$. Therefore we get $4^nR^2P(2^nR\leqq|X|<2^{n+1}R)<c$ for every $n\in\boldsymbol{N}$. Summing up for all n, we have

$$R^2P(|X|\geqq R)=\sum_{n=0}^{\infty}R^2P(2^nR\leqq|X|<2^{n+1}R)<\sum_{n=0}^{\infty}4^{-n}c<\infty. \qquad \square$$

PROPOSITION 4.2. *For an arbitrary non-negative right-continuous non-decreasing slowly varying function $f(x)$ on $[0,\infty)$, there exists a distribution μ on $[0,\infty)$ and a constant B such that*

$$f(x)=\int_{|t|<x}|t|^2\mu(dt) \qquad \text{for all} \quad x\geqq B.$$

Proof is straightforward.

In the following theorem, we give a sufficient condition in order that all non-trivial factors of a distribution belong to $\boldsymbol{D}(2)$.

We denote by C the class of distributions on $\boldsymbol{R}^1$ with dominatedly non-decreasing truncated variance. Note that any non-trivial distribution in C belongs to $\boldsymbol{D}(2)$.

THEOREM 4.3. *Let $\mu=\mu_1*\mu_2$. If both μ_1 and μ_2 belong to C, then μ belongs to C. Conversely, if μ is in C, then both μ_1 and μ_2 belong to C.*

PROOF. Let X and Y be independent random variables with distributions μ_1 and μ_2, respectively, and set $Z=X+Y$. Assume that μ_1 and μ_2 belong to C. Then,

$$P(|X+Y|>R)\leqq P(|X|\vee|Y|>R/2)\leqq P(|X|>R/2)+P(|Y|>R/2).$$

Hence we get

$$\limsup_{R\to\infty}R^2P(|X+Y|>R)\leqq 4\{\limsup_{R\to\infty}R^2P(|X|>R)+\limsup_{R\to\infty}R^2P(|Y|>R)\}.$$

By Proposition 4.1, $\mu_1*\mu_2$ belongs to C.

Conversely, assume that $\mu=\mu_1*\mu_2$ is in C. Since

$$P(|X|>R)P(|Y|<R/2)=P(|X|>R,\ |Y|<R/2)\leqq P(|X+Y|>R/2),$$

we have

$$\begin{aligned}&\limsup_{R\to\infty}R^2P(|X|>R)\leqq\limsup_{R\to\infty}P(|Y|<R/2)^{-1}R^2P(|X+Y|>R/2)\\&\leqq 4\limsup_{R\to\infty}R^2P(|X+Y|>R).\end{aligned}$$

By Proposition 4.1, μ_1 belongs to C, and similarly for μ_2. $\square$

The class C does not coincide with the class of distributions of which all non-trivial factors belong to $\boldsymbol{D}(2)$. In fact, any indecomposable distribution in $\boldsymbol{D}(2)\setminus C$ is an example of a distribution with this property (Example 1). There is also a decomposable distribution in $\boldsymbol{D}(2)\setminus C$ having this property (Example 2).

EXAMPLE 1. Define a discrete probability measure μ as follows:

$$\mu(\{2^k\}) = c\, k4^{-k} \quad \text{for} \quad k\in \boldsymbol{N} \text{ where } c = \Big(\sum_{k=1}^{\infty} k4^{-k}\Big)^{-1}.$$

Let $V(R)$ be the truncated variance of μ. Then,

$$V(R) = c \sum_{k<\log_2 R} k\,.$$

Hence

$$1 \leqq V(2R)/V(R) = 1+ \sum_{\log_2 R\leqq k<\log_2 R+1} k/ \sum_{k<\log_2 R} k = 1+n/\{n(n-1)/2\}\,,$$

where n is the positive integer such that $\log_2 R\leqq n<\log_2 R+1$. This shows that $V(R)$ is slowly varying. Further, $V(2R)-V(R)=cn$, hence $\lim_{R\to\infty}(V(2R)-V(R)) =\infty$. Thus μ is in $\boldsymbol{D}(2)\setminus C$. If $\mu=\mu_1*\mu_2$, then μ_1 or μ_2 is trivial, since the support $S=\{2^k : k\in \boldsymbol{N}\}$ of μ has the property that, if $S=S_1+S_2$, then S_1 or S_2 is a one-point set.

EXAMPLE 2. Let μ be the distribution in the Example 1. Then it is easy to see that $\mu*\mu$ is a distribution in $\boldsymbol{D}(2)$ with undominatedly non-decreasing truncated variance. Moreover, any non-trivial factor of this distribution is in $\boldsymbol{D}(2)$. In fact, we can prove that, if $\mu*\mu=\mu_1*\mu_2$ (neither μ_1 nor μ_2 is trivial), then $\mu=\mu_1*\delta_a=\mu_2*\delta_{-a}$ for some a. It is enough to prove $\mu=\mu_1=\mu_2$, assuming that the support of μ_1 and that of μ_2 both contain 2 as the smallest element. Comparing the support and the masses of $\{4, 6, 8, 10\}$ of $\mu*\mu$ with those of $\mu_1*\mu_2$ in detail, we can prove that both supports contain 4 as the next smallest element. Next we can show that the both supports coincide with the support of μ. Proof is complicated and omitted. After this is established, it is easy to prove that $\mu=\mu_1=\mu_2$.

The following theorem gives the relation between the truncated variance of a distribution in $\boldsymbol{D}(2)$ and those of its factors.

THEOREM 4.4. *Let $\mu_i\,(i=1, \cdots, n)$ be distributions on $\boldsymbol{R}^1$ with truncated variance $V_i(R)$. The convolution $\mu=\mu_1*\cdots*\mu_n$ belongs to $\boldsymbol{D}(2)$ if and only if $\sum_{i=1}^n V_i(R)$ is slowly varying. If the truncated variance $V(R)$ of μ is an unbounded slowly varying function, then*

$$\lim_{R\to\infty} \sum_{i=1}^{n} V_i(R)/V(R) = 1\,. \tag{4.3}$$

PROOF. Let X_i be independent random variables with distribution μ_i and set $S=\sum_{i=1}^n X_i$. If $V_S(R)$ converges to a finite limit, the assertion is trivial.

Let us assume that $V_S(R)$ is slowly varying and diverges to infinity. Notice that, for each $\varepsilon>0$, $E|S|^{2-\varepsilon}<\infty$ and, equivalently, $E|X_i|^{2-\varepsilon}<\infty$ for each i ([**2**]). Decompose $V_S(R)$ as

$$V_S(R)=V_S^1(R)+V_S^2(R),$$

where $V_S^1(R)=ES^2 1(|S|<R,$ the number of i such that $|X_i|\geqq R/n$ is at most one.) and $V_S^2(R)=ES^2 1(|S|<R,$ the number of i such that $|X_i|\geqq R/n$ is two or more.). Let $A_{ij}(R)$ be the set determined by the conditions: $|S|<R$, $|X_i|\geqq R/n$ and $|X_j|\geqq R/n$. Then

$$V_S^2(R)\leqq\sum_{i<j}ES^2 1(A_{ij}(R))\leqq\sum_{i<j}R^2P(|X_i|\geqq R/n)P(|X_j|\geqq R/n).$$

Integration by parts leads to $R^2P(|X_i|>R)=2\int_0^R tP(|X_i|>t)dt-EX_i^2 1(|X_i|\leqq R)$, and we get $\lim_{R\to\infty}R^{2-\varepsilon}P(|X_i|>R)=0$ for arbitrary $\varepsilon>0$. Hence we have $\lim_{R\to\infty}V_S^2(R)=0$. Therefore

$$\lim_{R\to\infty}V_S(R)/V_S^1(R)=1. \tag{4.4}$$

Set $W(R)=ES^2 1(\max_{1\leqq k\leqq n}|X_k|<R)$. Then

$$W(R/n)\leqq V_S^1(R)\leqq W((2n-1)n^{-1}R).$$

By this inequality and the slow variation of $V_S^1(R)$, we get

$$\lim_{R\to\infty}V_S^1(R)/W(R)=1. \tag{4.5}$$

Further,

$$\begin{aligned}W(R)&=\sum_{i=1}^n EX_i^2 1(\max_{1\leqq k\leqq n}|X_k|<R)+2\sum_{\substack{i,j=1\\ i<j}}^n EX_iX_j 1(\max_{1\leqq k\leqq n}|X_k|<R)\\&=\sum_{i=1}^n V_i(R)\prod_{\substack{k=1\\ k\neq i}}^n P(|X_k|<R)+2\sum_{\substack{i,j=1\\ i<j}}^n EX_iX_j 1(\max_{1\leqq k\leqq n}|X_k|<R).\end{aligned}$$

Hence

$$\lim_{R\to\infty}W(R)/\sum_{i=1}^n V_i(R)=1. \tag{4.6}$$

From (4.4)-(4.6), we get (4.3). This means that $\sum_{i=1}^n V_i(R)$ is slowly varying.

We can prove the converse assertion by following the reverse direction. Namely, assume that $\sum_{i=1}^n V_i(R)$ is unbounded slowly varying. Then, every μ_i has finite absolute moment of order $2-\varepsilon$. Now we get (4.6) and (4.5) is obtained from (4.6). (4.4) also holds. From (4.4)-(4.6), (4.3) is proved. □

REMARK. Tucker [**9**] proves that $\boldsymbol{D}(2)$ is closed under convolution. This result is a consequence of the above theorem since the sum of two slowly varying functions is also slowly varying ([**7**] p. 18).

Now we can prove the following theorem from Theorem 3.1, Proposition 3.3 and Theorem 4.4 easily.

THEOREM 4.5. *There exist distributions* μ_1 *and* μ_2 *such that none of them belongs to* $\boldsymbol{D}(2)$ *but* $\mu=\mu_1*\mu_2$ *belongs to* $\boldsymbol{D}(2)$. *In general, for each* $n\in\boldsymbol{N}$, *there exist distributions* $\mu_1, \mu_2, \cdots, \mu_n$ *such that* $\mu=\mu_1*\mu_2*\cdots*\mu_n$ *belongs to* $\boldsymbol{D}(2)$ *but, for each proper subset* S *of* $\{1, 2, \cdots, n\}$, *the convolution of* $\{\mu_j: j\in S\}$ *does not belong to* $\boldsymbol{D}(2)$.

PROOF. By Proposition 4.2, we can choose a distribution μ in $\boldsymbol{D}(2)$ such that $\limsup_{R\to\infty}(V(2R)-V(R))=\infty$. By Theorem 3.1, there exist measures μ_1^0 and μ_2^0 on $(0, \infty)$ such that $V(R)=V_1(R)+V_2(R)$, where $V_j(R)$ is not slowly varying and $V_j(R)=\int_{(0,R)}x^2\mu_j^0(dx)$ for $j=1, 2$ and $\mu(\boldsymbol{R}^1\setminus\{0\})=\sum_{j=1}^2\mu_j^0(0, \infty)$. We define probability measures $\mu_j\,(j=1, 2)$ on $[0, \infty)$ by $\mu_j=\mu_j^0+\delta_j$, where δ_j is a measure on $\{0\}$ with point mass $1-\mu_j^0(0, \infty)$. Then the truncated variance of μ_j is equal to $V_j(R)$. We define a probability measure $\tilde{\mu}$ by $\tilde{\mu}=\mu_1*\mu_2$. Then, by Theorem 4.4, $\tilde{\mu}$ belongs to $\boldsymbol{D}(2)$, but neither μ_1 nor μ_2 belongs to $\boldsymbol{D}(2)$. Similarly, Proposition 3.3 yields the latter half of the theorem. □

PROPOSITION 4.6. *Let* $0\leqq r<\infty$. *Then we can construct two distributions* μ_1 *and* μ_2 *such that* μ_1 *and* $\mu_1*\mu_2$ *belong to* $\boldsymbol{D}(2)$, μ_2 *does not belong to* $\boldsymbol{D}(2)$, *and moreover* $\limsup_{R\to\infty}V_2(R)/V_1(R)=r$, *where* $V_j(R)$ *is the truncated variance of* μ_j $(j=1, 2)$.

PROOF. Use Theorems 3.8 and 4.4 as in the proof of the above theorem. □

REMARK. We note that the example of Hahn and Klass [4] satisfies $\lim_{R\to\infty}V_2(R)/V_1(R)=0$.

We add a general result related to Tucker's conjecture.

THEOREM 4.7. *Any non-trivial factor of a distribution in* $\boldsymbol{D}(2)$ *belongs to the domain of partial attraction of Gaussian distributions.*

PROOF. Let μ be in $\boldsymbol{D}(2)$ and let $\mu=\mu_1*\mu_2$. Let X and Y be independent random variables with distribution μ_1 and μ_2, respectively. Set $Z=X+Y$. By Proposition 2.5, $E|Z|^\alpha$ is finite for every $\alpha\in(0, 2)$. Hence $E|X|^\alpha$ and $E|Y|^\alpha$ are finite for every $\alpha\in(0, 2)$ ([2]). Maller [5] shows that this implies that μ_1 and μ_2 belong to the domain of partial attraction of Gaussian distributions if they are non-trivial. □

Appendix.

Proposition 2.6 is found in Tucker [**9**], but his proof seems to be incomplete (the proof of the monotonicity is not understandable and the statement in p. 1384 l. 17-18, which is used in the construction, is erroneous). So we give a proof to the proposition and add some related remarks.

DEFINITION ([**1**] p. 15). A function $f(x)$ is called *normalized slowly varying function* if it has the form

$$f(x)=c\exp\left(\int_B^x \varepsilon(t)t^{-1}dt\right), \qquad x \geqq A,$$

where c is a constant $(0<c<\infty)$ and $\varepsilon(t)$ is a measurable function such that $\lim_{t\to\infty}\varepsilon(t)=0$.

PROOF OF PROPOSITION 2.6. Suppose that ν is in $\boldsymbol{D}(2)$. Let $h(x)$ be the slowly varying function part of the normalizing constant B_n in (2.8). We can assume that $h(x)$ is a normalized slowly varying function. The relation between B_n and the truncated variance of ν is that

$$\lim_{n\to\infty} n\, V(B_n)/B_n^2=1\,.$$

This is equivalent to that

$$\lim_{n\to\infty} V(n^{1/2}h(n))/h(n)^2=1\,. \tag{1}$$

By (1), if $h(x)$ is bounded, then ν has a finite variance and $h(x)$ converges to a positive constant, which implies that $h(x)$ is asymptotically equal to a non-decreasing one. Suppose that $h(x)$ is unbounded. Then $h(x)$ is asymptotically equal to a non-decreasing function if and only if $h(x)$ satisfies the following:

$$\lim_{x\to\infty}\max_{t\leq x} h(t)/h(x)=1\,. \tag{2}$$

Set $\bar{h}(x)=\max_{t\leq x}h(t)$. In order to prove (2), we assume $\liminf_{x\to\infty}h(x)/\bar{h}(x)=r<1$ and get a contradiction. There exist sequences x_k and y_k such that $\lim_{k\to\infty}x_k=\infty$, $\lim_{k\to\infty}y_k=\infty$, $\lim_{k\to\infty}h(x_k)/h(y_k)=r$, $\bar{h}(x_k)=h(y_k)$, and $y_k\leqq x_k$. By (1),

$$\lim_{k\to\infty}\{V(y_k^{1/2}h(y_k))/V(x_k^{1/2}h(x_k))\}\{h^2(x_k)/h^2(y_k)\}=1\,. \tag{3}$$

By a property of normalized slowly varying functions ([**1**] p. 24 Theorem 1.5.5), we have $y_k^{1/2}h(y_k)\leqq x_k^{1/2}h(x_k)$ for large k. Therefore

$$V(y_k^{1/2}h(y_k))/V(x_k^{1/2}h(x_k))\leqq 1 \qquad \text{for large } k\,.$$

Hence the left-hand side of (3) is not bigger than r^2, which is a contradiction.

Let us prove the converse. Let $h(x)$ be a non-decreasing slowly varying function and let $B(x)=x^{1/2}h(x)$. We set $V_0(x)=h^2(B^{-1}(x))$, where $B^{-1}(x)$ is an asymptotic inverse of $B(x)$ ([7] p. 23). It is easy to see that $V_0(x)$ is non-decreasing slowly varying. Therefore there exists a probability measure ν on $[0, \infty)$ satisfying $\lim_{x\to\infty} V_0(x)/\int_0^x t^2\nu(dt)=1$. This ν satisfies our condition.

REMARK. A similar proposition is correct in the case of d-dimensional Gaussian distributions. All one-dimensional Gaussian distributions have a common domain of attraction, but this is not true for Gaussian distributions on $\boldsymbol{R}^d$. Thus the "if" part of Proposition 2.6 for $\boldsymbol{R}^d$ should be as follows: if a slowly varying function $h(x)$ is asymptotically equal to a non-decreasing one, then, for any Gaussian distribution μ on $\boldsymbol{R}^d$, there exists a distribution in the domain of attraction of μ with a normalizing constant $B_n=n^{1/2}h(n)$. To prove this assertion, it is enough to construct the direct product of one-dimensional distributions constructed in the above proof, because the case of general Gaussian distributions is reduced by orthogonal transformations to the case of the direct products of one-dimensional Gaussian distributions. The "only if" part of Proposition 2.6 for $\boldsymbol{R}^d$ is a consequence of the case of $\boldsymbol{R}^1$ if we consider marginal distributions.

REMARK. It is well-known that the normalizing constant for a stable distribution with index α is represented as $B_n=n^{1/\alpha}h(n)$, where $h(x)$ is a slowly varying function. We have $\lim_{n\to\infty}(\max_{k\leq n}B_k)/B_n=1$ (see [1] p. 23 on monotone equivalents of regularly varying function). Notice that this fact does not mean monotonicity of the slowly varying function part. In fact, for any non-Gaussian stable distribution, every slowly varying function can appear as the slowly varying function part of the normalizing constant of some distribution in its domain of attraction ([9] and [8]). This is a big difference between Gaussian distributions and non-Gaussian stable distributions.

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