J. Math. Soc. Japan
Vol. 43, No. 3, 1991

On the relative p-capacity

By Toshio HORIUCHI

(Received Oct. 17, 1989)
(Revised Oct. 5, 1990)

1. Introduction.

Let E and F denote compact and open subsets of $\boldsymbol{R}^n$, respectively, $E\subset F$. The number

$$C_p(E, F)=\inf\left\{\int_F |\nabla u|^p dx:\ u\in C_E(F)\right\} \tag{1.1}$$

is called the p-capacity of a compactum E relative to F. Here $p\geqq 1$, $\nabla u=\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}\right)$ and $C_E(F)$ is the class of functions $u(x)\in C^{0,1}(F)$ with $u(x)\geqq 1$ for $x\in E$ and compact support contained in F. For the detailed, see § 2.

The purpose of this paper is to study the p-capacity $C_p(E, F)$ and make clear its behavior as a set function from the point of view of the relativity of E and F. For $p>1$ we shall show that the p-capacity of E relative to F can not remain bounded when E fills up F, or equivalently, when F shrinks away to E, if and only if $C_p(E, F)>0$. In other words, the p-capacity of the whole space F is naturally considered $+\infty$ provided $p>1$. By making use of this fact, we can give simple proofs of metric properties of the p-capacity in terms of Hausdorff measure, most of which are already known but the proofs in this paper seem to be more direct than those based on the non-linear potential theory initiated by V.G. Maz'ja and V.P. Havin [**13**], [**14**], N.G. Meyers [**15**]. This theory has been extensively developed during the last decade to fill the gap to a certain extent between the classical potential theory and non-linear counterparts of Newton and Riesz capacities (See [**4**], [**6**], [**9**] and [**10**]). However our methods in this paper are not based on potential theory but on the effective use of the theory of the Dirichlet problem for non-linear elliptic differential equations and the imbedding theorems of Sobolev type. Roughly speaking, Theorem 3.1 stated in § 3 and the Sobolev imbedding theorem give the upper and lower estimates for the p-capacity respectively. It is interesting that the methods in this paper can be applied to the study of the degenerated elliptic equations as well (See [**11**]).

Here we note that H. Federer and W.P. Ziemer also presented in [**8**] a direct treatment of this topic for $F=\boldsymbol{R}^n$, which was based on geometric measure theory. For the complete references, see the book by V.G. Maz'ja [**12**] (See

This research was partially supported by Grant-in-Aid for Scientific Research (No. 02854007), Ministry of Education, Science and Culture.

also the references in [**14**]). This paper is organized in the following way: In §2 we prepare notations and collect mostly without proofs the basic properties of the p-capacity. In §3 our main results will be stated, and the proof of Corollary 3.1 is also given there. §4 is devoted to preparing the proposition concerned with the classical Dirichlet problem for quasi-linear elliptic equations of second order. Under these preparations we shall establish Theorem 3.1 in §5. The proofs of Theorem 3.2 and Proposition 3.1 will be given in §6 and §7 respectively.

The author wishes to express his gratitude to Prof. M. Hasumi and Prof. N. Shimakura for their kind help and a number of useful conversations.

2. Preliminaries.

In this section we prepare notations to be used throughout the paper and present a very brief introduction to the p-capacity $C_p(E, F)$ defined by (1.1). We begin with recalling some simple properties of $C_p(E, F)$, which are mostly obvious consequences from the definition (cf. [**6**]):

PROPOSITION 2.1. *Let p satisfy $p \geqq 1$.*

(1) *Let E_1 and E_2 be compact sets $\subset F$. The inclusion $E_1 \subset E_2$ implies $C_p(E_1, F) \leqq C_p(E_2, F)$;*

(2) *The Choquet inequality*

$$C_p(E_1 \cap E_2, F) + C_p(E_1 \cup E_2, F) \leqq C_p(E_1, F) + C_p(E_2, F)$$

holds for any compact sets $E_1, E_2 \subset F$;

(3) *In the definition of $C_p(E, F)$, the space $C_E(F)$ can be replaced by the space*

$$D_E(F) = \{u \in C_E(F);\ 0 \leqq u \leqq 1,\ u = 1 \text{ in a neighborhood of } E\}.$$

(4) *For any compact set $E \subset F$ and $\varepsilon > 0$, there exists a bounded open set ω such that $E \subset \bar{\omega} \subset F$, $\partial\omega$ is smooth and*

$$C_p(E, F) \leqq C_p(\bar{\omega}, F) \leqq C_p(E, F) + \varepsilon .$$

It is very useful to know there exist extremal functions where the infimum in the definition of $C_p(E, F)$ is achieved. To this end we denote by $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the space $W^{1,p}(\Omega)$, where the space $W^{1,p}(\Omega)$ is the set of functions on Ω, whose generalized derivatives $\partial^\gamma u$ of order $\leqq 1$ satisfy

$$\|u : W^{1,p}(\Omega)\| = \sum_{|\gamma| \leqq 1} \Big(\int_\Omega |\partial^\gamma u|^p dx\Big)^{1/p} < +\infty . \tag{2.1}$$

Then it follows from the Clarkson inequalities and the Sobolev imbedding theorems that:

PROPOSITION 2.2. *Let p satisfy $p>1$. Assume that F is a bounded open set whose boundary is smooth. Then*

$$(2.2)\quad C_p(E, F)=\inf\left\{\int_F |\nabla u|^p dx;\ u\in W_0^{1,p}(F),\ u\geqq 1 \text{ on } E \text{ quasi-everywhere}\right\}.$$

Here by the term quasi-everywhere we mean that $u\geqq 1$ on E everywhere except possibly on a set of the p-capacity zero.

Moreover if $C_p(E, F)<+\infty$, then there exists $u\in W_0^{1,p}(F)$ such that $0\leqq u\leqq 1$, $u=1$ everywhere on E and

$$\int_F |\nabla u|^p dx = C_p(E, F).$$

This distribution u is called capacitary extremal of E relative to F, and it is essentially unique up to values on sets of vanishing p-capacity. Here we note that the assumptions on F may be avoided by the use of definition of $W_0^{1,p}(F)$ which does not require $u\in L^p(F)$ for $u\in W_0^{1,p}(F)$. For more precise informations, see Chapter 2 in [**16**] for example.

We also provide here the lower estimate for the p-capacity of compact set E assuming $p>1$. By $|E|$ we denote the n-dimensional Lebesgue measure of E. The proof is omitted (See p. 105, Corollary 2 in [**12**]).

PROPOSITION 2.3. *Let E and F be compact and bounded open subsets of $\boldsymbol{R}^n$ respectively, $E\subset F$. Assume that $p>1$. Then the following lower estimates hold:*

$$(2.3)\qquad C_p(E, F)\geqq C(n, p)\,\big|\,|F|^{(p-n)/n(p-1)}-|E|^{(p-n)/n(p-1)}\big|^{1-p},$$

for $p\neq n$ and

$$(2.4)\qquad C_p(E, F)\geqq C(n, p)\left(\log\frac{|F|}{|E|}\right)^{1-n},$$

for $p=n$, where $C(n, p)$ is a positive number depending only on n and p. In particular if $C_p(E, F)=0$ and $1<p\leqq n$, then we have $|E|=0$.

We note that if $|E|=0$, then these estimates become almost trivial. If $p=1$, then it holds that

$$(2.5)\qquad C_1(E, F)=\inf S(\partial g),$$

where by $S(\partial g)$ we denote the surface area of ∂g, and the infimum is taken over all bounded open sets g containing E such that $\bar{g}\subset F$ and ∂g is of class C^∞. For the detailed see §2.3 in [**12**]. In the rest of this section we prepare more notations including the definition of the Hausdorff measure. Let us set, for an arbitrary compact set E of $\boldsymbol{R}^n$,

$$(2.6)\qquad \mathrm{dist}(x, E)=\inf_{y\in E}\|x-y\|,$$

and

$$E_\eta = \{x \in \boldsymbol{R}^n : \mathrm{dist}(x, E) < \eta\}, \tag{2.7}$$

which is a tubular neighborhood of E in $\boldsymbol{R}^n$. If ∂E is smooth, then ∂E_η is also smooth for almost all $\eta>0$ by Sard's lemma. But even if ∂E_η is not smooth, we can always approximate ∂E_η by compact smooth manifolds. Therefore we assume throughout this paper that the family of tubular neighborhoods defined by (2.7) is smooth as well without loss of generality.

Lastly we give the definition of the d-dimensional Hausdorff measure. Let S be a bounded set in $\boldsymbol{R}^n$. Consider various coverings of S by balls B_j of radii r_j we put

$$h_d(S) = v_d \inf \sum r_j^d, \tag{2.8}$$

where v_d is the volume of the unit ball in $\boldsymbol{R}^d$ and the infimum is taken over all such coverings. It is of no importance if B_j are assumed open or closed. If we also assume $r_j \leqq \varepsilon$, we get a corresponding lower bound $H_{d,\varepsilon}(S)$. The limit

$$H_d(S) = \lim_{\varepsilon \to +0} H_{d,\varepsilon}(S) \tag{2.9}$$

clearly exists and is called the d-dimensional Hausdorff measure of S. Here we note that $h_d(S)$, $H_{d,\varepsilon}(S)$ are zero simultaneously.

3. Main results.

In this section we shall state our main results. First we give a theorem on the behavior of the p-capacity as E_η is shrinking away to E, which characterizes in some sense the sets of non-vanishing p-capacity and provides the upper estimates for the p-capacity. Then we state the result on the lower estimate which will be established later, by using the imbedding theorems of Sobolev type.

THEOREM 3.1. *Let p satisfy $p>1$. Let E be a compactum in $\boldsymbol{R}^n$. If $C_p(E, F)>0$ for some open set $F \subset \boldsymbol{R}^n$, then it holds that*

$$\lim_{\eta \to 0} C_p(E, E_\eta) = +\infty. \tag{3.1}$$

Moreover if F is bounded and smooth, then we have

$$C_p(E, F)^p \leqq C_p(E, E_\eta)\Big(\int_{E_\eta \setminus E} |\nabla u|^p dx\Big)^{p-1}, \tag{3.2}$$

for any $\eta \in (0, \mathrm{dist}(E, \partial F))$. Here u is the extremal function where the infimum in the definition of $C_p(E, F)$ is achieved.

Here we note that this inequality (3.2) becomes an equality when E and $\bar{F}$ are concentric closed balls, and that the auxiliary assumption on F is not essential (See Proposition 2.2 and the remark just after it). From this theorem we can easily derive the connection between the p-capacity and the Hausdorff measure. The following is a direct consequence, and the proof will be given in the last of this section.

COROLLARY 3.1. *Let p satisfy $1<p<n$. Let E be a compactum in $\boldsymbol{R}^n$. Assume that $H_{n-p}(E)<+\infty$. Then it holds that*

$$C_p(E, \boldsymbol{R}^n) = 0. \tag{3.3}$$

We also have

THEOREM 3.2. *Let p satisfy $1<p<n$. Let E be a compactum in $\boldsymbol{R}^n$. If $C_p(E, \boldsymbol{R}^n)=0$, then for any open set F containing E we have*

$$C_p(E, F) = 0. \tag{3.4}$$

Here we note that if $p\geqq n$, the assertion fails to hold. In fact $C_p(E, \boldsymbol{R}^n)=0$, for any compactum E. But from Proposition 2.3, $C_p(E, F)$ is away from 0 in general.

Secondly we give a proposition on the sets of vanishing p-capacity which is known but the proof in § 7 seems to be simpler than the ones based on the potential theory (cf. [**13**], [**14**]. See also [8]).

PROPOSITION 3.1. *Let p satisfy $1<p<n$. Assume that $C_p(E, F)=0$ for some open set $F\subset\boldsymbol{R}^n$. Then it holds that*

$$H_{n-p+\varepsilon}(E) = 0 \tag{3.5}$$

for an arbitrary $\varepsilon>0$.

In the rest of this section we shall establish Corollary 3.1 which is rather elementary if we admit Theorem 3.1.

PROOF OF COROLLARY 3.1. Let $\eta>0$. Since $H_{n-p}(E)<+\infty$, we can construct a locally finite open cover of E by balls $B_{r_j}(x_j)$ with radius r_j, center x_j ($j=1, 2, \cdots$) such that

$$B_{2r_j}(x_j)\subset E_\eta, \quad r_j\leqq \frac{\eta}{2} \quad \text{and} \quad \sum_j r_j^{n-p}<H, \tag{3.6}$$

where $H=v_{n-p}^{-1}\max(1, 2H_{n-p}(E))$. Let us choose a sequence of smooth functions φ_j ($j=1, 2, \cdots$) so that

$$\begin{aligned} &0\leqq \varphi_j(x)\leqq 1, \quad \varphi_j(x)=1 \quad \text{on} \quad B_{r_j}(x_j), \\ &\operatorname{supp}\varphi_j\subset B_{2r_j}(x_j) \quad \text{and} \quad |\nabla\varphi_j(x)|\leqq Cr_j^{-1}, \end{aligned} \tag{3.7}$$

where C is a positive number depending only on the dimension of the space. Then we immediately get

$$
\begin{aligned}
C_p(E, E_\eta) &\leqq \int |\nabla \sup_j \varphi_j(x)|^p dx \\
&\leqq \sum_j \int |\nabla \varphi_j|^p dx \leqq C' \sum_j r_j^{n-p} \leqq C'H < +\infty ,
\end{aligned}
\tag{3.8}
$$

where C' is a positive number depending only on the dimension of the space. Thus we know that (3.1) does not hold, hence we have $C_p(E, \boldsymbol{R}^n)=0$.

4. ε-Reguralization.

Throughout this section we assume that F denotes a bounded open subset of $\boldsymbol{R}^n$. We shall explain that the extremal function in Proposition 2.2 in §2 can be approximated by smooth solutions of regularized problems. Since this fact seems to be familiar, one may skip this section at first and return here if necessary when motivated by the use made later on.

For any $\varepsilon>0$, we set

$$
J_\varepsilon(u) = \int_F (|\nabla u|^2 + \varepsilon^2)^{p/2} dx \tag{4.1}
$$

and consider the variational problem

$$
\inf\{J_\varepsilon(u) : u \in C_E(F)\} . \tag{4.2}
$$

By $C_{p,\varepsilon}(E, F)$, we denote the minimal value of this problem. Then obviously we have

$$
C_p(E, F) \leqq C_{p,\varepsilon}(E, F) \leqq C(p)(C_p(E, F) + \varepsilon^p |F|), \tag{4.3}
$$

where $C(p)$ is a positive number depending only on p. This implies $C_p(E, F)$ and $C_{p,\varepsilon}(E, F)$ blow up simultaneously, provided F is bounded. We shall collect basic properties of this regularized p-capacity $C_{p,\varepsilon}(E, F)$ which are useful in this paper. The following is well-known. For the proof see Chapter 2 in [**16**] for instance (See also Proposition 2.3 and the remark just after Proposition 2.2).

LEMMA 4.1. *There exist extremals $u_\varepsilon \in W_0^{1,p}(F)$ for $\varepsilon \geqq 0$ such that*

$$
J_\varepsilon(u_\varepsilon) = C_{p,\varepsilon}(E, F) \qquad \text{for} \quad \varepsilon \geqq 0 , \tag{4.4}
$$

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) = C_p(E, F) , \\
&\lim_{\varepsilon\to 0} u_\varepsilon = u \qquad \text{in} \quad W_0^{1,p}(F) .
\end{aligned}
\tag{4.5}
$$

Now we assume that ∂E and ∂F are smooth manifolds and prepare precise imformations for the solutions of the variational problem (4.2). Let us suppose that u is the solution of this problem. From the assertion (3) in Proposition 2.1 and Proposition 2.2, it follows that $u=1$ on E. Note that

$$J_\varepsilon(u)=\int_{E\setminus F}(|\nabla u|^2+\varepsilon^2)^{p/2}dx+\varepsilon^p|E|\,. \tag{4.6}$$

Since the functional J_ε is smooth, this solution u satisfies the Euler-Lagrange equation in the weak sense:

$$\begin{gathered}-\operatorname{div}((|\nabla u|^2+\varepsilon^2)^{(p-2)/2}\nabla u)=0 \quad \text{in} \quad F\setminus E\,. \\ u=1 \quad \text{on} \quad E,\quad u=0 \quad \text{on} \quad \partial F.\end{gathered} \tag{4.7}$$

Moreover if u is of class C^2, then u is a solution of classical Dirichlet problem in $F\setminus E$ with boundary condition $u=1$ on ∂E, 0 on ∂F. Hence the solvability of (4.2) implies that of (4.7) with Dirichlet boundary condition (at least in the weak sense). On the other hand, the integrand in $J_\varepsilon(u)$ is of class C^2 and strictly convex with respect to $P=\nabla u$. Hence a solution of the problem (4.7) is consequently a solution of the variational problem (4.2). Moreover from the maximum principle for C^2-solution, the solution of (4.7) is unique (See Theorem 11.9, p. 289 in [9]). Therefore in order to determine the value $C_{p,\varepsilon}(E, F)$, it suffices to solve the Dirichlet problem (4.7) in the space $C^2(\overline{F\setminus E})$. To this end we shall prepare a proposition on the unique existence of the classical solution of the quasi-linear elliptic differential equation.

PROPOSITION 4.1. *The Dirichlet problem* (4.7) *is uniquely solvable in the space* $C^{2,\alpha}(\overline{F\setminus E})$ *for some* $\alpha\in(0, 1]$. *Here by* $C^{2,\alpha}(\overline{F\setminus E})$ *we denote an usual Hölder space, and* α *may depend possibly on* ε.

For the proof, see Theorem 15.11, p. 381 in [9] for example.

5. The proof of Theorem 3.1.

We shall show the inequality (3.2), assuming that

$$C_p(E, F)>0, \tag{5.1}$$

for some bounded smooth F.

For any $\eta>0$ and any $\varepsilon>0$, we can choose positive number δ and $\varepsilon(\eta)$ such that

$$\varepsilon(\eta)\leqq\min(\varepsilon, \eta), \tag{5.2}$$

$$E\subset E_\delta\subset E_{\varepsilon(\eta)}\subset E_\eta\,. \tag{5.3}$$

We may clearly assume that ∂E_δ, $\partial E_{\varepsilon(\eta)}$ and ∂E_η are smooth manifolds without loss of generality.

Let U_δ and V_η be solutions of the following Dirichlet problems:

$$(5.4)\qquad \begin{cases} -\operatorname{div}((|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}\nabla U_\delta)=0 \quad \text{in} \quad F\setminus E_\delta, \\ U_\delta|_{E_\delta}=1, \quad U_\delta|_{\partial F}=0, \\ U_\delta \in C^{2,\alpha}(\overline{F\setminus E_\delta})\cap C^{0,1}(\bar{F}), \end{cases}$$

and

$$(5.5)\qquad \begin{cases} -\operatorname{div}((|\nabla V_\eta|^2+\varepsilon^2)^{(p-2)/2}\nabla V_\eta)=0 \quad \text{in} \quad E_\eta\setminus E_{\varepsilon(\eta)}, \\ V_\eta|_{E_{\varepsilon(\eta)}}=1, \quad V_\eta|_{\partial E_\eta}=0, \\ V_\eta \in C^{2,\alpha}(\overline{E_\eta\setminus E_{\varepsilon(\eta)}})\cap C^{0,1}(\overline{E_\eta}), \end{cases}$$

where α depends possibly on the values of ε and δ. Then we have

$$(5.6)\qquad \begin{aligned} J_\varepsilon(U_\delta) &= C_{p,\varepsilon}(E_\delta, F), \\ J_\varepsilon(V_\eta) &= C_{p,\varepsilon}(E_{\varepsilon(\eta)}, E_\eta). \end{aligned}$$

We take a family of Lipschitz functions ψ_ρ for $\rho\in(0, 1/2)$ so that

$$(5.7)\qquad \psi_\rho(x)=\begin{cases} 1 & 0\leqq x\leqq \rho. \\ \dfrac{1-\rho-x}{1-2\rho} & \rho\leqq x\leqq 1-\rho, \\ 0 & 1-\rho\leqq x. \end{cases}$$

Then

$$(5.8)\qquad \begin{aligned} &\int_{E_\eta\setminus E_\delta}\nabla\psi_\rho(V_\eta)\cdot\nabla U_\delta(|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}dx \\ &=\int_{\partial(E_\eta\setminus E_\delta)}\psi_\rho(V_\eta)(|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}\frac{\partial}{\partial\nu}U_\delta dS \\ &=\int_{\partial E_\eta}(|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}\frac{\partial}{\partial\nu}U_\delta dS. \end{aligned}$$

Here we denote by S the $(n-1)$-dimensional Lebesgue measure, and ν is the unit outward normal. Let Ω be an arbitrary open set such that $\partial\Omega$ is smooth and $\Omega\subset F\setminus E_\delta$. Then we have

$$(5.9)\qquad \int_{\partial\Omega}(|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}\frac{\partial}{\partial\nu}U_\delta dS=\int_\Omega \operatorname{div}((|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}\nabla U_\delta)dx.$$

Therefore we have

$$(5.10)\qquad \int_{\partial E_\eta}(|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}\frac{\partial}{\partial\nu}U_\delta dS=\int_{\partial E_\delta}(|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}\frac{\partial}{\partial\nu}U_\delta dS.$$

Here we note that

$$(5.11)\qquad \begin{aligned} &\int_{F\setminus E_\delta}(|\nabla U_\delta|^2+\varepsilon^2)^{p/2}dx \\ &=\int_{\partial E_\delta}(|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}\frac{\partial}{\partial\nu}U_\delta dS+\varepsilon^2\int_{F\setminus E_\delta}(|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}dx. \end{aligned}$$

Combining this with (5.8) and (5.10) we have

$$C_{p,\varepsilon}(E_\delta, F)-\varepsilon^p|E_\delta|-\varepsilon^2\int_{F\setminus E_\delta}(|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}dx \tag{5.12}$$
$$=\int_{E_\eta\setminus E_\delta}\psi'_\rho(V_\eta)\nabla V_\eta\cdot\nabla U_\delta(|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}dx.$$

Now we choose a positive number δ_0 so that

$$C_{p,\varepsilon}(E_\delta, F)\leqq 2C_{p,\varepsilon}(E, F) \qquad \text{for any} \quad \delta\leqq\delta_0 \text{ and } \varepsilon\in[0, 1]. \tag{5.13}$$

Then a family of extremals $\{U_\delta\}_{0<\delta\leq\delta_0}$ is uniformly bounded in $W_0^{1,p}(F)$. Therefore it is weakly compact in $W_0^{1,p}(F)$ at least. Moreover from the Clarkson inequalities it follows that

$$\lim_{\delta\to 0} U_\delta = u_\varepsilon \qquad \text{in} \quad W_0^{1,p}(F), \tag{5.14}$$

where u_ε is the extremal function for $C_{p,\varepsilon}(E, F)$ as the one in Proposition 2.2 in §2. So that we can assume that when δ tends to 0 each component of

$$(|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}\nabla U_\delta$$

also converges to a correponding component of

$$F_\varepsilon=(|\nabla u_\varepsilon|^2+\varepsilon^2)^{(p-2)/2}\nabla u_\varepsilon \qquad \text{in} \quad [L^{p/p-1}(F)]^n \qquad \text{weakly}.$$

Since $\mathrm{supp}\psi_\rho(V_\eta)\subset E_\eta\setminus E_{\varepsilon(\eta)}$, we get as $\delta\to 0$,

$$\lim_{\delta\to 0}\int_{E_\eta\setminus E_\delta}\psi'_\rho(V_\eta)\nabla V_\eta\cdot\nabla U_\delta(|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}dx \tag{5.15}$$
$$=\int_{E_\eta}\psi'_\rho(V_\eta)\nabla V_\eta\cdot F_\varepsilon dx, \qquad \text{for any} \quad \eta\in(0, \mathrm{dist}(E, \partial F)).$$

Since $\lim_{\delta\to 0}J_\varepsilon(U_\delta)=C_{p,\varepsilon}(E, F)$, it follows from Hölder's inequality that

$$\limsup_{\delta\to 0}\int_{F\setminus E_\delta}(|\nabla U_\delta|^2+\varepsilon^2)^{(p-2)/2}dx\leqq\begin{cases}\varepsilon^{p-2}|F\setminus E|, & 1<p\leqq 2,\\ C_{p,\varepsilon}(E, F)^{1-(2/p)}|F\setminus E|^{2/p}, & p\geqq 2.\end{cases} \tag{5.16}$$

Therefore if $1<p\leqq 2$, we have

$$\begin{aligned} C_{p,\varepsilon}&(E, F) \\ &\leqq \varepsilon^p|F|+\int_{E_\eta\setminus E}|\psi'_\rho(V_\eta)||\nabla V_\eta||F_\varepsilon|dx \\ &\leqq \varepsilon^p|F|+\Big(\int_{E_\eta\setminus E}|\psi'_\rho(V_\eta)|^p|\nabla V_\eta|^p dx\Big)^{1/p}\cdot\Big(\int_{E_\eta\setminus E}|F_\varepsilon|^{p/(p-1)}dx\Big)^{(p-1)/p} \\ &\leqq \varepsilon^p|F|+\max|\psi'_\rho|C_{p,\varepsilon}(E_{\varepsilon(\eta)}, E_\eta)^{1/p}\Big(\int_{E_\eta\setminus E}(|\nabla u_\varepsilon|^2+\varepsilon^2)^{p/2}dx\Big)^{(p-1)/p}. \end{aligned} \tag{5.17}$$

If $p\geqq 2$, we can show in a similar way that

$$C_{p,\varepsilon}(E, F) \leqq \varepsilon^p |E| + \varepsilon^2 C_{p,\varepsilon}(E, F)^{1-(2/p)} \cdot |F \setminus E|^{2/p} \\ + \max|\psi'_\rho| C_{p,\varepsilon}(E_{\varepsilon(\eta)}, E_\eta)^{1/p} \Big(\int_{E_\eta \setminus E} (|\nabla u_\varepsilon|^2 + \varepsilon^2)^{p/2} dx \Big)^{(p-1)/p}, \tag{5.18}$$

for any $\eta \in (0, \operatorname{dist}(E, \partial F))$ and $\varepsilon > 0$. From Lemma 4.1 and (5.2) we have

$$\begin{cases} \lim\limits_{\varepsilon \to 0} C_{p,\varepsilon}(E_{\varepsilon(\eta)}, E_\eta) = C_p(E, E_\eta), \\ \lim\limits_{\varepsilon \to 0} u_\varepsilon = u \quad \text{in } W_0^{1,p}(F), \end{cases} \tag{5.19}$$

where u is the capacitary extremal defined in Proposition 2.2. Now letting $\varepsilon \to 0$ in (5.17) and (5.18), we have

$$C_p(E, F) \leqq \max|\psi'_\rho| C_p(E, E_\eta)^{1/p} \Big(\int_{E_\eta \setminus E} |\nabla u|^p dx \Big)^{1-(1/p)}. \tag{5.20}$$

Since ρ is an arbitrary positive number, we have the desired estimate by letting $\rho \to 0$.

6. The proof of Theorem 3.2.

By B_r we denote an open ball with radius r and center origin. Since $C_p(E, \boldsymbol{R}^n) = 0$, there is a sequence of $C_0^1(\boldsymbol{R}^n)$-functions $\{U_j\}_{j=0}^\infty$ such that:

$$\begin{cases} U_j(x) \geqq 1 \quad \text{for } x \in E, \quad j = 1, 2, \cdots, \\ \int_{\boldsymbol{R}^n} |\nabla U_j|^p dx \to 0 \quad \text{as } j \to +\infty. \end{cases} \tag{6.1}$$

Then we can show

LEMMA 6.1. *Let $\eta_0 > 0$. Then we have*

$$\lim_{j \to +\infty} \int_{E_{\eta_0}} |U_j|^p dx = 0. \tag{6.2}$$

THE PROOF OF THEOREM 3.2. Admitting this for a moment we first establish Theorem 3.2, which is rather elementary. Assume $\eta_0 > 0$ and let us set for $\eta_1 = \eta_0/2$,

$$\varphi_1(x) = \begin{cases} 1 & x \in E_{\eta_1}, \\ \dfrac{2}{\eta_0} \operatorname{dist}(x, E_{\eta_0}) & x \in E_{\eta_0} \setminus E_{\eta_1}, \\ 0 & \text{otherwise}. \end{cases} \tag{6.3}$$

Then we immediately get

$$\lim_{j \to +\infty} \int_{E_{\eta_0}} |\nabla \varphi_1|^p |U_j|^p dx = 0 \tag{6.4}$$

Since $\varphi_1 \cdot U_j \in C_E(E_{\eta_0})$, it holds that

$$C_p(E, E_{\eta_0})^{1/p} \leqq \left(\int_{E_{\eta_0}} |\nabla(\varphi_1 \cdot U_j)|^p dx\right)^{1/p} \tag{6.5}$$
$$\leqq \left(\int_{E_{\eta_0}} |\nabla U_j|^p \varphi_1^p dx\right)^{1/p} + \left(\int_{E_{\eta_0}} |\nabla \varphi_1|^p |U_j|^p dx\right)^{1/p}.$$

By letting $j \to +\infty$, we have

$$C_p(E, E_{\eta_0}) = 0. \tag{6.6}$$

This proves the assertion.

THE PROOF OF LEMMA 6.1. From the Sobolev imbedding theorem we have

$$\int_{E_{\eta_0}} |U_j|^p dx \leqq |E_{\eta_0}|^{p/n} \left(\int_{E_{\eta_0}} |U_j|^q dx\right)^{p/q} \tag{6.7}$$
$$\leqq C |E_{\eta_0}|^{p/n} \int_{R^n} |\nabla U_j|^p dx \to 0 \quad \text{as} \quad j \to +\infty.$$

Here we used the following (cf. Proposition 7.2): Assume that $1 \leqq p < n$. Then there exists a positive constant C such that

$$\left(\int_{R^n} |u|^q dx\right)^{1/q} \leqq C \left(\int_{R^n} |\nabla u|^p dx\right)^{1/p}, \quad \text{for} \quad q = \frac{np}{n-p}, \tag{6.8}$$

for any $u \in C^{0,1}(\boldsymbol{R}^n)$ with compact support. Here C is independent of each u.

7. The proof of Proposition 3.1.

We begin with preparing two propositions, one is seen in L. Carleson's famous book [**5**], and the other is due to D.R. Adams [**1**].

PROPOSITION 7.1. *Let d be a positive number $\leqq n$. Then there exists a constant C, only depending on the dimension, such that for every compact set E, there exists a nonnegative measure μ on $\boldsymbol{R}^n$ satisfying*

$$\begin{cases} \mu(B_\rho(x)) \leqq \rho^d & \textit{for every} \quad B_\rho(x), \\ \mu(E) \geqq C h_d(E). \end{cases} \tag{7.1}$$

Here $h_d(E)$ is defined by (2.8) *in* §2.

PROPOSITION 7.2. *Let $1 < p < q < +\infty$ and $p < n$. Let μ be a nonnegative measure on $\boldsymbol{R}^n$. Then the inequality*

$$\left(\int_{R^n} |u|^q d\mu\right)^{1/q} \leqq C \left(\int_{R^n} |\nabla u|^p dx\right)^{1/p}, \tag{7.2}$$

for all $u \in C_0^\infty(\boldsymbol{R}^n)$, holds if and only if

$$K = \sup_{x \in R^n,\, \rho > 0} \frac{\mu(B_\rho(x))}{\rho^{q(n/p-1)}} < +\infty, \tag{7.3}$$

where $B_\rho(x) = \{y \in \boldsymbol{R}^n : |x-y| < \rho\}$. Moreover K is equivalent to the best constant

C *in* (7.2).

THE PROOF OF PROPOSITION 3.1. Without loss of generality we assume that $0<\varepsilon<p$. By Proposition 7.1 there exists a nonnegative measure μ such that

$$\left\{\begin{array}{ll}\mu(B_\rho(x)) \leqq \rho^{n-p+\varepsilon}, & \text{for any } \rho \text{ and } x\in \boldsymbol{R}^n, \\ \mu(E) \geqq Ch_{n-p+\varepsilon}(E), & \text{for some constant } C>0.\end{array}\right. \tag{7.4}$$

On the other hand, it follows from Proposition 7.2 that the inequality

$$\left(\int_{\boldsymbol{R}^n}|u|^q d\mu\right)^{1/q} \leqq C\left(\int_{\boldsymbol{R}^n}|\nabla u|^p dx\right)^{1/p}, \qquad q=p\frac{n-p+\varepsilon}{n-p}>p, \tag{7.5}$$

holds for all $u\in C_0^\infty(\boldsymbol{R}^n)$. Since $C_p(E, F)=0$, for any $\delta>0$ we can find an element $u_\delta\in C^{0,1}(F)$ with compact support contained in F such that

$$u_\delta \geqq 1 \text{ on } E \quad \text{and} \quad \int_{\boldsymbol{R}^n}|u_\delta|^q d\mu \leqq C\delta. \tag{7.6}$$

Then $\mu(E)\leqq C\delta$. Thus we have $\mu(E)=0$, and hence $h_{n-p+\varepsilon}(E)=0$, which implies $H_{n-p+\varepsilon}(E)=0$.

References

[1] D. R. Adams, Traces and potentials arising from translation invariant operators, Ann. Scuola Norm. Sup., Pisa, **25** (1971), 203-217.

[2] D. R. Adams, A trace inequality for generalized potentials, Studia Math., **48** (1973), 139-158.

[3] D. R. Adams, Traces of potentials II, Indiana Univ. Math. J., **22** (1973), 907-918.

[4] D. R. Adams and N. G. Meyers, Thinness and Wiener criteria for non-linear potentials, Indiana Univ. Math. J., **22** (1972), 169-197.

[5] L. Carleson, Selected problems on exceptional sets, Toronto-London-Melbourne: Van Norstrand Co., 1967.

[6] G. Choquet, Theory of capacities, Ann. Inst. Fourier, **5** (1955), 131-295.

[7] H. Federer, Geometric measure theory, Springer, 1969.

[8] H. Federer and W. P. Ziemer, The Lebesgue set of a function whose distribution derivatives are p-th power summable, Indiana Univ. Math. J., **22** (1972), 139-158.

[9] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, second edition, Springer, 1983.

[10] L. I. Hedberg and T. H. Wolff, Thin sets in nonlinear potential theory, Ann. Inst. Fourier, **33** (1983), 161-187.

[11] T. Horiuchi, On the relative p-capacity and its applications to the degenerated elliptic equations, in preparation.

[12] V. G. Maz'ja, Sobolev spaces, Springer, 1985.

[13] V. G. Maz'ja and V. P. Havin, A nonlinear analogue of the Newton potential and metric properties of the (p, l)-capacity, Soviet Math. Dokl., **11** (1970), 1294-1298.

[14] V. G. Maz'ja and V. P. Havin, Nonlinear potential theory, Russian Math. Surveys, **27** (1972), 71-148.

[15] N. G. Meyers, A theory of capacities for potentials of functions in Lebesgue class, Math. Scand., **26** (1970), 255-292.

[16] E. W. Stredulinsky, Weighted inequalities and degenerated elliptic partial differential equations. Lecture Note in Mathematics, **1074** (1984) Springer.

Toshio HORIUCHI
Department of Mathematics
Ibaraki University
Mito, Ibaraki 310
Japan