J. Math. Soc. Japan
Vol. 43, No. 3, 1991

# Differentiable sphere theorem by curvature pinching

Dedicated to Professor Nobuyuki Ikeda on his 60th birthday

By Yoshihiko SUYAMA

(Received July 24, 1989)
(Revised Sept. 11, 1990)

## §0. Introduction.

An important problem in differential geometry is to characterize the global behaviour of a manifold in terms of local invariants. A result in this direction is given by the following theorem: If $M$ is a complete, simply connected riemannian manifold whose curvature tensor is close to the curvature tensor of the standard sphere $S$, then $M$ is diffeomorphic to $S$. This is called the differentiable sphere theorem. In this paper, we prove that 0.681-pinched riemannian manifold is diffeomorphic to the standard sphere.

The proximity of curvature tensors $R$ and $\bar{R}$ of the manifold $M$ and the standard sphere $S$ respectively is measured in terms of sectional curvature: A riemannian manifold whose sectional curvature $K$ satisfies the condition $\delta \leqq K \leqq 1$ is called $\delta$-pinched. For the first time, Gromoll [2], Calabi, and Shikata [**11**] gave some results on the differentiable sphere theorem. Later on, these results were improved: Sugimoto and Shiohama [**12**] found a pinching number $\delta(=0.87)$ independent of the dimension of $M$ such that a complete, simply connected and $\delta$-pinched riemannian manifold $M$ is diffeomorphic to the standard sphere. Im Hof and Ruh [**5**] gave a sequence $\delta_n$ of pinching numbers dependent on $n$ of dimension of $M$: A $\delta_n$-pinched manifold $M$ is not only diffeomorphic to the standard sphere, but the action of the isometry group of $M$ is also equivalent to the standard linear action of a subgroup of $O(n+1, \boldsymbol{R})$ on the sphere. The number $\delta_n$ is decreasing on $n$ and $\lim \delta_n = 0.68$ as $n$ tends to infinity. But, if we take the number $\delta$ independent of dimension of $M$ on Im Hof and Ruh's result, $\delta$ becomes considerably large, i.e., $\delta = 0.98$ for $n > 5$. It is unknown what number is the infimum of $\delta$ in order that a complete, simply connected and $\delta$-pinched riemannian manifold is diffeomorphic to the standard sphere.

Sugimoto and Shiohama's beginning idea was due to Omori [**7**], from which they derived that a complete, simply connected and $\delta$-pinched riemannian manifold $M^n$ is diffeomorphic to the standard sphere $S^n$ if a diffeomorphism $f$ of $S^{n-1}$, which is naturally defined for $\delta$-pinched manifold $M$, is diffeotopic to the identity map of $S^{n-1}$. We shall call this the diffeotopy idea. So the problems in their case were how to construct a diffeotopy, and how to find an explicit estimate for $\delta$ to guarantee such a diffeotopy. On the other hand, the main idea in a series of papers Ruh [8], Grove-Karcher-Ruh [3] and Im Hof-Ruh, was to lead from a connection with small curvature on the stabilized tangent bundle of $M$ to flat connection on this bundle. This first connection with small curture on the bundle was defined with relation to the pinching number $\delta$. We shall call this the flat connection idea. Using the resulting flat connection, they defined a generalized Gauss map $G: M^n \to S^n$, which gave a diffeomorphism. So the problems in this case were how to construct a flat connection from the connection with small curvature, and how to find an explicit estimate for $\delta$ in order that the Gauss map could be a diffeomorphism.

The emphasis of the present article is to combine these independent ideas from our viewpoint to obtain a new pinching constant.

THEOREM 1 (differentiable sphere theorem). *Suppose $\delta=0.681$. Then a complete, simply connected and $\delta$-pinched riemannian manifold is diffeomorphic to the standard sphere.*

Our pinching number 0.681 is almost same as the number $\lim \delta_n = 0.68$ given by Im Hof-Ruh. But their numbers are determined by different equations from each other. We use the diffeotopy idea in proof of the theorem, that is, we find a sufficient condition that the diffeomorphism $f$ of $S^{n-1}$ is diffeotopic to the identity map of $S^{n-1}$. But our diffeotopy is constructed in a quite different way from Sugimoto-Shiohama's. Our main idea is as follows: $f$ is homothetically extended to a diffeomorphism $F$ of $\boldsymbol{R}^n - \{0\}$. Then, the restriction of the differential $dF$ to $S^{n-1}$ becomes a map of $S^{n-1}$ into the space $M(n, R)$ of $n \times n$-matrices. We approximate $dF: S^{n-1} \to M(n, \boldsymbol{R})$ by a map $\alpha: S^{n-1} \to SO(n, \boldsymbol{R})$. For a differentiable map $\alpha: S^{n-1} \to SO(n, \boldsymbol{R})$, we denote by $\alpha_x$ the matrix correspondent to $x \in S^{n-1}$. Then, our diffeotopy is constructed by joining $\alpha_x$ to a constant matrix in $SO(n, \boldsymbol{R})$ for each $x \in S^{n-1}$. In particular, by our diffeotopy theorem below we can choose a neighborhood of the isometry $SO(n, \boldsymbol{R})$ of $S^{n-1}$ which is arcwise connected in the diffeomorphism group of $S^{n-1}$. [cf. Compare Theorem 2 with [**12**] § 5, Theorem.]

To state exactly our diffeotopy theorem, we explain some notations. Let $S^{n-1}$ be the standard sphere with curvature 1. Let $f$ be a diffeomorphism of $S^{n-1}$. We put $F(tx) = tf(x)$ for $t>0$. We define the norm of differential $d\alpha$ of $\alpha$ by

$$\|d\alpha\| = \max\{\|(d_X\alpha.)U\| \mid X \in T_x(S^{n-1}) \text{ and } U \in R^n \text{ with } \|X\| = \|U\| = 1\},$$

where $\|X\|$ denotes the euclidian norm of $X$.

DEFINITION 0.1. We say that $f$ is diffeotopic to the identity map of $S^{n-1}$, if there exists a differentiable map $H: [0, 1]\times S^{n-1}\rightarrow S^{n-1}$ satisfying the following (1) and (2):

(1) $H(1, x)=f(x)$ and $H(0, x)=x$.

(2) The map $H_t=H(t, \cdot)$ is a diffeomorphism of $S^{n-1}$ for each $t$.

DEFINITION 0.2. We say that $\alpha$ is an approximation of $df$ on $S^{n-1}$, if there exist real numbers $C_1$ and $N_1$ and they satisfy the following (1), (2), (3) and (4):

(1) $N_1<1$, (2) $\alpha_x(x)=(d_xF)(x)$ for $x\in S^{n-1}$.

(3) $\|\alpha-dF\|\leqq C_1$. (4) $\|d\alpha\|\leqq N_1$.

DEFINITION 0.3. For the approximation $\alpha$ of $df$, we define a positive function $P(t)$ for $t\in[0, \pi]$: We take $0\leqq t_0\leqq t_1\leqq\pi$ such that

$$\cos\left(\frac{3}{2}N_1(\pi-t_0)\right)=-1 \quad\text{and}\quad \cos\left(\frac{3}{2}N_1(\pi-t_1)\right)=0.$$

Then we put

$$P(t)^2=C_2^2\left[\frac{\sin\left(\frac{N_1}{2}t\right)}{\sin\left(\frac{N_1}{2}\pi\right)}\right]^2+C_3^2\left[\frac{\sin(N_1t)}{\sin(N_1\pi)}\right]^2+2C_2C_3\frac{\sin(N_1t)}{\sin(N_1\pi)}\varphi(t),$$

where $C_2=(N_1-C_1)/2$, $C_3=(N_1+C_1)/2$ and $\varphi(t)$ is given by

$$\varphi(t)=\begin{cases}\dfrac{\sin\left(\frac{N_1}{2}t\right)}{\sin\left(\frac{N_1}{2}\pi\right)} & (0\leqq t\leqq t_0)\\[2ex] -\dfrac{\sin\left(\frac{N_1}{2}t\right)}{\sin\left(\frac{N_1}{2}\pi\right)}\cos\left(\frac{3}{2}N_1(\pi-t)\right) & (t_0\leqq t\leqq t_1)\\[2ex] -\dfrac{t}{\pi}\cos\left(\frac{3}{2}N_1(\pi-t)\right) & (t_1\leqq t\leqq\pi).\end{cases}$$

THEOREM 2 (Diffeotopy theorem). *Let $f$ be a diffemorphism of $S^{n-1}$. Suppose that there exists an approximation $\alpha$ of $df$ such that $P(t)<1$ for $t\in[0, \pi]$. Then, $f$ is diffeotopic to the identity map of $S^{n-1}$.*

In our various procedure of the proof of sphere theorem, the first connection with small curvature on the stabilized tangent bundle $E$ due to Ruh plays an important role: We show that the diffeotopy idea is naturally introduced by using the connection on the bundle. A few estimates for $\alpha$ are obtained by using the connection. We shall construct a diffeotopy in the almost similar way to a construction of flat connection on $E$.

The contents of this paper are as follows:

§1. Diffeotopy theorem.

In this section, we prove the diffeotopy theorem.

§2. Preliminaries and formulation of problem.

In this and succeeding sections, we prove the differentiable sphere theorem. In this section, we define the stabilized tangent bundle $E$ of $M$ and a metric connection $\nabla$, which has a small curvature, on the bundle. We define the diffeomorphism $f$ of $S^{n-1}$ and explain the diffeotopy idea. Furthermore, we obtain a few results that are used later.

§3. Differential of $f$ and its approximation.

In this section, we define a map $\alpha: S^{n-1} \to SO(n, \boldsymbol{R})$ as an approximation of $dF|_{S^{n-1}}: S^{n-1} \to M(n, \boldsymbol{R})$ in two ways: First we define $\alpha$ by using the Levi-Civita connection $D$ of $M$. Second we define it by using local cross-sections $M \to P$, where $P$ is an $O(n+1, \boldsymbol{R})$-principal bundle over $M$ associated to $E$. The first definition of $\alpha$ seems to be natural in a viewpoint of approximation of $dF|_{S^{n-1}}$. So, we can estimate a norm $\|dF-\alpha\|$ on $S^{n-1}$. On the other hand, the second definition is useful to estimate differential of $\alpha$.

The first definition of $\alpha$ was also given by Sugimoto-Shiohama. But, our estimate $\|dF-\alpha\|$ is sharper than it. Furthermore, on the construction of diffeotopy we use the estimate in a quite different way from that of Sugimoto-Shiohama.

§4. Lemma necessary to estimate $\|d\alpha\|$.

In this section, we prepare to estimate the norm $\|d\alpha\|$ on $S^{n-1}$. Namely, for a map $\mathcal{A}: S^{n-1} \to SO(n+1, \boldsymbol{R})$, which is almost equal to $\alpha$, we estimate $\|d\mathcal{A}\|$. This map $\mathcal{A}$ is given in relation to the second definition of $\alpha$ in §3.

§5. Differentiable sphere theorem.

In this section, we first find the condition of $\delta$ in order that $E$ is a trivial bundle. Second, we estimate $\|d\alpha\|$. By this estimate together with the estimate $\|dF-\alpha\|$ in §3, we can obtain the condition of $\delta$ in order that $M$ is diffeomorphic to the standard sphere.

§6. Estimate of holonomy of principal bundle $P$.

Let $\tau=\tau(s)$, $0 \leqq s \leqq a$, be a piecewise differentiable loop in a normal coordinate neighborhood of $M$. In this section, we estimate a distance $\rho(u(0), u(a))$ for a horizontal lift $u(s)$ of $\tau$ in $P$. This estimate was already given by Ruh [8] in somewhat different form.

ACKNOWLEDGEMENT. The author wishes to thank professor T. Ochiai, professor T. Sakai and professor K. Shiohama for their kind advices.

## § 1. Diffeotopy Theorem.

DIFFEOTOPY THEOREM. *Let $f$ be a diffeomorphism of $S^{n-1}$. Suppose that there exists an approximation $\alpha$ of $df$ such that $P(t)<1$ for $t\in[0, \pi]$. Then, $f$ is a diffeotopic to the identity map of $S^{n-1}$.*

(**A**) Let $S^{n-1}$ be the standard sphere with curvature 1. We put $F(tx)=tf(x)$ for $t>0$. Then we have $(dF)_x(x)=f(x)$ for $x\in S^{n-1}$. The approximation $\alpha: S^{n-1}\to SO(n, \boldsymbol{R})$ of $df$ satisfies the following (1), (2) and (3):

$$(1)\quad \alpha_x(x)=f(x). \qquad (2)\quad \|\alpha-dF\|\leqq C_1. \qquad (3)\quad \|d\alpha\|\leqq N_1<1.$$

Then we have

$$(dF)_x X=(d_X\alpha.)x+\alpha_x(X) \qquad \text{for } X\in T_x(S^{n-1})$$

by $F(x)=\alpha_x(x)$. Therefore, we have $C_1\leqq N_1$. We already defined the function $P(t)$ for $t\in[0, \pi]$, with respect to $\alpha$, in the definition 0.3.

Now we start the proof of theorem. We define a norm $\|A\|$ of $A\in so(n, \boldsymbol{R})$ as follows.

$$\|A\|=\max\{\|AU\| \mid U\in \boldsymbol{R}^n \text{ with } \|U\|=1\}.$$

$A\in so(n, \boldsymbol{R})$ is equivalent, by $Ad(SO(n))$, to

$$\bar{A}=\begin{bmatrix} 0 & x^1 & & & \\ -x^1 & 0 & 0 & & \\ & & \ddots & & \\ & 0 & & 0 & x^m \\ & & & -x^m & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & x^1 & & & & \\ -x^1 & 0 & 0 & & & \\ & & \ddots & & & \\ & & & 0 & x^m & \\ & 0 & & -x^m & 0 & \\ & & & & & 0 \end{bmatrix}$$

for $m=n/2$ or $m=(n-1)/2$ respectively. Then we have $\|A\|=\max\{|x^i| \mid i=1, \cdots, m\}$. We denote above $\bar{A}$ by $\bar{A}=\sum x^i e_{2i-1,2i}$ for simplicity.

LEMMA 1. *Let $\alpha: S^{n-1}\to SO(n, \boldsymbol{R})$ be a differentiable map such that $\alpha_{x_0}=B$ for some $x_0\in S^{n-1}$. Suppose $\|d\alpha\|<1$, then the image of $\alpha$ is contained in a normal neighborhood of $B$ in $SO(n, \boldsymbol{R})$, where $SO(n, \boldsymbol{R})$ is equipped with a bi-invariant metric.*

PROOF. First, note that the tangent cut locus of unit $E$ in $SO(n, \boldsymbol{R})$ is given by

$$\cup\, Ad(SO(n))\Big(\sum_i \frac{\pi x^i}{\max|x^j|} e_{2i-1,2i}\Big),$$

where above sum $\cup$ is taken for $\sum x^i e_{2i-1,2i}$ with $\sum(x^i)^2=1$ [cf. **10**]. Second, let $\tau=\tau(t)$ be a geodesic joining $x_0$ to $-x_0$ in $S^{n-1}$. The length of $\tau$ is $\pi$. Thus, if $\|d\alpha\|<1$, then $\alpha_{\tau(t)}$ does not intersect with the cut locus of $B$ for every $t$ by $\|\alpha^{-1}d\alpha\|<1$. Q.E.D.

There exists a differentiable map $A: S^{n-1}\to so(n, \boldsymbol{R})$ such that $\alpha_x=B\exp(\pi A_x)$ by $\|d\alpha\|<1$.

We define a differentiable map $H: [0, \pi]\times S^{n-1}\to S^{n-1}$ as follows:

$$H(t, x)=B\exp(tA_x)x \qquad \text{for } (t, x)\in[0, \pi]\times S^{n-1}.$$

Then we have $H(\pi, x)=\alpha_x(x)=f(x)$ and $H(0, x)=Bx$. Now, we show that $H_t$ is a diffeomorphism of $S^{n-1}$ for each $t$ under the condition $P(t)<1$.

We have

$$dH_t(X)=B\, d_X[\exp tA.]x+B\exp(tA_x)X$$

for $X\in T_x(S^{n-1})$. Thus we have, for a unit vector $X$,

$$\|dH_t(X)\|\geqq 1-\|d_X[\exp tA.]x\|.$$

Therefore, if we have

$$\|d_X[\exp tA.]x\|<1 \qquad \text{for a unit vector } X,$$

then we have $\|dH_t(X)\|>0$. We show the following equation in (B) below:

$$\|d_X[\exp tA.]x\|\leqq P(t) \qquad \text{for } t\in[0, \pi]. \tag{1.1}$$

Furthermore, we can join $B$ to the unit $E$ in $SO(n, \boldsymbol{R})$. Thus, if we can prove the equation (1.1), then we have the diffeotopy theorem.

(**B**) Let $\alpha: S^{n-1}\to SO(n, \boldsymbol{R})$ be a differentiable map such that $\alpha_{x_0}=E$ and $\|d\alpha\|\leqq N_1(<1)$. So we can represent $\alpha_x=\exp(\pi A_x)$ by using a differentiable map $A: S^{n-1}\to so(n, \boldsymbol{R})$. Then we define $\alpha_t: S^{n-1}\to SO(n, \boldsymbol{R})$ for each $t\in[0, \pi]$ by

$$\alpha_{t,x}=\exp(tA_x).$$

The following lemma is a slight generalized form of (1.1).

LEMMA 2. *Let fix* $\boldsymbol{c}\in S^{n-1}$ *and a unit vector* $X\in T_x(S^{n-1})$. *Suppose* $\|(d_X\alpha.)\boldsymbol{c}\|\leqq C_1(\leqq N_1)$. *Then we have*

$$\|(d_X\alpha_t)\boldsymbol{c}\|\leqq P(t).$$

PROOF. The proof is divided into several steps. (a) We assume $n=2m$ for simplicity. We can assume $A_x=\sum y_i\boldsymbol{e}_{2i-1,2i}$ and $\boldsymbol{c}={}^t[c_1, 0, c_2, 0, \cdots, c_m, 0]$. In fact, we have

$$\alpha_x\boldsymbol{c}=\exp(\pi A_x)\boldsymbol{c}=\exp(\pi g^{-1}\bar{A}_xg)\boldsymbol{c}=g^{-1}\exp(\pi\bar{A}_x)g\boldsymbol{c}$$

for $g\in SO(n, \boldsymbol{R})$, and there exists $h\in SO(n, \boldsymbol{R})$ satisfying

$$h^{-1}\exp(\pi\bar{A}_x)h=\exp(\pi\bar{A}_x) \quad\text{and}\quad hg\boldsymbol{c}={}^t[c_1, 0, \cdots, c_m, 0].$$

We have $|y_i|\leqq N_1$ by the lemma 1, and

$$(d_X\alpha_t)=d(L_{\exp tA_x})_E\frac{E-\exp(-ad(tA_x))}{ad(tA_x)}(td_XA)$$

[cf. 4]. We denote $A_x=A$, $d_XA=Z$ and $(d_X\alpha_t)=\overline{Y}_t$ for the brevity. We put

$$Y_t=\frac{E-\exp(-ad(tA))}{ad(tA)}(tZ).$$

$\overline{Y}_t$ is a Jacobi field on $SO(n, \boldsymbol{R})$ along $k(t)=\exp(tA)$. So we have $\overline{Y}''+R(\overline{Y}, A)A=0$, where $\overline{Y}'$ is the covariant derivative of $\overline{Y}$ in the direction $dk/dt$. Then we have

$$\begin{aligned}(1.2)\qquad &(Y'\boldsymbol{c}, Y'\boldsymbol{c})+(Y''\boldsymbol{c}, Y\boldsymbol{c})=(\overline{Y}'\boldsymbol{c}, \overline{Y}'\boldsymbol{c})+(\overline{Y}''\boldsymbol{c}, \overline{Y}\boldsymbol{c})\\ &=(\overline{Y}'\boldsymbol{c}, \overline{Y}'\boldsymbol{c})-(R(\overline{Y}, A)A\boldsymbol{c}, \overline{Y}\boldsymbol{c})\geqq(\|Y\boldsymbol{c}\|')^2-(R(Y, A)A\boldsymbol{c}, Y\boldsymbol{c}).\end{aligned}$$

We denote

$$Y=\begin{bmatrix}u, & w\\ v, & z\end{bmatrix}\in so(n, \boldsymbol{R})$$

for the brevity. This implies $Y_{2i-1,2j-1}=u$, $Y_{2i,2j-1}=v$, $Y_{2i-1,2j}=w$ and $Y_{2i,2j}=z(i\neq j)$ for $Y=(Y_{ij})$. We have

$$(1.3)\qquad R(Y, A)A=-\frac{1}{4}[[Y, A], A]=\frac{1}{4}\left[(y_i^2+y_j^2)\begin{bmatrix}u, & w\\ v, & z\end{bmatrix}+2y_iy_j\begin{bmatrix}-z, & v\\ w, & -u\end{bmatrix}\right].$$

(b) From now on, we assume $y_i\geqq 0$ for simplicity. We divide $Z$ into two components $Z=Z_1+Z_2$: We define

$$Z_1=\frac{1}{2}\begin{bmatrix}a, & -b\\ b, & a\end{bmatrix}\quad\text{and}\quad Z_2=\frac{1}{2}\begin{bmatrix}c, & d\\ d, & -c\end{bmatrix}\qquad\text{for } Z=\begin{bmatrix}\alpha, & \gamma\\ \beta, & \delta\end{bmatrix},$$

where $a=\alpha+\delta$, $b=\beta-\gamma$, $c=\alpha-\delta$ and $d=\beta+\gamma$. Put

$$(Y_1)_t=\frac{E-\exp(-ad(tA))}{ad(tA)}(tZ_1),\qquad (Y_2)_t=\frac{E-\exp(-ad(tA))}{ad(tA)}(tZ_2).$$

Since we have

$$t\,\frac{E-\exp(-ad(tA))}{ad(tA)}=\int_0^t Ad(\exp(-tA))dt,$$

we obtain, if $y_i\neq y_j$ and $y_i+y_j\neq 0$,

$$(1.4)\qquad (Y_1)_t=\frac{1}{y_i-y_j}\sin\left(\frac{y_i-y_j}{2}t\right)\begin{bmatrix}\cos\left(\frac{y_i-y_j}{2}t\right), & -\sin\left(\frac{y_i-y_j}{2}t\right)\\ \sin\left(\frac{y_i-y_j}{2}t\right), & \cos\left(\frac{y_i-y_j}{2}t\right)\end{bmatrix}\begin{bmatrix}a & -b\\ b & a\end{bmatrix},$$

$$(1.5)\qquad (Y_2)_t=\frac{1}{y_i+y_j}\sin\Big(\frac{y_i+y_j}{2}t\Big)\begin{bmatrix}\cos\Big(\frac{y_i+y_j}{2}t\Big), & -\sin\Big(\frac{y_i+y_j}{2}t\Big)\\ \sin\Big(\frac{y_i+y_j}{2}t\Big), & \cos\Big(\frac{y_i+y_j}{2}t\Big)\end{bmatrix}\begin{bmatrix}c & d\\ d & -c\end{bmatrix}.$$

If $y_i=y_j$ in equation (1.4), then we have

$$(1.6)\qquad (Y_1)_t=\frac{1}{2}\begin{bmatrix}t & 0\\ 0 & t\end{bmatrix}\begin{bmatrix}a & -b\\ b & a\end{bmatrix}.$$

If we assume $\|(Y_1)_\pi \boldsymbol{c}\|=\|(Y_2)_\pi \boldsymbol{c}\|=1$, then we have

$$(1.7)\qquad (\|Y_1\boldsymbol{c}\|)'_{t=\pi}=\int_0^\pi\{(Y_1'\boldsymbol{c},\, Y_1'\boldsymbol{c})+(Y_1''\boldsymbol{c},\, Y_1\boldsymbol{c})\}dt$$

$$\geqq\int_0^\pi\Big\{(\|Y_1\boldsymbol{c}\|')^2-\frac{N_1^2}{4}\|Y_1\boldsymbol{c}\|^2\Big\}dt,$$

$$(1.8)\qquad (\|Y_2\boldsymbol{c}\|)'_{t=\pi}\geqq\int_0^\pi\{(\|Y_2\boldsymbol{c}\|')^2-N_1^2\|Y_2\boldsymbol{c}\|^2\}dt,$$

by (1.2), (1.3), (1.4) and (1.5). By (1.6), (1.7) and (1.8), we have

$$(1.9)\qquad \|(Y_1)_\pi\boldsymbol{c}\|\frac{t}{\pi}\leqq\|(Y_1)_t\boldsymbol{c}\|\leqq\|(Y_1)_\pi\boldsymbol{c}\|\frac{\sin\Big(\frac{N_1}{2}t\Big)}{\sin\Big(\frac{N_1}{2}\pi\Big)},$$

$$(1.10)\qquad \|(Y_2)_t\boldsymbol{c}\|\leqq\|(Y_2)_\pi\boldsymbol{c}\|\frac{\sin(N_1t)}{\sin(N_1\pi)}$$

[cf. **5**, Proof of Prop. 4.1]. The right hand side equation of (1.9) is increasing for $t\in[0,\pi]$. And the right hand side equation of (1.10) attains maximum at $t=\pi/(2N_1)$.

Put $\bar{\boldsymbol{c}}={}^t[0, c_1, 0, c_2, \cdots, 0, c_m]$. Then we have

$$\|(Y_1)_\pi\boldsymbol{c}\|=\|(Y_1)_\pi\bar{\boldsymbol{c}}\|,\qquad \|(Y_2)_\pi\boldsymbol{c}\|=\|(Y_2)_\pi\bar{\boldsymbol{c}}\|$$

and

$$((Y_1)_\pi\boldsymbol{c},\,(Y_2)_\pi\boldsymbol{c})=-((Y_1)_\pi\bar{\boldsymbol{c}},\,(Y_2)_\pi\bar{\boldsymbol{c}})$$

by (1.4) and (1.5). From the assumption, we have

$$(1.11)\qquad \begin{cases}\|Y_\pi\boldsymbol{c}\|^2=\|(Y_1)_\pi\boldsymbol{c}\|^2+\|(Y_2)_\pi\boldsymbol{c}\|^2+2((Y_1)_\pi\boldsymbol{c},\,(Y_2)_\pi\boldsymbol{c})\leqq C_1^2,\\ \|Y_\pi\bar{\boldsymbol{c}}\|^2=\|(Y_1)_\pi\bar{\boldsymbol{c}}\|^2+\|(Y_2)_\pi\bar{\boldsymbol{c}}\|^2+2((Y_1)_\pi\bar{\boldsymbol{c}},\,(Y_2)_\pi\bar{\boldsymbol{c}})\leqq N_1^2.\end{cases}$$

(c) We put $U(t)=\|Y_t\boldsymbol{c}\|$, $V(t)=\|(Y_1)_t\boldsymbol{c}\|$ and $W(t)=\|(Y_2)_t\boldsymbol{c}\|$ for simplicity. Then we must consider the case where $U(t)$ is maximal at each $t\in[0,\pi]$. First, we must take $V(\pi)^2+W(\pi)^2$ and $W(\pi)/V(\pi)$ as large as posible by (1.9) and (1.10). Therefore we have

$$(1.12)\qquad V(\pi)^2+W(\pi)^2=\frac{C_1^2+N_1^2}{2},$$

$$(1.13)\qquad ((Y_1)_\pi \boldsymbol{c},\ (Y_2)_\pi \boldsymbol{c})=-V(\pi)W(\pi)=\frac{C_1^2-N_1^2}{4},$$

by (1.11). So we have

$$V(\pi)=\frac{N_1-C_1}{2}\quad\text{and}\quad W(\pi)=\frac{N_1+C_1}{2}.$$

Finally, we consider the inner product $((Y_1)_t\boldsymbol{c}, (Y_2)_t\boldsymbol{c})$. We put $2\theta_{ij}=y_i-y_j$ at (1.4) and $2\eta_{ij}=y_i+y_j$ at (1.5). We study the case where $(0\leqq)\theta_{ij}(\leqq N_1/2)$ and $(0\leqq)\eta_{ij}(\leqq N_1)$ are considered as independent variables. We note that $U(t)$ increases as $W(t)$ becomes larger by $V(t)<W(t)$ (and (1.14) below). So we have $W(t)=W(\pi)\sin(N_1t)/\sin(N_1\pi)$. In this case we have $\eta_{ij}=N_1$ at (1.5). By (1.4), (1.5) and (1.13), we have

$$(1.14)\qquad ((Y_1)_t\boldsymbol{c}, (Y_2)_t\boldsymbol{c})\leqq\begin{cases} V(t)W(t) & \text{if } 0\leqq t\leqq t_0\\ -V(t)W(t)\cos((N_1+\theta)(\pi-t)) & \text{if } t_0\leqq t\leqq\pi,\end{cases}$$

where $\theta=\max|\theta_{ij}|$ and $\cos((N_1+\theta)(\pi-t_0))=-1$. Therefore we have

$$\frac{((Y_1)_t\boldsymbol{c}, (Y_2)_t\boldsymbol{c})}{V(\pi)W(\pi)}=\begin{cases} \dfrac{\sin\left(\frac{N_1}{2}t\right)\sin(N_1t)}{\sin\left(\frac{N_1}{2}\pi\right)\sin(N_1\pi)} & \text{if } 0\leqq t\leqq t_0,\\[2ex] -\dfrac{\sin\left(\frac{N_1}{2}t\right)\sin(N_1t)}{\sin\left(\frac{N_1}{2}\pi\right)\sin(N_1\pi)}\cos\left(\frac{3}{2}N_1(\pi-t)\right) & \text{if } t_0\leqq t\leqq t_1,\\[2ex] -\dfrac{t}{\pi}\dfrac{\sin(N_1t)}{\sin(N_1\pi)}\cos\left(\frac{3}{2}N_1(\pi-t)\right) & \text{if } t_1\leqq t\leqq\pi,\end{cases}$$

where $\cos(3N_1(\pi-t_0)/2)=-1$ and $\cos(3N_1(\pi-t_1)/2)=0$. Thus we have the lemma.

Q.E.D.

## §2. Preliminaries and formulation of problem.

Let $M$ be a complete, simply connected riemannian manifold of dimension $n$ with a riemannian metric $g$. We assume $M$ is $\delta$-pinched, that is, the sectional curvature $K$ satisfies $\delta\leqq K\leqq 1$. In particular, we assume $\delta>1/4$.

**(A)** The stabilized tangent bundle of $M$.

We denote by $E$ the stabilized tangent bundle of $M$, that is, $E=T(M)\oplus 1(M)$, where $T(M)$ and $1(M)$ are tangent bundle and trivial line bundle $M\times\boldsymbol{R}$ respectively. Let $\boldsymbol{e}: M\to E$ be a cross-section defined by $M\ni p\to(0, 1)_p\in T_p(M)\oplus\boldsymbol{R}$. The bundle $E$ has a natural fibre metric $h$ defined by $g$, i.e.,

$$h(X, Y) = g(X, Y), \qquad h(X, \boldsymbol{e}_p) = 0, \qquad h(\boldsymbol{e}_p, \boldsymbol{e}_p) = 1$$

for $X, Y \in T_p(M)$. We define a $h$-metric connection $\nabla$ on $E$ as follows:

$$\nabla_X Y = D_X Y - cg(X, Y)\boldsymbol{e}, \qquad \nabla_X \boldsymbol{e} = cX$$

for $X, Y \in T(M)$, where $c=\sqrt{(1+\delta)/2}$ and $D$ is the Levi-Civita connection on $M$ defined by $g$. The connection $\nabla$ has curvature tensor $R^{\nabla}=R-c^2\bar{R}$, where $R$ is the riemannian curvature tensor on $M$ and $\bar{R}$ is the algebraic expression of the curvature tensor on the unit sphere $S^n(1)$ in terms of the riemannian metric on $M$. In this and succeeding sections, we denote by $S^n(c^2)$ the standard sphere with curvature $c^2$.

We define a norm $\|R^{\nabla}\|$ of $R^{\nabla}$ by

$$\|R^{\nabla}\| = \max\{\|R^{\nabla}(X, Y)Z\| \mid X, Y \text{ and } Z \in T_p(M) \text{ with } \|X\| = \|Y\| = \|Z\| = 1\},$$

where, for a vector $X \in T_p(M)$, we denote by $\|X\|$ the norm of $X$ with respect to $h$. Then we have $\|R^{\nabla}\| \leqq 2(1-\delta)/3$ [cf. **9**].

Let $P$ be a principal bundle over $M$ of $(n+1)$-frames with structure group $O(n+1, \boldsymbol{R})$ associated to $E$, i.e.,

$$P = \{u=(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_{n+1}) \mid \boldsymbol{u}_i \in E_p \quad \text{for } p \in M \text{ with } h(\boldsymbol{u}_i, \boldsymbol{u}_j)=\delta_{ij}\}.$$

Then a connection form $\boldsymbol{\omega}$ and a curvature form $\Omega$ on $P$ are naturally defined by $\nabla$, and they satisfy the structure equation $d\boldsymbol{\omega}=-\boldsymbol{\omega}\wedge\boldsymbol{\omega}+\Omega$.

(**B**) The manifold $M$ is homeomorphic to the standard sphere by the sphere theorem [**1, 6**]. In particular, we use the following properties. Let $q_0$ and $q_1$ be a pair of points with maximal distance $d(q_0, q_1)$ on $M$, where $d$ denotes the distance function induced by the riemannian metric $g$. Put $M_0=\{p \in M \mid d(p, q_0) \leqq d(p, q_1)\}$, $M_1=\{p \in M \mid d(p, q_0) \geqq d(p, q_1)\}$ and $C=\{q \in M \mid d(q, q_0)=d(q, q_1)\}$. Then $C$ is diffeomorphic to the standard sphere $S^{n-1}$ and takes the place of the equator of $S^n$, while $M_0$ and $M_1$ take the place of upper and lower hemisphere respectively.

Let $S_{q_0}(M)$ and $S_{q_1}(M)$ denote unit spheres in the tangent space of points $q_0$ and $q_1$ respectively. The exponential maps $\mathrm{Exp}_{q_0}$ and $\mathrm{Exp}_{q_1}$ with centers at $q_0$ and $q_1$ respectively are bijective maps if restricted to an open ball of radius $\pi$. In particular, there exists the following diffeomorphism $f: S_{q_0}(M) \to S_{q_1}(M)$: $f$ is defined by requiring $\mathrm{Exp}_{q_0}(tx)$ and $\mathrm{Exp}_{q_1}(tf(x))$ to coincide for some $t=t(x)$ satisfying $\pi/2 \leqq t(x) \leqq \pi/(2\sqrt{\delta})$. Note that the point of intersection lies on the "equator" $C$. We denote $q=\mathrm{Exp}_{q_0}(t(x)x) \in C$ by $q(x)$.

(**C**) Cross-section $u^i: M_i \to P|_{M_i}$ $(i=0, 1)$.

We fix a minimal geodesic $\gamma=\gamma(t)$ joining $q_0=\gamma(0)$ to $q_1=\gamma(d(q_0, q_1))$. At first, we identify $T_{q_0}(M)$ with $T_{q_1}(M)$ as follows: Let $\{X_1, \cdots, X_{n-1}, X_n\}$ be an orthonormal basis of $T_{q_0}(M)$. Then we choose $X_n=\dot{\gamma}(0)$ particularly, where $\dot{\gamma}(0)=(d\gamma/dt)(0)$. The orthonormal basis $\{X_1, \cdots, X_{n-1}, X_n\}$ of $T_{q_1}(M)$ is now defined by the parallel translation with respect to $D$ of $\{X_1, \cdots, X_{n-1}, -X_n\}$ $(\subset T_{q_0}(M))$ along $\gamma$. Thus we can see $X_i$ $(i=1, \cdots, n)$ as a vector of both tangent spaces $T_{q_0}(M)$ and $T_{q_1}(M)$. Note $X_n \in T_{q_1}(M)$ is equal to $-\dot{\gamma}(d(q_0, q_1))$. Thus we can see the map $f: S_{q_0}(M) \to S_{q_1}(M)$ as a map $f: S^{n-1}(1) \to S^{n-1}(1)$, where $S^{n-1}(1)$ is the unit sphere in the enclidian space $R^n$ spanned by orthonormal basis $\{X_1, \cdots, X_n\}$.

Now, we define a cross-section $u^0: M_0 \to P|_{M_0}$ as follows: First we choose $u^0(q_0)=(X_1, \cdots, X_n, e_{q_0})$ over the center $q_0$ of $M_0$. Second we define a section $u^0$ on $M_0$ by moving the $(n+1)$-frame $u^0(q_0)$ by parallel translation with respect to $\nabla$ along geodesic from $q_0$ to points in $M_0$. Next, we choose $u^1(q_1)=(X_1, \cdots, X_n, -e_{q_1})$ over the center $q_1$ of $M_1$. Thus we can also define a cross-section $u^1: M_1 \to P|_{M_1}$ analogous to $u^0$.

We exactly write down these cross-sections $u^0$, $u^1$. Let $\tau^i(x)=\tau^i(x, t)$ denote a geodesic issuing from $q_i$ with direction $x$ $(i=0, 1)$. Let $[\tau^i(x)]_t^0 X$ denote a vector at $\tau^i(x, t)$ given by parallel translation of a vector $X$ at $q_i$ with respect to $D$ along $\tau^i(x)$. We denote $[\tau^i(x)]_t^0 X$ by $X$ for simplicity, in case where we might not confuse them. Put $u^i=(\boldsymbol{u}_1^i, \cdots, \boldsymbol{u}_{n+1}^i)$ $(i=0, 1)$, then we have the following:

$$\text{(2.1)} \qquad \begin{cases} (\boldsymbol{u}_i^0)_{\tau^0(x,t)} &= g(x, X_i)(q_0)\{\cos(ct)x+\sin(ct)\boldsymbol{e}\} \\ &\quad +\{X_i-g(x, X_i)(q_0)x\} \qquad \text{for } 1\leqq i\leqq n\,, \\ (\boldsymbol{u}_{n+1}^0)_{\tau^0(x,t)} &= \{\cos(ct)\boldsymbol{e}-\sin(ct)x\}. \end{cases}$$

$$\text{(2.2)} \qquad \begin{cases} (\boldsymbol{u}_i^1)_{\tau^1(x,t)} &= g(x, X_i)(q_1)\{\cos(ct)x+\sin(ct)\boldsymbol{e}\} \\ &\quad +\{X_i-g(x, X_i)(q_1)x\} \qquad \text{for } 1\leqq i\leqq n\,, \\ (\boldsymbol{u}_{n+1}^1)_{\tau^1(x,t)} &= \{-\cos(ct)\boldsymbol{e}-\sin(ct)x\}. \end{cases}$$

(**D**) Into diffeomorphism $F_i: M_i \to S^n(c^2)$ $(i=0, 1)$.

Let $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+1}\}$ be the standard basis of $\boldsymbol{R}^{n+1}$. Let $S^n(c^2) \subset \boldsymbol{R}^{n+1}$. We define a differentiable map $F_0: M_0 \to S^n(c^2)$ by

$$F_0(p)=\frac{1}{c}\langle \boldsymbol{e}, u^0\rangle(p) \qquad \text{for } p\in M_0\,,$$

where $\langle \boldsymbol{e}, u^0\rangle(p) \in \boldsymbol{R}^{n+1}$ denotes the components of $\boldsymbol{e}$ with respect to the frame $u^0$ at $p \in M_0$. In the same way, we also define a differentiable map $F_1: M_1 \to S^n(c^2)$ by

$$F_1(p)=\frac{1}{c}\langle \boldsymbol{e}, u^1\rangle(p) \qquad \text{for } p\in M_1\,.$$

The following lemmas 3 and 4 are easily shown by (2.1), (2.2) and the definition of $f$.

LEMMA 3. *We have the following:*

(1) $F_0(q_0)=\dfrac{1}{c}e_{n+1}=\left(0, \cdots, 0, \dfrac{1}{c}\right)$,

$F_1(q_1)=-\dfrac{1}{c}e_{n+1}=\left(0, \cdots, 0, -\dfrac{1}{c}\right)$.

(2) *For each* $x\in S_{q_i}(M)$, $F_i(\tau^i(x, t))$ *is a geodesic in* $S^n(c^2)$ *issuing from* $F_i(q_i)$.

(3) $(dF_i)_{q_i}: T_{q_i}(M)\to T_{F_i(q_i)}(S^n(c^2))$ *is isometric. In particular,* $(dF_i)_{q_i}(X_j)=e_j$ $(j=1, \cdots, n)$.

LEMMA 4. *Let* $q(x)=\tau^0(x, t(x))=\tau^1(f(x), t(x))\in C$. *Then we have*

$$F_0(q(x)) = \frac{1}{c}\begin{bmatrix}\sin(ct(x))\cdot g(x, X_i)(q_0)\\ \cos(ct(x))\end{bmatrix}, \qquad (1\leqq i\leqq n)$$

$$F_1(q(x)) = \frac{1}{c}\begin{bmatrix}\sin(ct(x))\cdot g(f(x), X_i)(q_1)\\ -\cos(ct(x))\end{bmatrix}. \qquad (1\leqq i\leqq n)$$

We identified the unit sphere $S_{q_0}(M)$ with the unit sphere $S_{q_1}(M)$ in $(C)$. So the diffeomorphism $f: S_{q_0}(M)\to S_{q_1}(M)$ is considered as a mapping $f: S^{n-1}(1)\to S^{n-1}(1)$. We defined in the definition 0.1 that $f$ is diffeotopic to the identity map. When $f$ is diffeotopic to the identity map of $S^{n-1}(1)$, we can construct a diffeomorphism $G: M\to S^n(c^2)$ by deforming $F_0$ and $F_1$ [cf. **12**, § 3].

PROPOSITION 1. *Suppose $f$ is diffeotopic to the identity map. Then $M$ is diffeomorphic to* $S^n(c^2)$.

(**E**) $E|_{M_i}$ and $P|_{M_i}$ as fibre bundles over $F_i(M_i)(\subset S^n(c^2))$.

Let $S^n(c^2)\subset \boldsymbol{R}^{n+1}=\{\sum x^i e_i \mid x^i\in \boldsymbol{R}\}$. We denote by $\bar{g}$ the canonical metric of $\boldsymbol{R}^{n+1}$ (or $S^n(c^2)$). The tangent bundle $\bar{E}$ of $\boldsymbol{R}^{n+1}$ restricted to $S^n(c^2)$ is given by

$$\bar{E} = T(\boldsymbol{R}^{n+1})|_{S^n(c^2)} = T(S^n(c^2))\oplus\nu(S^n(c^2)),$$

where $\nu(S^n(c^2))$ denotes the normal bundle. Let $\bar{P}$ denote a pricipal bundle of $(n+1)$-frames with structure group $O(n+1, \boldsymbol{R})$ associated to $\bar{E}$. The bundle $\bar{P}$ over $S^n(c^2)$ has a global cross-section $\bar{u}=(e_1, \cdots, e_{n+1})$ of $(n+1)$-frame at each point $p\in S^n(c^2)$. We identify respectively $M_i$, $E|_{M_i}$ and $P|_{M_i}$ with $F_i(M_i)$, $\bar{E}|_{F_i(M_i)}$ and $\bar{P}|_{F_i(M_i)}$ as follows:

$$M_i\ni p \longrightarrow F_i(p)\in F_i(M_i) \quad\text{and}\quad P|_{M_i}\ni(u^i)(p)\longrightarrow(\bar{u})(F_i(p))\in\bar{P}|_{F_i(M_i)}.$$

Then, by the definition of $F_i$ in $(D)$ and $E|_{M_i}=\bar{E}|_{F_i(M_i)}$, the cross-section $\boldsymbol{e}: M_i\to P|_{M_i}$ just corresponds to the outer unit normal vector of each point of $M_i$. So, we have $T(M)|_{M_i}=T(S^n(c^2))|_{F_i(M_i)}$. A connection form $\bar{\omega}$ on $P|_{M_i}$, which makes $u^i$ to a parallel field, induces the canonical flat connection $\bar{\nabla}$ on $E|_{M_i}$:

$$\begin{cases} \overline{\nabla}_X Y = \bar{D}_X Y - c\bar{g}(X, Y)e \\ \overline{\nabla}_X e = cX \end{cases} \qquad \textit{for } X, Y \in T(M),$$

where $\bar{D}$ is the canonical connection of $S^n(c^2)$. In particular, we have the following lemma by the above argument and the lemma 3.

LEMMA 5. *Let $\tau^i(x)=\tau^i(x, t)$ be a geodesic issuing from $q_i$ with direction $x$. Then, for a vector $Z\in T_{q_i}(M)$ with $g(Z, x)(q_i)=0$, two vectors given by both parallel translations of $\boldsymbol{Z}$ with respect to $\nabla(=D)$ and $\overline{\nabla}(=\bar{D})$ along $\tau^i(x)$ coincide at each point $\tau^i(x, t)$.*

## §3. Differential of $f$ and its approximation.

The purpose of this section is to study differential of the diffeomorphism $f$ of $S^{n-1}(1)$, where we identify $S_{q_0}(M)=S_{q_1}(M)=S^{n-1}(1)$ as in §2. We homothetically extend $f$ to a diffeomorphism $F$ of $\boldsymbol{R}^n-\{0\}(\supset S^{n-1}(1))$ so that $F(tx)=tf(x)$ for $x\in S^{n-1}(1)$ and $t>0$. Then the differential $(dF)_x$ at $x\in S^{n-1}(1)$ belongs to the space $M(n, \boldsymbol{R})$ of $n\times n$-matrices. In particular, we have $(dF)_x(x)=f(x)$ for $x\in S^{n-1}(1)$. In this viewpoint, we approximate $dF|_{S^{n-1}(1)}: S^{n-1}(1)\to M(n, \boldsymbol{R})$ by $\alpha: S^{n-1}(1)\to SO(n, \boldsymbol{R})$.

Through this section, we denote $x\in S^{n-1}(1)$, $q=q(x)=\tau^0(x, t(x))=\tau^1(f(x), t(x))\in C$, and $V\in T_q(C)$. Let $\dot{\tau}^i(x, t)=d\tau^i(x, t)/dt$. But we often denote $\dot{\tau}^i(x, t)$ by $\dot{\tau}^i(x)$ for short when we do not specialize $t$. Let $V^0$ and $V^1$ be Jacobi fields along the geodesics $\tau^0(x)$ and $\tau^1(f(x))$ respectively, satisfying $(V^0)_q=(V^1)_q=V$ and $(V^0)_{q_0}=(V^1)_{q_1}=0$. Then we denote by $W^0$ and $W^1$ Jacobi fields along $\tau^0(x)$ and $\tau^1(f(x))$ orthogonal to $\dot{\tau}^0(x)$ and $\dot{\tau}^1(f(x))$ respectively: $W^0=V^0-g(V^0, \dot{\tau}^0(x))\dot{\tau}^0(x)$ and $W^1=V^1-g(V^1, \dot{\tau}^1(f(x)))\dot{\tau}^1(f(x))$.

(**A**) Differential of $f$.

By the definition of $f$, we have

$$(df)_x(D_x W^0) = D_{f(x)} W^1 . \tag{3.1}$$

The estimate for the ratio $\|(df)X\|:\|X\|$ for $X\in T(S^{n-1}(1))$ is given by

$$\left[\sqrt{\delta}\sin\left(\frac{\pi}{2\sqrt{\delta}}\right)\right]^{-1} \geqq \frac{\|(df)X\|}{\|X\|} \geqq \sqrt{\delta}\sin\left(\frac{\pi}{2\sqrt{\delta}}\right). \tag{3.2}$$

The estimate follows from the Rauch comparison theorem [8].

(**B**) Approximation of $df$.

We define a map $\alpha: S^{n-1}(1)\to M(n, \boldsymbol{R})$ as follows:

$$\begin{cases} (1) \quad \alpha_x([\tau^0(x)]_0^{t(x)} W_q^0) = [\tau^1(f(x))]_0^{t(x)} W_q^1 \qquad \text{for } V\in T_q(C), \\ (2) \quad \alpha_x(x) = f(x), \end{cases}$$

where $[\tau^0(x)]_0^{t(x)}W_q^0$, $[\tau^1(f(x))]_0^{t(x)}W_q^1$, $x$ and $f(x)$ are the component vectors with respect to the basis $\{X_1, \cdots, X_n\}$.

PROPOSITION 2. *We have, for* $x\in S^{n-1}(1)$,

(1) $\alpha_x\in SO(n, \boldsymbol{R})$ *and* $\alpha_x(x)=f(x)$,

(2) $$\|(dF-\alpha)_x\| \leqq \frac{1-\delta}{1+c^2}\left\{\frac{c(e^{\pi/2\sqrt{\delta}}-e^{-\pi/2\sqrt{\delta}})}{2\sin\left(\frac{c\pi}{2\sqrt{\delta}}\right)}-1\right\}\left(\frac{1+\sqrt{\delta}\sin\left(\frac{\pi}{2\sqrt{\delta}}\right)}{2\sqrt{\delta}\sin\left(\frac{\pi}{2\sqrt{\delta}}\right)}\right).$$

PROOF. (1) First note, for $V\in T_q(C)$,

$$g(W^0, [\tau^0(x)]_{t(x)}^0 x) = g(W^1, [\tau^1(f(x))]_{t(x)}^0 f(x)) = 0\,.$$

Let $L$ be a subspace in $T_q(M)$ spanned by vectors $[\tau^0(x)]_{t(x)}^0 x$ and $[\tau^1(f(x))]_{t(x)}^0 f(x)$, and $L^\perp$ a subspace in $T_q(M)$ orthogonal to $L$ with respect to $g$. Second, note $(W^0)_q=(W^1)_q=V$ for $V\in T_q(M)\cap L^\perp$. Therefore, we only show $\|W^0\|=\|W^1\|$ for $V\in T_q(C)\cap L$. If this equation holds, we have $\alpha_x\in O(n, \boldsymbol{R})$. So, let $V\in T_q(C)\cap L$. We take a curve $x(s)$, $-\varepsilon<s<\varepsilon$, in $C$ with $\dot{x}(0)=V$. Since $d(q_0, x(s))=d(q_1, x(s))$, we have

$$g(\dot{\tau}^0(x, t(x)), V) = \frac{d}{ds}d(q_0, x(s))|_{s=0}$$

$$= \frac{d}{ds}d(q_1, x(s))|_{s=0} = g(\dot{\tau}^1(f(x), t(x)), V)$$

by the first variation formula of geodesic. Thus we have $\|W^0\|=\|W^1\|$ for $V\in T_q(C)\cap L$. In particular, we have

$$T_q(C)\cap L = \{y([\tau^0(x)]_{t(x)}^0 x+[\tau^1(f(x))]_{t(x)}^0 f(x)) \mid y\in\boldsymbol{R}\}. \tag{3.3}$$

Finally, since $\alpha_x$ is continuous for $x\in S^{n-1}(1)$ and $\alpha_{X_n}=E$ by the identification of $T_{q_0}(M)$ with $T_{q_1}(M)$, we have $\alpha_x\in SO(n, \boldsymbol{R})$ for each $x\in S^{n-1}(1)$.

(2) In this proof, we use the identifications of $M_0$, $M_1$, $T(M)|_{M_0}$ and $T(M)|_{M_1}$ with $F_0(M_0)$, $F_1(M_1)$, $T(S^n(c^2))|_{F_0(M_0)}$ and $T(S^n(c^2))|_{F_1(M_1)}$ respectively, that were given in § 2(E). The proof is divided into several steps.

(a) Let $V\in T_q(C)$. We put $V^\perp=V-g(V, \dot{\tau}^0(x, t(x)))\dot{\tau}^0(x, t(x))$ $(\in T_q(M))$. $W^0$ and $\overline{W}^0$ are the following Jacobi fields along the geodesic $\tau^0(x)$:

$$\begin{cases} D_{\dot{\tau}^0(x)}^2 W^0+R(W^0, \dot{\tau}^0(x))\dot{\tau}^0(x) = 0 \\ (W^0)_q = V^\perp, \qquad (W^0)_{q_0} = 0\,. \end{cases} \tag{3.4}$$

$$\begin{cases} \overline{D}_{\dot{\tau}^0(x)}^2 \overline{W}^0+c^2\overline{R}(\overline{W}^0, \dot{\tau}^0(x))\dot{\tau}^0(x) = 0 \\ (\overline{W}^0)_q = V^\perp, \qquad (\overline{W}^0)_{q_0} = 0\,, \end{cases} \tag{3.5}$$

where $\overline{R}$ is the curvature tensor of $S^{n-1}(1)$. For simplicity, we denote

$$[\tau^0(x)]_0^t W^0 = (W^0)_t\,, \qquad [\tau^0(x)]_0^t R[\tau^0(x)]_t^0 = R_t\,,$$
$$[\tau^0(x)]_0^t \overline{W}^0 = (\overline{W}^0)_t\,, \qquad [\tau^0(x)]_0^t \bar{R}[\tau^0(x)]_t^0 = \bar{R}_t\,.$$

Then equations (3.4) and (3.5) change into the following equations on $T_{q_0}(M)$:

$$(3.4)' \qquad \begin{cases} \dfrac{d^2}{dt^2}W^0+R(W^0,\, x)x = 0 \\ (W^0)_{t(x)} = [\tau^0(x)]_0^{t(x)}V^\perp\,, \qquad (W^0)_0 = 0\,, \end{cases}$$

$$(3.5)' \qquad \begin{cases} \dfrac{d^2}{dt^2}\overline{W}^0+c^2\bar{R}(\overline{W}^0,\, x)x = 0 \\ (\overline{W}^0)_{t(x)} = [\tau^0(x)]_0^{t(x)}V^\perp\,, \qquad (\overline{W}^0)_0 = 0\,. \end{cases}$$

Then, we have that the norm $\|d(W^0-\overline{W}^0)/dt\|_{t=0}$ for solutions of (3.4)′ and (3.5)′ is equal to $\|D_xW^0-\bar{D}_x\overline{W}^0\|$ for solutions (3.4) and (3.5) by the lemma 5. We estimate $\|d(W^0-\overline{W}^0)/dt\|_{t=0}$ in (b) and (c) below.

(b) We consider another Jacobi equation as follows:

$$(3.6) \qquad \begin{cases} \dfrac{d^2}{dt^2}\overline{\overline{W}}^0+c^2\bar{R}(\overline{\overline{W}}^0,\, x)x = 0 \\ \dfrac{d}{dt}\overline{\overline{W}}^0|_{t=0} = \dfrac{d}{dt}W^0|_{t=0}\,, \qquad (\overline{\overline{W}}^0)_0 = 0\,, \end{cases}$$

where $W^0$ is the solution of (3.4)′. Then we have

$$\left\|\frac{d}{dt}(W^0-\overline{W}^0)\right\|_{t=0} = \frac{c}{\sin(ct(x))}\|W^0-\overline{\overline{W}}^0\|_{t=t(x)}\,,$$

where $\overline{W}^0$ is the solution of (3.5)′.

Proof of (b). Since $\overline{W}^0$ and $\overline{\overline{W}}^0$ are Jacobi fields on $S^n(c^2)$, we have

$$(3.7) \qquad \overline{W}^0_{t(x)} = \frac{1}{c}\sin(ct(x))\frac{d}{dt}\overline{W}^0|_{t=0}\,, \qquad \overline{\overline{W}}^0_{t(x)} = \frac{1}{c}\sin(ct(x))\frac{d}{dt}\overline{\overline{W}}^0|_{t=0}\,.$$

Thus we have

$$\frac{d}{dt}(\overline{W}^0-W^0)|_{t=0} = \frac{d}{dt}(\overline{W}^0-\overline{\overline{W}}^0)|_{t=0}$$
$$= \frac{c}{\sin(ct(x))}(\overline{W}^0-\overline{\overline{W}}^0)_{t=t(x)} = \frac{c}{\sin(ct(x))}(W^0-\overline{\overline{W}}^0)_{t=t(x)}\,.$$

(c) We consider the following Jacobi equations:

$$(3.8) \qquad \frac{d^2}{dt^2}W^0+R(W^0,\, x)x = 0, \qquad (W^0)_0=0,$$

$$(3.9) \qquad \frac{d^2}{dt^2}\overline{\overline{W}}^0+c^2\bar{R}(\overline{\overline{W}}^0,\, x)x = 0, \qquad (\overline{\overline{W}}^0)_0 = 0,$$

under the condition $(dW^0/dt)_{t=0}=(d\overline{\overline{W}}^0/dt)_{t=0}$ and $\|dW^0/dt\|_{t=0}=1$. Then we have

$$(3.10) \qquad \|W^0-\overline{\overline{W}}{}^0\|_{t(x)} \leqq \frac{1}{2}\frac{1-\delta}{1+c^2}\left\{\frac{e^{t(x)}-e^{-t(x)}}{2}-\frac{1}{c}\sin(ct(x))\right\}.$$

PROOF OF (c). Integrating (3.8) and (3.9) with respect to $t$, we have

$$(W^0-\overline{\overline{W}}{}^0)_t+\int_0^t ds\int_0^s R_u(W^0-\overline{\overline{W}}{}^0,\ x)x du$$

$$+\int_0^t ds\int_0^s (R-c^2\bar{R})_u(\overline{\overline{W}}{}^0,\ x)x du=0\,.$$

Thus we have

$$(3.11) \qquad \|W^0-\overline{\overline{W}}{}^0\|_t \leqq \int_0^t ds\int_0^s \|W^0-\overline{\overline{W}}{}^0\|_u du+(1-c^2)\int_0^t ds\int_0^s \|\overline{\overline{W}}{}^0\|_u du\,.$$

From $\|\overline{\overline{W}}\|_u=(1/c)\sin(cu)$ and $c^2=(1+\delta)/2$ in (3.11), we have

$$(3.12) \qquad \|W^0-\overline{\overline{W}}{}^0\|_t \leqq \frac{1-\delta}{2}\left(\frac{t}{c^2}-\frac{1}{c^3}\sin(ct)\right)+\int_0^t ds\int_0^s \|W^0-\overline{\overline{W}}{}^0\|_u du\,.$$

Thus we have the statement (c) by applying ordinary iteration method to (3.12): We have

$$\|W-\overline{\overline{W}}\|_t \leqq \frac{1-\delta}{2}\left\{\frac{t^3}{3!}+\frac{t^5}{5!}(-c^2+1)+\frac{t^7}{7!}(c^4-c^2+1)+\cdots\right\}$$

$$=\frac{1-\delta}{2}\sum_{n=2}^{\infty}\frac{t^{2n-1}}{(2n-1)!}\left(\sum_{i=0}^{n-2}(-c^2)^i\right)=\frac{1-\delta}{2}\sum_{n=2}^{\infty}\frac{t^{2n-1}}{(2n-1)!}\frac{1-(-c^2)^{n-1}}{1+c^2}$$

$$=\frac{1}{2}\frac{1-\delta}{1+c^2}\left\{\sum_{n=2}^{\infty}\frac{t^{2n-1}}{(2n-1)!}-\frac{1}{c}\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(ct)^{2n-1}}{(2n-1)!}\right\}.$$

(d) Now, we prove (2). In the equations (3.4)′, (3.5)′ and (3.6), we respectively replace $\tau^0(x)$, $W^0$, $\overline{W}{}^0$ and $\overline{\overline{W}}{}^0$ by $\tau^1(f(x))$, $W^1$, $\overline{W}{}^1$ and $\overline{\overline{W}}{}^1$. Furthermore, in their equations, we choose

$$\begin{cases} W^0_{t(x)}=[\tau^0(x)]_0^{t(x)}[V-g(V,\ \dot\tau^0(x,\ t(x)))\dot\tau^0(x,\ t(x))] \\ W^1_{t(x)}=[\tau^1(f(x))]_0^{t(x)}[V-g(V,\ \dot\tau^1(f(x),t(x)))\dot\tau^1(f(x),t(x))] \end{cases}$$

for $V\in T_q(C)$. Then we have the following equation by (b) and (c):

$$(3.13) \qquad \left\|\frac{d}{dt}(W^0-\overline{\overline{W}}{}^0)\right\|_{t=0}=\frac{c}{\sin(ct(x))}\|W^0-\overline{\overline{W}}{}^0\|_{t=t(x)}$$

$$\leqq \frac{1}{2}\frac{1-\delta}{1+c^2}\left\{\frac{c}{2}\frac{e^{t(x)}-e^{-t(x)}}{\sin(ct(x))}-1\right\}\left\|\frac{dW^0}{dt}\right\|_{t=0}.$$

We also have the following equation by (3.2), (b) and (c):

$$(3.14) \qquad \left\|\frac{d}{dt}(W^1-\overline{\overline{W}}{}^1)\right\|_{t=0} \leqq \frac{1}{2}\frac{1-\delta}{1+c^2}\left\{\frac{c}{2}\frac{e^{t(x)}-e^{-t(x)}}{\sin(ct(x))}-1\right\}\frac{1}{\sqrt{\delta}\sin\frac{\pi}{2\sqrt{\delta}}}\left\|\frac{dW^0}{dt}\right\|_{t=0}.$$

On the other hand, we have

$$(3.15)\qquad \begin{aligned}(dF)_x(D_xW^0)-\alpha_x(D_xW^0) &= D_{f(x)}W^1-\alpha_x[\bar{D}_x\overline{W}^0+(D_xW^0-\bar{D}_x\overline{W}^0)]\\ &= D_{f(x)}W^1-\bar{D}_{f(x)}\overline{W}^1-\alpha_x(D_xW^0-\bar{D}_x\overline{W}^0).\end{aligned}$$

Thus, we have the assertion (2) by (a), (3.13), (3.14) and (3.15). Q.E.D.

(**C**) Another interpretation of $\alpha$.

Let $u^0: M_0\to P|_{M_0}$ and $u^1: M_1\to P|_{M_1}$ be the cross-sections that were defined in §2 (C). There exists a map $\mathcal{A}: C=M_0\cap M_1\to O(n+1, \boldsymbol{R})$ such that $u^0(q)\mathcal{A}(q)=u^1(q)$ for $q\in C$. The purpose of this section is to show that $\alpha_x$ is almost equal to $\mathcal{A}(q)$ for $q=q(x)$ in a sense. Note

$$\mathcal{A}(q)({}^t[z_1^1, z_1^2, \cdots, z_1^{n+1}])={}^t[z_0^1, z_0^2, \cdots, z_0^{n+1}]$$

for $Z=\sum_{i=1}^{n+1}z_0^i\boldsymbol{u}_i^0(q)=\sum_{i=1}^{n+1}z_1^i\boldsymbol{u}_i^1(q)\in E_{\pi^{-1}(q)}$ $(q\in C)$.

The following lemma is shown by using the exact forms of $u^0$ and $u^1$ in §2 (C).

LEMMA 6. *Let $q=q(x)=\tau^0(x, t(x))=\tau^1(f(x), t(x))\in C$. We represent $Z\in T_q(M)$ as*

$$\begin{aligned}Z &= \sum_{i=1}^{n}z_0^i[\tau^0(x)]_{t(x)}^0X_i=\sum_{i=1}^{n+1}\bar{z}_0^i\boldsymbol{u}_i^0(q)\\ &= \sum_{i=1}^{n}z_1^i[\tau^1(f(x))]_{t(x)}^0X_i=\sum_{i=1}^{n+1}\bar{z}_1^i\boldsymbol{u}_i^1(q).\end{aligned}$$

*Then we have*

$$(1)\quad\begin{cases}\text{(a)}\ \ \bar{z}_0^i=z_0^i\ \ (1\leqq i\leqq n), & \bar{z}_0^{n+1}=0\\ & \textit{if } g(Z, [\tau^0(x)]_{t(x)}^0x)=0.\\ \text{(b)}\ \ \bar{z}_0^i=\cos(ct(x))z_0^i\ \ (1\leqq i\leqq n), & \bar{z}_0^{n+1}=-\sin(ct(x))\\ & \textit{if } Z=[\tau^0(x)]_{t(x)}^0x.\end{cases}$$

$$(2)\quad\begin{cases}\text{(a)}\ \ \bar{z}_1^i=z_1^i\ \ (1\leqq i\leqq n), & \bar{z}_1^{n+1}=0\\ & \textit{if } g(Z, [\tau^1(f(x))]_{t(x)}^0f(x))=0.\\ \text{(b)}\ \ \bar{z}_1^i=\cos(ct(x))z_1^i\ \ (1\leqq i\leqq n), & \bar{z}_1^{n+1}=\sin(ct(x))\\ & \textit{if } Z=[\tau^1(f(x))]_{t(x)}^0f(x).\end{cases}$$

PROPOSITION 3. *Let $q=q(x)=\tau^0(x, t(x))=\tau^1(f(x), t(x))\in C$. We denote by $Z\in E_{\pi^{-1}(q)}$ the component vector of $Z$ with the basis $\{\boldsymbol{u}_1^1, \cdots, \boldsymbol{u}_1^{n+1}\}$ of $E_{\pi^{-1}(q)}$. Let put $\bar{\alpha}_x=\begin{bmatrix}\alpha_x, & 0\\ 0, & -1\end{bmatrix}\in O(n+1, \boldsymbol{R})$. Then we can consider $\bar{\alpha}_x\mathcal{A}(q)$ is a linear translation of $E_{\pi^{-1}(q)}$ and we have*

$$\bar{\alpha}_x\mathcal{A}(q)|_{w^\perp}=\textit{Identity and }\ \bar{\alpha}_x\mathcal{A}(q)\boldsymbol{w}=-\boldsymbol{w},$$

*where* $\boldsymbol{w}=\boldsymbol{w}(x)=[\tau^0(x)]^0_{l(x)}x-[\tau^1(f(x))]^0_{l(x)}f(x)$ $(\in E_{\pi^{-1}(q)})$ *and* $\boldsymbol{w}^{\perp}=\{\boldsymbol{u}\in E_{\pi^{-1}(q)}|$ $h(\boldsymbol{u}, \boldsymbol{w})(q)=0\}$.

PROOF. The proof is divided into several cases.

(a) We take $Z\in T_q(M)$ satisfying $g(Z, [\tau^0(x)]^0_{l(x)}x)=g(Z, [\tau^1(f(x))]^0_{l(x)}f(x))$ $=0$. Then $Z$ is represented as

$$Z=\sum_{i=1}^n \bar{z}_0^i \boldsymbol{u}_i^0(q)=\sum_{i=1}^n \bar{z}_0^i[\tau^0(x)]^0_{l(x)}X_i$$

$$=\sum_{i=1}^n \bar{z}_1^i \boldsymbol{u}_i^1(q)=\sum_{i=1}^n \bar{z}_1^i[\tau^1(f(x))]^0_{l(x)}X_i$$

by the lemma 6. By the definitions of $\mathcal{A}(q)$ and $\alpha_x$, we have

$$\mathcal{A}(q)({}^t[\bar{z}_1^1, \cdots, \bar{z}_1^n, 0])={}^t[\bar{z}_0^1, \cdots, \bar{z}_0^n, 0],$$

$$\bar{\alpha}_x({}^t[\bar{z}_0^1, \cdots, \bar{z}_0^n, 0])={}^t[\bar{z}_1^1, \cdots, \bar{z}_1^n, 0].$$

Thus $\bar{\alpha}_x\mathcal{A}(q)$ maps $(\bar{z}_1^1, \cdots, \bar{z}_1^n, 0)$ on itself.

(b) We put

$$x=\sum_{i=1}^n x^i X_i, \qquad f(x)=\sum_{i=1}^n f^i(x)X_i.$$

Then we have

$$\boldsymbol{e}_q=\sum_{i=1}^n x^i \sin(ct(x))\boldsymbol{u}_i^0(q)+\cos(ct(x))\boldsymbol{u}_{n+1}^0(q)$$

$$=\sum_{i=1}^n f^i(x)\sin(ct(x))\boldsymbol{u}_i^1(q)-\cos(ct(x))\boldsymbol{u}_{n+1}^1(q),$$

by (2.1) and (2.2). So, we have

$$\bar{\alpha}_x\mathcal{A}(q)({}^t[f^1(x)\sin(ct(x)), \cdots, f^n(x)\sin(ct(x)), -\cos(ct(x))])$$
$$={}^t[f^1(x)\sin(ct(x)), \cdots, f^n(x)\sin(ct(x)), -\cos(ct(x))].$$

(c) We put

$$\boldsymbol{v}=\frac{1}{2}\{[\tau^0(x)]^0_{l(x)}x+[\tau^1(f(x))]^0_{l(x)}f(x)\},$$

$$\boldsymbol{w}=\frac{1}{2}\{[\tau^0(x)]^0_{l(x)}x-[\tau^1(f(x))]^0_{l(x)}f(x)\}.$$

We have $[\tau^0(x)]^0_{l(x)}x=\boldsymbol{v}+\boldsymbol{w}$, $[\tau^1(f(x))]^0_{l(x)}f(x)=\boldsymbol{v}-\boldsymbol{w}$ and $\boldsymbol{v}\in T_q(C)$ by (3.3). Furthermore we have

$$(\boldsymbol{v}^0=)\quad \boldsymbol{v}-g(\boldsymbol{v}, [\tau^0(x)]^0_{l(x)}x)[\tau^0(x)]^0_{l(x)}x$$
$$=\frac{1}{2}\{(1-p)\boldsymbol{v}-(1+p)\boldsymbol{w}\},$$

$$(\boldsymbol{v}^1=)\quad \boldsymbol{v}-g(\boldsymbol{v},[\tau^1(f(x))]^0_{t(x)}f(x))[\tau^1(f(x))]^0_{t(x)}f(x)$$
$$=\frac{1}{2}\{(1-p)\boldsymbol{v}+(1+p)\boldsymbol{w}\},$$

where $p=\|\boldsymbol{v}\|^2-\|\boldsymbol{w}\|^2$. Since we have

$$\alpha_x([\tau^0(x)]^{t(x)}_0\boldsymbol{v}^0)=[\tau^1(f(x))]^{t(x)}_0\boldsymbol{v}^1 \quad\text{and}\quad \alpha_x(x)=f(x),$$

we have

$$\alpha_x([\tau^0(x)]^{t(x)}_0\boldsymbol{v})=[\tau^1(f(x))]^{t(x)}_0\boldsymbol{v}$$
$$\alpha_x([\tau^0(x)]^{t(x)}_0\boldsymbol{w})=-[\tau^1(f(x))]^{t(x)}_0\boldsymbol{w}.$$

On the other hand, putting

$$\boldsymbol{v}=\sum\bar{v}^i_0\boldsymbol{u}^0_i=\sum\bar{v}^i_1\boldsymbol{u}^1_i,\qquad \boldsymbol{w}=\sum\bar{w}^i_0\boldsymbol{u}^0_i=\sum\bar{w}^i_1\boldsymbol{u}^1_i,$$

we have

$$\mathcal{A}(q)({}^t[\bar{v}^1_1,\cdots,\bar{v}^{n+1}_1])={}^t[\bar{v}^1_0,\cdots,\bar{v}^{n+1}_0]$$
$$\mathcal{A}(q)({}^t[\bar{w}^1_1,\cdots,\bar{w}^{n+1}_1])={}^t[\bar{w}^1_0,\cdots,\bar{w}^{n+1}_0].$$

So we put

$$[\tau^0(x)]^{t(x)}_0\boldsymbol{v}=\sum v^i_0X_i,\qquad [\tau^1(f(x))]^{t(x)}_0\boldsymbol{v}=\sum v^i_1X_i,$$
$$[\tau^0(x)]^{t(x)}_0\boldsymbol{w}=\sum w^i_0X_i,\qquad [\tau^1(f(x))]^{t(x)}_0\boldsymbol{w}=\sum w^i_1X_i,$$

and study the relations between $v^j_i$ and $\bar{v}^j_i$, and between $w^j_i$ and $\bar{w}^j_i$.

Since the $x$-component of $[\tau^0(x)]^{t(x)}_0\boldsymbol{v}$ is equal to the $f(x)$-component of $[\tau^1(f(x))]^{t(x)}_0\boldsymbol{v}$, we denote by $m$ the common value:

$$m=g(\boldsymbol{v},[\tau^0(x)]^0_{t(x)}x)=g(\boldsymbol{v},[\tau^1(x)]^0_{t(x)}f(x)).$$

Then we have

$$\bar{v}^{n+1}_0=-m\sin(ct(x))=-\bar{v}^{n+1}_1$$

by the lemma 6. By the lemma 6 and

$$\boldsymbol{v}=m[\tau^0(x)]^0_{t(x)}x+\boldsymbol{v}^0=m[\tau^1(f(x))]^0_{t(x)}f(x)+\boldsymbol{v}^1,$$

we can see the relation between $v^j_i$ and $\bar{v}^j_i$ $(j=1,\cdots,n)$. This shows

$$\bar{\alpha}_x({}^t[\bar{v}^1_0,\cdots,\bar{v}^{n+1}_0])={}^t[\bar{v}^1_1,\cdots,\bar{v}^{n+1}_1].$$

Therefore we have

$$\bar{\alpha}_x\mathcal{A}(q)({}^t[\bar{v}^1_1,\cdots,\bar{v}^{n+1}_1])={}^t[\bar{v}^1_1,\cdots,\bar{v}^{n+1}_1].$$

In the same way, we have

$$\bar{\alpha}_x\mathcal{A}(q)({}^t[\bar{w}^1_1,\cdots,\bar{w}^{n+1}_1])=-{}^t[\bar{w}^1_1,\cdots,\bar{w}^{n+1}_1].$$

Q.E.D.

COROLLARY. *Let* $q=q(x)=\tau^0(x,t(x))=\tau^1(f(x),t(x))\in C$. *Let represent* ${}^t\mathcal{A}(q)$ *by column vectors* $\boldsymbol{a}_i(x)$ *of* $\boldsymbol{R}^{n+1}$ *as*

$$ {}^t\mathcal{A}(q) = [\boldsymbol{a}_1(x), \cdots, \boldsymbol{a}_n(x), \boldsymbol{a}_{n+1}(x)]. $$

*Let put*

$$ \boldsymbol{b}_i(x) = \boldsymbol{a}_i(x) - 2(\boldsymbol{a}_i, \boldsymbol{w})(x)\boldsymbol{w}(x), $$

*where* $\boldsymbol{w}(x)$ *is the unit vector satisfying* $\bar{\alpha}_x\mathcal{A}(q)\boldsymbol{w}(x) = -\boldsymbol{w}(x)$. *Then we have*

$$ \bar{\alpha}_x = [\boldsymbol{b}_1(x), \cdots, \boldsymbol{b}_n(x), \boldsymbol{b}_{n+1}(x)]. $$

*In particular, we have*

$$ -\boldsymbol{e}_{n+1} = \boldsymbol{a}_{n+1}(x) - 2(\boldsymbol{a}_{n+1}, \boldsymbol{w})(x)\boldsymbol{w}(x). $$

By the corollary, the unit vector $\boldsymbol{w}(x)$, which satisfies $\bar{\alpha}_x\mathcal{A}(q)\boldsymbol{w}(x) = -\boldsymbol{w}(x)$, is represented as

$$ \boldsymbol{w}(x) = \begin{bmatrix} \sin(u(x)/2)\boldsymbol{a}(x) \\ \cos(u(x)/2) \end{bmatrix} \qquad \text{for } \boldsymbol{a}_{n+1}(x) = \begin{bmatrix} \sin u(x)\boldsymbol{a}(x) \\ \cos u(x) \end{bmatrix}, $$

where $\boldsymbol{a}(x)$ is a unit column vector of $\boldsymbol{R}^n$.

## § 4. Lemma necessary for the estimate $\|d\alpha\|$.

To estimate the norm $\|d\alpha\|$ of differential of $\alpha: S^{n-1}(1) \to SO(n, \boldsymbol{R})$ in § 5, in this section we study the norm of differential $d\mathcal{A}$ of $\mathcal{A}: C \to O(n+1, \boldsymbol{R})$. Let $q(s)$ $(-\delta < s < \delta)$ be a curve in $C = M_0 \cap M_1$. Let $v^0(s)$ and $v^1(s)$ be horizontal lifts of $q(s)$ in $P$ with respect to $\omega$ with $v^0(0) = u^0(q(0))$ and $v^1(0) = u^1(q(0))$ respectively. Then there exist $O(n+1, \boldsymbol{R})$-valued functions $b^0(s)$ and $b^1(s)$ satisfying

$$ \begin{cases} v^0(s) = u^0(q(s))b^0(s) \\ b^0(0) = E \end{cases} \quad \text{and} \quad \begin{cases} v^1(s) = u^1(q(s))b^1(s) \\ b^1(0) = E. \end{cases} $$

LEMMA 7. *We have*

$$ \left\| \frac{d}{ds}\mathcal{A}(q(s)) \right\|_{s=0} \leqq \left\| \frac{d}{ds} b^0(s) \right\|_{s=0} + \left\| \frac{d}{ds} b^1(s) \right\|_{s=0}. $$

PROOF. Since $v^0(s)$ and $v^1(s)$ are horizontal lifts of $q(s)$, we have

$$ u^0(q(s))b^0(s)\mathcal{A}(q(0)) = u^1(q(s))b^1(s) = u^0(q(s))\mathcal{A}(q(s))b^1(s) $$

by $u^1(q(0)) = u^0(q(0))\mathcal{A}(q(0))$. Thus we have

$$ \mathcal{A}(q(s)) = b^0(s)\mathcal{A}(q(0))[b^1(s)]^{-1}. $$

Therefore we have

$$ \mathcal{A}(q(s)) - \mathcal{A}(q(0)) = [b^0(s) - E]\mathcal{A}(q(0))[b^1(s)]^{-1} + \mathcal{A}(q(0))[(b^1(s))^{-1} - E], $$

and

$$ \|\mathcal{A}(q(s)) - \mathcal{A}(q(0))\| \leqq \|b^0(s) - E\| + \|b^1(s) - E\|. $$

Q.E.D.

Let $x(s)$ be a curve in $S_{q_0}(M)$ such that $\|dx/ds\|=1$. Then we take a curve $q(s)=q(x(s))=\tau^0(x(s), t(x(s)))\in C$. For such a curve $q(s)$, we estimate $\|db^i(s)/ds\|_{s=0}$ in §6. The results are as follows:

$$(4.1)\qquad \begin{cases} \left\|\dfrac{d}{ds}b^0(s)\right\|_{s=0} \leqq \dfrac{2}{3}\dfrac{1-\delta}{\delta}, \\ \left\|\dfrac{d}{ds}b^1(s)\right\|_{s=0} \leqq \dfrac{2}{3}\dfrac{1-\delta}{\delta}\left[\sqrt{\delta}\sin\left(\dfrac{\pi}{2\sqrt{\delta}}\right)\right]^{-1}. \end{cases}$$

## §5. Differentiable sphere theorem.

(**A**) PROPOSITION 4. *Suppose $\delta=0.617$. Let $M^n$ be a simply connected, complete and $\delta$-pinched riemannian manifold, and $E$ the stabilized tangent bundle of $M$. Then $E$ is a trivial vector bundle of $M$, namely $E=M\times \boldsymbol{R}^{n+1}$.*

PROOF. Let $C\ni q\to \mathcal{A}(q)\in SO(n+1, \boldsymbol{R})$ be a differentiable map such that $u_q^0\mathcal{A}(q)=u_q^1$. We put $\beta_x=\mathcal{A}(q(x))=\mathcal{A}(q)$ for $q=q(x)=\tau^0(x, t(x))$. By the lemma 7 and (4.1), we have

$$\|d\beta\| \leqq \frac{2}{3}\frac{1-\delta}{\delta}\left\{1+\left(\sqrt{\delta}\sin\frac{\pi}{2\sqrt{\delta}}\right)^{-1}\right\}.$$

If $\|d\beta\|<1$, then there exists a differentiable map $B: S^{n-1}(1)\to so(n+1, \boldsymbol{R})$ such that $\beta_x=\beta_{x_0}\exp(B(x))$ for a fixed $x_0$ by the lemma 1. Therefore, first we can make new cross-section $\bar{u}^1: M_1\to P|_{M_1}$ such that $u_q^0\beta_{x_0}=\bar{u}_q^1$ for $q\in C$. Second we can make a global cross-section $u: M\to P$.

RESULT OF CALCULATION 1. We have

$$\frac{2}{3}\frac{1-\delta}{\delta}\left\{1+\left(\sqrt{\delta}\sin\frac{\pi}{2\sqrt{\delta}}\right)^{-1}\right\}=1 \qquad \text{at } \delta=0.616\cdots.$$

Thus, if $\delta=0.617$, then $E$ is a trivial bundle. Q.E.D.

(**B**) DIFFERENTIABLE SPHERE THEOREM. *Suppose $\delta=0.681$. Let $M^n$ be a simply connected, complete and $\delta$-pinched riemannian manifold. Then $M$ is diffeomorphic to the standard sphere.*

Let $q=q(x)=\tau^0(x, t(x))\in C$. For $\mathcal{A}(q)$ such that $u_q^0\mathcal{A}(q)=u_q^1$, we put $\beta_x=\mathcal{A}(q(x))=\mathcal{A}(q)$. We represent ${}^t\beta_x$ as ${}^t\beta_x=[\boldsymbol{a}_1(x), \boldsymbol{a}_2(x), \cdots, \boldsymbol{a}_n(x), \boldsymbol{a}_{n+1}(x)]$ by the column vectors. We denote $\boldsymbol{a}_{n+1}(x)$, with some vector $\boldsymbol{a}(x)\in\boldsymbol{R}^n$, by

$$(5.1)\qquad \boldsymbol{a}_{n+1}(x)={}^t[\sin u(x)\cdot\boldsymbol{a}(x), \cos u(x)],$$

and then we put

$$(5.2)\qquad \boldsymbol{w}(x)={}^t\left[\sin\frac{u(x)}{2}\cdot\boldsymbol{a}(x), \cos\frac{u(x)}{2}\right].$$

Let $\bar{\alpha}_x=\begin{bmatrix}\alpha_x & 0\\ 0 & -1\end{bmatrix}$. Then we have the following:

$$\bar{\alpha}_x = [\boldsymbol{b}_1(x), \boldsymbol{b}_2(x), \cdots, \boldsymbol{b}_n(x), -\boldsymbol{e}_{n+1}],$$

where $\boldsymbol{b}_i(x)=\boldsymbol{a}_i(x)-2(\boldsymbol{a}_i, \boldsymbol{w})(x)\boldsymbol{w}(x)$ by the corollary of proposition 3.

RESULT OF CALCULATION 2. We have

$$\begin{aligned}\cos u(x) &= h(\boldsymbol{u}^0_{n+1}, \boldsymbol{u}^1_{n+1})(q(x))\\ &= -\cos^2(ct(x))-\sin^2(ct(x))g([\tau^0(x)]^0_{t(x)}x, [\tau^1(f(x))]^0_{t(x)}f(x))\\ &\geqq -\cos^2(ct(x))-\sin^2(ct(x))\cos(\pi\sqrt{\delta})\\ &= -1+\sin^2(ct(x))[1-\cos(\pi\sqrt{\delta})]\end{aligned}$$

by (2.1) and (2.2). So, if $\delta\geqq 0.616$, we have $\cos u(x)\geqq 0.689$ and $\cos(u(x)/2)\geqq 0.9189$.

LEMMA 8. *We assume $\delta\geqq 0.616$. Then we have*

$$\|d\alpha\|_x \leqq \frac{1}{\cos^2(u(x)/2)}\frac{2}{3}\frac{1-\delta}{\delta}\left\{1+\left(\sqrt{\delta}\sin\frac{\pi}{2\sqrt{\delta}}\right)^{-1}\right\}.$$

PROOF. From $\|d\alpha\|=\|d\bar{\alpha}\|=\|d^t\bar{\alpha}\|$, we study $\|d^t\bar{\alpha}\|$ in this proof. Let $X\in T_x(S^{n-1}(1))$ and $x(s)$ be a curve in $S^{n-1}(1)$ such that $x(0)=x$ and $\dot{x}(0)=X$. We put $\|d\beta\|\leqq N$. The proof is divided into several steps.

(a) We can put $\boldsymbol{e}_{n+1}=\cos(u(x)/2)\cdot\boldsymbol{w}(x)+\sin(u(x)/2)\cdot Y(x)$, where $(\boldsymbol{w}, Y)(x)=0$ and $\sin(u(x)/2)>0$. Then we have

$$(d_X{}^t\bar{\alpha})\boldsymbol{w}(x) = -\frac{\sin(u(x)/2)}{\cos(u(x)/2)}(d_X{}^t\bar{\alpha})Y(x),$$

because of $(d_X{}^t\bar{\alpha})\boldsymbol{e}_{n+1}=0$.

(b) By the proposition 3, we have

$$\beta_x|_{\boldsymbol{w}(x)^\perp} = {}^t\bar{\alpha}_x|_{\boldsymbol{w}(x)^\perp}, \qquad \beta_x\boldsymbol{w}(x) = -{}^t\bar{\alpha}_x\boldsymbol{w}(x).$$

Let $Z\in\boldsymbol{R}^{n+1}$ be a unit vector such that $(Z, \boldsymbol{w}(x))=0$. We represent $Z$ as

$$Z = c_1(s)\boldsymbol{w}(x(s))+c_2(s)W(s)$$

along the curve $x(s)$, where $(\boldsymbol{w}(x(s)), W(s))=0$, $\|W(s)\|=1$ and $W(0)=Z$. For simplicity we use the following notations:

$$\boldsymbol{w} = \boldsymbol{w}(x),\quad \boldsymbol{w}(s) = \boldsymbol{w}(x(s)),\quad c_i' = d_Xc_i,\quad \boldsymbol{w}' = d_X\boldsymbol{w},\quad \text{and}\quad W' = d_XW.$$

We have

$$(5.3)\qquad (d_X{}^t\bar{\alpha})Z=\frac{d}{ds}[{}^t\bar{\alpha}_{x(s)}Z]_{s=0}$$

$$=\frac{d}{ds}[{}^t\bar{\alpha}_{x(s)}(c_1(s)\boldsymbol{w}(s))]_{s=0}+\frac{d}{ds}[{}^t\bar{\alpha}_{x(s)}(c_2(s)W(s))]_{s=0}$$

$$=\frac{d}{ds}[{}^t\bar{\alpha}_{x(s)}(c_1(s)\boldsymbol{w}(s))]_{s=0}+\frac{d}{ds}[\beta_{x(s)}(c_2(s)W(s))]_{s=0}$$

$$=c_1'{}^t\bar{\alpha}_x(\boldsymbol{w})+(d_X\beta)Z+\beta_x(W')=c_1'{}^t\bar{\alpha}_x(\boldsymbol{w})+(d_X\beta)Z-c_1'\beta_x(\boldsymbol{w})$$

$$=(d_X\beta)Z-2c_1'\beta_x(\boldsymbol{w})=(d_X\beta)Z-2(Z,\boldsymbol{w}')\beta_x(\boldsymbol{w}).$$

We take $Z=Z_1=\boldsymbol{w}'/\|\boldsymbol{w}'\|$ in (5.3), then we have

$$(5.4)\qquad (d_X{}^t\bar{\alpha})Z_1=(d_X\beta)Z_1-2\|\boldsymbol{w}'\|\beta_x(\boldsymbol{w}).$$

Furthermore, we have

$$(5.5)\qquad ((d_X{}^t\bar{\alpha})Z_1,\boldsymbol{e}_{n+1})=0,\qquad (\beta_x(\boldsymbol{w}),\boldsymbol{e}_{n+1})=\cos\Big(\frac{u(x)}{2}\Big)$$

by (5.1) and (5.2). We take $Z\in\{\boldsymbol{w},\boldsymbol{w}'\}^\perp$ in (5.3), then we have

$$(5.6)\qquad (d_X{}^t\bar{\alpha})Z=(d_X\beta)Z.$$

(c) We take a unit vector $W\in\boldsymbol{w}^\perp$, and put $W=c_1Z_1+c_2Z$, where $(Z_1, Z)=0$ and $\|Z\|=1$. We put $V^\perp=V-(V,\boldsymbol{e}_{n+1})\boldsymbol{e}_{n+1}$ in the calculation below, then we have

$$(5.7)\qquad ((d_X{}^t\bar{\alpha})W,(d_X{}^t\bar{\alpha})W)=((d_X\beta)W,((d_X\beta)W)^\perp)$$

$$-4c_1\|\boldsymbol{w}'\|((d_X\beta)W,\beta_x(\boldsymbol{w})^\perp)+4c_1^2\|\boldsymbol{w}'\|^2(\beta_x(\boldsymbol{w}),\beta_x(\boldsymbol{w})^\perp)$$

$$\leqq N^2-4c_1^2\|\boldsymbol{w}'\|^2\cos^2\Big(\frac{u(x)}{2}\Big)+4c_1^2\|\boldsymbol{w}'\|^2\sin^2\Big(\frac{u(x)}{2}\Big)$$

$$+4|c_1|\,\|\boldsymbol{w}'\|\sin\Big(\frac{u(x)}{2}\Big)\Big\{N^2-4c_1^2\|\boldsymbol{w}'\|^2\cos^2\Big(\frac{u(x)}{2}\Big)\Big\}^{1/2}$$

$$=N^2-4|c_1|\,\|\boldsymbol{w}'\|\Big\{|c_1|\,\|\boldsymbol{w}'\|\cos^2\Big(\frac{u(x)}{2}\Big)-|c_1|\,\|\boldsymbol{w}'\|\sin^2\Big(\frac{u(x)}{2}\Big)$$

$$-\sin\Big(\frac{u(x)}{2}\Big)\Big[N^2-4c_1^2\|\boldsymbol{w}'\|^2\cos^2\Big(\frac{u(x)}{2}\Big)\Big]^{1/2}\Big\}$$

by (5.4), (5.5) and (5.6). The last equation of (5.7) attains maximum $[N/\cos(u(x)/2)]^2$ at $2|c_1|\,\|\boldsymbol{w}'\|=N\tan(u(x)/2)$. Furthermore, since we have

$$\Big(d_1\frac{\sin(u(x)/2)}{\cos(u(x)/2)}+d_2\Big)^2\leqq(d_1^2+d_2^2)\Big(\frac{1}{\cos(u(x)/2)}\Big)^2,$$

we have $\|d\bar{\alpha}\|\leqq N(\cos(u(x)/2))^{-2}$ from the above argument and (a). Q.E.D.

RESULT OF CALCULATION 3. We put

$$N=\frac{2}{3}\frac{1-\delta}{\delta}\left\{1+\left(\sqrt{\delta}\sin\left(\frac{\pi}{2\sqrt{\delta}}\right)\right)^{-1}\right\}$$

and

$$N_1=\max\left\{\frac{1}{\cos^2(u(x)/2)}N \mid x\in S^{n-1}(1)\right\}.$$

We have the following results.

| $\delta$ | 0.643 | 0.644 | 0.645 |
|---|---|---|---|
| $N$ | $\leqq 0.8689$ | $\leqq 0.8644$ | $\leqq 0.8600$ |
| $\cos^2(u(x)/2)$ | $\geqq 0.8687$ | $\geqq 0.8696$ | $\geqq 0.8704$ |
| $N_1$ | $\leqq 1.00004$ | $\leqq 0.9940$ | $\leqq 0.9879$ |

We take $\delta\geqq 0.644$, then $\|d\alpha\|<1$. So there exists a differentiable map $A: S^{n-1}(1)\to so(n, \boldsymbol{R})$ such that $\alpha_x=\exp(\pi A_x)$. We put

$$C_1=\frac{1-\delta}{1+c^2}\left\{\frac{c(e^{\pi/2\sqrt{\delta}}-e^{-\pi/2\sqrt{\delta}})}{2\sin\left(\frac{c\pi}{2\sqrt{\delta}}\right)}-1\right\}\left(\frac{1+\sqrt{\delta}\sin\left(\frac{\pi}{2\sqrt{\delta}}\right)}{2\sqrt{\delta}\sin\left(\frac{\pi}{2\sqrt{\delta}}\right)}\right),$$

and

$$C_2=\frac{N_1-C_1}{2} \quad\text{and}\quad C_3=\frac{N_1+C_1}{2}.$$

In the calculation below, we have

$$\max_t P(t)=\max_t\left\{C_2^2\left[\frac{\sin\left(\frac{N_1}{2}t\right)}{\sin\left(\frac{N_1}{2}\pi\right)}\right]^2+C_3^2\left[\frac{\sin(N_1 t)}{\sin(N_1\pi)}\right]^2\right.$$
$$\left.-2C_2C_3\frac{\sin\left(\frac{N_1}{2}t\right)}{\sin\left(\frac{N_1}{2}\pi\right)}\frac{\sin(N_1 t)}{\sin(N_1\pi)}\cos\left(\frac{3}{2}N_1(\pi-t)\right)\right\}.$$

RESULT OF CALCULATION 4.

| $\delta$ | 0.680 | 0.681 | 0.682 |
|---|---|---|---|
| $C_1$ | $\leqq 0.4086$ | $\leqq 0.4061$ | $\leqq 0.4037$ |
| $N_1$ | $\leqq 0.7979$ | $\leqq 0.7931$ | $\leqq 0.7883$ |
| $P(t)$ | $\leqq 1.0225$ at $t\doteqdot\pi/1.7955$ | $\leqq 0.9936$ $t\doteqdot\pi/1.7909$ | $\leqq 0.9661$ $t\doteqdot\pi/1.7864$ |

PROOF OF THEOREM. We have $\|\alpha-dF\|\leqq C_1$ by the proposition 2. Therefore, if we take $\delta=0.681$, then $M$ is diffeomorphic to the standard sphere by the diffeotopy theorem and the proposition 1. Q.E.D.

## § 6. Estimate of holonomy of principal bundle $P$.

The setting of all notations in this section is the same as in § 2. $P$ is an $O(n+1, \boldsymbol{R})$-principal bundle over a $\delta$-pinched riemannian manifold $M$. $M$ is divided into $M_0$ and $M_1$ such that $M_0 \cap M_1 = C$. $P|_{M_0}$ is equiped with two connection forms $\omega$ and $\bar{\omega}$ that are defined by the connections $\nabla$ and $\bar{\nabla}$ on $E|_{M_0}$ respectively. In this section, we estimate holonomy determined by $(P, \omega)$.

Let $\tau=\tau(s)$ $(0 \leqq s \leqq a)$ be a piecewise differentiable curve in $M_0$. We take a horizontal lift $v(s)$ of $\tau$ in $P$ with respect to $\omega$ such that $v(0)=u^0(\tau(0))$, where $u^0$ is the cross-section $M_0 \to P|_{M_0}$ that was defined in § 2(C). Then there exists $b(s) \in O(n+1, \boldsymbol{R})$ for each $s$ satisfying $v(s)=u^0(\tau(s))b(s)$. From

$$0 = \omega(\dot{v}(s)) = ad(b(s)^{-1})\omega[u^0_*(\dot{\tau}(s))] + b(s)^{-1}\dot{b}(s),$$

we have

$$\dot{b}(s) = -\omega[u^0_*(\dot{\tau}(s))]b(s). \tag{6.1}$$

Let $D(s)$ be a surface that is made by geodesics joining $q_0$, which is the center of $M_0$, to $\tau(r)$ for $0 \leqq r \leqq s$. Integrating (6.1) with respect to $s$, we have

$$\begin{aligned} b(s)-E &= \int_0^s \dot{b}(r)dr \\ &= -\int_0^s \omega[u^0_*(\dot{\tau}(r))]dr - \int_0^s \omega[u^0_*(\dot{\tau}(r))](b(r)-E)dr \\ &= -\int_{D(s)} (u^0)^*\Omega - \int_0^s (\omega-\bar{\omega})[u^0_*(\dot{\tau}(r))](b(r)-E)dr, \end{aligned} \tag{6.2}$$

because $\bar{\omega}[u^0_*(\dot{\tau}(s))]=0$. Since $\omega-\bar{\omega}$ satisfies $R^*_a(\omega-\bar{\omega})=ad(a^{-1})(\omega-\bar{\omega})$, the norm $\|\omega-\bar{\omega}\|$ becomes a function on $M_0$: We define it by

$$\|\omega-\bar{\omega}\|_p = \max\{\|(\omega-\bar{\omega})(u^0_*X)\| \mid X \in T_p(M_0) \text{ with } \|X\|=1\}.$$

For $x \in S^n(1)$, we denote by $\eta_x$ a curve $b(r)x$ $(0 \leqq r \leqq a)$ in $S^n(1)$. The length $L(\eta_x)$ of $\eta_x$ in $S^n(1)$ holds the following equation:

$$L(\eta_x) = \int_0^a \|\dot{b}(r)x\|dr \leqq \int_0^a \|\dot{b}(r)\|dr. \tag{6.3}$$

We define a distance $\rho(b(s), b(t))$ in $SO(n, \boldsymbol{R})$ by

$$\rho(b(s), b(t)) = \|C\|, \tag{6.4}$$

where $C \in so(n, \boldsymbol{R})$ such that $b(t)=b(s)\exp(C)$ and $\|C\| \leqq \pi$. Then we have, for $s \leqq t$,

$$
(6.5)\qquad \begin{cases} \rho(b(s),\, b(t)) \leqq \max\{L(\eta_x|_{[s,t]}) \mid x \in S^n(1)\}, \\ \rho(b(s),\, b(t)) \geqq \|b(t)-b(s)\|\,. \end{cases}
$$

(**A**) PROOF OF (4.1).

Let $x(s)$ be a piecewise differentiable curve in $S_{q_0}(M)$ with $\|\dot{x}(s)\|=1$. Then $q(s)=\tau^0[x(s),\, t(x(s))]$ is a curve in $C$. We apply (6.2) to the curve $q(s)$. Then we have

$$
\left\|\frac{d}{ds}b(s)\right\|_{s=0} \leqq \|\Omega\|\frac{d}{ds}m(D(s))|_{s=0}\,,
$$

where $m(D(s))$ is the measure of $D(s)$. On the other hand, since $M$ is $\delta$-pinched and $d(q_0,\, q(s)) \leqq \pi/(2\sqrt{\delta})$, the Rauch comparison theorem yields the estimate $m(D(s)) \leqq s/\delta$. In fact, we arrive at the estimate if we observe the case where $M_0$ has the sectional curvature $\delta$.

(**B**) PROPOSITION 5. *Let $\tau=\tau(s)$ $(0\leqq s\leqq a)$ be a piecewise differentiable loop in a normal coordinate in $M$, and $v(s)$ be a horizontal lift of $\tau$ in $(P,\, \omega)$. Then we have*

$$
\rho(b,\, E) \leqq \|\Omega\| m(D) \exp[\|\Omega\| m(D)]\,,
$$

*where $v(0)b=v(a)$, $D$ is surface made by geodesics joining the center $p$ of the normal coordinate to each point of $\tau$.*

PROOF. We can suppose that the normal coordinate containing $\tau$ is $M_0$ and that the center $p$ of it is $q_0$. So, we can also apply (6.2) in this case under the condition $b(a)=b$. Furthermore, we assume that the parameter $s$ of $\tau$ is given by the arc-length for simplicity. By (6.2), (6.3) and (6.5), we have

$$
(6.6)\qquad \rho(b(s),\, E) \leqq \|\Omega\| m(D) + \int_0^s \|\omega-\bar{\omega}\|_{\tau(r)}\rho(b(r),\, E)dr\,.
$$

We estimate $\|\omega-\bar{\omega}\|$: Let fix $s\in(0,\, a)$, and $w(r)$ be a horizontal lift of $\tau(s+r)$ in $(P,\, \omega)$ satisfying $w(0)=u^0(\tau(s))$. Putting $w(r)=u^0(\tau(s+r))a(r)$, we apply (6.2) to this. Then we have

$$
\|\dot{a}(0)\| \leqq \|\Omega\|\frac{dm(D(s))}{ds}\,.
$$

On the other hand, we have

$$
\dot{a}(0) = -\omega[u^0_*(\dot{\tau}(s))] = -(\omega-\bar{\omega})[u^0_*(\dot{\tau}(s))]
$$

by (6.1). Therefore, we have

$$
(6.7)\qquad \rho(b(s),\, E) \leqq \|\Omega\| m(D) + \|\Omega\|\int_0^s \frac{dm(D(r))}{dr}\rho(b(r),\, E)dr\,.
$$

Finally, we obtain the assertion by applying the Gronwall's lemma to (6.7).

Q.E.D.

## References


[1] M. Berger, Les variétés riemanniennes (1/4)-pincées, Ann. Scuola Norm. Sup., Pisa, **14** (1960), 161-170.

[2] D. Gromoll, Differenzierbare Strukturen und Metriken positiver Krümmung auf Sphären, Math. Ann., **164** (1966), 351-371.

[3] K. Grove, H. Karcher and E. A. Ruh, Group actions and curvature, Invent. Math., **23** (1974), 31-48.

[4] S. Helgason, Differential Geometry, Lie Group, and Symmetric Spaces, Academic Press, New York, 1978.

[5] H. C. Im Hof and E. A. Ruh, An equivariant pinching theorem, Comment. Math. Helv., **50** (1975), 389-401.

[6] W. Klingenberg, Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung, Comment. Math. Helv., **35** (1961), 47-54.

[7] H. Omori, On the group of diffeomorphisms on a compact manifold, Global Analysis Symp. Pure Math., Berkeley, **15** (1968), 167-189.

[8] E. A. Ruh, Curvature and differentiable structure on spheres, Comment. Math. Helv., **46** (1971), 127-136.

[9] E. A. Ruh, Krummung und differentierbare Struktur auf Sphären, Comment. Math. Helv., **46** (1971), 127-136.

[10] T. Sakai, On cut loci of compact symmetric spaces, Hokkaido Math. J., **6** (1977), 136-161.

[11] Y. Shikata, On the differentiable pinching problem, Osaka Math. J., **4** (1967), 279-287.

[12] M. Sugimoto and K. Shiohama, On the differentiable pinching problem, Math. Ann., **195** (1971), 1-16.



Yoshihiko SUYAMA
Department of Applied Mathematics
Faculty of Science
Fukuoka University
Fukuoka, 814-01
Japan