

Signature defects and eta invariants of Picard modular cusp singularities

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Introduction.

A compact oriented framed manifold (N, α) of dimension $4k-1$ has an invariant called *signature defect* defined by Hirzebruch in [Hi] as follows: Since N is framed, there exists a compact oriented $4k$ dimensional manifold M with $\partial M = N$ and the tangent bundle of M restricted to ∂M is trivialized. Thus we can define the Pontrjagin classes of M as relative classes p_j in $H^{4j}(M, \partial M; \mathbf{Z})$. Hirzebruch defined the signature defect as

$$\sigma(N, \alpha) := L_k(p_1, \dots, p_k)[M, \partial M] - \text{sign}(M, \partial M),$$

where $L_k(p_1, \dots, p_k) \in H^{4k}(M, \partial M; \mathbf{Q})$ is the Hirzebruch L -polynomial with respect to p_j 's, $[M, \partial M]$ is the fundamental class of $(M, \partial M)$ and $\text{sign}(M, \partial M)$ is the signature of the bilinear form on $H^{2k}(M, \partial M; \mathbf{R})$ defined by cup product.

In [Hi] Hirzebruch showed that a Hilbert modular cusp singularity (X, p) has a compact neighborhood V of p such that the boundary ∂V is framed and conjectured that the signature defect of the singularity is equal to the special value of Shimizu's L -function. He proved the conjecture in the 2-dimensional case.

On the other hand, Atiyah, Patodi and Singer [APS1] defined the eta invariants of first order self-adjoint elliptic differential operators on compact manifolds, and derived the index theorem for manifolds with boundary. Their index theorem says that the difference between the integral of the closed differential form representing the L -genus and $\text{sign}(M, \partial M)$ is equal to the eta invariant of the tangential signature operator on the boundary manifold ∂M .

By using the index theorem for manifolds with boundary in [APS1] Atiyah, Donnelly and Singer proved Hirzebruch's conjecture in general ([ADS1], [ADS2]). And Müller also proved it ([Mu2]).

The purpose of this paper is to study the signature defects of Picard

modular cusp singularities and to relate them to the eta invariants of some operators.

In order to state our result, let K be an imaginary quadratic field with the ring of integers \mathcal{O} . We define an algebraic group G defined over \mathbf{Q} as follows:

$$G_{\mathbf{Q}} := \{g \in SL_{m+1}(K); {}^t \bar{g} I_{m,1} g = I_{m,1}\},$$

and $G_{\mathbf{R}} = SU(m, 1)$ with

$$I_{m,1} := \begin{pmatrix} E_m & 0 \\ 0 & -1 \end{pmatrix}.$$

The group $G_{\mathbf{R}}$ acts on the complex m -dimensional unit ball $B_m := \{z \in \mathbf{C}^m; |z| < 1\}$ as linear fractional transformations. The group $G_{\mathbf{Z}} := G_{\mathbf{Q}} \cap SL_{m+1}(\mathcal{O})$ is called the *Picard modular group*. Let Γ_0 be an arithmetic subgroup of $G_{\mathbf{R}}$. Γ_0 acts on B_m properly discontinuously. The factor space $\Gamma_0 \backslash B_m$ is called the *Picard modular variety*, and can be compactified (Satake compactification) by addition of finitely many points. We call these singular points *Picard modular cusps*. Let Γ be a neat normal subgroup of Γ_0 of finite index (such Γ exists by a theorem of Borel [B]). Then $\Gamma \backslash B_m$ is a manifold and its cusps are given by contraction of the zero-sections of negative line bundles over abelian varieties. Moreover, the cusps of $\Gamma_0 \backslash B_m$ are given as the quotients of those of $\Gamma \backslash B_m$ with respect to the action of the finite group $\Gamma_0 \backslash \Gamma$.

Let (V, p) be a cusp singularity of $\Gamma \backslash B_m$ such that it is the contraction of the zero-section (denoted by the same T as the base space of the bundle) of a negative line bundle L over an abelian variety T . We denote by $\sigma(V, p)$ the signature defect of (N, α) , where N is the boundary of a neighborhood of the singular point p and α is the natural frame induced from V . Let M be the disc bundle associated to L over T . By definition, the signature defect is

$$\sigma(V, p) = L_k(p_1, \dots, p_k)[M, \partial M] - \text{sign}(M, \partial M) \quad \text{when } m=2k.$$

In this situation we get the following theorems.

THEOREM 1. *Let A be the modified tangential signature operator on ∂M defined analogously as in [ADS1] and acting on the space of L^2 -forms with even degree. Then we have*

$$\eta_A(0) = L_k(p_1, \dots, p_k)[M, \partial M] = -\frac{2^{2k} T^{2k}}{(2k-1)!} \zeta(1-2k),$$

where T^m is the self-intersection number of T in M .

THEOREM 2. *Let A_1 be the tangential signature operator defined in [APS1]. Let H be the space of constant even forms on ∂M . Then we have*

$$\eta_{A_1|_H}(0) = -\text{sign}(M, \partial M).$$

THEOREM 3.

$$\text{sign}(M, \partial M) = (-1)^k C_{2k-1} - 2 \sum_{j=0}^{k-2} (-1)^j C_{2k-1-j}.$$

Barbasch and Moscovici [BM, Theorem 7.6] calculated the L^2 -index of the signature operator on $\Gamma \backslash X$, where X is a symmetric space of rank one, and obtained a formula for the difference, which we call “ L^2 -signature defect”, between the integral of the closed differential form representing the L -genus and this L^2 -index by using the Selberg trace formula. Stern [St1] calculated the L^2 -index of the signature operator on locally symmetric spaces in general. Müller [Mu1] calculated the L^2 -index of the signature operator on a Hilbert modular variety and in [Mu2] derived a more general formula for the L^2 -signature defect of one cusp on $\Gamma \backslash X$, where X is a product of copies of a symmetric space of rank one, by using wave operators and the Selberg trace formula. The L^2 -signature defect of a Hilbert modular cusp singularity coincides with the signature defect of the singularity in the sense of Hirzebruch ([ADS1], [ADS2], [Mu2]). In view of the conjecture of Hirzebruch on Hilbert modular cusps and the result of Stern [St1] it seems likely that the contribution from each cusp to the L^2 -signature defect coincides with a properly defined signature defect of the cusp. Recently Stern [St2] calculated the η -density in his formula of the L^2 -index, which coincides with our $\text{sign}(M, \partial M)$ in the case of $\Gamma \backslash B_m$ and vanishes otherwise, and pointed out that the formula for $\Gamma \backslash B_m$ in [BM, Theorem 7.6] need a correction term. Thus in the case of $\Gamma \backslash B_m$ [St2, Theorem 6.7] actually implies that the contribution from each cusp to the L^2 -signature defect coincides with our signature defect $\sigma(V, p)$ of a cusp (V, p) on $\Gamma \backslash B_m$ up to normalization of the measure on $\Gamma \backslash B_m$.

In Section 1 we recall the structure of Picard modular cusp singularities. In Section 2 we determine the eta invariant of the modified tangential signature operator on the nilmanifold covered by the Heisenberg group. We employ the method of Deninger and Singhof [DS]. In Section 3 we relate the eta invariant with the signature defect of Picard modular cusp singularities and the signature $\text{sign}(M, \partial M)$ of the bounding manifold M for the nilmanifold. In Section 4 we calculate $\text{sign}(M, \partial M)$.

§ 1. Picard modular cusps.

We review the structure of Picard modular cusp singularities following [He] and [Ho].

Let d be a square-free natural number and $K := \mathbb{Q}(\sqrt{-d})$ an imaginary quadratic number field with the ring of integers \mathcal{O} . Let $SU(m, 1) := \{g \in SL_{m+1}(\mathbb{C})$;

${}^t\bar{g}I_{m,1}g=I_{m,1}$ a real Lie group, where $I_{m,1}:=\begin{pmatrix} E_m & 0 \\ 0 & -1 \end{pmatrix}$. $SU(m, 1)$ acts on the complex m -dimensional unit ball $B_m:=\{z\in\mathbf{C}^m; |z|<1\}$ as linear fractional transformations. The group $SU(m, 1; K):=SU(m, 1)\cap SL_{m+1}(K)$ is a \mathbf{Q} -form of $SU(m, 1)$. Let Γ be an arithmetic subgroup of $SU(m, 1)$, that is, $\Gamma\subset SU(m, 1; K)$ and $\Gamma\cap SU(m, 1; \mathcal{O})$ is of finite index in both Γ and $SU(m, 1; \mathcal{O})$. The action of Γ on B_m is properly discontinuous. The factor space $\Gamma\backslash B_m$ is the *Picard modular variety*, and can be compactified (Satake compactification) by adding finitely many points. We call these singular points *Picard modular cusps*. In [Ho] these singularities are called ball cusps.

We are interested in one cusp singularity. We consider a Γ -rational boundary point $\kappa_0:={}^t(1, 0, \dots, 0)\in\partial_K B_m$. We can realize B_m as an unbounded domain by a biholomorphic mapping

$$\tau: B_m \ni {}^t(z_1, \dots, z_m) \longmapsto {}^t(\sqrt{-1}(1+z_1), \sqrt{2}z_2, \dots, \sqrt{2}z_m)/(1-z_1) \in \mathcal{D},$$

where $\mathcal{D}:=\{(z, u_1, \dots, u_{m-1})\in\mathbf{C}^m; 2\operatorname{Im} z - \sum_{i=1}^{m-1}|u_i|^2 > 0\}$. Let $\infty=\tau(\kappa_0)$. Denote the matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & -\sqrt{-2}E_{m-1} & 0 \\ \sqrt{-1} & 0 & -\sqrt{-1} \end{pmatrix}$$

of the transformation τ by the same symbol τ . By conjugation, the group $SU(m, 1)$ is transformed into the group $G:=\{g\in SL_{m+1}(\mathbf{C}); {}^t\bar{g}Hg=H\}$, where

$$H:=\begin{pmatrix} & & -\sqrt{-1} \\ & E_{m-1} & \\ \sqrt{-1} & & \end{pmatrix}.$$

The isotropy group $SU(m, 1)_{\kappa_0}$ is transformed into G_∞ , which is a parabolic subgroup $P\subset G$. P splits into $P=NAM$, where

$$A=\left\{\begin{pmatrix} \delta & & \\ & E_{m-1} & \\ & & \delta^{-1} \end{pmatrix}; \delta>0\right\}, \quad M=\left\{\begin{pmatrix} \beta & & \\ & B & \\ & & \beta \end{pmatrix}; B\in U(m-1), \det B=\beta^{-2}\right\}$$

and

$$N=\left\{[a, r]:=\begin{pmatrix} 1 & \sqrt{-1}{}^t\bar{a} & \sqrt{-1}|a|^2/2+r \\ 0 & E_{m-1} & a \\ 0 & 0 & 1 \end{pmatrix}; a\in\mathbf{C}^{m-1}, r\in\mathbf{R}\right\}.$$

Note that N is the Heisenberg group with the multiplication defined by $[a, r][b, s]=[a+b, r+s-\operatorname{Im}{}^t\bar{a}b]$. As a fundamental neighborhood system of ∞ in $\overline{\Gamma\backslash B_m}:=(\Gamma\backslash B_m)\cup\{\infty\}$ we can take the set $C(L)\cup\{\infty\}$, where for any positive L ,

$$C(L):=\Gamma^\tau\cap P\setminus\{(z, u)\in\mathcal{D}; 2\operatorname{Im} z - |u|^2 > L\} \quad \text{and} \quad \Gamma^\tau:=\tau^{-1}\Gamma\tau.$$

Suppose that Γ is torsion free and that $\Gamma^\tau \cap P = \Gamma^\tau \cap N$. Then $C(L) = \Gamma^\tau \cap N \setminus \{(z, u) \in \mathcal{D}; 2 \operatorname{Im} z - |u|^2 > L\}$ is a punctured disk bundle over an abelian variety as we explain in the following.

Let $L_2 := \Gamma^\tau \cap [N, N]$ and $q = q(\Gamma)$ the positive real number such that $[0, q] \in N$ generates L_2 . Hence there exist \mathbf{Z} -linearly independent vectors $\{e_1, \dots, e_{2m-2}\}$ in \mathbf{C}^{m-1} and real number r_x depending on $x = (x_i) \in \mathbf{Z}^{2m-2}$ such that

$$\Gamma^\tau \cap N = \left\{ \left[\sum_{i=1}^{2m-2} x_i e_i, r_x + yq \right]; x = (x_i) \in \mathbf{Z}^{2m-2}, y \in \mathbf{Z} \right\}.$$

Let $L(\Gamma)$ denote the lattice $\sum_{i=1}^{2m-2} \mathbf{Z} e_i$ in \mathbf{C}^{m-1} and let $E: \mathbf{C}^{m-1} \times \mathbf{C}^{m-1} \rightarrow \mathbf{R}$ be the nondegenerate alternating form $E(u, v) := (2/q) \operatorname{Im} {}^t \bar{u} v$. Then E restricted to $L(\Gamma) \times L(\Gamma)$ has values in \mathbf{Z} because $[u, r][v, s][u, r]^{-1}[v, s]^{-1} = [0, -2 \operatorname{Im} {}^t \bar{u} v]$. Thus E is a Riemann form on the complex torus $T := L(\Gamma) \backslash \mathbf{C}^{m-1}$, which is hence an abelian variety. Let $H(u, v) := E(u, \sqrt{-1}v) + \sqrt{-1}E(u, v)$ for $u, v \in \mathbf{C}^{m-1}$. Then $H(u, v) = (2/q) {}^t \bar{u} v$. Let $\mathcal{L} = \mathcal{L}(-H)$ be the line bundle over T with transition function

$$\exp\left(-\pi H(x, u) - \frac{\pi}{2} H(x, x) + 2\pi \sqrt{-1} r_x / q\right)$$

for $x \in L(\Gamma)$ and $u \in \mathbf{C}^{m-1}$.

LEMMA 1.1. $C(L)$ is a punctured disk bundle associated with the negative line bundle \mathcal{L} . If we identify the 0-section of \mathcal{L} with the base space T , then the self-intersection number of T in the total space of \mathcal{L} is given by

$$T^m = -(m-1)! \sqrt{\det E}.$$

PROOF. The first assertion is obvious. Let W be the Total space of the line bundle \mathcal{L} . Then \mathcal{L} is the normal bundle $N_{T/W} = \mathcal{O}_T(T)$ of $T \subset W$ and $T^m = c_1(N_{T/W})^{m-1}[T] = b_1(\mathcal{L})^{m-1}[T]$. For the dual bundle $\mathcal{L}^\vee = \mathcal{O}_T(-T)$, we have $\chi(\mathcal{L}^\vee) = -T^m / (m-1)!$ (see §16 in [AV]).

On the other hand, since the dual bundle \mathcal{L}^\vee has transition functions

$$\exp\left(\pi H(x, u) + \frac{\pi}{2} H(x, x) + 2\pi \sqrt{-1} r_x / q\right) \quad \text{for } u \in \mathbf{C}^{m-1}, x \in L(\Gamma),$$

we have $\chi(\mathcal{L}^\vee) = \sqrt{\det E}$ (see, for example, Section 3 in [AV]). q.e.d.

From Lemma 1.1 we see that the boundary of a compact neighborhood $V := \overline{C(L)} \cup \{\infty\}$ of a cusp ∞ is a circle bundle over the abelian variety T . We note that the disc bundle associated to \mathcal{L} is one of the bounding manifolds for ∂V . On the other hand, we can easily see that the boundary manifold ∂V is a compact nilmanifold $\Gamma \cap N \backslash N$, because the action of N on the set $\{(z, u) \in \mathcal{D}; 2 \operatorname{Im} z - |u|^2 = L\}$ is transitive. We use this fact in Section 2.

§2. Analysis on the Heisenberg group.

In this section we slightly modify the tangential signature operator on $I^\tau \cap N \setminus N$ defined by Atiyah, Patodi and Singer in [APS1], and calculate its η -invariant following the method used by Deninger and Singhof in [DS].

2.1. Heisenberg groups and representations. Let

$$N_n = \left\{ [a, r] = \begin{pmatrix} 1 & \sqrt{-1} {}^t a & \sqrt{-1} |a|^2/2 + r \\ 0 & E_n & a \\ 0 & 0 & 1 \end{pmatrix}; a \in \mathbf{C}^n, r \in \mathbf{R} \right\}$$

be the Heisenberg group with the multiplication law $[a, r][b, s] = [a+b, r+s - (q/2)E(a, b)]$, where $E(\cdot, \cdot)$ is the alternating form on $\mathbf{C}^n \times \mathbf{C}^n$ defined in Section 1. In the notation of Section 1, $N_n = N$ with $n = m-1$. Let

$$\mathfrak{N} = \left\{ X(a, r) := \begin{pmatrix} 0 & \sqrt{-1} {}^t \bar{a} & r \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \in M_{n+2}(\mathbf{C}); a \in \mathbf{C}^n, r \in \mathbf{R} \right\}$$

be the Lie algebra of N_n and $\mathfrak{Z} := [\mathfrak{N}, \mathfrak{N}]$ the center. Then we have $[X(a, r), X(b, s)] = X(0, -qE(a, b))$. In Section 1 we defined a \mathbf{Z} -basis $\{e_1, \dots, e_{2n}\}$ of $L(I)$. If necessary, we choose a new basis of $L(I)$ so that E satisfies the following condition:

$$E(e_i, e_{n+j}) = \delta_{ij} d_i, \quad d_i > 0,$$

and

$$E(e_i, e_j) = E(e_{n+i}, e_{n+j}) = 0 \quad \text{for } 1 \leq i, j \leq n.$$

Put $X_i := X(e_i, 0)$, $Y_i := X(e_{n+i}, 0)$ for $1 \leq i \leq n$ and $Z := X(0, q)$. Then the only nontrivial relations are

$$[X_i, Y_j] = -\delta_{ij} d_i Z.$$

The basis $\{X_1, Y_1, \dots, X_n, Y_n, Z\}$ of the Lie algebra \mathfrak{N} defines a frame α on the tangent space of the compact nilmanifold $I^\tau \cap N_n \setminus N_n$ so that it is an oriented orthonormal frame, and induces the left $I^\tau \cap N_n$ -invariant volume form on the nilmanifold. In this section we denote $I^\tau \cap N_n$ simply by I .

Let $L^2(I \setminus N_n)$ be the space of left I -invariant and square integrable functions on N_n with respect to the I -invariant volume form induced from α . The right quasi-regular representation R_I of N_n on $L^2(I \setminus N_n)$ decomposes discretely into the orthogonal direct sum $R_I = \bigoplus m(\pi) \pi$ ($\pi \in \hat{N}_n$) of irreducible representations, each occurring with finite multiplicity $m(\pi)$. We know unitary irreducible representations of the Heisenberg group N_n (see, for instance, [Mo]):

For $\tau \in \mathfrak{N}^*$ with $\tau|_{\mathfrak{Z}} = 0$, we define the one dimensional representation π_τ by

$$\pi_\tau([a, r]) = \exp(2\pi\sqrt{-1} \tau(X(a, r))) 1.$$

For $c \in \mathbf{R} \setminus \{0\}$, define a representation π_c on $\mathcal{H}_c := L^2(W_2)$ by

$$(\pi_c([a, r])f)(v_2) := \exp(2\pi\sqrt{-1}c(r - qE(w_1, v_2) + \frac{q}{2}E(w_1, w_2)))f(v_2 - w_2),$$

where $W_1 := \{\sum_{i=1}^n t_i X_i; t_i \in \mathbf{R}\} \cong \mathbf{R}^n$, $W_2 := \{\sum_{i=1}^n t_i Y_i; t_i \in \mathbf{R}\} \cong \mathbf{R}^n$ and $a = w_1 + w_2 \in W_1 + W_2 = \mathbf{C}^n$.

Let $L_1^* := \{\tau \in \mathfrak{N}^*; \tau(\mathfrak{Z}) = 0 \text{ and } \tau(\log \Gamma) \subset \mathbf{Z}\} = L(\Gamma)^*$ and $L_2^* := \{\lambda \in \mathfrak{Z}^*; \lambda(\log L_2) \subset \mathbf{Z}\}$. We identify \mathfrak{Z}^* with the multiplications of \mathbf{R} on $\mathfrak{Z} = \mathbf{R}$. Then $L_2^* = (1/q)\mathbf{Z}$ because $\log L_2 = q\mathbf{Z}$.

LEMMA 2.1. The representation R_Γ of N_n on $L^2(\Gamma \setminus N_n)$ decomposes as

$$R_\Gamma = \bigoplus_{\tau \in L_1^*} \pi_{\tau \oplus} \bigoplus_{c \in L_2^*} m(\pi_c) \pi_c,$$

where $m(\pi_c) = |l|^n d_1 \cdots d_n$ if $c = l/q$ for $l \in \mathbf{Z} \setminus \{0\}$.

For the proof see [R] or [Mo, Theorem 37].

2.2. Operator A. Let M be a $(4k-1)$ -dimensional compact oriented manifold without boundary. The tangential signature operator on M is a first order elliptic differential operator acting on square integrable differential forms of even degree defined on $2p$ -forms by $(-1)^{k+p+1}(*d - d*)$, where d is the exterior differential and $*$ is the Hodge star operator defined by the volume form on M .

In this section we define the operator A on $\Gamma \setminus N_{2k-1}$ slightly modifying the tangential signature operator, and in the next section we calculate its η -invariant. We define A on $2p$ -forms by

$$(-1)^{k+p+1}(*d^\nabla - d^\nabla*),$$

where d^∇ is the covariant differential of the flat connection ∇ defined by the frame α . The space of square integrable forms of even degree on $\Gamma \setminus N_{2k-1}$ is identified with $L^2(\Gamma \setminus N_{2k-1}) \otimes_{\mathbf{C}} (\wedge^{\text{ev}} \mathfrak{N}^* \otimes \mathbf{C})$, where $\wedge^{\text{ev}} \mathfrak{N}^* := \bigoplus_{p=0}^{2k-1} \wedge^{2p} \mathfrak{N}^*$ is the set of even degree alternating forms on \mathfrak{N} with values in \mathbf{R} . Put $\mathcal{M} := \wedge^{\text{ev}} \mathfrak{N}^* \otimes_{\mathbf{R}} \mathbf{C}$, which is identified with the space of constant forms of even degree on $\Gamma \setminus N_{2k-1}$.

PROPOSITION 2.1. $\eta(A, 0) = 2^{2k} \sqrt{\det E} \zeta(1-2k)$.

To the proof of the proposition we devote the rest of this section.

In what follows, we put $n = 2k-1$.

LEMMA 2.2. On $L^2(\Gamma \setminus N_n) \otimes_{\mathbf{C}} \mathcal{M}$, the operator A is written as

$$A = \sum_{i=1}^n Y_i \otimes F_i - \sqrt{-1} \sum_{i=1}^n X_i \otimes E_i - \sqrt{-1} Z \otimes E_0,$$

with $E_0, E_i, F_i \in \text{End}(\mathcal{M})$, E_0, E_i are Hermitian and F_i are skew Hermitian. Moreover $E_0^2 = E_i^2 = 1$, $F_i^2 = -1$, and any two distinct matrices among $\{E_0, E_i, F_i$

$(1 \leq i \leq n)$ anticommute.

PROOF. This follows from the fact that A is self-adjoint and that A^2 has the same principal symbol as that of the Laplace-Beltrami operator on forms.

Now put $\Gamma(\lambda) := \{[\lambda a, \lambda^2 r] \in N_n; [a, r] \in \Gamma\}$ for any positive real λ . On the compact nilmanifold $\Gamma(\lambda) \backslash N_n$ the frame α defines a metric, which we denote by g_λ , and an operator A , which we denote by $A(\lambda)$. Consider a diffeomorphism $\varphi_\lambda: \Gamma(\lambda) \backslash N_n \rightarrow \Gamma \backslash N_n$ defined by $\varphi_\lambda(\Gamma(\lambda)[a, r]) = \Gamma[a/\lambda, r/\lambda^2]$. Transform the operator $A(\lambda)$ on $\Gamma(\lambda) \backslash N_n$ to an operator D on $\Gamma \backslash N_n$: For $\Phi \in L^2(\Gamma \backslash N_n) \otimes \mathcal{M}$,

$$D(\Phi)(\Gamma g) := A(\lambda)(\Phi \circ \varphi_\lambda)(\varphi_\lambda^{-1}(\Gamma g)).$$

Then we have for $f \otimes \omega \in L^2(\Gamma \backslash N_n) \otimes \mathcal{M}$

$$D(f \otimes \omega) = \frac{1}{\lambda} \left\{ \sum_{i=1}^n Y_i f \otimes F_i \omega - \sqrt{-1} \sum_{i=1}^n X_i f \otimes E_i \omega \right\} - \frac{\sqrt{-1}}{\lambda^2} Z f \otimes E_0 \omega.$$

REMARK. The operator D is defined by the metric $(\varphi_\lambda^{-1})^* g_\lambda$ on $\Gamma \backslash N_n$, in other words, defined by the frame $\{X_1/\lambda, Y_1/\lambda, \dots, X_n/\lambda, Y_n/\lambda, Z/\lambda^2\}$. Hence the spectrum of D is equal to that of $A(\lambda)$.

Since D is an N_n -invariant operator, we can decompose D into the sum of the operators on the irreducible representation spaces of N_n on $L^2(\Gamma \backslash N_n)$. For the representation π_τ , the operator $D_\tau := \pi_\tau(D)$ on $C1 \otimes \mathcal{M}$ is written as

$$D_\tau = \frac{1}{\lambda} \left\{ \sum_{i=1}^n 2\pi \sqrt{-1} \tau(Y_i) F_i + \sum_{i=1}^n 2\pi \tau(X_i) E_i \right\}.$$

And for π_c the operator $D_c := \pi_c(D)$ on $\mathcal{H}_c \otimes \mathcal{M}$ is written as

$$D_c = -\frac{1}{\lambda} \left\{ \sum_{i=1}^n \frac{\partial}{\partial y_i} \otimes F_i + \sum_{i=1}^n 2\pi c q d_i y_i \otimes E_i \right\} + \frac{2\pi c q}{\lambda^2} \otimes E_0$$

on $L^2(\mathbf{R}^n) \otimes \mathcal{M}$. Thus the η -series of D is written as

$$\eta(D, s) = \sum_{\tau \in L_2^*} \eta(D_\tau, s) + \sum_{l \in \mathbf{Z} \setminus \{0\}} m(l/q) \eta(D_{l/q}, s)$$

for $\text{Re}(s)$ sufficiently large.

LEMMA 2.3. $\eta(D_\tau, s) = 0$ for all $\tau \in L_2^*$.

PROOF. Conjugating by the unitary matrix E_0 , we have $E_0^* D_\tau E_0 = -D_\tau$. Hence D_τ does not contribute to the η -series.

2.3. Determination of $\eta(A, 0)$. First we calculate the eigenvalues of D_c^2 . From the description of D_c in the previous subsection we have

$$D_c^2 = \frac{1}{\lambda^2} \sum_{i=1}^n \left\{ -\frac{\partial^2}{\partial y_i^2} + (2\pi c q d_i y_i)^2 \right\} \otimes \text{id} + \frac{4(\pi c q)^2}{\lambda^4} \otimes \text{id} - \frac{2\pi c q}{\lambda^2} \otimes \sum_{i=1}^n d_i E_i F_i.$$

Put $\Delta_0 := \sum_{i=1}^n \{-\partial^2/\partial y_i^2 + (2\pi c q d_i y_i)^2\}$.

LEMMA 2.4. Δ_0 on $L^2(\mathbf{R}^n)$ has eigenvalues

$$\left\{ 2\pi |c| q \sum_{i=1}^n d_i (2m_i + 1); m = (m_i) \in (\mathbf{Z}_{\geq 0})^n \right\}.$$

PROOF. Let $h_m(x) := (-1)^m e^{x^2} (d/dx)^m e^{-x^2}$ be the Hermite polynomial for nonnegative integer m , which satisfies the Hermite differential equation:

$$\left(\frac{d}{dx}\right)^2 h_m(x) - 2x \frac{d}{dx} h_m(x) + 2m h_m(x) = 0.$$

Set $f_m(x) := e^{-x^2/2} h_m(x)$. Then $\{f_m(x)\}_{m=0}^\infty$ forms a complete orthogonal basis of $L^2(\mathbf{R})$. Set $g_m^{(i)}(y) := f_m(\sqrt{2\pi|c|q d_i} y)$. Then $g_m^{(i)}(y)$ satisfies the differential equation

$$\left[(2\pi c q d_i)^2 y^2 - \left(\frac{d}{dy}\right)^2 \right] g_m^{(i)}(y) = 2\pi |c| q d_i (2m_i + 1) g_m^{(i)}(y).$$

Hence if we put

$$\Phi_m(y) := \prod_{i=1}^n g_{m_i}^{(i)}(y_i) / \prod_{i=1}^n \|g_{m_i}^{(i)}\|_{L^2}$$

for $m = (m_1, \dots, m_n) \in (\mathbf{Z}_{\geq 0})^n$, then $\{\Phi_m\}$ forms a complete orthonormal basis of $L^2(\mathbf{R}^n)$ and satisfies the equation

$$\Delta_0 \Phi_m = 2\pi |c| q \sum_{i=1}^n d_i (2m_i + 1) \Phi_m.$$

Next we diagonalize the operator $\sum_{i=1}^n d_i E_i F_i \in \text{End}(\mathcal{M})$. Since $(E_i F_i)^2 = 1$ and since $E_i F_i$ ($1 \leq i \leq n$) commute with one another, we can decompose \mathcal{M} into the direct sum of $V_\varepsilon := \{v \in \mathcal{M}; E_i F_i v = \varepsilon_i v \text{ for } 1 \leq i \leq n\}$ with $\varepsilon \in \{+1, -1\}^n$. Put $\varepsilon_0 = (+1, \dots, +1)$. Then for any $\varepsilon \in \{+1, -1\}^n$ the mapping

$$\prod_{1 \leq i \leq n, \varepsilon_i = -1} E_i : V_{\varepsilon_0} \longrightarrow V_\varepsilon$$

is bijective. Hence $\dim V_\varepsilon = 2^n$ because $\dim \mathcal{M} = 2^{2n}$. From Lemma 2.4 we have the following

LEMMA 2.5. D_c^2 restricted to $L^2(\mathbf{R}^n) \otimes V_\varepsilon$ has eigenvalues

$$\frac{2\pi |c| q}{\lambda^2} \sum_{i=1}^n d_i (2m_i + 1) + \frac{4\pi^2 c^2 q^2}{\lambda^4} - \frac{2\pi c q}{\lambda^2} \sum_{i=1}^n d_i \varepsilon_i.$$

LEMMA 2.6. If $\mathcal{E} \subset L^2(\mathbf{R}^n) \otimes \mathcal{M}$ is any eigenspace of D_c^2 , then

$$\text{Trace}(D_c|_{\mathcal{E}}) \in \frac{2\pi c q}{\lambda^2} \mathbf{Z}.$$

(Note that each eigenspace of D_c^2 is of finite dimension.)

PROOF. One has $E_0 V_\varepsilon = V_\varepsilon$ and $E_i V_\varepsilon = F_i V_\varepsilon = V_{\delta_i}$ with $\delta = (\delta_j)$, $\delta_i = -\varepsilon_i$, and $\delta_j = \varepsilon_j$ for $j \neq i$. Hence $\text{Trace}(D_c|_{\mathcal{E}}) = (2\pi c q / \lambda^2) \text{Trace}(E_0|_{\mathcal{E}})$. Since $E_0|_{V_\varepsilon} = \pm 1$, $\text{Trace}(E_0) \in \mathbb{Z}$. q.e.d.

By definition we see that for large $\text{Re}(s)$

$$\eta(D_c, s) = \sum_{\substack{\mathcal{E}: \text{eigenspace of } D_c^2 \\ \text{Trace}(D_c|_{\mathcal{E}}) \neq 0}} \eta(D_c|_{\mathcal{E}}, s).$$

We seek eigenspaces \mathcal{E} of D_c^2 with $\text{Trace}(D_c|_{\mathcal{E}}) \neq 0$. From Lemma 2.5 we have for some integer a

$$\text{Trace}(D_c|_{\mathcal{E}}) = a \left(\frac{2\pi|c|q}{\lambda^2} \left\{ \sum_{i=1}^n d_i(2m_i+1) - \frac{c}{|c|} \sum_{i=1}^n d_i \varepsilon_i \right\} + \frac{4\pi^2 c^2 q^2}{\lambda^4} \right)^{1/2}.$$

On the other hand, from Lemma 2.6 we have for some integer b

$$\text{Trace}(D_c|_{\mathcal{E}}) = b \frac{2\pi c q}{\lambda^2}.$$

If $c > 0$, then $a = b$, $m_i = 0$ and $\varepsilon_i = +1$. If $c < 0$, then $a = -b$, $m_i = 0$ and $\varepsilon_i = -1$. Thus the eigenspace \mathcal{E} of D_c^2 with $\text{Trace}(D_c|_{\mathcal{E}}) \neq 0$ is $\mathcal{E} = \mathcal{E}_c := \mathbf{C}\Phi_0 \otimes V_\varepsilon$, $\varepsilon = \varepsilon_0$ if $c > 0$ or $\varepsilon = -\varepsilon_0$ if $c < 0$, and has the eigenvalue $4\pi^2 c^2 q^2 / \lambda^4$. Hence we see that $\text{Trace}(D_c|_{\mathcal{E}_c}) = (2\pi c q / \lambda^2) \text{Trace}(E_0|_{V_\varepsilon})$.

$$\text{LEMMA 2.7.} \quad E_0 \prod_{i=1}^n E_i F_i = 1 \quad \text{on } \mathcal{M}.$$

For the proof see [ADS1, Lemma 10.2] which works also in our case. Or see the argument in p. 262 in [G].

From Lemma 2.7 we have

$$\text{Trace}(D_c|_{\mathcal{E}_c}) = \frac{2\pi|c|q}{\lambda^2} 2^n.$$

Hence we have

$$\begin{aligned} \eta(D, s) &= \sum_{l=1}^{\infty} 2m(l/q) (\lambda^2/2\pi)^s 2^n |l|^{-s} \\ &= 2^{n+1} (\lambda^2/2\pi)^s d_1 d_2 \cdots d_n \sum_{l=1}^{\infty} l^{n-s}. \end{aligned}$$

Thus we have

$$\eta(D, 0) = 2^{n+1} \sqrt{\det E} \zeta(-n),$$

which is independent of positive real λ . Hence $\eta(A, 0) = \eta(A(\lambda), 0) = \eta(D, 0)$. This completes the proof of Proposition 2.1.

REMARK. $\eta(A(\lambda), 0)$ is independent of positive real λ . In other words, $\eta(A, 0)$ is invariant under replacing the metric on $\Gamma \backslash N_n$ by $(\varphi_\lambda^{-1})^* g_\lambda$.

§ 3. Signature defects and eta invariants.

In this section we relate the eta invariant of A with the signature defect by using the similar argument in Sections 13, 14 and 15 in [ADS1].

Let M be an oriented compact manifold of dimension $4k=2n+2$ with $\partial M = \Gamma \setminus N_n$. Assume that M has the metric which is the product metric in a neighborhood of ∂M . From [ADS1, Theorem 13.1] we may write

$$\eta(A, 0) = \int_M \mathcal{D}_0 - l_0,$$

where l_0 is an integer and \mathcal{D}_0 is invariant under scaling of the metric on M . First we will identify the integer l_0 with the difference of the signature.

Let H be the subspace consisting of constant forms in $L^2(\Gamma \setminus N_n) \otimes \mathcal{M}$ and H^\perp the orthogonal complement. Then we have the decomposition $L^2(\Gamma \setminus N_n) \otimes \mathcal{M} = H \oplus H^\perp$.

LEMMA 3.1. $\text{Ker } A(\lambda) = H$, and $A(\lambda)^2 - 4\pi^2/\lambda^4 \geq 0$ on H^\perp .

PROOF. This follows from the estimates

$$D_\tau^2 = 4\pi^2 \|\tau\|^2/\lambda^2 \quad \text{and} \quad D_c^2 \geq 4\pi^2 c^2 q^2/\lambda^4.$$

Now let $B(\lambda)$ be the tangential signature operator on the compact nilmanifold $\Gamma(\lambda) \setminus N_n$. Then for $f \otimes \omega \in L^2(\Gamma \setminus N_n) \otimes \mathcal{M}$

$$B(\lambda)(f \otimes \omega) = A(\lambda)(f \otimes \omega) + f \otimes B_0 \omega,$$

where B_0 is the restriction of $B(\lambda)$ to H and is a constant matrix. Let us deform linearly from $B(\lambda)$ to $A(\lambda)$. Set

$$A_t(\lambda) := tB(\lambda) + (1-t)A(\lambda) = A(\lambda) + tB_0 \quad \text{for } 0 \leq t \leq 1.$$

LEMMA 3.2. We can suitably choose positive λ such that

$$\text{Ker } A_t(\lambda) = \begin{cases} H & \text{for } t=0, \\ \text{Ker } B_0 & \text{for } t>0. \end{cases}$$

PROOF. This follows from Lemma 3.1. We can choose $\lambda > 0$ such that $1/\lambda^2 \geq \|B_0\|/2\pi$.

Fix λ so that it satisfies the condition of Lemma 3.2 and denote $A_t(\lambda)$ by A_t . Then

$$\eta(A_t, 0) = \eta(A_t|_H, 0) + \eta(A_t|_{H^\perp}, 0).$$

LEMMA 3.3. $\eta(A_t|_{H^\perp}, 0)$ is continuous with respect to t .

PROOF. According to Lemma 3.2 we have $\text{Ker}(A_t) \cap H^\perp = 0$. The lemma follows from the proof of [APS2, Proposition 2.1] or from [ADS1, Proposition

5.1].

From [APS1, Theorem 4.2] we have

$$(3.4) \quad l_t = \int_M \mathcal{D}_t - \eta(A_t, 0),$$

where l_t is an integer and \mathcal{D}_t is continuous in t . We can write the equality (3.4) as

$$l_t + \eta(A_{t|H}, 0) = \int_M \mathcal{D}_t - \eta(A_{t|H^\perp}, 0),$$

whose right hand side is continuous in t from Lemma 3.3. But the left hand side has values in integers. Thus we see

$$l_0 + \eta(A_{0|H}, 0) = l_1 + \eta(A_{1|H}, 0).$$

Since $\eta(A_{0|H}, 0) = 0$ from Lemma 3.2 and since $l_1 = \text{sign}(M, \partial M)$ from [APS1, Theorem 4.14], we have

$$l_0 = \text{sign}(M, \partial M) + \eta(A_{1|H}, 0).$$

PROPOSITION 3.5. By $\text{sign}(T, c_1(\mathcal{L}))$ we denote the signature of the bilinear form on $H^{n-1}(T, \mathbf{R})$ defined by $(u \cup v \cup c_1(\mathcal{L}))[T]$ for $u, v \in H^{n-1}(T, \mathbf{R})$. Then we have

$$\eta(A_{1|H}, 0) = -\text{sign}(T, c_1(\mathcal{L})).$$

And hence $l_0 = \text{sign}(M, \partial M) - \text{sign}(T, c_1(\mathcal{L}))$.

PROOF. Applying the argument in pp. 67–68 [APS1] to our situation, we see that $\eta(A_{1|H}, 0)$ is identified with the signature of the bilinear form Q on the space of constant forms with degree $2k-1$ defined by

$$Q(\alpha, \beta) = \int_{\partial M} \alpha \wedge d\beta.$$

In the argument in p. 68 [APS1] the sign is not correct, and should be changed as follows. If $d\alpha$ is an eigenvector of d^* with eigenvalue λ , then

$$Q(\alpha, \alpha) = \langle \alpha, *d\alpha \rangle = \lambda^{-1} \langle \alpha, *d*d\alpha \rangle = \lambda^{-1} \langle d\alpha, d\alpha \rangle.$$

We will show that $\text{sign} Q = -\text{sign}(T, c_1(\mathcal{L}))$. Let ξ^i, η^i and ζ be the 1-forms forming the basis dual to X_i, Y_i and Z . Then $d\xi^i = d\eta^i = 0$ and $d\zeta = \sum_{i=1}^n d_i \xi^i \wedge \eta^i$. If α is a $(2k-1)$ -form in H with $d\alpha \neq 0$, then α has the form $\alpha = \alpha' \wedge \zeta$ for some $\alpha' \in H$. For $\alpha = \alpha' \wedge \zeta$ and $\beta = \beta' \wedge \zeta$ we have

$$Q(\alpha, \beta) = \int_{\partial M} \alpha' \wedge \beta' \wedge d\zeta \wedge \zeta.$$

If we identify $\{\xi^i, \eta^i\}$ with the frame fields of the cotangent bundle of the complex torus $T = L(\Gamma) \setminus \mathbf{C}^n$, then we can easily see that $c_1(\mathcal{L})$ is represented

by the 2-form $-\sum_{i=1}^n d_i \xi^i \wedge \eta^i$, and hence that the signature of Q is equal to $-\text{sign}(T, c_1(\mathcal{L}))$.
q. e. d.

COROLLARY. If we choose M to be the disc bundle associated to \mathcal{L} , then we have $\eta(A_{1|H}, 0) = -\text{sign}(M, \partial M)$, and hence $\int_M \mathcal{D}_0 = \eta(A, 0)$.

PROOF. From [HZ, Theorem 7 in Section 2] we see that $\text{sign}(M, \partial M) = \text{sign}(T, c_1(\mathcal{L}))$.

The first statement of the corollary is Theorem 2.

We want to know the integral $\int_M \mathcal{D}_0$. Unfortunately we can not directly show that the integrand $\mathcal{D}_0|_{\partial M \times I}$ vanishes in contrast with the case in [ADS1].

Let ∇ be the flat connection on the tangent bundle of $I(\lambda) \setminus N_n$ defined by the frame α and T_0 its torsion tensor. Choose a non-negative C^∞ -function f on $I = [0, 1]$ satisfying

$$0 \leq f \leq 1, \quad f([0, 1/4]) = 1 \quad \text{and} \quad f([3/4, 1]) = 0.$$

Define T to be $f(t)T_0$ on $\partial M \times I$ and 0 on $M \setminus \partial M \times I$. Then T is a tensor field on M . There uniquely exists the metric connection ϕ on TM with torsion tensor T (see, for instance, [KN, I]). We denote by $p_j(\phi)$ the j -th Pontrjagin form defined from the curvature form of ϕ by the Chern-Weil theory ([KN, II]). Then

$$L_k(p_1, \dots, p_k)[M, \partial M] = \int_M L_k(p_1(\phi), \dots, p_k(\phi)),$$

where $p_j \in H^{4j}(M, \partial M; \mathbf{Z})$ are the relative Pontrjagin classes associated to the frame α . We put $\mathcal{Q}(\phi) := L_k(p_1(\phi), \dots, p_k(\phi))$ for simplicity. The signature defect is

$$\begin{aligned} \sigma(I \setminus N_n, \alpha) &= \int_M \mathcal{Q}(\phi) - \text{sign}(M, \partial M) \\ &= \int_M \mathcal{Q}(\phi) - l_0 - \text{sign}(T, c_1(\mathcal{L})) \\ &= \int_M \mathcal{Q}(\phi) - \int_M \mathcal{D}_0 + \eta(A, 0) - \text{sign}(T, c_1(\mathcal{L})). \end{aligned}$$

We may choose the connection ϕ so that ϕ defines the integrand \mathcal{D}_0 as in [ADS1, Theorem 13.2]. Since the integrands $\mathcal{Q}(\phi)$ and \mathcal{D}_0 restricted to $M \setminus \partial M \times I$ coincide, the integrals turn out

$$\int_M (\mathcal{Q}(\phi) - \mathcal{D}_0) = \int_{\partial M \times I} (\mathcal{Q}(\phi) - \mathcal{D}_0).$$

Up to now we have seen that

$$(3.6) \quad \int_{\partial M \times I} (\Omega(\phi) - \mathcal{D}_0) = \sigma(\Gamma(\lambda) \setminus N_n, \alpha) - \eta(A(\lambda), 0) + \text{sign}(T, c_1(\mathcal{L})) \\ = \sigma(\Gamma \setminus N_n, \alpha) - \eta(A, 0) + \text{sign}(T, c_1(\mathcal{L})).$$

The right hand side is independent of λ . We will consider the behavior of the left hand side of (3.6) under changing the metric of ∂M by $(\varphi_\lambda^{-1})^* g_\lambda$. The integrand is a $O(4k)$ -invariant $4k$ -form and has weight zero under scaling the metric $g \rightarrow \mu^2 g$. Moreover we have the following

LEMMA 3.7. *On $\partial M \times I$ we have*

$$\Omega(\phi) - \mathcal{D}_0 = \sum a_i(f) P_i(T_0),$$

where $a_i(f)$ is a polynomial in f and in the derivatives of f with values in 1-forms on I , and $P_i(T_0)$ is an $O(4k-1)$ -invariant $(4k-1)$ -form valued polynomial in the components of T_0 and in its covariant derivatives with respect to the flat connection ∇ . Moreover each P_i has nonnegative weight.

For the proof see [ADS1, Proposition 13.5].

Every invariant polynomial is a finite linear combination of elementary monomials $m(T_0)$ in the torsion tensor T_0 with values in q -forms defined in [ABP] by

$$m(T_0) = \sum_q^* T_{\alpha_1} \cdots T_{\alpha_r}.$$

Here α_i are multi-indices, and the sum goes over alternation of precisely q indices and contraction of the remaining ones. Since the torsion tensor T_0 of canonical connection on any reductive homogeneous space of Lie group is parallel (see Chapter X of [KN, II]), the length $|\alpha_i|$ of multi-indices is 3 and $m(T)$ has weight $q-r$. We are concerned with the case $q=4k-1$ and $q \geq 0$.

Therefore we consider only elementary monomials with $r \leq 4k-1$. Let us rename them as $e_{2i-1} := X_i$, $e_{2i} := Y_i$ ($i=1, \dots, n$) and $e_{2n+1} := Z$.

LEMMA 3.8. *With respect to the frame field $\{e_1, \dots, e_{2n+1}\}$ the only non-vanishing components of T are*

$$T_{ij}^{2n+1} = -T_{ji}^{2n+1} = d_i \delta_{j, n+i} \quad \text{for } i=1, \dots, n.$$

LEMMA 3.9. *If we change the metric on $\Gamma \setminus N_n$ by $(\varphi_\lambda^{-1})^* g_\lambda$ for any $\lambda > 0$, then elementary monomials in the torsion tensor with values in $(4k-1)$ -forms change as multiplication by λ^{2n+2} .*

PROOF. Let $m(T_0) = \sum_q^* T_{\alpha_1} \cdots T_{\alpha_r}$ be an elementary monomial under consideration with $q=4k-1$ and $r \leq 4k-1$. T_{α_i} are contracted by g with respect to e_{2n+1} -component. In $m(T_0)$ the numbers of contractions by g^{-1} with respect to e_{2n+1} -component and the other components are $(r-1)/2$ and $r-n$ respectively.

Since $(\varphi_\lambda^{-1})^* g_\lambda$ is the diagonal matrix $\text{diag}(\lambda^2, \dots, \lambda^2, \lambda^4)$ with respect to the frame $\{e_1, \dots, e_{2n+1}\}$, from the number of contractions we see that $m(T_0)$ is multiplied by

$$(\lambda^4)^r (\lambda^{-4})^{(r-1)/2} (\lambda^{-2})^{r-n} = \lambda^{2n+2}$$

under changing of the metric.

q. e. d.

PROPOSITION 3.10. $\int_M \mathcal{D}_0 = L_k(p_1, \dots, p_k)[M, \partial M]$.

PROOF. From Lemmas 3.7 and 3.9 we see that the left hand side of (3.6) vanishes.

q. e. d.

Proposition 3.10 and the corollary to Proposition 3.5 imply the first equality of Theorem 1.

§ 4. Estimate of $\text{sign}(T, c_1(\mathcal{L}))$.

In this section we prove Theorem 3, that is, we calculate $\eta(A_{1|H}, 0) = -\text{sign}(T, c_1(\mathcal{L}))$.

Put $\omega^i := \xi^i \wedge \eta^i$ for $i=1, \dots, n$ and denote $\omega^{i_1} \wedge \dots \wedge \omega^{i_s}$ by ω^I with $I = \{i_1, \dots, i_s\}$, $1 \leq i_1 < \dots < i_s \leq n$. Let F be the complex vector space spanned by ω^I with $|I|=k$. Then $\dim_{\mathbb{C}} F = {}_n C_k = {}_{2k-1} C_k := (2k-1)!/k!(k-1)!$.

PROPOSITION 4.1. $\eta(A_{1|H}, 0)$ is equal to the signature of d^* restricted to F .

PROOF. Since A_1^2 is the Laplace-Beltrami operator acting on even forms, which preserves the degree of forms, we restrict the operator d^* only to $2k$ -forms as far as we are concerned with the eta invariant. Moreover $(A_{1|H})^2$ preserves the types of forms, that is,

$$A_1^2 \xi^I \wedge \eta^J = \sum_i a_i \xi^{I_i} \wedge \eta^{J_i} \quad \text{with } |I_i| = |I|, |J_i| = |J|.$$

and

$$A_1^2 \xi^I \wedge \eta^J \wedge \zeta = \sum_j b_j \xi^{I_j} \wedge \eta^{J_j} \wedge \zeta \quad \text{with } |I_j| = |I|, |J_j| = |J|.$$

Therefore eigenspaces $E \subset H$ of A_1^2 with $\text{Trace}(A_{1|E}) \neq 0$ are generated by the linear combinations of $\xi^I \wedge \eta^J$ with $|I| = |J| = k$.

We define the lexicographic order in the set $\{I \subset \{1, \dots, n\}; |I|=k\}$ by $I < J$ if $i_1 < j_1$ or there exists s with $1 \leq s \leq k-1$ such that $i_1 = j_1, \dots, i_s = j_s$ and $i_{s+1} < j_{s+1}$ for $I = \{i_1, \dots, i_k\}$, $J = \{j_1, \dots, j_k\}$ with $1 \leq i_1 < \dots < i_k \leq n$ and $1 \leq j_1 < \dots < j_k \leq n$. Then two complex vector spaces spanned by $\{\xi^I \wedge \eta^J + \xi^J \wedge \eta^I; I < J\}$ and $\{\xi^I \wedge \eta^J - \xi^J \wedge \eta^I; I < J\}$ have the same dimension and are stable under the operation of d^* , while the matrices of d^* restricted to these spaces have opposite sign because $J^c < I^c$ for $I < J$. Here I^c is the complement of I . Thus we proved the proposition.

From Proposition 4.1 we see that $-\text{sign}(T, c_1(\mathcal{L}))$ is equal to the signature of the operator on $F_R := \sum_{|I|=k} \mathbf{R}\omega^I \subset H^{k,k}(T) \cap H^{2k}(T, \mathbf{R})$ defined by the composition of the exterior product of $\omega_0 := \sum_{i=1}^{2k-1} d_i \omega^i$ after $*$. Since $\text{sign}(T, c_1(\mathcal{L}))$ is an topological invariant, by changing the metric we may identify ω_0 with the Kähler form $\Omega_0 := \sum_{i=1}^{2k-1} \omega^i$ on T .

LEMMA 4.2. *The signature of $\text{ext}(\Omega_0)*$ on F_R is equal to*

$$1 + \sum_{j=1}^{k-1} (-1)^{k+j} \{ {}_{2k-1}C_{k-j} - {}_{2k-1}C_{k-j-1} \}.$$

PROOF. Let F^j be the real vector space generated by $\{\omega^I; |I|=j\}$ and L the exterior multiplication by Ω_0 . Since $F^j \subset H^{j,j}(T) \cap H^{2j}(T, \mathbf{R})$ and since $LF^j \subset F^{j+1}$, from the Hard Lefschetz theorem we can decompose $F^k = F_R$ as

$$\bigoplus_{j=0}^k L^j(P_j \cap F^{k-j}),$$

where $P_j := \text{Ker}(L^{j+1}) \cap H^{k-j, k-j}(T)$. According to the Hodge-Riemann relations, L^* has the sign $(-1)^{k+j}$ on $L^j(P_j \cap F^{k-j})$ for $j \geq 1$ and vanishes on $P_0 \cap F^k$. Thus we have

$$\begin{aligned} \text{sign}(L^*|_{F^k}) &= \sum_{j=1}^k (-1)^{k+j} \dim P_j \cap F^{k-j} \\ &= (-1)^{2k} + \sum_{j=1}^{k-1} (-1)^{k+j} \{ {}_{2k-1}C_{k-j} - {}_{2k-1}C_{k-j-1} \}. \end{aligned}$$

q. e. d.

From Proposition 4.1 and Lemma 4.2 we complete the proof of Theorem 3.

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