# On the Milnor number of a generic hyperplane section 

By Shintaro MIMA

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## § 0. Introduction.

Let $F\left(z_{1}, \cdots, z_{n}\right)$ be an analytic function on an open neighbourhood of the origin $\overrightarrow{0}$ in $\boldsymbol{C}^{n}$ with $F(\overrightarrow{0})=0$ and let $V=F^{-1}(0)$. Suppose that $F(z)$ has an isolated critical point at the origin. Then for sufficiently small $\varepsilon>0$, the map $\phi: S_{\varepsilon}-$ $K_{\mathrm{s}} \rightarrow S^{1}$ which is defined by $\phi(z)=F(z) /|F(z)|$ gives a smooth fiber bundle, which is called the Milnor fibration. Here $S_{\varepsilon}=\left\{z \in \boldsymbol{C}^{n}| | z \mid=\varepsilon\right\}, K_{\varepsilon}=S_{\varepsilon} \cap V$ and $S^{1}=$ $\left\{z \in \boldsymbol{C}||z|=1\}\right.$. Moreover the fiber $X_{t}=\phi^{-1}(t)$ is an ( $n-2$ )-connected $2(n-1)$ dimensional smooth manifold and has the homotopy type of a bouquet $S^{n-1} \vee \cdots$ $\vee S^{n-1}$ of $(n-1)$-spheres ([3]). $\mu^{(n)}=$ the $(n-1)$-th Betti number of $X_{t}$ is usually called the Milnor number of $F$ (or $V$ ). It is important to calculate the Milnor number in order to study topological properties of $V$. Suppose that $F$ is nondegenerate and convenient, then the beautiful formula by Kouchnirenko ([2]) says that $\mu^{(n)}=\nu^{(n)}$, where $\nu^{(n)}$ is the Newton number of $F(\S 1)$. By this formula, we can calculate the Milnor number via the Newton boundary of $F$.

Let $L=\left\{z_{n}=a_{1} z_{1}+\cdots+a_{n-1} z_{n-1}\right\}$ be a generic hyperplane through the origin $\overrightarrow{0}$. $V \cap L=f^{-1}(0)$ is called a generic hyperplane section, where $f\left(z_{1}, \cdots, z_{n-1}\right)=$ $F\left(z_{1}, \cdots, z_{n-1}, a_{1} z_{1}+\cdots+a_{n-1} z_{n-1}\right)$. $f$ has also an isolated critical point at the origin and its Milnor number $\mu^{(n-1)}$ is independent of the choice of $L$. Similarly $\mu^{(i)}(1 \leqq i \leqq n-1)$ can be defined and we define $\mu^{*}$ by $\mu^{*}=\left(\mu^{(n)}, \mu^{(n-1)}, \cdots, \mu^{(1)}\right)$. It is known that $\mu^{*}$ is determined by $F([7])$. However it is not known how $\mu^{*}$ can be calculated for a given $F$. Because, even if $F$ is non-degenerate, $f$ is not necessarily non-degenerate. Hence we cannot apply Kouchnirenko's formula even to $\mu^{(n-1)}$. If $f$ is degenerate, then $\mu^{(n-1)} \geqq \nu^{(n-1)}([2])$ and similarly $\mu^{(i)} \geqq \nu^{(i)}$ ( $1 \leqq i \leqq n-1$ ). Thus in order to calculate $\mu^{*}$, we want to know how the degeneracy index $\alpha^{(i)}=\mu^{(i)}-\nu^{(i)}([4])$ is determined by $F$. In this paper, we will show the following result.

Theorem A. Let $F(x, y, z)$ be an analytic function on an open neighbourhood of the origin in $\boldsymbol{C}^{3}$ with $F(\overrightarrow{0})=0$. Suppose that $F$ has an isolated critical point at the origin and that $F$ is non-degenerate and convenient. Let $z=a x+b y$ be a generic hyperplane and let $f(x, y)=F(x, y, a x+b y)$. Let $\mu^{(2)}$ and $\nu^{(2)}$ be the Milnor number and the Newton number of $f$ respectively. Then the degeneracy
index $\alpha^{(2)}=\mu^{(2)}-\nu^{(2)}$ is given by $\alpha^{(2)}=\tilde{\nu}\left(F^{\prime}\right)$, where $F^{\prime}$ is the associated function of $F$ and $\tilde{\Sigma}$ is the reduced Newton number ( $\S 1$ ).

By this formula, we can calculate the Milnor number of $f$.
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## § 1. Preliminaries.

(1) The Newton boundary ([2], [5]). Let $f\left(z_{1}, \cdots, z_{n}\right)$ be an analytic function on an open neighbourhood of the origin in $\boldsymbol{C}^{n}$ with $f(\overrightarrow{0})=0$. Let $f\left(z_{1}, \cdots, z_{n}\right)=\sum_{\nu} a_{\nu} z^{\nu}\left(z^{\nu}=z_{1}^{\nu} \cdots z_{n}^{\nu n}\right)$ be the Taylor expansion of $f$. We define the polyhedron $\Gamma_{+}(f)$ in $\boldsymbol{R}_{+}^{n}=\left\{\left(x_{1}, \cdots, x_{n}\right) \in \boldsymbol{R}^{n} \mid x_{i} \geqq 0(1 \leqq i \leqq n)\right\}$ by the convex hull of $\left\{\nu+\boldsymbol{R}_{+}^{n} \mid a_{\nu} \neq 0\right\}$. Let $\Gamma(f)$ be the compact polyhedron which is the union of the compact faces of $\Gamma_{+}(f)$ and we call $\Gamma(f)$ the Newton boundary of $f$. We also define $\Gamma_{-}(f)$ by the cone of $\Gamma(f)$ with the origin $\overrightarrow{0}$. For any face $\Delta$ of $\Gamma(f)$, we associate a weighted homogeneous polynomial $f_{\Delta}(z)=\sum_{\nu \in \Delta} a_{\nu} z^{\nu}$. We say that $f$ is non-degenerate on $\Delta$ if $\partial f_{\Delta} / \partial z_{1}=\cdots=\partial f_{\Delta} / \partial z_{n}=0$ has no solution in $\left(\boldsymbol{C}^{*}\right)^{n}$. We say that $f$ is non-degenerate if $f$ is non-degenerate on any face of $\Gamma(f)$. We say that $f$ is convenient if for $i=1, \cdots, n, \Gamma(f)^{(i)}=\left\{\left(x_{1}, \cdots, x_{n}\right)\right.$ $\in \Gamma(f) \mid x_{j}=0$ for $\left.j \neq i\right\}$ is non-empty. In other words, it means that for every $i(1 \leqq i \leqq n), f$ has some monomials $z_{i}^{m i}$ with the non-zero coefficient.

Remark. In general, a different coordinate gives a different Newton boundary. Therefore if we want to specify the coordinate, we denote the Newton boundary by $\Gamma\left(f ;\left(z_{1}, \cdots, z_{n}\right)\right)$ ([4]).
(2) The Newton number ([2], [6]). Let $W$ be a polyhedron in $\boldsymbol{R}_{+}^{n}$. The Newton number $\nu(W)$ is defined by $\Sigma_{I}(-1)^{n-|I|}|I|$-dim. volume $\left(W^{I}\right)$, where the sum is taken for every subset $I$ of $\{1, \cdots, n\}$ and $W^{I}=\left\{\left(x_{1}, \cdots, x_{n}\right) \in W \mid x_{i}=0\right.$ for $i \notin I\}$. The corresponding term for $I=\varnothing$ is $(-1)^{n}$ or 0 according to $\overrightarrow{0} \in W$ or not. Notice that by definition for $W=W_{1} \cup W_{2}, \nu(W)=\nu\left(W_{1}\right)+\nu\left(W_{2}\right)-\nu\left(W_{1} \cap W_{2}\right)$. For a complex analytic function $f\left(z_{1}, \cdots, z_{n}\right)$ with $f(\overrightarrow{0})=0$, the Newton number $\nu^{(n)}$ (or $\nu(f)$ ) is defined by $\nu\left(\Gamma_{-}(f)\right)$.
(3) The associated function and the reduced Newton number. Let $F(x, y, z)$ be a complex analytic function of three variables and convenient ( $F(\overrightarrow{0})=0) . \quad F$ has an expansion in homogeneous polynomials of the form $F=$ $\Sigma_{i} F_{i}$, where $F_{i}$ is the $i$-th degree homogeneous polynomial. We define the degree of $F$ by $d=d(F)=\min \left\{i \mid F_{i} \neq 0\right\}$ and call $F_{d}$ the principal part of $F$. For
$F$, we associate the function $F^{\prime}(u, v)$ of two complex variables which is defined by $F^{\prime}(u, v)=F(u, u, u v)$ and we call it the associated function with respect to $z$ coordinate of $F$. By definition, we can write $F^{\prime}(u, v)=u^{d} F^{\prime \prime}(u, v)$, where $d=d(F)$ and $F^{\prime \prime}(u, v)$ is convenient. For this $F^{\prime}$, we define the polyhedron $\tilde{\Gamma}_{-}\left(F^{\prime}\right)$ in $\boldsymbol{R}_{+}^{2}$ by the cone of $\Gamma_{-}\left(F^{\prime}\right)$ with the vertex $(d, 0)$. Then $\tilde{\nu}\left(F^{\prime}\right)$ is defined by $\tilde{\mathcal{L}}\left(F^{\prime}\right)=\nu\left(\tilde{\Gamma}_{-}\left(F^{\prime}\right)\right)$ and we call it the reduced Newton number.

## § 2. Examples.

(1) $\quad F(x, y, z)=x^{l}+y^{m}+z^{n}+x^{p} y^{q} z^{r}$, where $\frac{p}{l}+\frac{q}{m}+\frac{r}{n}<1$.

Then

$$
f(x, y)=F(x, y, a x+b y)=x^{l}+y^{m}+(a x+b y)^{n}+x^{p} y^{q}(a x+b y)^{r} .
$$

Thus the Newton boundary is as in Figure 2.1. It follows that the Newton number $\nu^{(2)}$ is

$$
\begin{aligned}
\nu^{(2)}=\nu(f) & =r(p+q+r)+p \cdot \min (m, n)+q \cdot \min (l, n)-\min (m, n)-\min (l, n)+1 \\
& =r d+\min (m, n) \cdot(p-1)+\min (l, n) \cdot(q-1)+1,
\end{aligned}
$$

where $d=d(F)=p+q+r$. On the other hand, the associated function

$$
F^{\prime}(u, v)=F(u, u, u v)=u^{l}+u^{m}+u^{n} v^{n}+u^{d} v^{r} .
$$

Thus the Newton boundary of $F^{\prime}$ is as in Figure 2.2.


Figure 2.1.


Figure 2.2.

Hence

$$
\alpha^{(2)}=\mu^{(2)}-\nu^{(2)}=\tilde{\nu}\left(F^{\prime}\right)=(\min (l, m)-d) \cdot(r-1) .
$$

Consequently,

$$
\begin{aligned}
\mu^{(2)} & =r d+\min (m, n) \cdot(p-1)+\min (l, n) \cdot(q-1)+1+(\min (l, m)-d) \cdot(r-1) \\
& =\min (m, n) \cdot(p-1)+\min (l, n) \cdot(q-1)+\min (l, m) \cdot(r-1)+d+1 .
\end{aligned}
$$

(2) (See [6] example 3.3.)

$$
F(x, y, z)=x^{l}+y^{l}+z^{8}+x^{2} z^{5}+x^{3} y z^{3} \quad(l \geqq 16) .
$$

Then

$$
f(x, y)=F\left(x, y^{\prime}, a x+b y\right)=x^{l}+y^{l}+(a x+b y)^{8}+x^{2}(a x+b y)^{5}+x^{3} y(a x+b y)^{3} .
$$

Thus the Newton boundary of $f$ is given in Figure 2.3. It follows that the Newton number $\nu^{(2)}$ is

$$
\nu^{(2)}=\nu(f)=35+16-7-8+1=37 .
$$

On the other hand, the associated function

$$
F^{\prime}(u, v)=F(u, u, u v)=2 u^{l}+u^{8} v^{8}+u^{7} v^{5}+u^{7} v^{3} .
$$

Thus the Newton boundary of $F^{\prime}$ is as in Figure 2.4.


Figure 2.3.


Figure 2.4.

Hence

$$
\alpha^{(2)}=\mu^{(2)}-\nu^{(2)}=\tilde{\nu}\left(F^{\prime}\right)=2(l-7) .
$$

Consequently,

$$
\mu^{(2)}=37+2(l-7)=2 l+23 .
$$

$$
\begin{align*}
F(x, y, z)= & x^{l}+y^{l}+z^{l}+x^{5} y^{5} z^{5}+(x y z)^{3}\left(x^{7}+y^{7}+z^{7}\right)  \tag{3}\\
& +(x y z)^{2}\left(x^{11}+y^{11}+z^{11}\right) \quad(l \geqq 19) .
\end{align*}
$$

The Newton boundary of $F$ is given in Figure 2.5. On the other hand, the associated function

$$
F^{\prime}(u, v)=F(u, u, u v)=2 u^{l}+u^{l} v^{l}+u^{15} v^{5}+u^{16} v^{3}\left(2+v^{7}\right)+u^{17} v^{2}\left(2+v^{11}\right) .
$$

Thus the Newton boundary of $F^{\prime}$ is as in Figure 2.6.


Figure 2.5.


Figure 2.6.

Hence

$$
\alpha^{(2)}=\mu^{(2)}-\nu^{(2)}=\tilde{\nu}\left(F^{\prime}\right)=5+4+2(l-15)-(l-15)=l-6 .
$$

This example shows that the degeneracy index $\alpha^{(2)}$ depends also on the outside faces.

## § 3. Proof of Theorem A.

Let $F(x, y, z)$ be an analytic function on an open neighbourhood of the origin in $\boldsymbol{C}^{3}$ with $F(\overrightarrow{0})=0$. Assume that $F$ has an isolated critical point at the origin and that $F$ is non-degenerate and convenient. $V=F^{-1}(0)$. Let $L=$ $\{z=a x+b y\}$ be a generic hyperplane and let $V^{\prime}=V \cap L=f^{-1}(0)$, where $f(x, y)$ $=F(x, y, a x+b y)$. Let

$$
\begin{aligned}
& \pi: \tilde{V}^{\prime} \longrightarrow V^{\prime} \\
& \tilde{\pi}: \tilde{\boldsymbol{C}}^{2} \longrightarrow \cap^{2}
\end{aligned}
$$

be the good minimal resolution of $V^{\prime}, ~ \tilde{\pi}^{-1}(\overrightarrow{0})=D_{1} \cup \cdots \cup D_{r}$ is the irreducible decomposition of the exceptional divisors $\tilde{\pi}^{-1}(\overrightarrow{0})$. Let $m_{i}$ be the multiplicity of $D_{i}$ in the divisor $\tilde{\pi}^{-1}(\overrightarrow{0})$. Then we use the following lemma.

Lemma 3.1 ([1]). The Euler characteristic $\chi$ of the Milnor fiber determined by $f$ is given by

$$
\begin{equation*}
\chi=\sum_{i=1}^{r} m_{i}\left(2-r_{i}\right), \tag{3.2}
\end{equation*}
$$

where $r_{i}$ is the number of the curves which meet the divisor $D_{i}$.

Furthermore, since the second Betti number of two dimensional connected non-compact manifold is zero, $\chi=1-\mu$ by the Euler-Poincaré formula. Therefore combining this with (3.2), we get

$$
\begin{equation*}
\mu=1+\sum_{i=1}^{r} m_{i}\left(r_{i}-2\right) . \tag{3.3}
\end{equation*}
$$

By this formula, the Milnor number can be calculated via the resolution graph. We define the Milnor number $\mu_{i}$ which is determined by the Euler characteristic of each irreducible component of the exceptional divisors by

$$
\begin{equation*}
\mu_{i}=m_{i}\left(r_{i}-2\right) \tag{3.4}
\end{equation*}
$$

Then by (3.3), we have

$$
\begin{equation*}
\mu=1+\sum_{i=1}^{r} \mu_{i} \tag{3.5}
\end{equation*}
$$

Now we consider the Newton boundary of $f$. The monomial $x^{p} y^{q} z^{r}$ of $F$ transforms into $x^{p} y^{q}(a x+b y)^{r}$ by $z=a x+b y$ and represents the lattice points on the segment connecting two points $(p+r, q),(p, q+r)$ in $\Gamma_{+}(f)$. Let $\Delta_{i}(0 \leqq i \leqq k)$ be the faces of $\Gamma(f)$ and let $\Delta_{0}$ be the face corresponding to the weight $P_{0}=$ $\binom{1}{1}$. In general, $f$ is not necessarily non-degenerate on $\Delta_{0}$ (for example, $F_{d}=$ $x^{p} y^{q} z^{r}(r \geqq 2)$ ). However on the other faces $\Delta_{i}(1 \leqq i \leqq k) f$ is non-degenerate.

Lemma 3.6. $f$ is non-degenerate on $\Delta_{i}(1 \leqq i \leqq k)$.
Proof. We will show that for any face $\Delta_{i}(1 \leqq i \leqq k)$, the equation

$$
\begin{equation*}
\frac{\partial f_{\Lambda_{i}}}{\partial x}=\frac{\partial f_{\Lambda_{i}}}{\partial y}=0 \tag{3.7}
\end{equation*}
$$

has no solution in $\left(\boldsymbol{C}^{*}\right)^{2}$. We denote the dual vector (the weight vector) of $\Delta_{i}$ of $\Gamma(f)$ by $P_{i}=\binom{\alpha}{\beta}$, where we assume $\alpha>\beta$. Let $\tilde{\Delta}_{i}$ be the face of $\Gamma(F)$ corresponding to the dual vector $\tilde{P}_{i}=\left(\begin{array}{l}\boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\beta}\end{array}\right)$. Notice that $f_{\Lambda_{i}}(x, y)=F_{\tilde{\Lambda}_{i}}(x, y, b y)$. (3.7) implies that

$$
\begin{align*}
& \frac{\partial f_{\Lambda_{i}}}{\partial x}=\frac{\partial F_{\tilde{\Lambda}_{i}}}{\partial x}=0,  \tag{3.8}\\
& \frac{\partial f_{\Lambda_{i}}}{\partial y}=\frac{\partial F_{\tilde{\Lambda}_{i}}}{\partial y}+b \frac{\partial F_{\tilde{\Delta}_{i}}}{\partial z}=0 . \tag{3.9}
\end{align*}
$$

Since $f_{\Delta_{i}}$ is weighted homogeneous with the weight $P_{i}$, the Euler equation of $f_{\Delta_{i}}$ says that $c \cdot f_{A_{i}}=\alpha x\left(\partial f_{\Delta_{i}} / \partial x\right)+\beta y\left(\partial f_{A_{i}} / \partial y\right)$, where $c$ is a positive constant. By (3.7), $f_{\Lambda_{i}}(x, y)=0$. Thus

$$
\begin{equation*}
F_{\tilde{\Delta}_{i}}(x, y, b y)=0 . \tag{3.10}
\end{equation*}
$$

Suppose that for any $b$, the system of equations (3.8), (3.9), (3.10) has a solution in $\left(\boldsymbol{C}^{*}\right)^{3}$. Then by Curve Selection Lemma ([3]), we can find a real analytic curve $p(t)=(x(t), y(t), b(t) y(t))(0 \leqq t \leqq \varepsilon)$ such that

$$
\begin{equation*}
F_{\tilde{\beth}_{i}}(p(t))=F_{\tilde{y}_{i}}(x(t), y(t), b(t) y(t)) \equiv 0 \quad \text { and } \quad \frac{d b}{d t} \not \equiv 0 . \tag{3.11}
\end{equation*}
$$

Differentiating (3.11) in $t$, we get

$$
\frac{\partial F_{\tilde{\mathcal{A}}_{i}}}{\partial t}=\frac{\partial F_{\tilde{a}_{i}}}{\partial x} \frac{d x}{d t}+\frac{\partial F_{\tilde{\Lambda}_{i}}}{\partial y} \frac{d y}{d t}+\frac{\partial F_{\tilde{\Lambda}_{i}}}{\partial z}\left(\frac{d b}{d t} y(t)+b(t) \frac{d y}{d t}\right) \equiv 0 .
$$

By (3.8) and (3.9), this implies that $\left(\partial F_{\tilde{\boldsymbol{u}}_{i}} / \partial z\right)(d b / d t) y(t) \equiv 0$. Since $d b / d t \not \equiv 0$, we get $\partial F_{\tilde{d}_{i}} / \partial z=0$ for some $t_{0} \in[0, \varepsilon]$. By $(3.9)$, the last equality implies that $\partial F_{\tilde{\Lambda}_{i}} / \partial y=0$. Therefore on the curve $p(t) \in\left(\boldsymbol{C}^{*}\right)^{3}$, we have $\partial F_{\tilde{\Lambda}_{i}} / \partial x=\partial{\tilde{\tilde{u}_{i}}}_{i} / \partial y=$ $\partial F_{\tilde{y}_{i}} / \partial z=0$. However this contradicts the assumption that $F$ is non-degenerate. It follows that for some $b$ the system of equations (3.8), (3.9), (3.10) has no solution in $\left(\boldsymbol{C}^{*}\right)^{3}$. Therefore (3.7) has no solutions in $\left(\boldsymbol{C}^{*}\right)^{2}$ either. In the case of $\alpha<\beta$, the proof can be done similarly.
Q.E.D.

We consider a toroidal blowing-up $\hat{\pi}: \widetilde{\boldsymbol{C}}^{2} \rightarrow \boldsymbol{C}^{2}$ which is associated with the dual Newton diagram of $f$. (We use the same terminology of M. Oka [5] unless otherwise stated.) It is obvious by Lemma 3.6 that $\hat{\pi}^{-1}(0)$ has only normal crossing singularities except on the divisor $\hat{E}\left(P_{0}\right)$ where $P_{0}=\binom{1}{1}$. Therefore by Lemma 3.1, the non-degenerate face of $\Gamma(f)$ does not contribute to the degeneracy index and in order to calculate $\alpha^{(2)}$ it suffices to calculate the contribution of $\Delta_{0}$, on which $f$ is degenerate, to $\mu^{(2)}$ and $\nu^{(2)}$. We denote these contributions by $\mu_{0}$ and $\nu_{0}$ respectively. To carry out calculation, we distinguish two cases.

Case 1) $\Delta_{0} \cap\{$ the $x$-axis $\}=\varnothing$ and $\Delta_{0} \cap\{$ the $y$-axis $\}=\varnothing$. Assume that the principal part of $F_{d}$ of $F$ has the form

$$
\begin{equation*}
F_{d}=a_{\nu_{1}} x^{p_{1}} y^{q_{1}} z^{r_{1}}+\cdots+a_{\nu_{t}} x^{p_{t}} y^{q} z^{q_{t}}, \quad p_{j}+q_{j}+r_{j}=d \quad(1 \leqq j \leqq t) . \tag{3.12}
\end{equation*}
$$

Define $p_{\min }=\min \left\{p_{j}\right\}, q_{\min }=\min \left\{q_{j}\right\}$ and $r_{\min }=\min \left\{r_{j}\right\}$. Then the face $\Delta_{0}$ of $\Gamma(f)$ is given in Figure 3.13. From this

$$
\nu_{0}=\operatorname{det}\left(\begin{array}{lr}
d-q_{\min } & p_{\min }  \tag{3.14}\\
q_{\min } & d-p_{\min }
\end{array}\right)=d\left(d-p_{\min }-q_{\min }\right) .
$$

Next we will study the divisor $\hat{\pi}^{-1}(0)$ on $\hat{E}\left(P_{0}\right)$. By (3.12), the dual vector of $\Delta_{0}$ is $P_{0}=\binom{1}{1}$. Let $Q$ be the vertex of $\Sigma^{*}$ which is adjacent to $P_{0}$ as in Figure 3.15.


Figure 3.13.


Figure 3.15.
By the assumption, $P_{0}$ is not adjacent to $\binom{0}{1}$ or $\binom{1}{0}$. Let $Q=\binom{\alpha-1}{\alpha}$ and let $\left(y_{1}, y_{2}\right)$ be the coordinate of the affine space $\boldsymbol{C}_{\sigma}^{2}$ where $\sigma=\left(P_{0}, Q\right)$. Then $(x, y)$ can be written as follows in the local coordinate ( $y_{1}, y_{2}$ )

$$
\left\{\begin{array}{l}
x=y_{1} y_{2}^{\alpha-1}  \tag{3.16}\\
y=y_{1} y_{2}^{\alpha} .
\end{array}\right.
$$

On the other hand,

$$
\begin{aligned}
f_{\Lambda_{0}}(x, y) & =F_{d}(x, y, a x+b y) \\
& =a_{\nu_{1}} x^{p_{1}} y^{q_{1}}(a x+b y)^{r_{1}}+\cdots+a_{\nu_{t}} x^{p_{t}} y^{q_{t}}(a x+b y)^{r_{t}} .
\end{aligned}
$$

We want to write $f_{A_{0}}$ in the coordinate $\left(y_{1}, y_{2}\right)$. Define $M_{j}=a_{\nu_{j}} x^{p_{j}} y^{q_{j}}(a x+b y)^{r_{j}}$ $(1 \leqq j \leqq t)$. By (3.16), in the local coordinate ( $y_{1}, y_{2}$ )

$$
\begin{align*}
M_{j} & =a_{\nu j} y_{1}^{d} y_{2}{ }^{p_{j}(\alpha-1)+q_{j} \alpha+r_{j}(\alpha-1)}\left(a+b y_{2}\right)^{r_{j}}  \tag{3.17}\\
& =a_{\nu j} y_{1}^{d} y_{2}^{d(\alpha-1)+q_{j}}\left(a+b y_{2}\right)^{r_{j}} \quad(\text { by } 3.12) . \tag{3.18}
\end{align*}
$$

Since $q_{\min } \leqq q_{j}$ and $r_{\min } \leqq r_{j}$, we can write

$$
f_{A_{0}}(\hat{\pi}(y))=\sum_{j=1}^{t} M_{j}(\hat{\pi}(y))=y_{1}^{d} y_{2}^{d(\alpha-1)+q_{\min }\left(a+b y_{2}\right)^{r_{\min }} g\left(y_{2}\right), ~, ~, ~}
$$

where $g\left(y_{2}\right)$ is the polynomial with $g(0) \neq 0$. Now we will calculate the degree of $g\left(y_{2}\right)$. By (3.18), the degree of $M_{j}$ in $y_{2}$ is equal to $d(\alpha-1)+q_{j}+r_{j}=d \alpha-p_{j}$ (by 3.12). Since $p_{\min } \leqq p_{j}$, the degree of $g\left(y_{2}\right)$ is equal to $d \alpha-p_{\min }-(d(\alpha-1)$ $\left.+q_{\min }+r_{\min }\right)=d-p_{\min }-q_{\min }-r_{\min }$. Since $a$ and $b$ are generic, we may assume that $g\left(y_{2}\right)=0$ has only simple roots. Then in the local coordinate $\left(y_{1}, y_{2}\right)$, we can write

$$
\begin{equation*}
f_{\Delta_{0}}=y_{1}^{d} y_{2}^{d(\alpha-1)+q_{\min }\left(a+b y_{2}\right)^{r_{\min }} \prod_{j=1}^{d-p_{\min }-q_{\min }-r_{\min }}\left(y_{2}+\xi_{j}\right) . . . . .} \tag{3.19}
\end{equation*}
$$

Consequently the graph of $\hat{\pi}^{-1}(0)$ is given in Figure (3.20).


Figure 3.20.
However, since this still has a singularity at $y_{1}=0, a+b y_{2}=0$, we need to resolve it. Put $a+b y_{2}=b \tilde{y}_{2},-a / b=a^{\prime}$, then $y_{2}=\tilde{y}_{2}+a^{\prime}$. Therefore in the local coordinate ( $y_{1}, \tilde{y}_{2}$ ),

$$
\begin{aligned}
f(x, y) & =F(x, y, a x+b y) \\
& =F\left(y_{1} y_{2}^{\alpha-1}, y_{1} y_{2}^{\alpha}, y_{1} y_{2}^{\alpha-1}\left(a+b y_{2}\right)\right) \\
& =F\left(y_{1}\left(\tilde{y}_{2}+a^{\prime}\right)^{\alpha-1}, y_{1}\left(\tilde{y}_{2}+a^{\prime}\right)^{\alpha}, y_{1}\left(\tilde{y}_{2}+a^{\prime}\right)^{\alpha-1} \tilde{y}_{2}\right) .
\end{aligned} \text { by (3.16) }
$$

Now consider the monomial $h(x, y, z)=a_{s} x^{p} y^{q} z^{r}(p+q+r=s)$ of $F$.

$$
h\left(y_{1}\left(\tilde{y}_{2}+a^{\prime}\right)^{\alpha-1}, y_{1}\left(\tilde{y}_{2}+a^{\prime}\right)^{\alpha}, y_{1}\left(\tilde{y}_{2}+a^{\prime}\right)^{\alpha-1} \tilde{y}_{2}\right)=a_{s} y_{1}^{s} \tilde{y}_{2}^{r}\left(\tilde{y}_{2}+a^{\prime}\right)^{d(\tilde{P})},
$$

where $d(\tilde{P})=p(\alpha-1)+q \alpha+r(\alpha-1)$. Since this polynomial becomes $d(\tilde{P})+1$ vertices $(s, r),(s, r+1), \cdots,(s, r+d(\tilde{P}))$ in $\Gamma_{+}\left(h ;\left(y_{1}, \tilde{y}_{2}\right)\right)$, the vertices other than $(s, r)$ do not influence the Newton boundary of $h$. Then

$$
\begin{gathered}
\Gamma\left(h\left(y_{1}\left(\tilde{y}_{2}+a^{\prime}\right)^{\alpha-1}, y_{1}\left(\tilde{y}_{2}+a^{\prime}\right)^{\alpha}, y_{1}\left(\tilde{y}_{2}+a^{\prime}\right)^{\alpha-1} \tilde{y}_{2}\right) ;\left(y_{1}, \tilde{y}_{2}\right)\right) \\
=\Gamma\left(h\left(a^{\prime \alpha-1} y_{1}, a^{\prime \alpha} y_{1}, a^{\alpha-1} y_{1} \tilde{y}_{2}\right) ;\left(y_{1}, \tilde{y}_{2}\right)\right) .
\end{gathered}
$$

Therefore

$$
\begin{align*}
\Gamma\left(f ;\left(y_{1}, \tilde{y}_{2}\right)\right) & =\Gamma\left(F\left(y_{1}\left(\tilde{y}_{2}+a^{\prime}\right)^{\alpha-1}, y_{1}\left(\tilde{y}_{2}+a^{\prime}\right)^{\alpha}, y_{1}\left(\tilde{y}_{2}+a^{\prime}\right)^{\alpha-1} \tilde{y}_{2}\right) ;\left(y_{1}, \tilde{y}_{2}\right)\right) \\
& =\Gamma\left(F\left(a^{\prime \alpha-1} y_{1}, a^{\prime \alpha} y_{1}, a^{\prime \alpha-1} y_{1} \tilde{y}_{2}\right) ;\left(y_{1}, \tilde{y}_{2}\right)\right) . \tag{3.21}
\end{align*}
$$

Since the coordinate change (linear transformation) $(x, y, z) \mapsto\left(a^{\prime \alpha-1} x, a^{\prime \alpha} y, a^{\alpha-1} z\right)$ in $\boldsymbol{C}^{3}$ does not change the Newton boundary, by (3.21)

$$
\begin{equation*}
\Gamma\left(f ;\left(y_{1}, \tilde{y}_{2}\right)\right)=\Gamma\left(F\left(y_{1}, y_{1}, y_{1} \tilde{y}_{2}\right) ;\left(y_{1}, \tilde{y}_{2}\right)\right) . \tag{3.22}
\end{equation*}
$$

Consider the associated function of $F(\S 1) F^{\prime}(u, v)=F(u, u, u v)$. Then by (3.22), $\Gamma\left(f ;\left(y_{1}, \tilde{y}_{2}\right)\right)=\Gamma\left(F^{\prime}(u, v) ;(u, v)\right)$. By definition, we can write $F^{\prime}(u, v)$ $=u^{d} F^{\prime \prime}(u, v)$ where $d=d(F)$ and $F^{\prime \prime}$ is convenient. Moreover the principal part $F_{d}$ can be written $F_{d}(u, u, u v)=u^{d} v^{r_{\min }} k(v)$ where $k(v)$ is the polynomial with $k(0) \neq 0$. Therefore $\Gamma\left(f ;\left(y_{1}, \tilde{y}_{2}\right)\right)$ is as in Figure (3.23).


Figure 3.23.
Lemma 3.24. $f\left(y_{1}, \tilde{y}_{2}\right)$ is non-degenerate.
Proof. By (3.21) it suffices to show that $f_{P}\left(y_{1}, \tilde{y}_{2}\right):=F\left(c^{\alpha-1} y_{1}, c^{\alpha} y_{1}, c^{\alpha-1} y_{1} \tilde{y}_{2}\right)$ is non-degenerate. (We denote $a^{\prime}$ by $c$.) In other words, we will show that for any face $\bar{\Delta}$ of $\Gamma\left(f_{P} ;\left(y_{1}, \tilde{y}_{2}\right)\right)$,

$$
\begin{equation*}
\frac{\partial f_{P, \bar{U}}}{\partial y_{1}}=\frac{\partial f_{P, \bar{U}}}{\partial \tilde{y}_{2}}=0 \tag{3.25}
\end{equation*}
$$

has no solution in $\left(\boldsymbol{C}^{*}\right)^{2}$. We denote the dual vector of $\bar{\Delta}$ by $\binom{\boldsymbol{\sigma}}{\tau}$, where we assume $\sigma<\tau$. Let $\tilde{\Delta}$ be the face of $\Gamma(F)$ corresponding to the dual vector $\left(\begin{array}{l}\boldsymbol{\sigma} \\ \tau \\ \boldsymbol{\sigma}\end{array}\right)$. Then (3.25) implies that

$$
\frac{\partial f_{P, \bar{U}}}{\partial y_{1}}=c^{\alpha-1} \frac{\partial F_{\tilde{A}}}{\partial x}+c^{\alpha} \frac{\partial F_{\tilde{A}}}{\partial y}+c^{\alpha-1} \tilde{y}_{2} \frac{\partial F_{\tilde{A}}}{\partial z}=0, \quad \frac{\partial f_{P, \tilde{I}}}{\partial \tilde{y}_{2}}=c^{\alpha-1} y_{1} \frac{\partial F_{\tilde{\mathcal{A}}}}{\partial z}=0 .
$$

We may assume $c \neq 0$. Thus we get on $\left(C^{*}\right)^{3}$

$$
\begin{gather*}
\frac{\partial F_{\tilde{\sim}}}{\partial x}+c \frac{\partial F_{\tilde{\sim}}}{\partial y}=0,  \tag{3.26}\\
\frac{\partial F_{\tilde{\sim}}}{\partial z}=0 . \tag{3.27}
\end{gather*}
$$

As in the proof of Lemma 3.6,

$$
\begin{equation*}
F_{\tilde{\partial}}\left(c^{\alpha-1} y_{1}, c^{\alpha} y_{1}, c^{\alpha-1} y_{1} \tilde{y}_{2}\right)=10 . \tag{3.28}
\end{equation*}
$$

Suppose that for any $c$ the system of equations (3.26), (3.27), (3.28) has a solution in $\left(\boldsymbol{C}^{*}\right)^{3}$. Then by the Curve Selection Lemma ([3]) we can find a real analytic curve $p(t)=\left(c(t)^{\alpha-1} y_{1}(t), c(t)^{\alpha} y_{1}(t), c(t)^{\alpha-1} y_{1}(t) \tilde{y}_{2}(t)\right)(0 \leqq t \leqq \varepsilon)$ such that

$$
\begin{equation*}
F_{\tilde{y}}(p(t))=F_{\tilde{X}}\left(c(t)^{\alpha-1} y_{1}(t), c(t)^{\alpha} y_{1}(t), c(t)^{\alpha-1} y_{1}(t) \tilde{y}_{2}(t)\right) \equiv 0 \tag{3.29}
\end{equation*}
$$

and $d c / d t \not \equiv 0$. Differentiating (3.29) in $t$, we get

$$
\begin{aligned}
\frac{\partial F_{\tilde{u}}}{\partial t}= & \frac{\partial F_{\tilde{d}}}{\partial x}\left(c(t)^{\alpha-1} \frac{d y_{1}}{d t}+(\alpha-1) c(t)^{\alpha-2} y_{1}(t) \frac{d c}{d t}\right) \\
& +\frac{\partial F_{\tilde{\tilde{d}}}}{\partial y}\left(c(t)^{\alpha} \frac{d y_{1}}{d t}+\alpha c(t)^{\alpha-1} y_{1}(t) \frac{d c}{d t}\right)+\frac{\partial F_{\tilde{u}}}{\partial z} \frac{d}{d t}\left(c(t)^{\alpha-1} y_{1}(t) \tilde{y}_{2}(t)\right) \equiv 0 .
\end{aligned}
$$

Using (3.27), we rewrite this as follows:

$$
c^{\alpha-1} \frac{d y_{1}}{d t}\left(\frac{\partial F_{\tilde{A}}}{\partial x}+c \frac{\partial F_{\tilde{A}}}{\partial y}\right)+\alpha c^{\alpha-1} y_{1} \frac{d c}{d t}\left(\frac{\partial F_{\tilde{A}}}{\partial x}+c \frac{\partial F_{\tilde{A}}}{\partial y}\right)-c^{\alpha-2} y_{1} \frac{\partial F_{\tilde{J}}}{\partial x} \frac{d c}{d t} \equiv 0 .
$$

By (3.26), we obtain $c(t)^{\alpha-2} y_{1}(t)\left(\partial F_{\tilde{\mu}} / \partial x\right)(d c / d t) \equiv 0$. Since $c(t)^{\alpha-2} y_{1}(t) d c / d t \equiv 0$, for some $t_{0} \in[0, \varepsilon]$, we get $\partial F_{\tilde{\jmath}} / \partial x=0$. By (3.26), we have $\partial F \tilde{y} / \partial y=0$. Therefore on the curve $p(t) \in\left(C^{*}\right)^{3}$, we obtain $\partial F_{\tilde{\jmath}} / \partial x=\partial F_{\tilde{\jmath}} / \partial y=\partial F_{\tilde{\jmath}} / \partial z=0$. However this contradicts the assumption that $F$ is non-degenerate. It follows that for some $c$ the system of equations (3.26), (3.27), (3.28) has no solution in $\left(\boldsymbol{C}^{*}\right)^{3}$. Therefore (3.25) has no solutions in $\left(\boldsymbol{C}^{*}\right)^{2}$, either.
Q.E.D.

Using again the troidal blowing-up for $f\left(y_{1}, \tilde{y}_{2}\right)$, we can see that the resolution graph is given in Figure 3.30. By Lemma 3.24, the Milnor number of the divisors $\hat{E}\left(P_{0}\right)$ is the Newton number of $\Gamma\left(F^{\prime}\right)$, that is $\nu\left(F^{\prime}\right)$. By (3.4), we get $\mu_{0}=\nu\left(F^{\prime}\right)+d\left(d-p_{\min }-q_{\text {min }}-r_{\text {min }}+1\right)$. Then

$$
\begin{aligned}
\alpha^{(2)} & =\mu^{(2)}-\nu^{(2)}=\mu_{0}-\nu_{0} \\
& =\nu\left(F^{\prime}\right)+d\left(d-p_{\min }-q_{\min }-r_{\min }+1\right)-d\left(d-p_{\min }-q_{\min }\right) \\
& =\nu\left(F^{\prime}\right)-\left(d r_{\min }-d\right)=\nu\left(F^{\prime}\right)-\nu(W)=\tilde{\nu}\left(F^{\prime}\right) .
\end{aligned}
$$



Figure 3.30 .
Here the polyhedron $W=\left\{(u, v) \in \boldsymbol{R}_{+}^{2} \mid r_{\min } u \geqq d v, 0 \leqq u \leqq d, v \geqq 0\right\}$ (see Figure 3.31) and $\nu\left(W \cap \tilde{\Gamma}_{-}\left(F^{\prime}\right)\right)=0$. We complete the proof of Case 1 ).

Case 2) $\Delta_{0} \cap\{$ the $x$-axis $\} \neq \varnothing$ or $\Delta_{0} \cap\{$ the $y$-axis $\} \neq \varnothing$. We may assume that $\Delta_{0} \cap\{$ the $x$-axis $\} \neq \varnothing$ and $\Delta_{0} \cap\{$ the $y$-axis $\}=\varnothing$. The other cases can be proved similarly. The face $\Delta_{0}$ of $\Gamma(f)$ is given in Figure 3.32.


Figure 3.31 .


Figure 3.32.

From this

$$
\nu_{0}=\operatorname{det}\left(\begin{array}{cc}
d & p_{\min }  \tag{3.33}\\
0 & d-p_{\min }
\end{array}\right)-d=d\left(d-p_{\min }\right)-d .
$$

The dual Newton diagram $\Gamma^{*}(f)$ of $\Gamma(f)$ is as in Figure (3.34). Recall that the
dual vector of $\Delta_{0}$ is $P_{0}=\binom{1}{1}$.


Figure 334.
By the assumption $P_{0}$ is adjacent to $\binom{0}{1}$, but not to $\binom{1}{0}$. Notice that $q_{\text {min }}=0$, as in the case 1) (see Figure 3.30) we draw the resolution graph as in Figure 3.35.


Figure 3.35 .
Since $P_{0}$ is adjacent to $\binom{0}{1}$, the exceptional divisor $D$ corresponding to $\binom{0}{1}$ contributes to $\mu_{0}$. Therefore as in the case 1 ), by (3.4)

$$
\mu_{0}=\nu\left(F^{\prime}\right)+d\left(d-p_{\min }-r_{\min }+1\right)-d
$$

Since $\Gamma\left(f ;\left(y_{1}, \tilde{y}_{2}\right)\right)$ is the same as that of the case 1 ) (see Figure 3.23), by (3.33)

$$
\begin{aligned}
\alpha^{(2)}=\mu_{0}-\nu_{0} & =\nu\left(F^{\prime}\right)+d\left(d-p_{\min }-r_{\min }+1\right)-d-\left(d\left(d-p_{\min }\right)-d\right) \\
& =\nu\left(F^{\prime}\right)-\left(d r_{\min }-d\right)=\tilde{\nu}\left(F^{\prime}\right) . \quad \text { Q.E.D. }
\end{aligned}
$$

Corollary 3.36. Suppose that $F(x, y, z)$ is a complex analytic function with an isolated critical point at the origin $\overrightarrow{0} \in \boldsymbol{C}^{3}$ and that $F$ is non-degenerate and
convenient. Then $\mu^{(2)}=\nu^{(2)}$ if and only if the principal part $F_{d}$ of $F$ is the polynomial whose degree in $z$ is equal or less than one (where the 0-th degree polynomial in $z$ means that it is independent of $z$ ).

Proof. We first show sufficiency. We consider two cases.

1) $F_{d}$ is the 1 -st degree polynomial in $z$.
2) $F_{d}$ is the 0-th degree polynomial in $z$.

Case 1) Since we can write $F(x, y, z)=a x^{\alpha} y^{\beta} z+[$ degree $\geqq \alpha+\beta+1] . F^{\prime}(u, v)$ $=F(u, u, u v)=a u^{\alpha+\beta+1} v+[$ higher terms $]$. The Newton boundary of $F^{\prime}$ is given in Figure 3.37. Then $\alpha^{(2)}=\mu^{(2)}-\nu^{(2)}=\tilde{\nu}\left(F^{\prime}\right)=(m-(\alpha+\beta+1)) \cdot 1-(m-(\alpha+\beta+1))$ $=0$.

Case 2) We can write $F(x, y, z)=b x^{d}+[$ degree $\geqq d]$ or $b y^{d}+[$ degree $\geqq d]$. Then $F^{\prime}(u, v)=b u^{d}+[$ higher terms $]$. By definition, $\alpha^{(2)}=0$.

Next we show necessity. Assume that the Newton boundary of $F^{\prime}$ is given in Figure 3.38.


Figure 3.37


Figure 3.38.

Take the cone of each face $\Delta_{i}^{\prime}(1 \leqq i \leqq k)$ with the vertex ( $d, 0$ ). We denote its Newton number by $\nu_{i}^{\prime}$ respectively. By the definition of the Newton number

$$
\begin{equation*}
\nu_{i}^{\prime}>0 \quad(1 \leqq i \leqq k-1), \quad \nu_{k}^{\prime}=\left(v_{k}-1\right)\left(u_{k+1}-d\right) \geqq 0 \tag{3.39}
\end{equation*}
$$

On the other hand, $\alpha^{(2)}=\tilde{\nu}\left(F^{\prime}\right)=\nu_{1}^{\prime}+\cdots+\nu_{k}^{\prime}$. Suppose that $\alpha^{(2)}=0$, then it follows that $\Gamma\left(F^{\prime}\right)$ cannot have the faces $\Delta_{1}^{\prime}, \cdots, \Delta_{k-1}^{\prime}$ by (3.39) and that $\nu_{k}^{\prime}=$ $\left(v_{k}-1\right)\left(u_{k+1}-d\right)=0$. Therefore we get $v_{k}=1$ or $u_{k+1}=d$, which correspond to Case 1) and Case 2) respectively.
Q.E.D.

So far we have fixed the generic hyperplane $z=a x+b y$. However if the principal part $F_{d}$ has the variable whose degree is equal or less than one, say $x$, then we are well to think of $x=b y+c z$ as the generic hyperplane. In this case, since by Lemma 3.36 the degeneracy index is equal to zero, we can calculate $\mu^{(2)}$ directly via the Newton boundary (of $F(b y+c z, y, z)$ ).

Remark. (1) Recall that the resolution graph of $V^{\prime}$ is as in Figure 3.30. There we took $z=a x+b y$ as the generic hyperplane. The resolution graph remains the same as Figure 3.30, however, in fact, even if we take $x=b y+c z$


Figure 3.40 .


$$
\begin{aligned}
& \text { where }\left(l_{x}, 0,0\right)=\Gamma(F) \cap\{y=z=0\} \\
& \left(0, l_{y}, 0\right)=\Gamma(F) \cap\{x=z=0\}, \quad\left(0,0, l_{z}\right)=\Gamma(F) \cap\{x=y=0\}
\end{aligned}
$$

Figure 3.41.
or $y=a x+c z$ as the generic hyperplane. Recall the resolution of $V^{\prime}$. Even often we perform the toroidal blowing-up from $\Gamma(f)$, the singularity still remains because $\Gamma(f)$ has the degenerate face. And again we perform the toroidal blowing-up. Then we obtain the resolution graph of Figure 3.30.
(2) We simplify Figure 3.30 and denote the exceptional divisors by $D_{x}, D_{y}$, $D_{z}$ as Figure 3.40. Then the troidal resolutions are as Figure 3.41 according as we take $x=b y+c z, y=a x+c z$ or $z=a x+b y$ as the generic hyperplane respectively.

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Shintaro Mima<br>Investment Technology and Research Division The Nikko Securities Co., Ltd.<br>3-1, Marunouchi 3-chome<br>Chiyoda-ku, Tokyo 100<br>Japan

