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A graph for Kleinian groups

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Introduction.

A finitely generated Kleinian group G has some special kind of subgroups. They are, for example, component subgroups, web and nest subgroups. By means of those subgroups we shall construct a graph on which G acts without inversion. This is down by a special choice of the vertex set. An edge is a separator of web type, that is a separator which lies on the boundary of only one component of G. The extremities of an edge are different ones, one is associated with components and the other is a web. This would imply G acts without inversion.

The most part of this article is devoted to construct the vertices and to determine their stabilizers, which are associated with components. Here we outline the idea of this procedure. To each component Δ of G we associate a set $\mathcal{L}(\Delta)$ which consists of components of G linked by separators of nest type to Δ. A separator is of nest type if it lies on the boundaries of two components of G. The linkage means a sequence of components interleaved with separators of nest type. Then we show that the stability subgroup of $\mathcal{L}(\Delta)$ in G is either a component subgroup or a nest subgroup. This is in Section 3 following the preliminary Sections 1 and 2. The construction of the graph is in Section 4.

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1. Known results and residual limit points.

Let G be a finitely generated Kleinian group and denote by $\Omega(G)$ and $\Lambda(G)$ the region of discontinuity and the limit set of G, respectively. A component of $\Omega(G)$ is called a component of G. Let Δ be a component of G and denote by G_{Δ} the stabilizer of Δ in G, that is, $G_{\Delta}=\{g\in G \mid g(\Delta)=\Delta\}$. It is well known that G_{Δ} is a finitely generated function group having Δ as an invariant component and is called a component subgroup of Δ. Assuming $\Omega(G_{\Delta})\neq\Delta$, let Δ^* be a component of G_{Δ} different from Δ. Then the boundary $\partial\Delta^*$ of Δ^* is a quasiconformal image of a circle and is called a separator for G. We denote by $S(G)$

the set of all separators for G. The following are well known.

PROPOSITION 1.1. *Separators form a null sequence with respect to the spherical metric.*

PROPOSITION 1.2. *The number of G-inequivalent elements of $S(G)$ is finite.*

PROPOSITION 1.3. *Let Δ_1^* and Δ_2^* be distinct non-invariant components of G_Δ. If $\partial\Delta_1^* \cap \partial\Delta_2^*$ is not empty, then it consists of one point and is the fixed point of a parabolic element of G_Δ.*

Let $s \in S(G)$ and let E and E' be subsets of $\hat{C} = C \cup \{\infty\}$ not contained in s. We say that s separates E from E' if E lies in the closure of one component of $\hat{C} \setminus s$ and E' lies in the closure of the other component of $\hat{C} \setminus s$. A separator s for G is of web type if there is only one component of G on whose boundary s lies, and s is of nest type if there are two components of G on whose boundaries s lies. We denote by $S_w(G)$ and $S_n(G)$ the sets of all separators for G of web type and of nest type, respectively. Then $S(G)$ is the union of $S_w(G)$ and $S_n(G)$. A sequence $\{s_n\}_{n=1}^\infty$, $s_n \in S(G)$, is a nest sequence if s_n separates s_{n-1} from s_{n+1} for each $n > 1$. A point p of $\Lambda(G)$ is called a residual limit point if there is no component of G on whose boundary p lies. The set of all the residual limit points of G is denoted by $\Lambda_0(G)$ and is called the residual limit set of G. There are two classifications of the residual limit set. One is the following ([1]):

Let $p \in \Lambda_0(G)$; p is of the first kind, denoted by $p \in L_1(G)$, if there is a nested sequence of separators converging to p; p is of the second kind, denoted by $p \in L_2(G)$, if $p \notin L_1(G)$.

The other is the following ([4]):

Let $p \in \Lambda_0(G)$; p is of the web kind, denoted by $p \in L_w(G)$, if $p \in L_2(G)$; p is of the nest kind, denoted by $p \in L_n(G)$, if $p \notin L_w(G)$ and if there is no nested sequence of separators of nest type converging to p; p is of the general kind, denoted by $p \in L_g(G)$, if $p \notin L_w(G) \cup L_n(G)$.

The relation between two classifications is the following.

$$L_1(G) = L_n(G) + L_g(G), \qquad L_2(G) = L_w(G).$$

We prove here the following, which are well known but the author can not find in the literature.

LEMMA 1.4. *Let G be a finitely generated Kleinian group with $\Lambda_0(G) \neq \emptyset$ and let $K = K(G)$ be the number of inequivalent components of G. Then for any collection of inequivalent components $\{\Delta_1, \cdots, \Delta_k\}$, $k < K$, there is a collection of inequivalent components $\{\Delta_{k+1}, \cdots, \Delta_K\}$ satisfying the following:*

i) *$\{\Delta_1, \cdots, \Delta_K\}$ is a complete list of inequivalent components of G,*

ii) $\partial\Delta_i \cap \partial\Delta_j = \emptyset$ $(i \leqq k < j)$, *and*

iii) $\partial\Delta_i \cap \partial\Delta_j = \emptyset$ $(k < i < j)$.

PROOF. Let $\{g_{k+1}, \cdots, g_K\}$ be a set of loxodromic elements of G having the fixed points on $\Lambda_0(G)$ such that the fixed points of g_i and g_j are distinct $(k<i<j)$. Let ξ_i and ξ_i' be the attractive and the repelling fixed points of g_i, respectively, and let U_i be a closed neighborhood of ξ_i such that $U_i \cap (\bar{\Delta}_1 \cup \cdots \cup \bar{\Delta}_k) = \emptyset$ and that $U_i \cap U_j = \emptyset$ $(i \neq j)$. Let $\{\Delta_{k+1}', \cdots, \Delta_K'\}$ be a set of components of G such that $\{\Delta_1, \cdots, \Delta_k, \Delta_{k+1}', \cdots, \Delta_K'\}$ is a complete list of inequivalent components of G. Since ξ_i and ξ_i' do not lie on the boundary of any component of G, there is a set of positive integers $\{n_{k+1}, \cdots, n_K\}$ such that $g_i^{n_i}(\Delta_i') \subset U_i$ $(i = k+1, \cdots, K)$. Putting $\Delta_i = g_i^{n_i}(\Delta_i')$, we have the desired collection $\{\Delta_{k+1}, \cdots, \Delta_K\}$.

LEMMA 1.5. *Let G be a finitely generated function group and let $K' = K'(G)$ be the number of inequivalent non-invariant components of G. Then for any collection of inequivalent non-invariant components $\{\Delta_1, \cdots, \Delta_k\}$, $k < K'$, there is a collection of inequivalent non-invariant components $\{\Delta_{k+1}, \cdots, \Delta_{K'}\}$ satisfying the following:*

i) *$\{\Delta_1, \cdots, \Delta_{K'}\}$ is a complete list of inequivalent non-invariant components of G,*

ii) $\partial\Delta_i \cap \partial\Delta_j = \emptyset$ $(i \leqq k < j)$, *and*

iii) $\partial\Delta_i \cap \partial\Delta_j = \emptyset$ $(k < i < j)$.

PROOF. Let $\{\Delta_1, \cdots, \Delta_k, \Delta_{k+1}', \cdots, \Delta_{K'}'\}$ be a complete list of inequivalent non-invariant components of G and let Δ be a non-invariant component of G different from any element of $\{\Delta_1, \cdots, \Delta_k, \Delta_{k+1}', \cdots, \Delta_{K'}'\}$. Then by Proposition 1.3 we can find a set of loxodromic elements $\{g_{k+1}, \cdots, g_{K'}\}$ of G_Δ such that their fixed points lie on $\partial\Delta \setminus \{\partial\Delta_1 \cup \cdots \cup \partial\Delta_k \cup \partial\Delta_{k+1}' \cup \cdots \cup \partial\Delta_{K'}'\}$ and are distinct to each other. Hence by the same argument of the proof of Lemma 1.4, we can find a set of integers $\{n_{k+1}, \cdots, n_{K'}\}$ so that $\{g_{k+1}^{n_{k+1}}(\Delta_{k+1}'), \cdots, g_{K'}^{n_{K'}}(\Delta_{K'}')\}$ is the desired collection.

2. Web subgroup and nest subgroup.

A finitely generated Kleinian group is called a web group if each component subgroup is quasi-Fuchsian [2]. Let G be a finitely generated Kleinian group with $L_w(G) \neq \emptyset$ and let $q \in L_w(G)$ and $z \in \Omega(G)$. Writing by $S(z, q)$ the set of all separators for G separating q from z, we denote by $s(z, q)$ the separator of $S(z, q)$ such that there is no separator separating q from $s(z, q)$. This $s(z, q)$ is called a maximal separator in $S(z, q)$ for q and we denote by $M(q)$ the set of all maximal separators for q, that is, $M(q) = \{s(z, q) \mid z \in \Omega(G)\}$. The set $\Phi(q) = \overline{\bigcup_{s \in M(q)} s}$ is called the web of q and the web subgroup $W(q)$ of q is the stabilizer of $\Phi(q)$ in G ([1]). It is shown in [2] that the web subgroups are web groups. We recall some properties of web subgroups. Writing by $d(z, q)$ the component of $\hat{C}\setminus s(z, q)$ containing z, we set $D(q)=\{d(z, q)|z\in\Omega(G)\}$.

PROPOSITION 2.1 ([1]). (1) $\Omega(W(q))=D(q)$,
(2) $\Phi(q)=\Lambda(W(q))=L_w(W(q))+M(q)$, *and*
(3) $M(q)=S(W(q))\subset S_w(G)$.

PROPOSITION 2.2 ([4]). *For each separator s of $S_w(G)$ there is the web subgroup having s as a separator.*

A finitely generated Kleinian group G is called a nest group if $\Lambda_0(G)=L_1(G)\neq\emptyset$ ([3]). It is equivalent to say that $\Lambda_0(G)=L_n(G)\neq\emptyset$ ([4]). Let G be a finitely generated Kleinian group with $\Lambda_0(G)\supsetneqq L_n(G)\neq\emptyset$. Then $L_w(G)\neq\emptyset$. Let $p\in L_n(G)$ and let $q\in L_w(G)$. Writing by $S_w(q, p)$ the set of all separators for G of web type separating p from q, we denote by $\sigma(q, p)$ the separator of $S_w(q, p)$ such that there is no separator in $S_w(q, p)$ separating p from $\sigma(q, p)$. The set $\Psi(p)=\overline{\bigcup_{q\in L_w(G)}\sigma(q, p)}$ is called the nest of p and the nest subgroup $N(p)$ of p is the stabilizer of $\Psi(p)$ in G ([4]). It is shown in [4] that the nest subgroups are nest groups. Writing by $c(q, p)$ the component of $\hat{C}\setminus\sigma(q, p)$ containing q, we set $C(p)=\{c(q, p)|q\in L_w(G)\}$. We say that a component Δ' of G is linked to a component Δ of G by separators of nest type if there are components $\Delta_1, \Delta_2, \cdots, \Delta_n$ such that $\partial\Delta\cap\partial\Delta_1, \partial\Delta_1\cap\partial\Delta_2, \cdots, \partial\Delta_n\cap\partial\Delta'$ are separators of nest type. By Propositions 1.1 and 2.2 it is equivalent to saying that there is no separator of web type separating Δ from Δ'. We denote by $\mathcal{L}(\Delta)$ the set of all components of G which are linked to Δ by separators of nest type. Clearly $\mathcal{L}(\Delta')=\mathcal{L}(\Delta)$ whenever $\Delta'\in\mathcal{L}(\Delta)$. The following are shown in [4].

THEOREM 2.3. $\Omega(N(p))=\mathcal{L}(\Delta)\cup C(p)$ *for some component Δ.*

LEMMA 2.4. *Let $g\in G$. If there is an element Δ' of $\mathcal{L}(\Delta)$ such that $g(\Delta')\in\mathcal{L}(\Delta)$, then $g\in N(p)$.*

PROPOSITION 2.5. *The component containing $\sigma(q, p)$ on its boundary is an element of $\mathcal{L}(\Delta)$, where Δ is the component in Theorem 2.3.*

PROPOSITION 2.6. *A component Δ of G is a component of $N(p)$ if and only if there is no separator of web type separating Δ from p.*

3. The subgroup $N(\Delta)$.

In this section we shall introduce and examine a subgroup $N(\Delta)$ whose elements keep $\mathcal{L}(\Delta)$ invariant, that is, $N(\Delta)=\{g\in G|g(\mathcal{L}(\Delta))=\mathcal{L}(\Delta)\}$. It is shown implicitly in [4] that if Δ is a component of both G and the nest subgroup $N(p)$ of $p\in L_n(G)$, then $N(\Delta)=N(p)$. In general, we show the following

PROPOSITION 3.1. *Let G be a finitely generated Kleinian group with $\Lambda_0(G)\neq\emptyset$ and let Δ be a component of G. Then $N(\Delta)$ is either a component subgroup $G_{\Delta'}$, $\Delta'\in\mathcal{L}(\Delta)$, or the nest subgroup $N(p)$, $p\in L_n(G)$.*

PROOF. Let Δ_1 and Δ_2 be elements of $\mathcal{L}(\Delta)$. We shall define the length of linkage between Δ_1 and Δ_2 to be the number of separators separating them. If $\Delta_1=\Delta_2$, we interpret the length of linkage as zero.

We first treat the case in which there are no two elements of $\mathcal{L}(\Delta)$ such that the length of linkage between them is 3. In this case we show that $N(\Delta)$ is a component subgroup. If $\mathcal{L}(\Delta)=\{\Delta\}$, then clearly we have $N(\Delta)=G_\Delta$. So we assume that $\mathcal{L}(\Delta)\neq\{\Delta\}$. We assert that there is an element $\Delta'\in\mathcal{L}(\Delta)$ such that $N(\Delta)=G_{\Delta'}$. In order to prove the assertion we first show that there is an element of $\mathcal{L}(\Delta)$ whose component subgroup is not quasi-Fuchsian. If G_Δ is not quasi-Fuchsian, we have shown with this Δ. If G_Δ is quasi-Fuchsian, then let Δ' be the element of $\mathcal{L}(\Delta)$ such that the length of linkage between Δ' and Δ is 1. If $G_{\Delta'}$ is quasi-Fuchsian, then $\hat{C}=\Delta\cup\Delta'\cup\partial\Delta$ so $\Lambda_0(G)=\emptyset$, which contradicts the assumption that $\Lambda_0(G)\neq\emptyset$. Hence Δ' is a component having the desired property. Thus we have shown that there is an element of $\mathcal{L}(\Delta)$ whose component subgroup is not quasi-Fuchsian. We denote this component by Δ'. By this Δ' we show our assertion that $N(\Delta)=G_{\Delta'}$. Let Δ'' be an element of $\mathcal{L}(\Delta)$ with the length of linkage between Δ' and Δ'' is 1 and let Δ^* be the non-invariant component of $G_{\Delta'}$ containing Δ''. Note that $\partial\Delta'\cap\partial\Delta''=\partial\Delta^*$. We assert that $G_{\Delta''}$ is quasi-Fuchsian so that $\Delta''=\Delta^*$. Contrary to our assertion, if $G_{\Delta''}$ is not quasi-Fuchsian, then the complement of the closure of Δ^* is a non-invariant component of $G_{\Delta''}$. Let g and h be elements of $G_{\Delta'}$ and $G_{\Delta''}$, respectively, such that they do not keep Δ^* invariant. Then $g(\Delta'')$ is linked to Δ' by a separator of nest type $g(\partial\Delta^*)$ and $h(\Delta')$ is linked to Δ'' by a separator of nest type $h(\partial\Delta^*)$. Hence $g(\Delta'')$ and $h(\Delta')$ belong to $\mathcal{L}(\Delta)$ and the length of linkage between them is 3. This contradicts our assumption on the length of linkage. Thus we have our assertion that $G_{\Delta''}$ is quasi-Fuchsian so that $\Delta''=\Delta^*$. Clearly this implies $N(\Delta)=G_{\Delta'}$.

Next we treat the case where there are two elements of $\mathcal{L}(\Delta)$ such that the length of linkage between them is 3. In this case we show that $N(\Delta)=N(p)$, $p\in L_n(G)$. Let Δ_1, Δ_2, Δ_3, Δ_4 and s_1, s_2, s_3 be elements of $\mathcal{L}(\Delta)$ and separators of nest type, respectively, such that $\partial\Delta_i\cap\partial\Delta_{i+1}=s_i$ ($i=1, 2, 3$) so that the length of linkage between Δ_1 and Δ_4 is 3. Since both G_{Δ_2} and G_{Δ_3} are not quasi-Fuchsian, we can find $g\in G_{\Delta_2}\setminus G_{\Delta_3}$ and $h\in G_{\Delta_3}\setminus G_{\Delta_2}$ such that $g(s_2)\cap(s_1\cup s_2)=h(s_2)\cap(s_2\cup s_3)=\emptyset$. For example, let g (or h) be a loxodromic element of G_{Δ_2} (or G_{Δ_3}) having the fixed points on s_1 (or s_3). Let p and q be the attractive and the repelling fixed points of hg, respectively. It is not difficult to see that s_2 separates p from q, so $\{(hg)^n(s_2)\}$ is a nest sequence of the separators converging to p so that $p\in \Lambda_0(G)\setminus L_w(G)$. More precisely, we show that $p\in L_n(G)$. For each positive integer n, $(hg)^n(\Delta_3)$ is linked to Δ_2 by separators of nest type s_2, $h(s_2)$, $hg(s_2)$, $\cdots$, $(hg)^{n-1}(s_2)$, $(hg)^n(s_2)$ such that

$$\begin{aligned}
&\partial\Delta_2\cap\partial\Delta_3 = s_2, \quad \partial\Delta_3\cap\partial h(\Delta_2) = \partial h(\Delta_3)\cap\partial h(\Delta_2) = h(s_2),\\
&\partial h(\Delta_2)\cap\partial hg(\Delta_3) = \partial hg(\Delta_2)\cap\partial hg(\Delta_3) = hg(s_2), \cdots,\\
&\partial(hg)^{n-1}(\Delta_3)\cap\partial(hg)^{n-1}h(\Delta_2) = (hg)^{n-1}h(s_2),\\
&\partial(hg)^{n-1}h(\Delta_2)\cap\partial(hg)^n(\Delta_3) = (hg)^n(s_2).
\end{aligned}$$

This implies that there is no separator of web type separating p from Δ_2. This clearly implies that there is no nest sequence of separators of web type converging to p, so $p\in L_n(G)$. By Theorem 2.3 and Proposition 2.6 we see that $\mathcal{L}(\Delta)$ is the set of all components of both G and $N(p)$. Since each element of $N(p)$ keeps $\mathcal{L}(\Delta)$ invariant, we have $N(p)\subset N(\Delta)$. Conversely, since each element of $N(\Delta)$ maps Δ to an element of $\mathcal{L}(\Delta)$, we see by Lemma 2.4 that $N(\Delta)\subset N(p)$. Hence $N(\Delta)=N(p)$. q. e. d.

In view of Proposition 3.1, we can generalize some results in § 2. The generalizations are just for component subgroups, so their proofs are clear. We denote by $C(\Delta)$ the set of all components of $N(\Delta)$ which are not components of G.

THEOREM 3.2. $\Omega(N(\Delta)) = \mathcal{L}(\Delta)\cup C(\Delta)$.

LEMMA 3.3. *Let g be an element of G. If there is an element Δ' of $\mathcal{L}(\Delta)$ such that $g(\Delta')\in\mathcal{L}(\Delta)$, then $g\in N(\Delta)$.*

PROPOSITION 3.4. *A component of G which contains the boundary of an element of $C(\Delta)$ on its boundary is an element of $\mathcal{L}(\Delta)$.*

PROPOSITION 3.5. *For each separator s of $S_w(G)$ there is an $N(\Delta)$ having s as a separator.*

4. Graph.

First we recall some terminology and definitions from the graph theory in [5]. A graph Γ consists of a set $X=\mathrm{vert}\,\Gamma$, a set $Y=\mathrm{edge}\,\Gamma$ and two maps $Y\to X\times X$, $y\mapsto(o(y), t(y))$ and $Y\to Y$, $y\mapsto\bar{y}$ which satisfy the following condition: for each $y\in Y$ we have $\bar{\bar{y}}=y$, $\bar{y}\neq y$ and $o(y)=t(\bar{y})$. Each element of X is called a vertex, an element $y\in Y$ is called an (oriented) edge, and $\bar{y}$ is called the inverse edge. The vertices $o(y)$ and $t(y)$ are called the origin and the terminus of y, respectively. The extremities of y are $o(y)$ and $t(y)$. Two vertices of Γ are adjacent if they are the extremities of some edge. An orientation of Γ is a subset Y_+ of Y such that Y is the disjoint union Y_+ and $\bar{Y}_+$. An oriented graph is defined by giving two sets X and Y_+ and a map $Y_+ \to X \times X$.

Let G be a group acting on Γ. An invertion is a pair consisting of an element $g \in G$ and an edge $y \in Y$ such that $gy = \bar{y}$; if there is no such pair G acts without inversion; this is equivalent to say that an orientation Y_+ is preserved by G. Using the quotients of X and Y under the action of G, the quotient graph $G \backslash \Gamma$ is defined. A tree of representatives of $\Gamma \bmod G$ is any subtree T of Γ which is the lift of a maximal tree in $G \backslash \Gamma$. A fundamental domain of $\Gamma \bmod G$ is a subgraph U of Γ such that $U \to G \backslash \Gamma$ is an isomorphism.

Now we define a graph. Let G be a finitely generated Kleinian group with $L_g(G) \neq \emptyset$. We set

$$X = \{\mathcal{L}(\Delta) \mid \Delta \in \Omega(G)\} \cup \{\Phi(q) \mid q \in L_w(G)\},$$
$$Y_+ = S_w(G) \quad \text{and} \quad Y = Y_+ \cup \bar{Y}_+.$$

The subset $\{\mathcal{L}(\Delta)\} \times \{\Phi(q)\}$ of $X \times X$ is associated with $S_w(G)$ by Propositions 3.5 and 2.2. Hence there is a natural map $Y_+ \to X \times X$, $y \mapsto (o(y), t(y)) \in \{\mathcal{L}(\Delta)\} \times \{\Phi(q)\}$. Thus we have a graph and denote it by Γ.

PROPOSITION 4.1. *Γ is a tree and G acts on Γ without inversion.*

PROOF. Since $\{\mathcal{L}(\Delta)\}$ and $\{\Phi(q)\}$ consist of subsets of $\Omega(G)$ and $\Lambda(G)$, respectively, G acts on $\{\mathcal{L}(\Delta)\} \times \{\Phi(q)\}$ preserving the order of factors. Hence the orientation Y_+ is preserved by G and G acts on Γ without inversion. To see Γ is a tree we must show that Γ is connected and there is no circuit in Γ. Proposition 1.1 implies that, for any two separators, there are only a finite number of separators separating them. Hence, in view of the definitions of the web $\Phi(q)$ and the nest $\Psi(p)$, we see that there are a finite number of edges connecting two vertices. Therefore Γ is connected. Since separators are Jordan curves lying in $\hat{C}$ and they do not cross each other, it is clear that there is no circuit. Hence Γ is a tree.

Next we investigate the quotient graph $G \backslash \Gamma$ and the structure of G. The following is adequate to our purpose.

LEMMA 4.2 ([**5**]). *Let G be a group acting without inversion on a connected graph Γ, and let T be a tree of representatives of $\Gamma \bmod G$. Let U be a subgraph of Γ containing T, each edge of which has an extremity in T, and such that $G \cdot U = \Gamma$. For each edge y of U with origin in T, let g_y be an element of G such that $g_y t(y) \in \operatorname{vert} T$. Then the group generated by the elements g_y and the stabilizers G_x $(x \in \operatorname{vert} T)$ is equal to G.*

We denote by l, m, n the numbers of the following:

$l=$ the number of the edges of $G\backslash\Gamma$,
$m=$ the number of the vertices of $G\backslash\Gamma$ associated with components, and
$n=$ the number of the vertices of $G\backslash\Gamma$ which are webs.

By the Ahlfors finiteness theorem and by the assumption on $L_g(G)$, those three numbers are positive integers. Since a tree of representatives T of $\Gamma \bmod G$ has the same number of vertices as that of $G\backslash\Gamma$, T consists of $m+n$ vertices and $m+n-1$ edges. We denote by $x_1, \cdots, x_m$ the vertices associated with components and by $x_{m+1}, \cdots, x_{m+n}$ the vertices which are webs. Let y be an edge in T. Then there are vertices x_i and x_{m+j} such that they are adjacent by y. Let N_i and W_{m+j} be the nest or the component and the web subgroups of G which are stabilizers of x_i and x_{m+j}, respectively. Since the stabilizer G_y of y in G is the stability subgroup of a separator of web type, both N_i and W_{m+j} contain G_y as a subgroup. This implies that the group generated by $N_1, \cdots, N_m, W_{m+1}, \cdots, W_{m+n}$ is the amalgam of the $N_1, \cdots, N_m, \cdots, W_{m+1}, \cdots, W_{m+n}$ along the G_y (cf. page 37 in [5]). We denote this amalgam by G_T. In a special case that $m+n=l+1$, we have the following

PROPOSITION 4.3. *If $m+n=l+1$, then T is a fundamental domain of $\Gamma \bmod G$ so that G is equal to the amalgam G_T.*

PROOF. The condition $m+n=l+1$ implies that the quotient graph $G\backslash\Gamma$ is a tree. Hence the injection $T\rightarrow G\backslash\Gamma$ is surjective. Therefore T is a fundamental domain of $\Gamma \bmod G$. Lemma 4.2 implies that $G=G_T$.

Now we consider the case in which T is not a fundamental domain. This implies that $m+n\leqq l$. Let y_i $(i=m+n, \cdots, l)$ be the edge in $G\backslash\Gamma$ but not in T. Then one of the extremities of y_i does not lie in T. Let g_i be an element of G such that g_i maps the extremity of y_i which does not lie in T to a vertex of T. We denote by S the group generated by those g_i. Then Lemma 4.2 tells us the following

PROPOSITION 4.4. *If $m+n\leqq l$, then G is generated by the amalgam G_T and $S=\langle g_{m+n}, \cdots, g_l\rangle$.*

REMARK 1. Using Lemmas 1.4 and 1.5, we can choose a tree of representative T such that any two edges of T are disjoint as the Jordan curves in $\hat{\boldsymbol{C}}$. Moreover, one can choose the set $\{g_{m+n}, \cdots, g_l\}$ such that S is a Schottky group with $\{y_{m+n}, g_{m+n}(y_{m+n}), \cdots, y_l, g_l(y_l)\}$ as the defining curves.

REMARK 2. The path connecting y_i to $g_i(y_i)$ in the tree T has length greater than zero. The straight path consisting of the transforms of this path and y_i by the cyclic group $\langle g_i\rangle$ has the ends in $L_g(G)$.

REMARK 3. The graph Γ is not locally finite. A vertex is called a terminal vertex if it is the extremity of only one edge in Y_+. Each terminal vertex of Γ is a component of both G and a web subgroup of a point in $L_w(G)$.

REMARK 4. If we adopt the definition that G is an amalgam if it can be written $G \simeq G_1 *_A G_2$ with $G_1 \neq A \neq G_2$ (cf. page 58 in [5]), then we should abandon some N_i in the amalgam G_T. This does happen for terminal vertices which are components of both G and web subgroups.

References

[1] W. Abikoff, The residual limit sets of Kleinian groups, Acta Math., **130** (1973), 127-144.

[2] W. Abikoff and B. Maskit, Geometric decompositions of Kleinian groups, Amer. J. Math., **99** (1977), 687-697.

[3] T. Sasaki, The residual limit points of the first kind and the nest groups, Tôhoku Math. J., **32** (1980), 35-48.

[4] T. Sasaki, The nest subgroups of Kleinian groups, Trans. Amer. Math. Soc., **278** (1983), 389-399.

[5] J.-P. Serre, Trees, Springer, 1980.

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