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Geometry of anti-self-dual connections and Kuranishi map

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1. Introduction.

Let (M, h) be a compact, connected oriented C^∞ Riemannian 4-manifold. Let P be a C^∞ $SU(n)$-principal bundle over M, $n \geqq 2$, and $\mathcal{A}^-$ the set of anti-self-dual Yang-Mills connections on P. The quotient space $\mathcal{M}^-$ of $\mathcal{A}^-$ by the action of gauge transformations is called the moduli space of anti-self-dual connections on P.

It is a known fact that the moduli space is a finite dimensional C^∞ manifold (with singularities). This fact is originally due to [**3**].

On the other hand we have an example of the moduli space over a particular 4-manifold with a complete description. Over the 4-sphere S^4 with the standard metric the moduli space of anti-instantons of index $k=1$ and of group $SU(2)$, namely the set of equivalence classes of BPST-anti-instantons is given by the upper half space $\boldsymbol{R}^5_+=\{(\xi, \lambda), \xi \in \boldsymbol{R}^4, \lambda>0\}$ where ξ and λ are the center and the scale of an anti-instanton ([**2**]). Here the index k denotes $c_2(E)$, $E=P\times_\rho \boldsymbol{C}^2$.

Moreover the moduli space $\mathcal{M}^-$ of $SU(2)$-anti-self-dual connections of index 2 over the complex projective plane $P_2(\boldsymbol{C})$ with the Fubini-Study metric is identified, by theorems of Kobayashi and Lübke ([**16**], [**21**]) and also a theorem of Donaldson ([**9**]), with the moduli of stable holomorphic structures on the associated vector bundle E, which turns out to be the complement of a singular hypersurface of degree 3 in $P_5(\boldsymbol{C})$; $\{(z_0 : z_1 : z_2 : z_3 : z_4 : z_5) \in P_5(\boldsymbol{C})$; $z_0z_1z_2-z_0z_5^2-z_2z_3^2-z_1z_4^2+2z_3z_4z_5=0\}$ by a result of [**5**] (see also [**24**], p. 352).

The moduli space over a 4-manifold M is considered to reflect topological properties of the base space M ([**8**], [**10**], [**11**]).

Moreover, since the definition of anti-self-dual connection requires the presence of a Riemannian structure on M, the space $\mathcal{M}^-$ must admit a Riemannian structure, for instance, a metrical structure, curvature properties, etc.

When the base space is a complex surface with a Kähler metric, the anti-self-duality of a connection gives an integrability condition of a holomorphic structure on the associated vector bundle and the moduli space enjoys a structure of complex manifold (with singularities) of complex dimension

$4k-p_a(M)\dim_R SU(n)$, where $p_a(M)$ is the arithmetic genus ([**14**]). It seems to be natural that the moduli space over a Kähler manifold is endowed with an intrinsic Kähler structure. In their paper Atiyah and Bott mentioned the Kähler structure on the moduli space of Yang-Mills connections over a Riemann surface ([**4**]). However we will show this by our method in § 4.

Since the moduli space $\mathcal{M}^-$ over a Riemannian 4-manifold M is a subspace of the orbit space $\mathcal{B}=\mathcal{A}/\mathcal{G}$ ($\mathcal{A}$ is the space of C^∞ connections on P and $\mathcal{G}$ is the group of gauge transformations), the canonical $\mathcal{G}$-invariant inner product of $\mathcal{A}$ descends to an inner product of $\mathcal{B}$ to define a Riemannian metric on $\mathcal{M}^-$.

In fact we decompose the tangent space at a connection A to $\mathcal{A}$ into the direct sum of the vertical subspace which is tangential to the $\mathcal{G}$-orbit through A and the horizontal subspace (see Figure 1 in § 3) and obtain the Riemannian metric on $\mathcal{B}$ by using this splitting in such a way that the projection $\pi:\mathcal{A}\to\mathcal{B}$ is a Riemannian submersion. We shall investigate this Riemannian structure in a way analogous to that used in [**19**].

For each anti-self-dual connection A the covariant exterior derivatives define an elliptic complex $0\longrightarrow\Omega^0(\mathfrak{g}_p)\xrightarrow{d_A}\Omega^1(\mathfrak{g}_p)\xrightarrow{d_A^+}\Omega^2_+(\mathfrak{g}_p)\longrightarrow 0$. The first cohomology group H^1_A is identified with the tangent space to $\mathcal{M}^-$ if A is generic. Furthermore a map, called the Kuranishi map which sends a slice neighborhood in $\mathcal{A}^-$ representing a neighborhood of $\pi(A)$ in $\mathcal{M}^-$ into H^1_A, defines a local coordinate of the moduli space of "generic" anti-self-dual connections $\hat{\mathcal{M}}^-$, i.e., the smooth part of $\mathcal{M}^-$ ([**20**]).

This coordinate is shown to be Gaussian normal with respect to the Riemannian metric on $\hat{\mathcal{M}}^-$ (Proposition 3.4 in § 3).

By making use of this fact we compute the Riemannian curvature tensor and sectional curvature. Actually, to obtain a formula on the curvature tensor amounts to getting a formula fulfilled by the second order derivatives of inner products given by the metric.

The formula is in fact expressed in terms of the Green operators of the Laplacians relative to the elliptic complex and it exhibits the similarity to the results due to Ahlfors on the Weil-Petersson metric of the Teichmüller space ([**1**], [**25**]).

The Hodge decomposition of Lie algebra valued forms associated to the elliptic complex and also a nice property of the Kuranishi map play an essential role in obtaining the formula.

If the base space M is Kähler, the almost complex structure I of M;

$$I(\alpha^{1,0}+\alpha^{0,1})=\sqrt{-1}(\alpha^{1,0}-\alpha^{0,1}),\qquad \alpha^{1,0}\in T^{1,0}M,\quad \alpha^{0,1}\in T^{0,1}M,$$

canonically induces an almost complex structure I on H^1 for each anti-self-dual connection which defines the complex structure of the complex manifold $\hat{\mathcal{M}}^-$.

We extend I on the tangent space at a single point over a slice neighborhood compatible with the Kuranishi map and prove that the canonically defined Riemannian metric is Kähler.

Formulae of holomorphic sectional curvature and holomorphic bisectional curvature are computed in § 6.

Finally we consider the moduli space of Einstein-Hermitian connections on an $U(n)$-principal bundle $P\to M$, where M is a compact Riemann surface. We are able to apply our method also to this moduli space and obtain a canonical Kähler metric and its Riemannian curvature tensor, since an Einstein-Hermitian connection relates to an elliptic complex. As a consequence it can be proven that the holomorphic sectional curvature of this space is nonnegative (Theorem 7.1 in § 7). This nonnegative curvature property is compatible with results on the moduli of stable vector bundles with fixed determinant bundle obtained by Morse theoretic method and a method of algebraic geometry ([**4**],[**23**]).

We exhibits moreover infinitely many compact simply-connected complex Kähler manifolds of complex dimension $(g-1)\dim_{\boldsymbol{R}} SU(n)$ with nonnegative holomorphic sectional curvature (g is the genus of M) by considering the cases of $(n, l)=1$, where l is the degree of $E=P\times_{\rho}\boldsymbol{C}^n$.

2. Kuranishi map and local coordinates.

Let (M, h) and P be the same as in § 1. A connection A is called anti-self-dual if its curvature F_A is anti-self-dual (i.e., $*F_A=-F_A$ or the self-dual part F_+ of F_A vanishes). The group $\mathcal{G}$ leaves invariant the set of anti-self-dual connections $\mathcal{A}^-$. We call the quotient space $\mathcal{M}^-=\mathcal{A}^-/\mathcal{G}$ the moduli space of anti-self-dual connections.

In this section we define the Kuranishi map and introduce a local coordinate system on the smooth part of $\mathcal{M}^-$ by using this map. For this, we recall some definitions and notations ([**13**], [**14**]).

DEFINITIONS AND NOTATIONS. A vector bundle $\mathfrak{g}_P=P\times_{Ad}\mathfrak{su}(n)$ ($\mathfrak{su}(n)$ is the Lie algebra of $SU(n)$) associated to P is called the adjoint bundle of P. Denote by Λ^l the vector bundle of l-forms ($l=0, 1, 2$) and by Λ^2_+ the vector bundle of self-dual 2-forms on M. Denote by $\Omega^l(\mathfrak{g}_P)$ the vector space of adjoint bundle-valued C^∞ l-forms over M; $\Gamma(M; \Lambda^l\otimes\mathfrak{g}_P)$, and by $\Omega^2_+(\mathfrak{g}_P)$ the space of adjoint bundle-valued C^∞ self-dual 2-forms over M; $\Gamma(M; \Lambda^2_+\otimes\mathfrak{g}_P)$.

The bracket product on the Lie algebra $\mathfrak{su}(n)$ defines bilinear maps on

$$\Omega^l(\mathfrak{g}_P);\ [\cdot\wedge\cdot];\ \Omega^k(\mathfrak{g}_P)\times\Omega^l(\mathfrak{g}_P)\to\Omega^{k+l}(\mathfrak{g}_P);\ k, l\geqq 0.$$

The space $\mathcal{A}$ of connections is an affine space associated to the vector space $\Omega^1(\mathfrak{g}_P)$. The action of $\mathcal{G}$ on $\mathcal{A}$ is given by $g(A)=g^{-1}\cdot dg+g^{-1}\cdot A\cdot g$, $g\in\mathcal{G}$, $A\in\mathcal{A}$

and its action on $\Omega^l(\mathfrak{g}_P)$ is defined by $g(\phi)=g^{-1}\cdot\phi\cdot g$, where $\cdot$ implies the matrix multiplication.

For each $A\in\mathcal{A}$ we have an adjoint bundle-valued 2-form $F_A=dA+1/2[A\wedge A]$, called the curvature. The curvature satisfies $F_{g(A)}=g(F_A)$ for the action of $\mathcal{G}$.

Each connection A defines covariant exterior derivatives d_A; $\Omega^l(\mathfrak{g}_P)\to\Omega^{l+1}(\mathfrak{g}_P)$, $l=0, 1, 2$ with identities $d_Ad_A\phi=[F_A\wedge\phi]$ (Ricci formula) and $d_AF_A=0$ (Bianchi identity). From its definition we have $F_{(A+\alpha)}=F_A+d_A\alpha+1/2[\alpha\wedge\alpha]$, $\alpha\in\Omega^1(\mathfrak{g}_P)$. By using the Riemannian metric h of M together with the invariant inner product $(X, Y)=-\mathrm{Tr}\,X\cdot Y$, $X, Y\in\mathfrak{su}(n)$, we define a $\mathcal{G}$-invariant inner product (,) on $\Omega^l(\mathfrak{g}_P)$;

$$(\phi, \psi)=\int_M(\phi(x), \psi(x))d\mathrm{vol}, \qquad \phi, \psi\in\Omega^l(\mathfrak{g}_P).$$

We denote by d_A^* the adjoint operator of d_A with respect to the inner product (,). We notice that $d^*_{g(A)}\ g(\phi)=g(d_A^*\phi)$ holds since the gauge transformation g acts on $d_A\phi$ equivariantly.

We call a connection to be irreducible when it satisfies

$$\mathrm{Ker}\{d_A;\Omega^0(\mathfrak{g}_P)\longrightarrow\Omega^1(\mathfrak{g}_P)\}=\{0\}.$$

The tangent space at A to the $\mathcal{G}$-orbit through A is given by $\mathrm{Im}\{d_A;\Omega^0(\mathfrak{g}_P)\to\Omega^1(\mathfrak{g}_P)\}$. This space is a subspace of the tangent space to $\mathcal{A}$, $T_A\mathcal{A}=\Omega^1(\mathfrak{g}_P)$. A linear subspace $\mathrm{Ker}\{d_A^*;\Omega^1(\mathfrak{g}_P)\to\Omega^0(\mathfrak{g}_P)\}$ of $\Omega^1(\mathfrak{g}_P)$ is called a slice at A.

The spaces $\Omega^l(\mathfrak{g}_P)$ carry Sobolev L^2_k-norms $|\cdot|_k$ (k is a sufficiently large integer with respect to the dimension of M).

PROPOSITION 2.1 ([**3**]). *Let A be an anti-self-dual connection on P. Then the following sequence is an elliptic complex;*

$$0\longrightarrow\Omega^0(\mathfrak{g}_P)\xrightarrow{d^0=d_A}\Omega^1(\mathfrak{g}_P)\xrightarrow{d^1=d_A^+}\Omega^2_+(\mathfrak{g}_P)\longrightarrow 0, \tag{2.1}$$

here $d_A^+=p_+d_A$ and $p_+;\Omega^2(\mathfrak{g}_P)\to\Omega^2_+(\mathfrak{g}_P)$ is the canonical projection.

Relative to the elliptic complex the Laplacians Δ^0, Δ^1 and Δ^2_+ are defined; $\Delta^0=d^{0*}d^0$, $\Delta^1=d^0d^{0*}+d^{1*}d^1$, $\Delta^2_+=d^1d^{1*}$. Vector spaces $H^0=\mathrm{Ker}\,\Delta^0$, $H^1=\mathrm{Ker}\,\Delta^1$ and $H^2_+=\mathrm{Ker}\,\Delta^2_+$ are called 0-th, first and second cohomology groups associated to the anti-self-dual connection A, respectively. If we are required to emphasize the presence of connection, then the Laplacians and the cohomology groups are written as Δ^1_A and H^1_A, for examples. Moreover we define linear mappings G_A from $\Omega^l(\mathfrak{g}_P)$ to $\mathrm{Im}\,\Delta^l\subset\Omega^l(\mathfrak{g}_P)$, called the Green operators, as follows;

$$\phi=\boldsymbol{H}_A\phi+\Delta^l_AG_A\phi, \tag{2.2}$$

$\phi\in\Omega^l(\mathfrak{g}_P)$, here $\boldsymbol{H}_A;\Omega^l(\mathfrak{g}_P)\to H^l_A$ is the canonical projection. We notice that G_A is self-adjoint and commutes with d_A and d_A^* ([**18**]).

We have the following Hodge decomposition

$$\Omega^1(\mathfrak{g}_P) = H^1_A \oplus \mathrm{Im}\{d^0 = d_A\} \oplus \mathrm{Im}\{d^{1*} = d_A^{+*}\},$$

and hence every $\alpha \in \Omega^1(\mathfrak{g}_P)$ has an orthogonal expression;

$$\alpha = \alpha^1 + d_A\phi + d_A^{+*}\Phi,$$

$\alpha^1 \in H^1_A$, $\phi \in \Omega^0(\mathfrak{g}_P)$ and $\Phi \in \Omega^2_+(\mathfrak{g}_P)$.

We now assume that there exists near an anti-self-dual connection A another anti-self-dual connection. We denote it by $A+\alpha$, $\alpha \in \Omega^1(\mathfrak{g}_P)$. Since $F_{(A+\alpha)} = F_A + d_A\alpha + 1/2[\alpha \wedge \alpha]$, that $(F_{(A+\alpha)})_+ = 0$ holds if and only if α satisfies the non-linear equation

$$d_A^+\alpha + 1/2[\alpha \wedge \alpha]^+ = 0, \tag{2.3}$$

$([\alpha \wedge \alpha]^+ = p_+[\alpha \wedge \alpha])$.

Now we consider the following map, called the Kuranishi map;

$$\begin{aligned} \Xi_A;\ \Omega^1(\mathfrak{g}_P) &\longrightarrow \Omega^1(\mathfrak{g}_P), \\ \alpha &\longmapsto \alpha + d_A^{+*}G_A(1/2[\alpha \wedge \alpha]^+). \end{aligned} \tag{2.4}$$

We have then

PROPOSITION 2.2. *The equation* (2.3) *is equivalent to that* $d_A^+(\Xi_A(\alpha)) = 0$ *and* $\boldsymbol{H}_A([\alpha \wedge \alpha]^+) = 0$.

We introduce a topology to the affine space $\mathcal{A}$ by the aid of the norm $|\cdot|_k$ of $\Omega^1(\mathfrak{g}_P)$ and define the quotient topology on $\mathcal{B}$ by using the projection $\pi;\ \mathcal{A} \to \mathcal{B};\ A \mapsto [A]$, the gauge equivalence class containing A.

The following is known as a slice lemma.

PROPOSITION 2.3. *Let A be an irreducible connection. Then there is an $\varepsilon > 0$ depending on A such that the restriction of π*

$$\pi|_{S_{A,\varepsilon}};\ S_{A,\varepsilon} = \{A+\alpha;\ d_A^*\alpha = 0,\ |\alpha|_k < \varepsilon\} \longrightarrow \mathcal{B}$$

gives a homeomorphism onto $\pi(S_{A,\varepsilon})$ and $\pi(S_{A,\varepsilon})$ is an open subset of $\mathcal{B}$.

For the proof of this proposition, refer to [**3**] and [**11**].

It is necessary to argue this proposition over the completion with respect to the Sobolev norm. But we have no difficulty in dealing with it in C^∞ category, because the moduli space $\mathcal{M}^-$ is considered in some sense as a solution space of ε certain system of elliptic equations ([**11**]).

The space $\mathcal{M}^-$ is endowed with a topology as a subspace of $\mathcal{B}$. From Proposition 2.3 the subset $S_{A,\varepsilon} \cap \mathcal{A}^- = \{A+\alpha;\ \alpha \in \Omega^1(\mathfrak{g}_P),\ |\alpha|_k < \varepsilon,\ d_A^*\alpha = 0$ and α is a solution of (2.3)$\}$ gives a neighborhood of $\mathcal{M}^-$ around $[A]$ through π.

We have from the Kuranishi map

$$A+\alpha\in S_{A,\varepsilon}\cap\mathcal{A}^- \Longleftrightarrow \Xi_A(\alpha)\in H_A^1 \quad \text{and} \quad \boldsymbol{H}_A([\alpha\wedge\alpha]^+)=0.$$

Hence for an anti-self-dual connection A with $H_+^2=\{0\}$, $A+\alpha\in S_{A,\varepsilon}\cap\mathcal{A}^- \Leftrightarrow \Xi_A(\alpha)\in H_A^1$. Since the Frechet differential $(d/d\alpha)\Xi_A$ of Ξ_A at $\alpha=0$ is the identity map, we have by the inverse mapping theorem of Banach spaces the inverse $(\Xi_A)^{-1}$ defined on an ε-neighborhood $\{\alpha\,;\,|\alpha|_k<\varepsilon\}\subset\Omega^1(\mathfrak{g}_P)$. Restricting $(\Xi_A)^{-1}$ to $\{\alpha\,;\,|\alpha|_k<\varepsilon\}\cap H_A^1$ we obtain

PROPOSITION 2.4. *Let A be an anti-self-dual connection which is irreducible (i.e., $H^0=\{0\}$) and satisfies $H_+^2=\{0\}$. Then a neighborhood of $\mathcal{M}^-$ containing $[A]$ is homeomorphic to an open ball $\{\alpha\in H_A^1\,;\,|\alpha|<\varepsilon\}$.*

We call an anti-self-dual connection with $H^0=\{0\}$ and $H_+^2=\{0\}$ generic. Then $\mathcal{M}^-=\{$generic anti-self-dual connections on $P\}/\mathcal{G}$ is an open, dense subset of $\hat{\mathcal{M}}^-$ (see for example [**11**]). It is verified that this is a C^∞ Hausdorff manifold. The smooth structure is given by the collection of slice neighborhoods $S_{A,\varepsilon}\cap\mathcal{A}^-$ together with the Kuranishi map $\Xi_A\,;\,S_{A,\varepsilon}\cap\mathcal{A}^-\to H_A^1$, where A is a generic anti-self-dual connection. From the Atiyah-Singer index theorem the index of the complex (2.1) gives its dimension.

3. Riemannian metric on the moduli space.

The space $\mathcal{A}$ is an affine space with vector space $\Omega^1(\mathfrak{g}_P)$ which is endowed with the inner product $(\ ,\ )$. Since $(\ ,\ )$ is $\mathcal{G}$-invariant, it induces naturally an inner product on the quotient space $\mathcal{B}$.

In order to obtain an explicit expression of this inner product it suffices to give an inner product to each slice neighborhood gauge invariantly by using a decomposition of the tangent spaces of $\mathcal{A}$.

Denote by $\mathcal{A}^*$ the set of all irreducible connections on the bundle P and by $\mathcal{B}^*$ the quotient space $\mathcal{A}^*/\mathcal{G}$.

For each $A\in\mathcal{A}^*$ the tangent space at A to $\mathcal{A}^*$ $T_A\mathcal{A}^*=\Omega^1(\mathfrak{g}_P)$ has a unique orthogonal decomposition

$$\Omega^1(\mathfrak{g}_P)=\operatorname{Im} d_A\oplus\operatorname{Ker} d_A^* \tag{3.1}$$

with respect to the inner product $(\ ,\ )$. Then each $\beta\in\Omega^1(\mathfrak{g}_P)$ splits into the vertical part and the horizontal part

$$\beta=\beta^v+\beta^h, \tag{3.2}$$

$\beta^v\in\operatorname{Im} d_A$, $\beta^h\in\operatorname{Ker} d_A^*$.

We define a positive semi-definite inner product $\langle\ ,\ \rangle_A$ on $\Omega^1(\mathfrak{g}_P)$ by

$$\langle\beta,\beta_1\rangle_A=(\beta^h,\beta_1^h)=\int_M(\beta^h(x),\beta_1^h(x))d\mathrm{vol}. \tag{3.3}$$

Note that this inner product depends on a connection.

Choose a slice neighborhood at A $S_{A,\varepsilon}=\{A+\alpha\in\mathcal{A},\ d_A^*\alpha=0,\ |\alpha|_k<\varepsilon\}$ and fix it. To the tangent space to $S_{A,\varepsilon}$ at $A+\alpha$ we give an inner product by the restriction of the above to $T_{A+\alpha}S_{A,\varepsilon}$, namely

$$\langle\beta,\ \beta_1\rangle_{A+\alpha}=(\beta^h,\ \beta_1^h), \tag{3.4}$$

$\beta,\ \beta_1\in T_{A+\alpha}S_{A,\varepsilon}$. Here β^h and β_1^h are the horizontal parts with respect to the splitting $\Omega^1(\mathfrak{g}_P)=\mathrm{Im}\, d_{A+\alpha}\oplus\mathrm{Ker}\, d_{A+\alpha}^*$ (Figure 1).

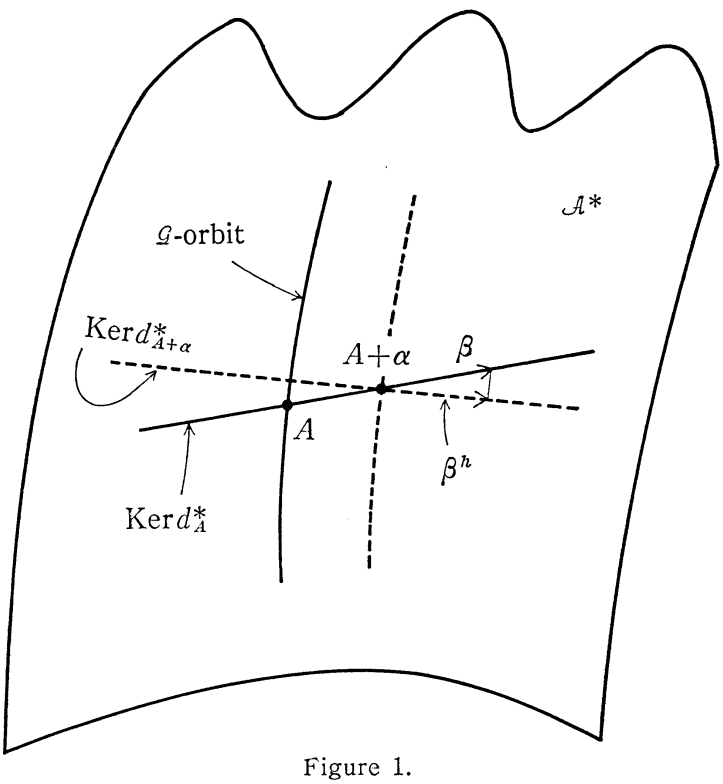

Figure 1.

We remark that the choice of a slice neighborhood is not unique for $\mathcal{B}$. We assume that there exists another slice neighborhood $S_{A',\varepsilon'}$ satisfying $\pi(S_{A,\varepsilon})\cap\pi(S_{A',\varepsilon'})\neq\emptyset$. We have a C^∞ map g from the subset $S_{A,\varepsilon}\cap\pi^{-1}(\pi(S_{A',\varepsilon'}))$ in $S_{A,\varepsilon}$ into $\mathcal{G}$ such that $g_\alpha(A+\alpha)=A'+\alpha'$ belongs to $S_{A',\varepsilon'}$. Then we obtain the following map which gives a transformation between the slice neighborhoods

$$\begin{aligned}\Psi;\ S_{A,\varepsilon}\cap\pi^{-1}(\pi(S_{A',\varepsilon'}))&\longrightarrow S_{A',\varepsilon'}\cap\pi^{-1}(\pi(S_{A,\varepsilon}))\,,\\ A+\alpha&\longmapsto g_\alpha(A+\alpha)\,.\end{aligned}$$

The derivative of Ψ is represented by

$$\begin{aligned}&\Psi_*;\ T_{A+\alpha}S_{A,\varepsilon}\longrightarrow T_{A'+\alpha'}S_{A',\varepsilon'}\,,\\ &(\Psi_*)_{A+\alpha}(\beta)=g_\alpha(\beta)+d_{A'+\alpha'}\phi,\ \ \phi\in\Omega^0(\mathfrak{g}_P)\,,\end{aligned} \tag{3.5}$$

$(A'+\alpha'=g_\alpha(A+\alpha))$. The formula (3.5) is derived in the following way. Since $\Psi_*(\beta)=d/dt|_{t=0}\Psi(A+\alpha+t\beta)=d/dt|_{t=0}g_{(\alpha+t\beta)}(A+\alpha+t\beta)$ and $g_{(\alpha+t\beta)}=g_\alpha\cdot\exp\psi_t$ for a curve ψ_t in $\Omega^0(\mathfrak{g}_P)$ with $\psi_0=0$, $\Psi_*(\beta)$ reduces to

$$d/dt|_{t=0}g_{(\alpha+t\beta)}(A+\alpha)+d/dt|_{t=0}g_{(\alpha+t\beta)}(t\beta)=d_{A'+\alpha'}\psi+g_\alpha(\beta),$$

where we set $\psi=d/dt|_{t=0}\psi_t$.

We observe from (3.5) that the horizontal part of $\Psi_*(\beta)$ coincides with $g_\alpha(\beta^h)$. Thus by the $\mathcal{G}$-invariance of (,) we have

PROPOSITION 3.1. *The restriction to a slice neighborhood of the inner product given by* (3.4) *defines a positive definite inner product on* $\mathcal{B}^*$, *which is independent of the choice of slice neighborhoods.*

We restrict this inner product to the moduli space $\mathcal{M}^-$ to define a C^∞ Riemannian metric.

By the way the Kuranishi map Ξ_A with an open neighborhood U $(=\Xi_A(S_{A,\varepsilon}\cap\mathcal{A}^-))\subset H_A^1$ gives a local coordinate around $[A]$ in $\mathcal{M}^-$. So we will give a further consideration to the Riemannian metric on U.

We remark first that the derivative of the Kuranishi map is given by the following

$$\begin{aligned}&\Xi_{A*};\ T_\alpha\Omega^1(\mathfrak{g}_P)\longrightarrow T_{\Xi_A(\alpha)}\Omega^1(\mathfrak{g}_P),\\&\Xi_{A*}(\beta)=\beta+d_A^{+*}G_A([\alpha\wedge\beta]^+)\end{aligned}\tag{3.6}$$

and the tangent space to a slice neighborhood at $A+\alpha$ is characterized as

$$T_{A+\alpha}(S_{A,\varepsilon}\cap\mathcal{A}^-)=\{\beta\in\Omega^1(\mathfrak{g}_P);\ d_A^*\beta=0,\ d_A^+\beta+[\alpha\wedge\beta]^+=d_{A+\alpha}^+\beta=0\}.\tag{3.7}$$

Here and in what follows we identify $A+\alpha$ with α for a fixed connection A unless any confusion occurs, and we denote by $S_{A,\varepsilon}^-$ a slice neighborhood $S_{A,\varepsilon}\cap\mathcal{A}^-$. We have obviously $T_AS_{A,\varepsilon}^-=H_A^1$ from (3.7). Note that $\langle\ ,\ \rangle=(\ ,\ )$ on H_A^1.

By (3.6) and (3.7) the restriction of Ξ_{A*} to $S_{A,\varepsilon}^-$ has the following form.

PROPOSITION 3.2. *The differential of the Kuranishi map* $\Xi_{A*\alpha}$; $T_\alpha S_{A,\varepsilon}^-\to T_{\Xi_A(\alpha)}U=H_A^1$ *is written by*

$$\Xi_{A*\alpha}(\beta)=\boldsymbol{H}_A\beta.\tag{3.8}$$

PROOF. Since $\beta\in T_\alpha S_{A,\varepsilon}^-$ satisfies $d_A^+\beta+[\alpha\wedge\beta]^+=0$ and d_A^{+*} commutes with the Green operator G_A, $\Xi_{A*\alpha}(\beta)$ reduces to $\beta-\Delta_A^1G_A\beta$. By the definition of the Green operator we have $\Xi_{A*\alpha}(\beta)=\boldsymbol{H}_A\beta$. Q.E.D.

We will now solve conversely an equation $\Xi_{A*\alpha}(\beta)=\gamma$ for given $\gamma\in H_A^1$ and $\alpha\in S_{A,\varepsilon}^-$ with respect to $\beta\in T_\alpha S_{A,\varepsilon}^-$. We decompose β into $\beta=d_A\gamma^0+\gamma^1+d_A^{+*}\gamma^2$, $\gamma^0\in\Omega^0(\mathfrak{g}_P)$, $\gamma^1\in H_A^1$ and $\gamma^2\in\Omega_+^2(\mathfrak{g}_P)$. From Proposition 3.2 γ^1 coincides with γ. Moreover from $d_A^*\beta=0$ we have that $\gamma^0=0$ and hence $\beta=\gamma+d_A^{+*}\gamma^2$. The element

γ^2 satisfies

$$d_A^+ d_A^{+*}\gamma^2 + [\alpha \wedge (\gamma + d_A^{+*}\gamma^2)]^+ = 0$$

or

$$d_A^+ d_A^{+*}\gamma^2 + [\alpha \wedge d_A^{+*}\gamma^2]^+ = -[\alpha \wedge \gamma]^+. \tag{3.9}$$

PROPOSITION 3.3. *Given $\gamma \in H_A^1$, the inverse image $\beta = (\Xi_{A*\alpha})^{-1}(\gamma) \in T_\alpha S_{A,\varepsilon}^-$ is represented by*

$$\beta = \gamma + d_A^{+*}\gamma^2, \tag{3.10}$$

where $\gamma^2 \in \Omega_+^2(\mathfrak{g}_P)$ is a solution of (3.9).

NOTE. At $\alpha = 0$ Ξ_{A*} reduces to the identity map.

We split $\beta \in T_\alpha S_{A,\varepsilon}^-$ into $\beta = \beta^v + \beta^h$ to see that β^h belongs to $H_{A+\alpha}^1$. Then the inner product $\langle\ ,\ \rangle$ on $T_\alpha S_{A,\varepsilon}^-$ is written as

$$\langle \beta, \beta_1 \rangle_{A+\alpha} = (\boldsymbol{H}_\alpha \beta, \boldsymbol{H}_\alpha \beta_1).$$

Let X, Y and Z be tangent vectors at A to the slice neighborhood $S_{A,\varepsilon}^-$ ($X, Y, Z \in T_A S_{A,\varepsilon}^- = H_A^1$). Since H_A^1 is affine, these vectors represent also vector fields canonically extending on $U \subset H_A^1$.

Because the Kuranishi map is bijective from $S_{A,\varepsilon}^-$ onto U, we get vector fields $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ along $S_{A,\varepsilon}^-$ so that for example the value of $\bar{X}$ at $\alpha \in S_{A,\varepsilon}^-$ is given by $(\Xi_{A*\alpha})^{-1}(X)$. Then we have from Proposition 3.3 $\bar{X}_\alpha = X + d_A^{+*}\gamma^2$, here $\gamma^2 = \gamma^2(\alpha, X) \in \Omega_+^2(\mathfrak{g}_P)$ is a solution of the equation

$$d_A^+ d_A^{+*}\gamma^2 + [\alpha \wedge d_A^{+*}\gamma^2]^+ = -[\alpha \wedge X]^+. \tag{3.11}$$

Obviously $\bar{X}_\alpha = X$ at $\alpha = 0$ (Figure 2).

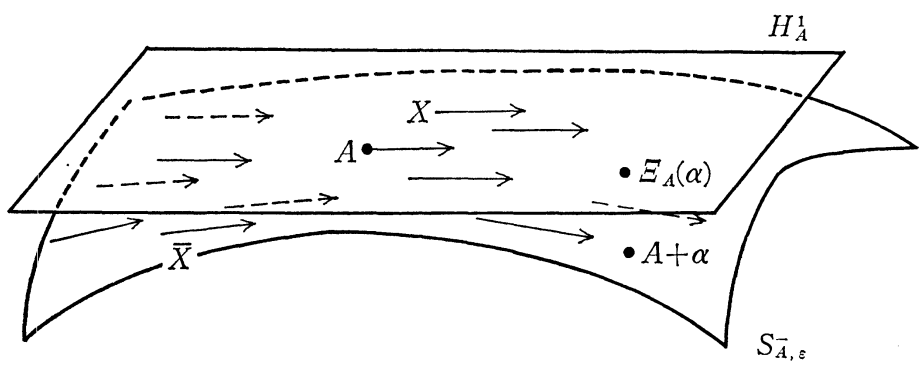

Figure 2.

We let $\alpha(t)$ be a curve in $S_{A,\varepsilon}^-$ corresponding to a straight line tX in H_A^1, namely, $\alpha(t)$ is defined by $\Xi_A(\alpha(t)) = tX$. Then we have $\alpha(0) = 0$ and $d/dt|_{t=0}\alpha(t) = X \in T_A S_{A,\varepsilon}^-$.

PROPOSITION 3.4. *The Riemannian metric* $\langle\ ,\ \rangle$ *on* $\mathfrak{M}^-$ *satisfies at* $\alpha=0$ *in a slice neighborhood* $S^-_{A,\varepsilon}$

$$X\langle Y, Z\rangle = 0 \tag{3.12}$$

for every $X, Y, Z\in H^1_A$.

From this proposition we see that the local coordinate associated to Ξ_A and $U\subset H^1_A$ is normal at A for the metric.

PROOF. By the definition of the metric we have at $\alpha=0$

$$\begin{aligned} X\langle Y, Z\rangle &= d/dt|_{t=0}((\overline{Y}_{\alpha(t)})^h, (\overline{Z}_{\alpha(t)})^h) \\ &= (d/dt|_{t=0}(\overline{Y}_{\alpha(t)})^h, Z)+(Y, d/dt|_{t=0}(\overline{Z}_{\alpha(t)})^h). \end{aligned}$$

Differentiating $(\overline{Y}_{\alpha(t)})^h=\boldsymbol{H}_{\alpha(t)}\overline{Y}_{\alpha(t)}$, we get

$$d/dt|_{t=0}(\boldsymbol{H}_{\alpha(t)}\overline{Y}_{\alpha(t)}) = \boldsymbol{H}_{\alpha(0)}(d/dt|_{t=0}\overline{Y}_{\alpha(t)})+(d/dt|_{t=0}\boldsymbol{H}_{\alpha(t)})Y. \tag{3.13}$$

We show first two lemmas.

LEMMA 3.5. $$\boldsymbol{H}_{\alpha(0)}(d/dt|_{t=0}\overline{Y}_{\alpha(t)}) = 0. \tag{3.14}$$

PROOF OF LEMMA 3.5. We have from Proposition 3.3

$$\begin{aligned} d/dt|_{t=0}(\overline{Y}_{\alpha(t)}) &= d/dt|_{t=0}(Y+d_A^{+*}\gamma^2(\alpha, Y)) \\ &= d_A^{+*}(d/dt|_{t=0}\gamma^2(\alpha(t), Y)). \end{aligned}$$

On the other hand we differentiate at $\alpha=0$

$$d_A^+d_A^{+*}\gamma^2+[\alpha\wedge d_A^{+*}\gamma^2]^+ = -[\alpha\wedge Y]^+$$

to obtain

$$d_A^+d_A^{+*}(d/dt|_0\gamma^2) = -[X\wedge Y]^+. \tag{3.15}$$

Since $H^2_{+,A}=\{0\}$, $d/dt|_0\gamma^2=-G_A([X\wedge Y]^+)$, namely

$$d/dt|_0(\overline{Y}_{\alpha(t)}) = -d_A^{+*}G_A([X\wedge Y]^+). \tag{3.16}$$

Then the lemma is obtained by the Hodge decomposition. Q. E. D.

LEMMA 3.6. *For* $Y\in H^1_A$ *we have*

$$(d/dt|_0\boldsymbol{H}_{\alpha(t)})Y = d_AG_A(\{X, Y\})-d_A^{+*}G_A([X\wedge Y]^+). \tag{3.17}$$

Here $\{\ ,\ \}$; $\Omega^1(\mathfrak{g}_P)\times\Omega^1(\mathfrak{g}_P)\to\Omega^0(\mathfrak{g}_P)$ *is a skew-symmetric bilinear map defined by*

$$\{X, Y\} = \sum_{i,j=1}^{4} h^{ij}[X_i, Y_j], \tag{3.18}$$

$X=\sum X_idx^i$, $Y=\sum Y_jdx^j\in\Omega^1(\mathfrak{g}_P)$.

PROOF OF LEMMA 3.6. Because $(d/dt|_0\boldsymbol{H}_{\alpha(t)})(Y)=d/dt|_0(\boldsymbol{H}_{\alpha(t)}Y)$ and $\boldsymbol{H}_{\alpha(t)}Y=Y-G_{\alpha(t)}\Delta^1_{\alpha(t)}Y$, we obtain

$$\begin{aligned}(d/dt|_0\boldsymbol{H}_{\alpha(t)})Y &= -d/dt|_0(G_{\alpha(t)}\Delta^1_{\alpha(t)}Y)\\ &= -(d/dt|_0G_{\alpha(t)})(\Delta^1_AY)-G_A(d/dt|_0\Delta^1_{\alpha(t)}Y)\\ &= -G_A(d/dt|_0\Delta^1_{\alpha(t)}Y).\end{aligned}$$

We differentiate at $t=0$

$$\Delta^1_{\alpha(t)}Y = d_{A+\alpha(t)}d^*_{A+\alpha(t)}Y+d^{+*}_{A+\alpha(t)}d^+_{A+\alpha(t)}Y.$$

Then $d/dt|_0(\Delta^1_{\alpha(t)}Y)$ reduces to

$$d/dt|_0(\Delta^1_{\alpha(t)}Y) = [X\wedge d^*_AY]-d_A(\{X, Y\})+d^{+*}_A([X\wedge Y]^+)-[(X\wedge d^+_AY)].$$

Here $[(\cdot\wedge\cdot)]$; $\Omega^1(\mathfrak{g}_P)\times\Omega^2_+(\mathfrak{g}_P)\to\Omega^1(\mathfrak{g}_P)$ denotes the natural contraction with respect to the metric h on the base space M together with the Lie bracket operation. Hence, because $Y\in H^1_A$ we get

$$d/dt|_0(\Delta^1_{\alpha(t)}Y) = -d_A(\{X, Y\})+d^{+*}_A([X\wedge Y]^+). \tag{3.19}$$

Therefore we have

$$\begin{aligned}(d/dt|_0\boldsymbol{H}_{\alpha(t)})Y &= -G_A(-d_A\{X, Y\}+d^{+*}_A[X\wedge Y]^+)\\ &= d_AG_A\{X, Y\}-d^{+*}_AG_A[X\wedge Y]^+. \qquad \text{Q.E.D.}\end{aligned}$$

The proof of Proposition 3.4 is easily derived from these lemmas.

4. Complex structure and Kähler metric on $\mathcal{M}^-$.

We assume in this section that the base space M is a compact complex surface and the metric h is Kähler. The almost complex structure of M induces in a natural way an almost complex structure I on $\Omega^1(\mathfrak{g}_P)$ as

$$I\alpha = \sqrt{-1}\,\alpha^{1,0}-\sqrt{-1}\,\alpha^{0,1} \tag{4.1}$$

where $\alpha=\alpha^{1,0}+\alpha^{0,1}\in\Omega^1(\mathfrak{g}_P)$, $\alpha^{1,0}=\sum_{\mu=1}^2\alpha_\mu dz^\mu$ and $\alpha^{0,1}=\sum_{\mu=1}^2\alpha_{\bar\mu}dz^{\bar\mu}$ ((z^1, z^2) is a complex coordinate). The restriction of I to the first cohomology group H^1_A for $A\in\mathcal{A}^-$ is well defined ([**14**], Proposition 2.4).

We show first that $I|_{H^1_A}$ gives indeed the complex structure of the moduli space $\mathcal{M}^-$.

For this purpose we recall the moduli space $\mathcal{M}_h$ of holomorphic (0, 1)-connections and the canonical map from $\mathcal{M}^-$ to $\mathcal{M}_h$ ([**14**]).

We call the (0, 1)-component of a connection a (0, 1)-connection. Each (0, 1)-connection B has the curvature $F_B=\bar\partial B+1/2[B\wedge B]\in\Omega^{0,2}(\mathfrak{g}^C_P)$, where $\mathfrak{g}^C_P$ is the complexification of $\mathfrak{g}_P$. A (0, 1)-connection is called holomorphic when $F=0$

identically. The group of complex gauge transformations $\mathcal{G}^C=\Gamma(M; P\times_{Ad}SL(n; \boldsymbol{C}))$ acts on the set of all C^∞ (0, 1)-connections and fix the subset of holomorphic (0, 1)-connections. So we have the quotient space {holomorphic (0, 1)-connections}$/\mathcal{G}^C$, called the moduli space $\mathcal{M}_h$ of holomorphic (0, 1)-connections.

Analogously to the moduli space of anti-self-dual connections, to each holomorphic (0, 1)-connection B we associate a slice neighborhood $S^{0,1}_{B,\varepsilon}=\{\alpha\in\Omega^{0,1}(\mathfrak{g}^C_P);\ |\alpha|<\varepsilon,\ \bar\partial^*_B\alpha=0,\ \bar\partial_B\alpha+1/2[\alpha\wedge\alpha]=0\}$ which represents a neighborhood of $[B]$ in $\mathcal{M}_h$ and we define the Kuranishi map $\Theta_B; S^{0,1}_{B,\varepsilon}\to H^{0,1}_B=\mathrm{Ker}\,\bar\partial^*_B\cap\mathrm{Ker}\,\bar\partial_B$ so that $\hat{\mathcal{M}}_h=\{$generic, holomorphic (0, 1)-connections$\}/\mathcal{G}^C$, an open dense subset of $\mathcal{M}_h$, carries the complex structure with respect to the coordinate system $\{(\Theta_B$, an open ball in $H^{0,1}_B)\}_B$, where B varies in the set of all generic, holomorphic (0, 1)-connections.

Let A be an anti-self-dual connection. Then its curvature is of type (1, 1). So a (0, 1)-connection which is given by the (0, 1)-component $A^{0,1}$ of A is holomorphic, since $F_{A^{0,1}}$ is the (0, 2)-component of F_A. Moreover we have $\mathcal{G}\subset\mathcal{G}^C$ so that the canonical map f is well defined from $\mathcal{M}^-$ to $\mathcal{M}_h$ by assigning $[A^{0,1}]$ to $[A]$. The complex structure of $\mathcal{M}^-$ is defined by the natural complex structure of $\mathcal{M}_h$ through f. The map f induces a linear isomorphism between tangent spaces of $\mathcal{M}^-$ and $\mathcal{M}_h$; $f_{*A}; H^1_A\to H^{0,1}_{A^{0,1}}; \beta\mapsto\beta^{0,1}$, the (0, 1)-component of β. Hence we have $f_{*A}(I\beta)=-\sqrt{-1}f_{*A}(\beta)$.

REMARK. Although the complex Kuranishi map Θ_B defines naturally a complex holomorphic coordinate on a slice neighborhood $S^{0,1}_{B,\varepsilon}$, the real Kuranishi map Ξ_A does not yield directly a holomorphic coordinate on a slice neighborhood $S^-_{A,\varepsilon}$.

We notice that the Riemannian structure $\langle\ ,\ \rangle$ on $\mathcal{M}^-$ introduced in §3 is Hermitian with respect to the almost complex structure $I|_{H^1_A}$.

We will show that $\langle\ ,\ \rangle$ is a Kähler metric. For this aim it is necessary to extend over a slice neighborhood the almost complex structure, which we abbreviate as I, defined on the tangent space H^1_A at just a single point $[A]$.

For a tangent vector $\beta\in T_\alpha S^-_{A,\varepsilon}=\mathrm{Ker}\,d^*_A\cap\mathrm{Ker}\,d^+_{A+\alpha}$ the horizontal part β^h belongs to $H^1_{A+\alpha}$. We express the almost complex structure at α in $S^-_{A,\varepsilon}$ in the form

$$\begin{aligned}&\mathcal{J};\ T_\alpha S^-_{A,\varepsilon}\longrightarrow T_\alpha S^-_{A,\varepsilon},\\&\mathcal{J}\beta=I\beta+\delta(\beta),\end{aligned}\tag{4.2}$$

here $\delta(\beta)$ is a linear transformation depending on α, which represents the difference between $\mathcal{J}\beta$ and $I\beta$.

We will find an explicit form of $\mathcal{J}$.

LEMMA 4.1. *The almost complex structure $\mathcal{J}$ has the following expression,*

$$\mathcal{J}\beta = I(\beta^h)+d_{A+\alpha}\phi\,, \tag{4.3}$$

for some $\phi\in\Omega^0(\mathfrak{g}_P)$.

PROOF. The condition $\mathcal{J}\beta\in\mathrm{Ker}\, d^+_{A+\alpha}$ is equivalent to that $\mathcal{J}\beta$ has the form

$$\mathcal{J}\beta = I(\beta^h)+\delta^1(\beta)+d_{A+\alpha}\phi \tag{4.4}$$

with $\delta^1(\beta)\in H^1_{A+\alpha}$ and $\phi\in\Omega^0(\mathfrak{g}_P)$.

In fact, since $I(\beta^h)\in\mathrm{Ker}\, d^+_{A+\alpha}$, we have

$$\mathcal{J}\beta\in\mathrm{Ker}\, d^+_{A+\alpha} \Longleftrightarrow I(\beta^v)+\delta(\beta)\in\mathrm{Ker}\, d^+_{A+\alpha}\,.$$

If we decompose $\delta(\beta)$ into $\delta(\beta)=\delta^1(\beta)+d_{A+\alpha}\phi+d^{+*}_{A+\alpha}\Psi$ as the Hodge decomposition, the above is equivalent to

$$I(\beta^v)+d^{+*}_{A+\alpha}\Psi = I(d_{A+\alpha}\phi')+d^{+*}_{A+\alpha}\Psi \in \mathrm{Ker}\, d^+_{A+\alpha} \tag{4.5}$$

where $\beta^v=d_{A+\alpha}\phi'$, $\phi'\in\Omega^0(\mathfrak{g}_P)$. From the formula $I(d_{A+\alpha}\phi')=d^{+*}_{A+\alpha}(\phi'\otimes\omega_h)$, where ω_h is the Kähler form of h ([**15**, § 2]), we have

$$\begin{aligned}(4.5) &\Longleftrightarrow d^{+*}_{A+\alpha}(\phi'\otimes\omega_h+\Psi) \in \mathrm{Ker}\, d^+_{A+\alpha}\\ &\Longleftrightarrow \phi'\otimes\omega_h+\Psi \in H^2_{+,A}=\{0\},\end{aligned}$$

hence we get $\Psi=-\phi'\otimes\omega_h$. Thus,

$$\mathcal{J}\beta \in \mathrm{Ker}\, d^+_{A+\alpha} \Longleftrightarrow \delta(\beta) = \delta^1(\beta)+d_{A+\alpha}\phi-I(\beta^v)\,,$$

which is equivalent to $\mathcal{J}\beta=I(\beta^h)+\delta^1(\beta)+d_{A+\alpha}\phi$.

We make a further restriction of the form of $\mathcal{J}$ such that $\mathcal{J}^2=-\mathrm{id}$ holds. We recall that the lift of the canonical map f from the slice neighborhood $S^-_{A,\varepsilon}$ of $\hat{\mathcal{M}}^-$ to the slice neighborhood $S^{0,1}_{A^{0,1},\varepsilon'}$ of the moduli space $\mathcal{M}_h$ is defined by

$$f(A+\alpha) = g_\alpha(A^{0,1}+\alpha^{0,1}) = A^{0,1}+\alpha^{0,1}+g_\alpha^{-1}\cdot(\bar\partial_{A+\alpha}g_\alpha) \tag{4.6}$$

here a complex gauge transformation g_α is required in order to satisfy a slice condition ([**14**], p. 857). From (4.6) the differential $f_*: T_\alpha S^-_{A,\varepsilon}\longrightarrow T_{\alpha^{0,1}}S^{0,1}_{A^{0,1},\varepsilon'}$ is given by

$$f_{*\alpha}(\beta) = \beta^{0,1}+\bar\partial_{A+\alpha}\psi\,,$$

where $\psi\in\Omega^0(\mathfrak{g}^C_P)$. Since $\mathcal{J}$ is compatible with the canonical complex structure of $\mathcal{M}_h$, it must satisfy

$$f_{*\alpha}(\mathcal{J}\beta) = -\sqrt{-1}\,f_{*\alpha}(\beta)\,, \tag{4.7}$$

$\beta\in T_\alpha S^-_{A,\varepsilon}$.

We substitute (4.4) to (4.7) and obtain

$$f_{*\alpha}(\mathcal{J}\beta)=(\mathcal{J}\beta)^{0,1}+\bar{\partial}_{A+\alpha}\psi'=(I(\beta^h))^{0,1}+(\delta^1(\beta))^{0,1}+\bar{\partial}_{A+\alpha}\phi+\bar{\partial}_{A+\alpha}\psi',$$

$\psi'\in\Omega^0(\mathfrak{g}_P^C)$. Since this must be equal to $-\sqrt{-1}(\beta^{0,1}+\bar{\partial}_{A+\alpha}\psi)=-\sqrt{-1}(\beta^v+\beta^h)^{0,1}$ $-\sqrt{-1}\bar{\partial}_{A+\alpha}\psi$, it follows from the Hodge decomposition associated to the $\bar{\partial}_A$-complex that $(\delta^1(\beta))^{0,1}=0$, in other words, $\delta^1(\beta)=0$. Q. E. D.

REMARK. The condition for $\mathcal{J}\beta$ to be tangent to the slice is that ϕ satisfies the equation;

$$d^*_{A+\alpha}d_{A+\alpha}\phi+\{\alpha, I(\beta^h)+d_{A+\alpha}\phi\}=0. \tag{4.8}$$

Obviously $\mathcal{J}=I$ at $\alpha=0$.

PROPOSITION 4.2. *The fundamental form ω, defined by $\omega(X, Y)=\langle\mathcal{J}X, Y\rangle$, is closed. Namely the metric $\langle\ ,\ \rangle$ is Kähler.*

PROOF. Let X, Y and Z be tangent vectors at A to $S^-_{A,\varepsilon}$ ($X, Y, Z\in T_AS^-_{A,\varepsilon}=H^1_A$). Let $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ be vector fields along the slice neighborhood given at § 3. The vectors X, Y and Z also denote canonically extended vector fields over an open subset $U\subset H^1_A$.

What to show is that at $\alpha=0$

$$X\omega(Y, Z)=0. \tag{4.9}$$

If (4.9) is proven, then the form ω is closed, since $(d\omega)(X, Y, Z)=X\omega(Y, Z)+Y\omega(Z, X)+Z\omega(X, Y)-\omega([X, Y], Z)-\omega([Y, Z], X)-\omega([Z, X], Y)$.

The left hand side of (4.9) is given by

$$d/dt|_0\langle\mathcal{J}Y, Z\rangle=d/dt|_0((\mathcal{J}\bar{Y})^h, \bar{Z}^h)$$

which is from (3.16) equal to $(d/dt|_0(\mathcal{J}\bar{Y})^h, Z)+(IY, d/dt|_0\bar{Z}^h)=(d/dt|_0(\mathcal{J}\bar{Y})^h, Z)$. By Lemma 4.1 we have that $(\mathcal{J}\bar{Y})^h=I(\bar{Y}^h)$, from which it follows that $(d/dt|_0(\mathcal{J}\bar{Y})^h, Z)=(I(d/dt|_0\bar{Y}^h), Z)=-(d/dt|_0\bar{Y}^h, IZ)=0$. Q. E. D.

5. Riemannian curvature tensor.

Let the base space M be in this section a compact, connected oriented Riemannian 4-manifold. Let $\langle\ ,\ \rangle$ be the Riemannian metric on the moduli space $\hat{\mathcal{M}}^-$ of generic anti-self-dual connections on the bundle P. The Riemannian curvature tensor R is defined as $R(X, Y)Z=[\nabla_X, \nabla_Y]Z-\nabla_{[X,Y]}Z$ in terms of the Levi-Civita connection ∇ of the metric $\langle\ ,\ \rangle$.

We exhibit first the following theorem.

THEOREM 5.1. *The Riemannian curvature tensor is represented at $[A]$ in $\hat{\mathcal{M}}^-$ as*

$$\begin{aligned}\langle R(X, Y)Z, W\rangle = &-(\{X, W\}, G_A\{Z, Y\})-2(\{Z, W\}, G_A\{X, Y\})\\ &+(\{Y, W\}, G_A\{Z, X\})+([X\wedge W]^+, G_A[Z\wedge Y]^+) \qquad (5.1)\\ &-([Y\wedge W]^+, G_A[Z\wedge X]^+),\end{aligned}$$

$X, Y, Z, W\in H_A^1$ $(\cong T_{[A]}\hat{\mathcal{M}}^-)$ *(for the definition of* $[X\wedge Y]^+$ *and* $\{X, Y\}$, *see* (2.3) *in* § 2 *and* (3.18) *in* § 3).

We have in general the following formula of the Riemannian curvature tensor. If a local coordinate $\{x^1, \cdots, x^n\}$ is normal for a Riemannian manifold at a point p (i.e., $h_{ij}(p)=\delta_{ij}$ and $\partial/\partial x^i\, h_{jk}(p)=0$ for the metric h), then the tensor R is written at p by

$$\begin{aligned}R_{lkij}(p) = 1/2\{&\partial^2 h_{lj}/\partial x^i\partial x^k(p)-\partial^2 h_{li}/\partial x^j\partial x^k(p)+\partial^2 h_{ki}/\partial x^j\partial x^l(p)\\ &-\partial^2 h_{kj}/\partial x^i\partial x^l(p)\}.\end{aligned} \qquad (5.2)$$

Here R_{lkij} denotes the component $h(R(\partial/\partial x^i, \partial/\partial x^j)\partial/\partial x^k, \partial/\partial x^l)$ ([**17**]).

Since from Proposition 3.4 the local coordinate given by the Kuranishi map is normal, it suffices to derive an explicit expression of the second order partial derivatives $\partial^2 h_{lj}/\partial x^i\partial x^k(p)$.

In our case the second order derivative reduces to

$$\begin{aligned}\partial^2/\partial s\partial t|_{(0,0)}(\boldsymbol{H}_\alpha\bar{Y}, \boldsymbol{H}_\alpha\bar{Z}) = &(\partial^2/\partial s\partial t|_{(0,0)}(\boldsymbol{H}_\alpha\bar{Y}), Z)+(\partial/\partial t|_{(0,0)}\boldsymbol{H}_\alpha\bar{Y}, \partial/\partial s|_{(0,0)}\boldsymbol{H}_\alpha\bar{Z})\\ &+(\partial/\partial s|_{(0,0)}\boldsymbol{H}_\alpha\bar{Y}, \partial/\partial t|_{(0,0)}\boldsymbol{H}_\alpha\bar{Z}) \qquad (5.3)\\ &+(Y, \partial^2/\partial s\partial t|_{(0,0)}(\boldsymbol{H}_\alpha\bar{Z})),\end{aligned}$$

where $\alpha(s, t)$ denotes a surface in a slice neighborhood, $\alpha(0, 0)=0$. We let the surface $\alpha(s, t)$ correspond to a plane in $U\subset H_A^1$, namely $\alpha(s, t)$ satisfies that

$$\Xi_A(\alpha(s, t)) = sX+tW \qquad (5.4)$$

($|s|$ and $|t|$ are sufficiently small).

LEMMA 5.2. *The second term together with the third term of* (5.3) *is written in the form*

$$\begin{aligned}&(\partial/\partial t|_{(0,0)}\boldsymbol{H}_\alpha\bar{Y}, \partial/\partial s|_{(0,0)}\boldsymbol{H}_\alpha\bar{Z})+(\partial/\partial s|_{(0,0)}\boldsymbol{H}_\alpha\bar{Y}, \partial/\partial t|_{(0,0)}\boldsymbol{H}_\alpha\bar{Z})\\ &= (G_A\{W, Y\}, \{X, Z\})+(G_A\{X, Y\}, \{W, Z\})\\ &\quad +(G_A[W\wedge Y]^+, [X\wedge Z]^+)+(G_A[X\wedge Y]^+, [W\wedge Z]^+). \qquad (5.5)\end{aligned}$$

PROOF. From (3.16) in § 3 we have

$$\begin{aligned}\partial/\partial s|_{(0,0)}(\boldsymbol{H}_\alpha\bar{Y}) &= (\partial/\partial s|_{(0,0)}\boldsymbol{H}_\alpha)Y+\boldsymbol{H}_A(\partial/\partial s|_{(0,0)}\bar{Y})\\ &= (\partial/\partial s|_{(0,0)}\boldsymbol{H}_\alpha)Y+\boldsymbol{H}_A d_A^{+*}G_A(-[X\wedge Y]^+)\\ &= (\partial/\partial s|_{(0,0)}\boldsymbol{H}_\alpha)Y.\end{aligned}$$

Since $Y\in H_A^1$, it follows from Lemma 3.6 that

$$\partial/\partial s|_{(0,0)}(\boldsymbol{H}_\alpha\overline{Y}) = d_A G_A\{X, Y\} - d_A^{+*}G_A[X\wedge Y]^+ .$$

Therefore the left hand side of (5.5) is given by

$$\begin{aligned}&(d_A G_A\{W, Y\} - d_A^{+*}G_A[W\wedge Y]^+,\ d_A G_A\{X, Z\} - d_A^{+*}G_A[X\wedge Z]^+) + \{X\leftrightarrow W\}\\ &= (d_A G_A\{W, Y\},\ d_A G_A\{X, Z\}) + (d_A^{+*}G_A[W\wedge Y]^+,\ d_A^{+*}G_A[X\wedge Z]^+) + \{X\leftrightarrow W\}\\ &= (G_A\{W, Y\},\ \{X, Z\}) + (G_A[W\wedge Y]^+,\ [X\wedge Z]^+) + \{X\leftrightarrow W\},\end{aligned}$$

where we used $\Delta_A^0 G_A = \mathrm{id}$ and $\Delta_{+,A}^2 G_A = \mathrm{id}$. Q. E. D.

Next we compute the derivative $\partial^2/\partial s\partial t|_{(0,0)}(\boldsymbol{H}_\alpha\overline{Y})$. We have

$$\begin{aligned}\partial^2/\partial s\partial t|_{(0,0)}(\boldsymbol{H}_\alpha\overline{Y}) &= (\partial^2/\partial s\partial t|_{(0,0)}\boldsymbol{H}_\alpha)Y + (\partial/\partial t|_{(0,0)}\boldsymbol{H}_\alpha)(\partial/\partial s|_{(0,0)}\overline{Y})\\ &\quad + (\partial/\partial s|_{(0,0)}\boldsymbol{H}_\alpha)(\partial/\partial t|_{(0,0)}\overline{Y}) + \boldsymbol{H}_A(\partial^2/\partial s\partial t|_{(0,0)}\overline{Y}) .\end{aligned} \tag{5.6}$$

LEMMA 5.3. $(\partial/\partial t|_{(0,0)}\boldsymbol{H}_\alpha)(\partial/\partial s|_{(0,0)}\overline{Y}) = \boldsymbol{H}_A([(W\wedge G_A[X\wedge Y]^+)])$. (5.7)

REMARK. It follows from (5.7) that

$$((\partial/\partial t|_{(0,0)}\boldsymbol{H}_\alpha)(\partial/\partial s|_{(0,0)}\overline{Y}),\ Z) = (G_A[X\wedge Y]^+,\ [W\wedge Z]^+) . \tag{5.8}$$

PROOF. From (3.16) we obtain that $\partial/\partial s|_{(0,0)}\overline{Y} = -d_A^{+*}G_A[X\wedge Y]^+$. We denote this by V. Let $\{\beta_i(t)\}$ $(1\leqq i\leqq \dim H_A^1)$ be an orthonormal frame of $H_{A+\alpha(t)}^1$ $(\alpha(t)=\alpha(0, t))$ parametrized by t. Assume that $\beta_i(t)$ is C^∞ with respect to t. Then we get $\boldsymbol{H}_{\alpha(t)}V = \sum_i (V, \beta_i(t))\beta_i(t)$.

We differentiate this at $t=0$ to obtain

$$(\partial/\partial t|_{(0,0)}\boldsymbol{H}_\alpha)V = d/dt|_0(\boldsymbol{H}_{\alpha(t)}V) = \sum_i (V,\ d/dt\ \beta_i(0))\beta_i(0) + \sum_i (V,\ \beta_i(0))d/dt\ \beta_i(0) .$$

Since $V\in \mathrm{Im}\, d_A^{+*}$, this reduces to $\sum_i (V,\ d/dt\ \beta_i(0))\beta_i(0)$. On the other hand, differentiating $\Delta_{\alpha(t)}^1\beta_i(t)=0$ at $t=0$, we have

$$\Delta_A^1((d/dt\ \beta_i(0)) + (d/dt|_0\Delta_{\alpha(t)}^1)\beta_i(0) = 0 ,$$

which is by the aid of (3.19) written as

$$\Delta_A^1(d/dt\ \beta_i(0)) - d_A\{W,\ \beta_i(0)\} + d_A^{+*}[W\wedge\beta_i(0)]^+ = 0 . \tag{5.9}$$

Then because $V\in \mathrm{Im}\, d_A^{+*}$ we have

$$(V,\ d/dt\ \beta_i(0)) = (V,\ \boldsymbol{H}_A(d/dt\ \beta_i(0))) + (V,\ G_A\Delta_A^1(d/dt\ \beta_i(0)))$$

which reduces to $-(V,\ G_A d_A^{+*}[W\wedge\beta_i(0)]^+)$. Hence we get

$$(V,\ d/dt\ \beta_i(0)) = (G_A[X\wedge Y]^+,\ [W\wedge\beta_i(0)]^+) ,$$

that is,

$$\sum_i (V,\ d/dt\ \beta_i(0))\beta_i(0) = \sum_i (G_A[X\wedge Y]^+,\ [W\wedge\beta_i(0)]^+)\beta_i(0) . \tag{5.10}$$

Thus the lemma and the formula (5.8) are derived. Q. E. D.

For the second order derivative $\partial^2/\partial s\partial t|_{(0,0)}\overline{Y}_\alpha$ we have $\partial^2/\partial s\partial t|_{(0,0)}\overline{Y}=d_A^{+*}(\partial^2/\partial s\partial t|_{(0,0)}\gamma^2(\alpha, Y))$ and hence

$$\boldsymbol{H}_A(\partial^2/\partial s\partial t|_{(0,0)}\overline{Y}_\alpha)=0. \tag{5.11}$$

What to do next is to compute the second order derivative of the projection $(\partial^2/\partial s\partial t|_{(0,0)}\boldsymbol{H}_\alpha)Y$.

LEMMA 5.4. *We have*

$$\begin{aligned}((\partial^2/\partial s\partial t|_{(0,0)}\boldsymbol{H}_\alpha)Y, Z)=&-((\partial/\partial t|_{(0,0)}G_\alpha)(\partial/\partial s|_{(0,0)}\Delta_\alpha^1)Y, Z)\\&-((\partial/\partial s|_{(0,0)}G_\alpha)(\partial/\partial t|_{(0,0)}\Delta_\alpha^1)Y, Z).\end{aligned} \tag{5.12}$$

PROOF. By the definition of the canonical projection $\boldsymbol{H}_\alpha$,

$$\begin{aligned}(\partial^2/\partial s\partial t|_{(0,0)}\boldsymbol{H}_\alpha)Y&=\partial^2/\partial s\partial t|_{(0,0)}(\boldsymbol{H}_\alpha Y)\\&=-\{(\partial^2/\partial s\partial t|_{(0,0)}G_\alpha)\Delta_A^1Y+(\partial/\partial t|_{(0,0)}G_\alpha)(\partial/\partial s|_{(0,0)}\Delta_\alpha^1)Y\\&\quad+(\partial/\partial s|_{(0,0)}G_\alpha)(\partial/\partial t|_{(0,0)}\Delta_\alpha^1)Y+G_A(\partial^2/\partial s\partial t|_{(0,0)}\Delta_\alpha^1)Y\}.\end{aligned}$$

The formula (5.12) is derived from the fact $Y\in H_A^1$ and $G_A(\partial^2/\partial s\partial t|_{(0,0)}\Delta_\alpha^1Y)\in(H_A^1)^\perp$. Q. E. D.

Since the Green operator is self-adjoint, we have

$$\begin{aligned}((\partial/\partial t|_{(0,0)}G_\alpha)(\partial/\partial s|_{(0,0)}\Delta_\alpha^1)Y, Z)&=d/dt|_0(G_{\alpha(0,t)}(\partial/\partial s|_{(0,0)}\Delta_\alpha^1)Y, Z)\\=d/dt|_0((\partial/\partial s|_{(0,0)}\Delta_\alpha^1)Y, G_{\alpha(0,t)}Z)&=((\partial/\partial s|_{(0,0)}\Delta_\alpha^1)Y, (\partial/\partial t|_{(0,0)}G_\alpha)Z).\end{aligned}$$

LEMMA 5.5.

$$((\partial/\partial s|_{(0,0)}\Delta_\alpha^1)Y, (\partial/\partial t|_{(0,0)}G_\alpha)Z)=(\{X, Y\}, G_A\{W, Z\})+([X\wedge Y]^+, G_A[W\wedge Z]^+). \tag{5.13}$$

PROOF. We have from (3.19)

$$(\partial/\partial s|_{(0,0)}\Delta_\alpha^1)Y=-d_A\{X, Y\}+d_A^{+*}[X\wedge Y]^+.$$

For brevity we set $\alpha(t)=\alpha(0, t)$. We compute $(d/dt|_0G_{\alpha(t)})Z$. Because $\boldsymbol{H}_\alpha Z=Z-\Delta_\alpha^1G_\alpha Z$, $Z\in H_A^1$, we get

$$\begin{aligned}(d/dt|_0\boldsymbol{H}_{\alpha(t)})Z&=-(d/dt|_0\Delta_{\alpha(t)}^1)G_AZ-\Delta_A^1(d/dt|_0G_\alpha)Z\\&=-\Delta_A^1((d/dt|_0G_\alpha)Z.\end{aligned}$$

We operate the green operator G_A to both sides and obtain

$$(d/dt|_0G_\alpha)Z=\boldsymbol{H}_A(d/dt|_0G_\alpha)Z-G_A((d/dt|_0\boldsymbol{H}_\alpha)Z), \tag{5.14}$$

since $(d/dt|_0 G_\alpha)Z = \boldsymbol{H}_A((d/dt|_0 G_\alpha)Z) + G_A\Delta_A^1((d/dt|_0 G_\alpha)Z)$. Hence by using the fact that $(\partial/\partial s|_{(0,0)}\Delta_\alpha^1)Y \in (H_A^1)^\perp$ we have

$$((\partial/\partial s|_{(0,0)}\Delta_\alpha^1)Y,\ (d/dt|_0 G_{\alpha(t)})Z) = ((\partial/\partial s|_{(0,0)}\Delta_\alpha^1)Y,\ -G_A((d/dt|_0 \boldsymbol{H}_\alpha)Z)).$$

From (3.17) the left hand side of (5.13) can then be expressed as

$$\begin{aligned}&(-d_A\{X, Y\} + d_A^{+*}[X\wedge Y]^+,\ -G_A(d_A G_A\{W, Z\} - d_A^{+*} G_A[W\wedge Z]^+))\\ &= (d_A\{X, Y\},\ G_A d_A G_A\{W, Z\}) + (d_A^{+*}[X\wedge Y]^+,\ G_A d_A^{+*} G_A[W\wedge Z]^+),\end{aligned}$$

from which the lemma is derived. Q. E. D.

Thus it follows from these lemmas that

$$\begin{aligned}((\partial^2/\partial s\partial t|_{(0,0)}\boldsymbol{H}_\alpha)Y,\ Z) = &-(\{W, Y\},\ G_A\{X, Z\}) - ([W\wedge Y]^+,\ G_A[X\wedge Z]^+)\\ &-(\{X, Y\},\ G_A\{W, Z\}) - ([X\wedge Y]^+,\ G_A[W\wedge Z]^+).\end{aligned} \tag{5.15}$$

Therefore by the aid of (5.8), (5.11) and (5.15)

$$\begin{aligned}(\partial^2/\partial s\partial t|_{(0,0)}(\boldsymbol{H}_\alpha\overline{Y}),\ Z) = &((\partial^2/\partial s\partial t|_{(0,0)}\boldsymbol{H}_\alpha)Y,\ Z) + ((\partial/\partial t|_{(0,0)}\boldsymbol{H}_\alpha)(\partial/\partial s|_{(0,0)}\overline{Y}),\ Z)\\ &+((\partial/\partial s|_{(0,0)}\boldsymbol{H}_\alpha)(\partial/\partial t|_{(0,0)}\overline{Y}),\ Z) + (\boldsymbol{H}_A(\partial^2/\partial s\partial t|_{(0,0)}\overline{Y}),\ Z)\end{aligned}$$

can be written in the following form.

LEMMA 5.6.

$$(\partial^2/\partial s\partial t|_{(0,0)}(\boldsymbol{H}_\alpha\overline{Y}),\ Z) = -(\{X, Y\},\ G_A\{W, Z\}) - (\{W, Y\},\ G_A\{X, Z\}). \tag{5.16}$$

REMARK. The formula is symmetric also with respect to Y and Z.

PROPOSITION 5.7. *For the second order derivative* $\partial^2/\partial s\partial t|_{(0,0)}(\boldsymbol{H}_\alpha\overline{Y},\ \boldsymbol{H}_\alpha\overline{Z})$, *we have*

$$\begin{aligned}\partial^2/\partial s\partial t|_{(0,0)}(\boldsymbol{H}_\alpha\overline{Y},\ \boldsymbol{H}_\alpha\overline{Z}) = &-(\{X, Y\},\ G_A\{W, Z\}) - (\{W, Y\},\ G_A\{X, Z\})\\ &+([X\wedge Y]^+,\ G_A[W\wedge Z]^+) + ([W\wedge Y]^+,\ G_A[X\wedge Z]^+).\end{aligned} \tag{5.17}$$

PROOF. From (5.3), (5.5) and (5.16) we obtain

$$\begin{aligned}\partial^2/\partial s\partial t|_{(0,0)}(\boldsymbol{H}_\alpha\overline{Y},\ \boldsymbol{H}_\alpha\overline{Z}) = &-2(\{X, Y\},\ G_A\{W, Z\}) - 2(\{W, Y\},\ G_A\{X, Z\})\\ &+(G_A\{W, Y\},\ \{X, Z\}) + (G_A[W\wedge Y]^+,\ [X\wedge Z]^+)\\ &+(G_A\{X, Y\},\ \{W, Z\}) + (G_A[X\wedge Y]^+,\ [W\wedge Z]^+).\end{aligned}$$

Q. E. D.

We will obtain from this proposition the components of the Riemannian curvature tensor. Since the components of the metric $\langle\ ,\ \rangle$ are given by

$g_{lj}=\langle X_l, X_j\rangle=(\boldsymbol{H}_\alpha\bar{X}_l, \boldsymbol{H}_\alpha\bar{X}_j)$ at $\alpha\in S^-_{A,\varepsilon}$ (here we assume that $\{X_j\}$ be an orthonormal basis of H^1_A with respect to (,)), the second order derivatives of g_{lj} at A (i.e., at $\alpha=0$) are written as

$$\begin{aligned}\partial^2 g_{lj}/\partial x^i\partial x^k(0) &= \partial^2/\partial s\partial t|_{(0,0)}(\boldsymbol{H}_\alpha\bar{X}_l, \boldsymbol{H}_\alpha\bar{X}_j)\\ &= -(\{X_i, X_l\}, G_A\{X_k, X_j\})+([X_i\wedge X_l]^+, G_A[X_k\wedge X_j]^+)\\ &\quad -(\{X_k, X_l\}, G_A\{X_i, X_j\})+([X_k\wedge X_l]^+, G_A[X_i\wedge X_j]^+),\end{aligned} \tag{5.18}$$

where we set $\alpha(s, t)=(\Xi_A)^{-1}(sX_i+tX_k)$. From the expression of (5.18) we see $\partial^2 g_{lj}/\partial x^i\partial x^k(0)=\partial^2 g_{ki}/\partial x^j\partial x^l(0)$. Then the component $R_{lkij}(0)$ is given from formula (5.2) as

$$\begin{aligned}R_{lkij}(0) &= \partial^2 g_{lj}/\partial x^i\partial x^k(0)-\partial^2 g_{li}/\partial x^j\partial x^k(0)\\ &= -(\{X_i, X_l\}, G_A\{X_k, X_j\})-(\{X_k, X_l\}, G_A\{X_i, X_j\})\\ &\quad +([X_i\wedge X_l]^+, G_A[X_k\wedge X_j]^+)+([X_k\wedge X_l]^+, G_A[X_i\wedge X_j]^+)\\ &\quad +(\{X_j, X_l\}, G_A\{X_k, X_i\})+(\{X_k, X_l\}, G_A\{X_j, X_i\})\\ &\quad -([X_j\wedge X_l]^+, G_A[X_k\wedge X_i]^+)-([X_k\wedge X_l]^+, G_A[X_j\wedge X_i]^+).\end{aligned}$$

Since $\{X, Y\}$ is skew-symmetric and $[X\wedge Y]^+$ is symmetric, we obtain

LEMMA 5.8. *At $\alpha=0$ in a slice neighborhood the Riemannian curvature tensor is written as*

$$\begin{aligned}R_{lkij}(0) &= -(\{X_i, X_l\}, G_A\{X_k, X_j\})-2(\{X_k, X_l\}, G_A\{X_i, X_j\})\\ &\quad +(\{X_j, X_l\}, G_A\{X_k, X_i\})+([X_i\wedge X_l]^+, G_A[X_k\wedge X_j]^+)\\ &\quad -([X_j\wedge X_l]^+, G_A[X_k\wedge X_i]^+).\end{aligned} \tag{5.19}$$

From this lemma the explicit representation of the curvature tensor $R(X, Y)Z$ is derived in the form of Theorem 5.1.

6. Sectional curvature and holomorphic sectional curvature.

If we set $Z=Y$ and $W=X$ in (5.1), then we get

$$\begin{aligned}\langle R(X, Y)Y, X\rangle &= -(\{X, X\}, G_A\{Y, Y\})-2(\{Y, X\}, G_A\{X, Y\})\\ &\quad +(\{Y, X\}, G_A\{Y, X\})+([X\wedge X]^+, G_A[Y\wedge Y]^+)\\ &\quad -([Y\wedge X]^+, G_A[Y\wedge X]^+).\end{aligned}$$

THEOREM 6.1. *At $\alpha=0$ in a slice neighborhood $S^-_{A,\varepsilon}$ the sectional curvature of a plane spanned by X and Y ($|X|=|Y|=1$, $\langle X, Y\rangle=0$) is written as*

$$\begin{aligned}\langle R(X, Y)Y, X\rangle &= 3(\{X, Y\}, G_A\{X, Y\})-([X\wedge Y]^+, G_A[X\wedge Y]^+)\\ &\quad +([X\wedge X]^+, G_A[Y\wedge Y]^+).\end{aligned} \tag{6.1}$$

From this theorem we obtain a formula on the holomorphic sectional curvature.

THEOREM 6.2. *Let $\mathfrak{M}^-$ be the moduli space of generic anti-self-dual connections on an $SU(n)$-principal bundle P over a compact complex Kähler surface. The holomorphic sectional curvature of $\mathfrak{M}^-$ is given by*

$$\langle R(X, IX)IX, X\rangle = 4(\{X, IX\}, G_A\{X, IX\})-2([X\wedge X]^2, G_A[X\wedge X]^2) \tag{6.2}$$

$(X\in H_A^1,\ |X|=1)$ *where* $[X\wedge X]^2=[X\wedge X]^{2,0}+[X\wedge X]^{0,2}$.

PROOF. Setting $Y=IX$ in (6.1), we have

$$\langle R(X, IX)IX, X\rangle = 3(\{X, IX\}, G_A\{X, IX\})-([X\wedge IX]^+, G_A[X\wedge IX]^+) \\ +([X\wedge X]^+, G_A[IX\wedge IX]^+). \tag{6.3}$$

A self-dual 2-form $[X\wedge Y]^+$ decomposes into components of type (2, 0), (0, 2) and (1, 1);

$$[X\wedge Y]^+ = [X\wedge Y]^{2,0}+[X\wedge Y]^{0,2}+[X\wedge Y]^{1,1}.$$

The components of type (2, 0) and (0, 2) satisfy a certain relation because of the reality condition. Since $[X\wedge Y]^{2,0}=([X_1, Y_2]-[X_2, Y_1])dz^1\wedge dz^2$ $(X=X_1dz^1+X_2dz^2+X_{\bar 1}dz^{\bar 1}+X_{\bar 2}dz^{\bar 2},\ Y=Y_1dz^1+Y_2dz^2+Y_{\bar 1}dz^{\bar 1}+Y_{\bar 2}dz^{\bar 2})$, we have $[X\wedge IY]^{2,0}=\sqrt{-1}[X\wedge Y]^{2,0}$ and $[IX\wedge IY]^{2,0}=-[X\wedge Y]^{2,0}$. The (1, 1)-component $[X\wedge Y]^{1,1}$ is given by $[X\wedge Y]^{1,1}=1/2([X\wedge Y], \omega_h)\,\omega_h$ ([**15**]). We denote by $[X\wedge Y]^0$ the $\mathfrak{g}_P$-valued 0-form $([X\wedge Y], \omega_h)$. Then we have

$$[X\wedge Y]^0 = -\sqrt{-1}\{[X_1, Y_{\bar 1}]+[X_2, Y_{\bar 2}]+[Y_1, X_{\bar 1}]+[Y_2, X_{\bar 2}]\}, \tag{6.4}$$

here we normalize the Kähler metric at each point of M. We use the formula

$$\{X, Y\} = [X_1, Y_{\bar 1}]+[X_2, Y_{\bar 2}]+[X_{\bar 1}, Y_1]+[X_{\bar 2}, Y_2] \tag{6.5}$$

to get

$$[X\wedge Y]^0 = \{X, IY\}. \tag{6.6}$$

Hence

$$[X\wedge X]^+ = [X\wedge X]^{2,0}+[X\wedge X]^{0,2}+1/2\{X, IX\}\omega_h \tag{6.7}$$

and

$$[IX\wedge IX]^+ = -[X\wedge X]^{2,0}-[X\wedge X]^{0,2}+1/2\{X, IX\}\omega_h. \tag{6.8}$$

Moreover, since $[X\wedge IX]^0=-\{X, X\}=0$ we have

$$[X\wedge IX]^+ = \sqrt{-1}[X\wedge X]^{2,0}-\sqrt{-1}[X\wedge X]^{0,2}. \tag{6.9}$$

Combining these formulas we obtain

$$\langle R(X, IX)IX, X\rangle = 3(\{X, IX\}, G_A\{X, IX\})-([X\wedge X]^2, G_A[X\wedge X]^2)$$
$$-([X\wedge X]^2, G_A[X\wedge X]^2)+(\{X, IX\}, G_A\{X, IX\}).$$

Here we used the Hermitian property of the inner product (,) ([**15**]) and also the following formula (6.11). Therefore the holomorphic sectional curvature can be expressed as (6.2). Q. E. D.

LEMMA 6.3. *Let A be an irreducible anti-self-dual connection. Then*

$$d_A^+ d_A^{+*}(\phi\otimes\omega_h) = 1/2(d_A^* d_A\phi)\otimes\omega_h \tag{6.10}$$

and

$$G_A(\phi\otimes\omega_h) = 2(G_A\phi)\otimes\omega_h\,, \quad \phi\in\Omega^0(\mathfrak{g}_P)\,. \tag{6.11}$$

PROOF. Since $d_A^{+*}(\phi\otimes\omega_h)=I d_A\phi$ ([**15**], §2), we have $d_A^+ d_A^{+*}(\phi\otimes\omega_h)=(d_A I d_A\phi)^+$. It follows from the anti-self-duality that $d_A I d_A\phi$ is of type (1, 1) and hence $(d_A I d_A\phi)^+$ is proportional to ω_h, that is, $(d_A I d_A\phi)^+=\psi\otimes\omega_h$ for some $\psi\in\Omega^0(\mathfrak{g}_P)$ ([**14**]). To evaluate ψ we consider the inner product $(\psi\otimes\omega_h, \eta\otimes\omega_h)$ for all $\eta\otimes\omega_h$. This inner product becomes $(d_A\phi, d_A\eta)=(d_A^* d_A\phi, \eta)$ which is equal to $1/2((d_A^* d_A\phi)\otimes\omega_h, \eta\otimes\omega_h)$. Hence we have (6.10). (6.11) is derived as follows. Since A is irreducible, the Laplacians are invertible so that for any $\phi\in\Omega^0(\mathfrak{g}_P)$ we have $\psi\in\Omega^0(\mathfrak{g}_P)$ satisfying $d_A^* d_A\psi=\phi$, that is, $\psi=G_A\phi$. Then $G_A(\phi\otimes\omega_h)=G_A(d_A^* d_A\psi\otimes\omega_h)=2G_A d_A^+ d_A^{+*}(\psi\otimes\omega_h)=2(\psi\otimes\omega_h)$.

PROPOSITION 6.4. *Holomorphic bisectional curvature with respect to X and Y in H_A^1 ($|X|=|Y|=1$) is represented by the following*

$$\langle R(X, IX)IY, Y\rangle = 3(\{X, Y\}, G_A\{X, Y\})+3(\{X, IY\}, G_A\{X, IY\})$$
$$-([X\wedge Y]^+, G_A[X\wedge Y]^+)-([X\wedge IY]^+, G_A[X\wedge IY]^+)$$
$$+2(\{X, IX\}, G_A\{Y, IY\}). \tag{6.12}$$

PROOF. The holomorphic bisectional curvature is defined by $\langle R(X, IX)IY, Y\rangle$ and this is equal by the first Bianchi identity to the sum $\langle R(X, Y)Y, X\rangle+\langle R(X, IY)IY, X\rangle$. From (6.1) we have

$$\langle R(X, Y)Y, X\rangle = 3(\{X, Y\}, G_A\{X, Y\})-([X\wedge Y]^+, G_A[X\wedge Y]^+)$$
$$+([X\wedge X]^+, G_A[Y\wedge Y]^+)$$

and

$$\langle R(X, IY)IY, X\rangle = 3(\{X, IY\}, G_A\{X, IY\})-([X\wedge IY]^+, G_A[X\wedge IY]^+)$$
$$+([X\wedge X]^+, G_A[IY\wedge IY]^+).$$

Since $[Y\wedge Y]^+=[Y\wedge Y]^2+1/2\{Y, IY\}\omega_h$ and $[IY\wedge IY]^+=-[Y\wedge Y]^2+1/2\{Y, IY\}\omega_h$, we obtain by using (6.7) and (6.8)

$$\begin{aligned}\langle R(X, IX)IY, Y\rangle &= 3(\{X, Y\}, G_A\{X, Y\})+3(\{X, IY\}, G_A\{X, IY\})\\ &\quad -([X\wedge Y]^+, G_A[X\wedge Y]^+)-([X\wedge IY]^+, G_A[X\wedge IY]^+)\\ &\quad +(\{X, IX\}, G_A\{Y, IY\})+(\{X, IX\}, G_A\{Y, IY\})\\ &= 3(\{X, Y\}, G_A\{X, Y\})+3(\{X, IY\}, G_A\{X, IY\})\\ &\quad -([X\wedge Y]^+, G_A[X\wedge Y]^+)-([X\wedge IY]^+, G_A[X\wedge IY]^+)\\ &\quad +2(\{X, IX\}, G_A\{Y, IY\}). \qquad \text{Q.E.D.}\end{aligned}$$

7. The curvature of the moduli space over a Riemann surface.

Let P be a C^∞ $U(n)$-bundle over a Riemann surface M with a hermitian metric h. We call a connection on P Einstein-Hermitian if its curvature is equal to $\lambda\,\mathrm{id}_E\otimes\omega_h$ with a constant λ, where id_E denotes the identity endomorphism of the bundle $E=P\times_\rho \boldsymbol{C}^n$. The constant λ is the invariant of P, $\lambda=-2\pi\sqrt{-1}/\mathrm{vol}(M)\cdot k/n$, $k=c_1(E)$. Like the moduli space of anti-self-dual connections we define the moduli space of Einstein-Hermitian connections on P and denote it by $\mathcal{M}_E$.

By theorems of Narasimhan-Seshadri and Donaldson the moduli space $\hat{\mathcal{M}}_E$ of irreducible Einstein-Hermitian connections is identified with the moduli $N(n, k)$ of stable holomorphic vector bundle structures on E ([**22**], [**4**], [**7**]).

Let A be an Einstein-Hermitian connection. Then we have an elliptic complex

$$0 \longrightarrow \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^2(\mathfrak{g}_P) \longrightarrow 0. \tag{7.1}$$

We have here a decomposition; $\mathfrak{g}_P=\mathfrak{g}_P^c\oplus\mathfrak{g}_P^s$ into the center part $\mathfrak{g}_P^c=P\times_{U(n)}\mathfrak{c}$, which is isomorphic to the product bundle $M\times\boldsymbol{R}$, and the traceless part $\mathfrak{g}_P^s=P\times_{U(n)}\mathfrak{su}(n)$. This corresponds to the Lie algebra decomposition $\mathfrak{u}(n)=\mathfrak{c}\oplus\mathfrak{su}(n)$. Hence the above elliptic complex splits as

$$0 \longrightarrow \Omega^0\oplus\Omega^0(\mathfrak{g}_P^s) \xrightarrow{d\oplus d_A} \Omega^1\oplus\Omega^1(\mathfrak{g}_P^s) \xrightarrow{d\oplus d_A} \Omega^2\oplus\Omega^2(\mathfrak{g}_P^s) \longrightarrow 0$$

with cohomology groups $H_A^i=H_{\mathrm{deR}}^i\oplus H_A^{s,i}$, $i=0, 1, 2$. We notice by the Hodge duality that $H_{\mathrm{deR}}^2=H_{\mathrm{deR}}^0$ and $H_A^{s,2}=H_A^{s,0}$. We call a connection A on P to be irreducible when $\mathrm{Ker}\{d_A; \Omega^0(\mathfrak{g}_P)\to\Omega^1(\mathfrak{g}_P)\}$ is one dimensional. For each $[A]\in\hat{\mathcal{M}}_E$ there exists a neighborhood of $\hat{\mathcal{M}}_E$ around $[A]$ represented by the quotient of a slice

$$S_{A,\varepsilon}=\{A+\alpha;\ \alpha\in\Omega^1(\mathfrak{g}_P),\ |\alpha|_k<\varepsilon,\ d_A^*\alpha=0,\ d_A\alpha+1/2[\alpha\wedge\alpha]=0\}$$

with respect to the isotropy group Γ_A. The quotient $S_{A,\varepsilon}/\Gamma_A$ reduces to $\{\alpha^c\in H_{\mathrm{deR}}^1;\ |\alpha^c|<\varepsilon'\}\times\{\alpha^s\in\Omega^1(\mathfrak{g}_P^s);\ |\alpha^s|<\varepsilon'',\ d_A^*\alpha^s=0,\ d_A\alpha^s+1/2[\alpha^s\wedge\alpha^s]=0\}$. Then $\hat{\mathcal{M}}_E$ has locally a product manifold structure, that is, it is written locally

as a product of H^1_{deR} and the moduli space $\hat{\mathcal{M}}_{E,0}$ of irreducible connections whose traceless part is flat. It is also seen that $\hat{\mathcal{M}}_E$ admits a canonical holomorphic structure compatible with this product structure by observing the elliptic complex:

$$0 \longrightarrow \Omega^0(\mathfrak{g}_P^C) \xrightarrow{\bar{\partial}_A} \Omega^{0,1}(\mathfrak{g}_P^C) \longrightarrow 0 \qquad ([\mathbf{14}]).$$

This fact corresponds to the product structure of $N(n, k)=(N_0(n, k)\times J)/\Sigma_n$, by the finite quotient, where J is a Jacobian variety and $N_0(n, k)$ is the moduli of holomorphic vector bundle structures whose determinant bundle is isomorphic to a fixed line bundle ([**4**]).

We define canonically a Riemannian metric on $\hat{\mathcal{M}}_E$ in an analogous way to the case of anti-self-dual connections. Relative to this metric we obtain

THEOREM 7.1. *The metric is Kähler. It is flat on the Jacobian variety part and is of nonnegative holomorphic sectional curvature on* $\hat{\mathcal{M}}_{E,0}$. *Moreover the scalar curvature is nonnegative.*

PROOF. By the aid of the same arguments as in the proof of Proposition 4.3, we can show that the metric is Kähler. Since the Kuranishi map on $\Omega^1(\mathfrak{g}_P^C)$ reduces to the identity, this metric is flat over H^1_{deR}. We use (6.3) in § 6 with a slight modification to obtain a formula of holomorphic sectional curvature of $\hat{\mathcal{M}}_{E,0}$ as

$$\begin{aligned}\langle R(X, IX)IX, X\rangle = 3(\{X, IX\}, G_A\{X, IX\})-([X\wedge IX], G_A[X\wedge IX]) \\ +([X\wedge X], G_A[IX\wedge IX]).\end{aligned} \tag{7.2}$$

Since $\dim_C M=1$ and 2-form $[X\wedge Y]$ is of type (1, 1), this form is written as $[X\wedge Y]=[X\wedge Y]^0\omega_h$, where $[X\wedge Y]^0=([X\wedge Y], \omega_h)$. Because $[X\wedge Y]^0=\{X, IY\}$, we have $[X\wedge X]=[IX\wedge IX]=\{X, IX\}\omega_h$ and $[X\wedge IX]=0$. Hence we obtain

$$\langle R(X, IX)IX, X\rangle =4(\{X, IX\}, G_A\{X, IX\})= 4([X\wedge X], G_A[X\wedge X]) \geqq 0. \tag{7.3}$$

The nonnegativity of the scalar curvature is easily derived from the formula (2.11) in [**12**] (see also [**6**]), since the scalar curvature is represented by an average of holomorphic sectional curvature in some sense. Q. E. D.

REMARK. In the case $(n, k)=1$ the moduli space $\hat{\mathcal{M}}_{E,0}$ is a simply connected compact Kähler manifold with complex dimension $(g-1)\dim_R SU(n)$ ([**22**], [**4**]) (g is the genus of M). It is shown that the scalar curvature is strictly positive at a point. From this scalar curvature property we have $H^0(\hat{\mathcal{M}}_{E,0}; K^l)=0$, $l>0$ with respect to the canonical line bundle K. This fact is compatible with the unirationality of $N_0(n, k)$ ([**23**], [**4**]).

Typical examples of simply connected compact Kähler manifold with non-negative holomorphic sectional curvature are Hermitian symmetric spaces of compact type. However, consider the moduli space of Einstein-Hermitian connections of $n=2$ and $l=1$ with the fixed determinant bundle over a Riemann surface of genus 2. Then it is 3-dimensional and its Poincaré polynomial is $P(t)=1+t^2+4t^3+t^4+t^6$ ([4], p. 593). Only $P_3(\boldsymbol{C})$ and the hyperquadric $Q_3(\boldsymbol{C})$ are 3-dimensional Hermitian symmetric spaces of $b_2=1$. Hence this moduli space does not admit any structure of Hermitian symmetric space.

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