J. Math. Soc. Japan
Vol. 39, No. 3, 1987

Moishezon threefolds homeomorphic to P^3

By Iku NAKAMURA

(Received May 21, 1986)

Introduction.

A compact complex threefold is called a Moishezon threefold if it has three algebraically independent meromorphic functions on it. The purpose of this article is to prove

THEOREM. *A Moishezon threefold homeomorphic to complex projective space* P^3 *is isomorphic to* P^3 *if the Kodaira dimension of it is less than three.*

As a corollary to it, we obtain,

THEOREM. *An arbitrary complex analytic (global) deformation of* P^3 *is isomorphic to* P^3.

As for the (topological) characterization of P^n, it is known that an arbitrary Kählerian complex manifold homeomorphic to P^n is isomorphic to P^n by Hirzebruch-Kodaira [**9**] and Yau [**24**] (see also [**17**]). However neither of the above theorems are entirely clear from this because both a Moishezon threefold and a complex analytic deformation of a compact Kählerian threefold can be nonKählerian as Hironaka's example shows [**6**]. Recently Tsuji [**23**] claims that he is able to prove the second theorem for P^n, whereas Peternell [**19**] asserts both of the above theorems in a stronger form. However there is a gap in the proof of [**19**], as the author of [**19**] himself admits at the end of the article. After completing this article, I received two preprints of Peternell [**20**], [**21**] via Tsunoda and Nishiguchi, in which Peternell claims that he completes the proof of [**19**]. See (3.3).

In this article, we make an approach different from theirs and give an elementary proof of the above theorems.

Our idea of the proof of the first theorem is as follows. Let X be a Moishezon threefold homeomorphic to P^3 whose Kodaira dimension is less than three. Let L be the generator of $\mathrm{Pic}X\,(\cong \boldsymbol{Z})$ with L^3 equal to one. First we notice that $K_X=-4L$ [8], [**17**] and that $\dim|L|$ is not less than three. For an arbitrary pair D and D' in the complete linear system $|L|$, the scheme-theoretic complete intersection l of D and D' is a pure one dimensional connected closed analytic subspace of X with no embedded components. We show that l is a nonsingular rational curve with Ll equal to one whose normal bundle is isomorphic to $\mathcal{O}_l(1)\oplus\mathcal{O}_l(1)$ for arbitrary D and D'. We also see that the base locus of the linear system $|L|$ is the same as that of $|L_l|$ and $\dim|L|$ is equal to $2+\dim|L_l|$, L_l being the restriction of L to l. Since L_l is isomorphic to $\mathcal{O}_l(1)$, $|L|$ is therefore base point free and $\dim|L|$ is equal to three. Thus we have a bimeromorphic morphism f of X onto $\boldsymbol{P}^3$ associated with the linear system $|L|$. The exceptional set of f is a Cartier divisor of X whose image in $\boldsymbol{P}^3$ is zero or one dimensional. Since f_* induces an isomorphism of $\mathrm{Pic}X$ onto $\mathrm{Pic}\boldsymbol{P}^3$, this shows that f is an isomorphism of X onto $\boldsymbol{P}^3$.

The outline of the article is as follows. In sections 1 and 2, we consider a compact complex threefold X with a line bundle L such that $\mathrm{Pic}X=\boldsymbol{Z}L$, L^3 is positive, $K_X=-dL$ $(d\geqq 4)$ and $\kappa(X, L)\geqq 1$. (See [10] for the definition of $\kappa(X, L)$.) The last condition $\kappa(X, L)\geqq 1$ is equivalent to the existence of a positive integer m such that $\dim|mL|$ is positive. In section 1, we prove that $\dim|L|$ is not less than three. We also prove some vanishing lemmas of certain cohomology groups. In section 2, we study the scheme-theoretic complete intersection l of two distinct members D and D' of the linenar system $|L|$. In view of the vanishing lemmas in section 1, l_{red} consists of nonsingular rational curves (intersecting transversally), among which there is a unique irreducible component C of l_{red} such that LC is positive (indeed, equal to one). We shall show in (2.7) that l is isomorphic to C for an arbitrary pair D and D' and that $|L|$ is base point free. We shall prove in (2.8) that X is isomorphic to $\boldsymbol{P}^3$ and therefore $L^3=1$, $d=4$. We remark that (1.1) gives a characterization of $\boldsymbol{P}^3$ in arbitrary characteristic by a slight modification, see (2.10). In section 3, we complete the proofs of the theorems mentioned above by applying the results in section 2.

ACKNOWLEDGEMENT. We are very grateful to A. Fujiki, T. Fujita, F. Hidaka, T. Suwa and K. Ueno for their encouragement and valuable advices. Fujiki kindly pointed out an error in the first version of the article. This article is dedicated to my wife and children.

List of notations and terminologies.

$\boldsymbol{Z}$	the ring of integers or the infinite cyclic group
threefold	a connected complex manifold of three dimension
$\kappa(X, L)$	L-dimension of X, L being a line bundle on X [10]
$H^q(X, \mathcal{F})$	the q-th cohomology group of X with coefficients in a coherent sheaf $\mathcal{F}$

$h^q(X, \mathcal{F})$	$\dim_C H^q(X, \mathcal{F})$
$\chi(X, \mathcal{F})$	$\sum_{q\in Z}(-1)^q h^q(X, \mathcal{F})$
$\mathrm{Bs}\|L\|$	the set of base points of the linear system $\|L\|$
$\mathcal{O}_X, \mathcal{O}_X^*$	the sheaf of germs over X of holomorphic (resp. nonvanishing holomorphic) functions
Ω_X^p	the sheaf of germs over X of holomorphic p-forms
K_X	the canonical line bundle of X
$[D]$	the line bundle associated with a Cartier divisor D
c_q	the q-th Chern class (of X)
$c_1(E)$	the first Chern class of a vector bundle E
b_q	the q-th Betti number (of X)

§1. Threefolds with $K_X=-dL$ ($d\geqq 4$).

Our first aim is to prove

(1.1) THEOREM. *Let X be a compact complex threefold with* $\mathrm{Pic}X=\boldsymbol{Z}$. *Assume that there is a complex line bundle L on X such that $L^3>0$, $K_X=-dL$ ($d\geqq 4$) and $\kappa(X, L)\geqq 1$. Then $L^3=1$, $d=4$ and X is isomorphic to complex projective space $\boldsymbol{P}^3$.*

Compare [4] and [14].

Sections 1 and 2 are devoted to proving (1.1). The proof of (1.1) is completed in (2.8).

Throughout sections 1 and 2, we always assume that X is a compact complex threefold satisfying the conditions in (1.1). By taking thereby a generator of $\mathrm{Pic}X$ for L if necessary, we may assume L generates $\mathrm{Pic}X$.

(1.2) LEMMA. *$H^0(X, -mL)=0$ for $m>0$.*

PROOF. Suppose $H^0(X, -mL)\neq 0$ for some $m>0$. Then there is an effective divisor D on X such that $[D]=-mL$. Since $\kappa(X, L)\geqq 1$, there are an m_0 (>0) and an effective divisor D_0 such that $[D_0]=m_0L$. Hence mD_0+m_0D is linearly equivalent to zero, which contradicts $h^0(X, \mathcal{O}_X)=1$. q.e.d.

(1.3) LEMMA. *$H^q(X, \mathcal{O}_X)=0$ for $q=1, 3$ and $c_1c_2\geqq 24$, $\chi(X, mL)\geqq (m+1)(m+2)(m+3)/6$.*

PROOF. Since $\mathrm{Pic}X$ is discrete, we have $H^1(X, \mathcal{O}_X)=0$. By (1.2), $h^3(X, \mathcal{O}_X) =h^0(X, K_X)=h^0(X, -dL)=0$. Hence $\chi(X, \mathcal{O}_X)\geqq 1$. By Riemann-Roch-Hirzebruch formula, $c_1c_2=24\chi(X, \mathcal{O}_X)\geqq 24$ and $\chi(X, mL)=\chi(X, \mathcal{O}_X)+m(c_1^2+c_2)L/12+m^2c_1L^2/4 +m^3L^3/6$. Hence $c_2L\geqq 24/d$ and $\chi(X, mL)\geqq 1+m(d^2+(24/d))/12+m^2d/4+m^3/6$ by $L^3\geqq 1$. For $d\geqq 4$, we have $d^2+(24/d)\geqq 22$, whence $\chi(X, mL)\geqq\binom{m+3}{3}$. q.e.d.

(1.4) LEMMA. *Let D be a reduced and connected effective divisor on X. Then $H^1(X, \mathcal{O}_X(-D))=0$.*

PROOF. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

It follows from this that $0 \to H^1(X, \mathcal{O}_X(-D)) \to H^1(X, \mathcal{O}_X)$ is exact. Hence by (1.3), we have $H^1(X, \mathcal{O}_X(-D))=0$. q.e.d.

(1.5) LEMMA. $h^0(X, L) \geqq 4$.

PROOF. The proof is given in a series of sublemmas.

(1.5.1) SUBLEMMA. *Suppose $h^0(X, L) \geqq 1$. Then any member of $|L|$ is reduced and irreducible.*

PROOF. Let D be an arbitrary member of $|L|$. Let $D=a_1D_1+\cdots+a_rD_r$ with $a_i>0$, D_i irreducible. Since L is a generator of $\operatorname{Pic} X$, then there are $b_i>0$ such that $[D_i]=b_iL$ by (1.2). Hence $a_1b_1+\cdots+a_rb_r=1$, whence $r=1$, $a_1=b_1=1$. Therefore D is reduced and irreducible. □

(1.5.2) SUBLEMMA. *Assume $h^0(X, L) \geqq 2$. Then $h^0(X, L) \geqq 4$.*

PROOF. Since $h^0(X, L) \geqq 2$, we can choose infinitely many distinct D_i's from $|L|$. Then by (1.5.1), D_i is reduced and connected. Since $D_iD_jD_k=L^3 \geqq 1$, D_i's intersect each other. Hence $D_1+\cdots+D_m$ is reduced and connected. Hence $H^1(X, -mL)=H^1(X, -(D_1+\cdots+D_m))=0$ in view of (1.4). In particular, $h^2(X, L) =h^1(X, K_X-L)=h^1(X, -(d+1)L)=0$. Consequently, by (1.3), $h^0(X, L) \geqq \chi(X, L) \geqq 4$. □

(1.5.3) SUBLEMMA. $h^0(X, L) \geqq 2$.

PROOF. Suppose $h^0(X, L) \leqq 1$ to derive a contradiction. By $\kappa(X, L) \geqq 1$, there exists p $(\geqq 2)$ such that $h^0(X, L) \leqq 1$ for $1 \leqq k \leqq p-1$ and $h^0(X, pL) \geqq 2$. Then any general member of $|pL|$ is reduced and irreducible. Indeed, otherwise, $D \in |pL|$ is written as $D=D'+D''$ with D', D'' effective. Since $h^0(X, kL) \leqq 1$ $(1 \leqq k \leqq p-1)$, we see that $D' \in |aL|$, $D'' \in |bL|$ are the unique members for some $a, b>0$, $a+b=p$. Hence $h^0(X, pL)=1$, which is absurd. Hence any general member of $|pL|$ is reduced and irreducible. Therefore by taking distinct members $D_1, \cdots, D_m$ of $|pL|$, we apply (1.3), (1.4) and the proof of (1.5.2) so as to show $H^1(X, -pmL)=0$ for any $m \geqq 0$. Hence $H^2(X, (pm-d)L)=0$ for any $m \geqq 0$. Let $d=pa+b$, $0 \leqq b \leqq p-1$. If $b>0$, then $h^0(X, (p-b)L) \geqq \chi(X, (p-b)L) \geqq \binom{p-b+3}{3} \geqq 4$. This contradicts $h^0(X, kL) \leqq 1$ for $k \leqq p-1$. When $b=0$, we *assume* moreover *that there is q not divisible by p such that $q>p$, $h^0(X, qL) \geqq 2$.*

We take minimal such q. Then *any general member of* $|qL|$ *is reduced and connected.* In fact, let D be a general member of $|qL|$ and assume that $D=D'+D''$, $D'\in|q'L|$, $D''\in|q''L|$. By the choice of q, there are two possibilities; *Case* 1. $q'<p$, $q''<p$, *Case* 2. $q'<p$, $p|q''$. In Case 1, $h^0(X, q'L)=h^0(X, q''L)=1$, whence $h^0(X, qL)=1$. This is absurd. In Case 2, $h^0(X, q'L)=1$. By the choice of q and $h^0(X, pL)\geqq 2$, we see that $q=q'+p$, $q''=p$ and $h^0(X, sL)=0$ for $s<q'$. Clearly the unique member of $|q'L|$ is reduced and irreducible. Therefore any general member of $|qL|$ $(=D'+|pL|)$ is reduced and connected. By applying (1.4) to a sum of members of $|qL|$ and $|pmL|$, we have $H^1(X, -(pm+q)L)=0$. Hence $H^2(X, (pm+q-d)L)=0$ for $m\geqq 0$. Letting $m=a-1$ $(\geqq 0)$, we obtain $h^2(X, (q-p)L)=0$, whence $h^0(X, q'L)=h^0(X, (q-p)L)\geqq 4$. This contradicts $h^0(X, q'L)=1$.

Thus in order to complete the proof of (1.5.3), it suffices to prove

(1.5.4) SUBLEMMA. *There exists q not divisible by p such that $q>p$, $h^0(X, qL)\geqq 2$.*

PROOF. Any general member D of $|pL|$ is reduced and irreducible. Since another general member of $|pL|$ gives a nontrivial element of $H^0(D, pL)$, we have $H^0(D, -sL)=0$ for any $s>0$. Since the dualising sheaf ω_D of D is given by $(p-d)L\otimes\mathcal{O}_D$, we have $h^2(D, kL)=h^0(X, (p-d-k)L)=0$ for $k>p-d$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X((k-p)L) \longrightarrow \mathcal{O}_X(kL) \longrightarrow \mathcal{O}_D(kL) \longrightarrow 0\,.$$

Then it follows that $h^2(X, kL)\leqq h^2(X, (k-p)L)$ for $k>p-d$. Let $A=\max\{h^2(X, jL);\ 0\leqq j\leqq p-1\}$. Then $h^2(X, kL)\leqq A$ for $k>0$. Hence $h^0(X, kL)\geqq\binom{k+3}{3}-A$ for $k>0$. Consequently there exists k_0 such that $h^0(X, kL)>1$ for $k>k_0$. This guarantees the existence of the desired q. □

Combining (1.5.1)-(1.5.4), we obtain (1.5). q. e. d.

(1.6) LEMMA. *Let D and D' be distinct members of $|L|$, $l:=D\cap D'$ the scheme-theoretic intersection of D and D'. Then $0\to\mathcal{O}_D(-L)\to\mathcal{O}_D\to\mathcal{O}_l\to 0$ is exact.*

PROOF. Let I_D (resp. $I_{D'}$) be the ideal sheaf of D (resp. D') and $I_l:=I_D+I_{D'}$. Then $\mathcal{O}_D=\mathcal{O}_X/I_D$, $\mathcal{O}_l=\mathcal{O}_X/I_l$. We have an exact sequence

$$0 \longrightarrow I_D+I_{D'}/I_D\ (=I_{D'}/I_D\cap I_{D'}) \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_l \longrightarrow 0\,.$$

Once one shows $I_D\cap I_{D'}=I_DI_{D'}$, we see $I_{D'}/I_D\cap I_{D'}=I_{D'}/I_DI_{D'}=I_{D'}\otimes_{\mathcal{O}_X}\mathcal{O}_X/I_D=\mathcal{O}_D(-L)$. It suffices to prove that D and D' have no common locally irreducible components anywhere on X. Let $\Phi=\{U_j\}$ be an open covering of an open neighborhood of $D'\setminus\mathrm{Sing}\,D'$ by open balls U_j and let f_j be a generator of I_D

on U_j. We assume that D and D' have a common irreducible component at $p\in X$, $p\in$the closure of U_0 for some $U_0\in\Phi$ and that f_0 vanishes identically on $U_0\cap D'$. By the connectedness of $D'\setminus\mathrm{Sing}\,D'$ (1.5.1), there is $U_1\in\Phi$ such that $U_0\cap U_1\neq\emptyset$. Then f_0 vanishes identically on $U_0\cap U_1\cap D'$, so that f_1 vanishes identically there. Hence f_1 vanishes identically on $U_1\cap D'$ by Hartog's continuation theorem. By repeating the argument, we see that in view of (1.5.1), D contains $D'\setminus\mathrm{Sing}\,D'$, hence D'. Conversely D' contains D, whence $D=D'$.

q. e. d.

We also notice that for any point p of X, the defining equations f and f' of D and D' form a regular sequence in the local ring $\mathcal{O}_{X,p}$ and therefore the intersection l is Gorenstein and has no embedded components [1, pp. 54-55].

(1.7) LEMMA. *Let D, D' and $l=D\cap D'$ be the same as in* (1.6). *Then we have,*

(1.7.1) $H^q(X, -mL)=0$ *for* $q=0, 1, m>0$; $q=2, 0\leqq m\leqq d$; $q=3, 0\leqq m\leqq d-1$,

(1.7.2) $H^q(D, -mL_D)=0$ *for* $q=0, m>0$; $q=1, 0\leqq m\leqq d-1$; $q=2, 0\leqq m\leqq d-2$,

(1.7.3) $H^0(l, -mL_l)=0$ *for* $1\leqq m\leqq d-2$; $H^1(l, -mL_l)=0$ *for* $0\leqq m\leqq d-3$,

(1.7.4) $H^0(X, \mathcal{O}_X)=H^0(D, \mathcal{O}_D)=H^0(l, \mathcal{O}_l)=\boldsymbol{C}$,

(1.7.5) $H^3(X, -dL)=H^2(D, -(d-1)L_D)=H^1(l, -(d-2)L_l)=\boldsymbol{C}$.

PROOF. By (1.5.1), any member of $|L|$ is reduced and irreducible. Since $h^0(X, L)\geqq 4$, we can choose distinct D_i's $(i=1, \cdots, m)$ from $|L|$. Hence $D_1+\cdots+D_m$ is reduced and connected. Hence by (1.4), $H^1(X, -mL)=0$ for any $m>0$. Hence $h^2(X, -mL)=h^1(X, -(d-m)L)=0$ for $0\leqq m\leqq d-1$. It follows from (1.3) that $h^2(X, -dL)=h^1(X, \mathcal{O}_X)=0$. The rest of (1.7.1) is clear from (1.2). (1.7.2) follows from (1.7.1) and the exact sequence $0\to\mathcal{O}_X(-D)\to\mathcal{O}_X\to\mathcal{O}_D\to 0$. (1.7.3) follows from (1.7.2) and (1.6). The remaining assertions can be proved similarly.

q. e. d.

(1.8) COROLLARY. $H^2(X, \mathcal{O}_X)=0$ *and* $\chi(X, \mathcal{O}_X)=1$.

PROOF. Clear from (1.7.1) and (1.3). q. e. d.

§2. Base points of the linear system $|L|$.

(2.1) Here we recall the intersection theory in analytic geometry briefly from [2], [3] and [12]. To an arbitrary closed complex analytic subset A of pure complex dimension m in a compact complex manifold X, one associates a (Borel-Moore) homology class $\mathrm{cl}(A)\in H_{2m}(X, \boldsymbol{Z})$. Two analytic subset D and D' of X are said to intersect properly if any irreducible component B of $D\cap D'$

has the same dimension $\dim D+\dim D'-\dim X$ anywhere on $(D\cap D')_{\text{red}}$. Given two analytic subsets D and D' intersecting properly, the intersection cycle $D\cap D'$ is defined so that $\mathrm{cl}(D\cap D')=\mathrm{cl}(D)\cap\mathrm{cl}(D')$, where the right hand side is the cap product. If we are given another topological cycle γ, then we have $\gamma\cap\mathrm{cl}(D\cap D')=(\gamma\cap\mathrm{cl}(D))\cap\mathrm{cl}(D')$ by the topological associativity. In the sequel, we omit the symbol $\cap$ for the cap product for brevity.

We notice that three kinds of intersection theory—topological [**2**], analytic [**3**] and current-theoretic [**12**]—are the same by [**12**, p. 211]. We also note that the associativity law and the projection formula in the intersection theory are true [**2**], [**3**].

Coming back to our situation where $l=D\cap D'$ in (1.6), we see that there are positive integers n_i such that $\mathrm{cl}(l)=n_1\,\mathrm{cl}(A_1)+\cdots+n_s\,\mathrm{cl}(A_s)\in H_2(X, \boldsymbol{Z})$ where A_i ranges over all the irreducible components of l_{red}.

(2.2) LEMMA. $d=4$, $K_X=-4L$.

PROOF. Let $l_{\text{red}}=A_1+\cdots+A_s$ be the decomposition into irreducible components. Let I_l (resp. I_j) be the ideal sheaf in $\mathcal{O}_X$ defining l (resp. A_j). By definition, $\mathcal{O}_l=\mathcal{O}_X/I_l$, $\mathcal{O}_{A_j}=\mathcal{O}_X/I_j$. It follows from $H^1(\mathcal{O}_l)=0$ that $H^1(\mathcal{O}_{A_j})=0$. Hence A_j is a nonsingular rational curve. Suppose $d>4$. Then by (1.7), $H^1(\mathcal{O}_l(-2L))=0$, whence $H^1(\mathcal{O}_{A_j}(-2L))=0$ for any j. Therefore $c_1(L_{A_j})\leqq 0$ for any j. However $\mathrm{cl}(L)(n_1\,\mathrm{cl}(A_1)+\cdots+n_s\,\mathrm{cl}(A_s))=\mathrm{cl}(L)\,\mathrm{cl}(l)(=: Ll)=L\cdot D\cdot D'=L^3\geqq 1$, whence there exists i such that $LA_i:=\mathrm{cl}(L)\,\mathrm{cl}(A_i)>0$. This shows that $c_1(L_{A_i})=\mathrm{cl}(L_{A_i})=\mathrm{cl}(L)\mathrm{cl}(A_i)$ $(\in H_0(A_i, \boldsymbol{Z})\cong\boldsymbol{Z})$ is positive. This is a contradiction. Therefore $d=4$ and $K_X=-4L$. q. e. d.

(2.3) LEMMA. *Let D and D' be two distinct members of $|L|$, $l:=D\cap D'$, $A=l_{\text{red}}=A_1+\cdots+A_{a+b}$ the decomposition of l_{red} into irreducible components, and let $B=A_{a+1}+\cdots+A_{a+b}$ be the one dimensional part of* $\mathrm{Bs}|L|$, *$C=A-B=A_1+\cdots+A_a$. Then there is a unique irreducible component A_i $(1\leqq i\leqq a)$ of C such that $LA_i>0$, say, $LA_1>0$. Moreover*

(2.3.1) *each A_i is a nonsingular rational curve,*

(2.3.2) $LA_1=1$, $LA_i=0$ $(2\leqq i\leqq a)$, $LA_j\leqq 0$ $(a+1\leqq j\leqq a+b)$,

(2.3.3) *A, B and C are connected and if moreover $B\neq\emptyset$, then B and C intersect at a unique point of A_1.*

PROOF. The assertion (2.3.1) is clear from the proof of (2.2). By (2.1), we set $\mathrm{cl}(l)=n_1\mathrm{cl}(A_1)+\cdots+n_{a+b}\mathrm{cl}(A_{a+b})$ for $n_i>0$. Since $Ll\geqq 1$, there is at least one i such that $LA_i=\mathrm{cl}(L)\mathrm{cl}(A_i)>0$. By the exact sequences

$$0 \longrightarrow (I_A/I_l)\otimes\mathcal{O}_X(-2L) \longrightarrow \mathcal{O}_l(-2L) \longrightarrow \mathcal{O}_A(-2L) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_A(-2L) \longrightarrow \nu_*(\oplus\mathcal{O}_{A_j}(-2L)) \longrightarrow \mathcal{H} \longrightarrow 0$$

where $\nu: \bigcup_j A_j \to A$ is the normalization and $\mathcal{H}=\nu_*(\oplus_j\mathcal{O}_{A_j})/\mathcal{O}_A$ is a sheaf supported by $\mathrm{Sing}\,A$, we see that $h^1(\mathcal{O}_l(-2L))\geqq h^1(\mathcal{O}_A(-2L))\geqq\sum_{j=1}^{a+b}h^1(\mathcal{O}_{A_j}(-2L))$ and $h^1(\mathcal{O}_l(-2L))=1$ in view of (1.7). Since A_j is a nonsingular rational curve, this proves that $LA_i=1$ and $LA_j\leqq 0$ for $j\neq i$. In order to complete the proof of (2.3.2), we prove

(2.3.4) SUBLEMMA. *The unique irreducible component A_i of A with $LA_i>0$ is not contained in B.*

PROOF OF (2.3.4). First we notice that if $LA_j<0$, then $A_j\subset B$. In fact, since $H^0(\mathcal{O}_{A_j}(L))=0$, any element of $H^0(X, L)$ vanishes identically on A_j. Hence $A_j\subset B$. Next we notice that if $LA_j=0$, and if A_j intersects B, then $A_j\subset B$. Indeed, then $H^0(\mathcal{O}_{A_j}(L))=H^0(\mathcal{O}_{A_j})=\boldsymbol{C}$. Any element of $H^0(X, L)$ vanishes at $A_j\cap B$ $(\neq\emptyset)$, whence it vanishes identically on A_j. Therefore $A_j\subset B$. Hence in particular, if $LA_j=0$, and $A_j\subset C$, then $A_j\cap B=\emptyset$. Suppose that A_i (the unique component of A with $LA_i>0$) is contained in B. Then since A is connected by $H^0(\mathcal{O}_l)=\boldsymbol{C}$ in (1.7), any irreducible component A_j of A is contained in B by the above argument. Namely, $A=B$. Notice that this is valid for any pair of D and D' if the unique component A' of $(D\cap D')_{\mathrm{red}}$ with $LA'>0$ is contained in B. Let D'' be an arbitrary member of $|L|$, $D''\neq D$. Then since $(D\cap D'')_{\mathrm{red}}\supset B\supset A_1$, we have therefore $(D\cap D'')_{\mathrm{red}}=B$. Since $\mathrm{Im}(H^0(X, L)\to H^0(D, L_D))$ $(=H^0(D, L_D))$ is at least 3 dimensional, and since D is reduced irreducible, the curves $(D\cap D')_{\mathrm{red}}$, $D'\in|L|$ covers D. This is a contradiction. □

By (2.3.4), we have $C\neq\emptyset$ and $A_i\subset C$, so we may assume $i=1$ without loss of generality. This completes the proof of (2.3.2). It remains to prove (2.3.3). By the proof of (2.3.4), no irreducible components A_j $(2\leqq j\leqq a)$ of C except A_1 intersect B. Clearly A_1 intersects B at exactly one point if $B\neq\emptyset$. Since A is connected by (1.7.4), both B and C are connected. This completes the proof of (2.3.3). q. e. d.

(2.4) LEMMA. *Let A_j and B be the same as in* (2.3). *Suppose that* $\mathrm{Bs}|L|$ *has no one dimensional components, i. e., $B=\emptyset$. Then* $\mathrm{Bs}|L|$ *consists of at most one point p of $A_1\setminus(\bigcup_{j\geqq 2}A_j)$.*

PROOF. By (2.3), $LA_j=0$ $(2\leqq j\leqq a)$. If $\mathrm{Bs}|L|\cap A_j=\{q, \cdots\}\neq\emptyset$ for some $j\geqq 2$, then $\mathrm{Bs}|L|$ contains $\mathrm{Bs}|L_{A_j}-q|=A_j$ by $H^0(\mathcal{O}_{A_j}(L))=H^0(\mathcal{O}_{A_j})=\boldsymbol{C}$. This contradicts $B=\emptyset$. Hence $\mathrm{Bs}|L|\cap A_j=\emptyset$. Suppose $\mathrm{Bs}|L|=\{p, q, \cdots\}$, $p\neq q$. Then A_1 contains p and q. Therefore $\mathrm{Bs}|L|$ contains $\mathrm{Bs}|L_{A_1}-p-q|=A_1$, which is absurd. Hence if $\mathrm{Bs}|L|\neq\emptyset$, then $\mathrm{Bs}|L|=\{p\}$ where $p\in A_1$, $p\notin A_j$

$(j \geqq 2)$. q. e. d.

(2.5) LEMMA. *Let l, A, B and C be the same as in* (2.3). *Let D'' be a member of $|L|$ other than D and D', and let $l'=D\cap D''$, $A'=(l')_{\mathrm{red}}$, $A'=C'+B$. Let A_1' be the unique irreducible component of A' with $LA_1'=1$. Assume that* $\mathrm{Bs}|L|\neq\emptyset$ *and $A\neq A'$. Then we have*

(2.5.1) *$A_1\neq A_1'$ and $A_1\cap A_1'=$*$\mathrm{Bs}|L|$ *(resp. $A_1\cap A_1'\cap B$) if $B=\emptyset$ (resp. if $B\neq\emptyset$),*

(2.5.2) *no irreducible components of C (resp. of C') distinct from A_1 (resp. from A_1') intersect C' (resp. C).*

PROOF. Let C_j be an irreducible component of $C-A_1$ (:=the closure of $C\setminus A_1$). Suppose C_j intersects C'. Then C_j meets D''. Since $D''C_j=LC_j=0$ by (2.3), D'' contains C_j. Hence any irreducible component C_k of C intersecting C_j meets D'', hence it is contained in D'' if $LC_k=0$. *If A_1 intersects C_j, then A_1 is also contained in D''.* In fact, by the assumption $\mathrm{Bs}|L|\neq\emptyset$, D'' contains a point $\mathrm{Bs}|L|$ (resp. $B\cap A_1$) if $B=\emptyset$ (resp. if $B\neq\emptyset$) by (2.4) and (2.3.3). If A_1 intersects C_j, then $A_1\cap C_j$ is contained in D'', where $A_1\cap C_j$ is disjoint from $\mathrm{Bs}|L|$ or $B\cap A_1$. Therefore D'' contains at least two points of A_1. However $D''A_1=LA_1=1$, which implies that D'' contains A_1. Since C is connected, it can be shown by repeating this argument that D'' contains C and that A' contains A. By the uniqueness of A_1 and A_1', we have $A_1=A_1'$. Hence D' contains A_1'. By the same argument as above, any irreducible component of C' is contained in D'. Hence $A'\subset A$ whence $A=A'$. This contradicts our assumption. This proves that no irreducible components of $C-A_1$ meet C'. By the symmetry of the roles of D' and D'', we complete the proof of (2.5.2).

Next we shall show (2.5.1). Suppose $A_1=A_1'$. Hence either $C\neq A_1$ or $C'\neq A_1$ because $A\neq A'$. By the symmetry of roles of C and C', we may assume $C\neq A_1$. Then $(C-A_1)\cap C'$ contains $(C-A_1)\cap A_1'=(C-A_1)\cap A_1\neq\emptyset$ whence $(C-A_1)\cap C'$ is not empty. This contradicts (2.5.2). Hence $A_1\neq A_1'$. If $A_1\cap A_1'=\{p, q, \cdots\}$, $p\neq q$, then D'' contains two points p and q of A_1, hence D'' contains A_1 by $D''A_1=LA_1''=1$. By the uniqueness (2.3) of A_1 and A_1', we have $A_1=A_1'$, which is absurd. Hence $A_1\cap A_1'$ consists of at most one point. If $B=\emptyset$, then $A_1\cap A_1'$ $=A\cap A'=\mathrm{Bs}|L|$ by (2.4). If $B\neq\emptyset$, and if A_1 intersects A_1' outside B, then $D''\supset A_1$ and $A_1=A_1'$ because D'' contains $B\cap A_1\neq\emptyset$ by (2.3.3). But $A_1=A_1'$ is absurd. Therefore if $B\neq\emptyset$, then A_1 intersects A_1' only in B. This proves $A_1\cap A_1'=A_1\cap A_1'\cap B$. This completes the proof of (2.5.1). q. e. d.

(2.6) LEMMA. *Let D, $D'\in|L|$ and let l, A, B and C be the same as in* (2.3). *Suppose* $\mathrm{Bs}|L|\neq\emptyset$. *Then by choosing a sufficiently general pair D and D', C is irreducible.*

PROOF. Assume that C is reducible for any pair D, D' $(D\neq D')$. Choose a pair D and D' such that $l=D\cap D'$ has the minimum number of irreducible components. Take a one parameter family $D'_t\in|L|$ $(t\in\boldsymbol{P}^1)$ such that $D'_0=D'$, $l_t:=D\cap D'_t$ is one dimensional for any $t\in\boldsymbol{P}^1$. Let $l_{t,\mathrm{red}}=C_t+B$, $C_t=A_{t,1}+\cdots+A_{t,a(t)}$ with $LA_{t,1}=1$, $LA_{t,j}=0$ $(j\geqq 2)$ and $B=B_1+\cdots+B_b$ $(=A_{a+1}+\cdots+A_{a+b})$, $a_{\min}=\min a(t)$. Then there is an open dense subset U of $\boldsymbol{P}^1$ such that $a(t)=a_{\min}$ $(\geqq 2)$ for any $t\in U$. Let d (resp. d'_t) be the equation defining D (resp. D'_t) and define an analytic subset Z of $X\times\boldsymbol{P}^1$ by $Z=\{(x,t)\in X\times\boldsymbol{P}^1;\ d(x)=d'_t(x)=0\}$. Let p_j $(j=1,2)$ be the j-th projection of $X\times\boldsymbol{P}^1$. Then Z is a proper flat family by p_2, whose fiber $p_2^{-1}(t)$ is l_t. The analytic space Z is therefore two dimensional. Let Z_j $(1\leqq j\leqq k)$ be all the irreducible components of Z_{red}, Y_j the normalization of Z_j, ϕ_j the natural map of Y_j into $X\times\boldsymbol{P}^1$. Let $Y_j\xrightarrow{\pi_j}U_j\xrightarrow{h_j}\boldsymbol{P}^1$ be the Stein factorization of $p_2\phi_j$. Then U_j is a nonsingular curve. Since Y_j is Cohen-Macaulay (normal and two dimensional), and since π_j is equidimensional, π_j is flat. Therefore there exists a Zariski dense open subset V_j of U_j such that $\pi_j^{-1}(v)$ is irreducible nonsingular for any $v\in V_j$.

Since ϕ_j is a birational map of Y_j onto Z_j, we may assume, by taking a smaller Zariski open subset of V_j if necessary, that $\pi_j^{-1}(v)$ is mapped birationally onto an irreducible component of $(l_t)_{\mathrm{red}}$ where $t=h_j(v)$. By (2.3), any irreducible component of $(l_t)_{\mathrm{red}}$ is non-singular, so $\pi_j^{-1}(v)$ is isomorphic to the image $p_1\phi_j(\pi_j^{-1}(v))$, a reduced irreducible component of $(l_t)_{\mathrm{red}}$. Since π_j is flat, the images of fibers of π_j over V_j by $p_1\phi_j$ are algebraically equivalent. Choosing $A_{t_1,1}$ for A_1 for a generic $t_1\in\boldsymbol{P}^1$ if necessary, we may assume $A_1=p_1\phi_1(\pi_1^{-1}(v_1))$ for some $v_1\in V_1$. Hence the image by $p_1\phi_1$ of any fiber of π_1 over V_1 is algebraically equivalent to A_1, so that it is just the unique irreducible component $A_{t,1}$ of C_t with $LA_{t,1}=1$ for some $t\in\boldsymbol{P}^1$.

Hence irreducible components of $C_t-A_{t,1}$ can appear only in the image $p_1\phi_1\pi_1^{-1}(U_1\setminus V_1)$ or in $p_1\phi_j(Y_j)$ $(j\geqq 2)$, hence those $A_{t,j}$ $(j\geqq 2)$ which are contained in $p_1\phi_1(Y_1)$ are only finitely many. Therefore there exists $\pi_2: Y_2\to U_2$ such that $p_1\phi_2(\pi_2^{-1}(v_2))=A_{t_2,2}$ for some $v_2\in V_2$ and $t_2=h_2(v_2)\in\boldsymbol{P}^1$. Here we may assume $A_{t_2,2}=A_2$ without loss of generality. The image $p_1\phi_2(\pi_2^{-1}(v))$ of a fiber $\pi_2^{-1}(v)$, $v\in V_2$ is therefore algebraically equivalent to A_2, whence $Lp_1\phi_2(\pi_2^{-1}(v))=LA_2=0$. Hence $p_1\phi_2(\pi_2^{-1}(v))$ $(v\in V_2)$ is contained in $C_t-A_{t,1}+B$. Since $A_2\cap B=\emptyset$, we may assume, by taking a smaller V_2 if necessary, that $p_1\phi_2(\pi_2^{-1}(v))$ is an irreducible component, say, $A_{t,j(v)}$ of $C_t-A_{t,1}$ where $t=h_2(v)$, $j(v)\geqq 2$.

Since $A_{t,1}\cap A_{s,j}=\emptyset$ for $t\neq s$, t, $s\in\boldsymbol{P}^1$ and $j\geqq 2$ by (2.5), $A_{t,1}$ $(t\in\boldsymbol{P}^1)$ can intersect $p_1\phi_2(\pi_2^{-1}(V_2))$ only along $A_{t,1}\cap(C_t-A_{t,1})$ by (2.3), whose cardinality is bounded by $a_{\min}-1$.

This shows that $p_1\phi_1(\pi_1^{-1}(V_1))$ and $p_1\phi_2(\pi_2^{-1}(V_2))$ intersect along at most a (possibly reducible) curve, hence that the intersection of $p_1\phi_1(Y_1)$ and $p_1\phi_2(Y_2)$

is at most one dimensional. However since the irreducible surface $p_1\psi_j(Y_j)$ is contained in an irreducible surface D, we have $D=p_1\psi_1(Y_1)=p_1\psi_2(Y_2)$. This is a contradiction. Therefore for a sufficiently general pair D and D', C is irreducible. q. e. d.

(2.7) LEMMA. $\mathrm{Bs}|L|$ *is empty and the complete intersection* $l:=D\cap D'$ *is an irreducible nonsingular rational curve with* $Ll=1$ *for any pair* D *and* $D'\in|L|$ *with* $D\neq D'$.

PROOF. We first assume $\mathrm{Bs}|L|\neq\emptyset$ to derive a contradiction. By (2.6), C is irreducible by choosing a general pair D and D'. Since D (and D') are reduced, the movable part of $D\cap D'$ is reduced somewhere, hence reduced over a Zariski open subset U of C (see [**11**, Theorem 7.18]). This implies that l is reduced and isomorphic to C over U.

Let I_l (resp. I_C) be the ideal sheaf in $\mathcal{O}_X$ defining l (resp. C). Then $I_l\subset I_C$ and the natural inclusion of I_l into I_C induces an isomorphism of $(I_l/I_l^2)\otimes\mathcal{O}_C$ into I_C/I_C^2 because I_l/I_l^2 is locally $\mathcal{O}_l$-free and $\mathcal{O}_C$ is torsion free. By Grothendieck's theorem, we write $I_C/I_C^2=\mathcal{O}_C(-p)\oplus\mathcal{O}_C(-q)$ for some integers p and q. Since $LC=1$ and $I_l/I_l^2=\mathcal{O}_l(-L)\oplus\mathcal{O}_l(-L)$, we have $(I_l/I_l^2)\otimes\mathcal{O}_C=\mathcal{O}_C(-1)\oplus\mathcal{O}_C(-1)$ Hence we have $p\leqq1$, $q\leqq1$. From the exact sequence

$$0\longrightarrow I_C/I_C^2\longrightarrow\Omega_X^1\otimes\mathcal{O}_C\longrightarrow\Omega_C^1\longrightarrow0,$$

we infer $c_1(I_C/I_C^2)+c_1(\Omega_C^1)=c_1(\Omega_X^1\otimes\mathcal{O}_C)=K_XC=-4LC=-4$. Hence $p+q=2$. This shows $p=q=1$ and that $(I_l/I_l^2)\otimes\mathcal{O}_C\cong I_C/I_C^2$. But when $B\neq\emptyset$, either of the two generators (chosen suitably) of I_l/I_l^2 at $p:=C\cap B$ vanishes at p, whence $(I_l/I_l^2)\otimes\mathcal{O}_C$ is not isomorphic to I_C/I_C^2. Hence $B=\emptyset$. By (2.4), $\mathrm{Bs}|L|$ is one point. Consider the exact sequence

$$0\longrightarrow I_C/I_l\longrightarrow\mathcal{O}_l\longrightarrow\mathcal{O}_C\longrightarrow0.$$

Since $l_{\mathrm{red}}=C$, the support of I_C/I_l is isolated. Since $H^0(l,\mathcal{O}_l)=H^0(C,\mathcal{O}_C)=\boldsymbol{C}$ by (1.7.4), we have $I_C/I_l=0$, whence $I_l=I_C$, $l\cong C$. By (1.7), $\mathrm{Bs}|L|=\mathrm{Bs}|L_D|=\mathrm{Bs}|L_l|=\mathrm{Bs}|L_C|=\mathrm{Bs}|\mathcal{O}_{P^1}(1)|=\emptyset$. This contradicts $\mathrm{Bs}|L|\neq\emptyset$.

Now we consider the case $\mathrm{Bs}|L|=\emptyset$. Then by Bertini's theorem, any general member D of $|L|$ is nonsingular. The divisor D is irreducible by (1.5.1). The linear system $|L_D|$ is base point free because $\mathrm{Bs}|L|=\mathrm{Bs}|L_D|$ in view of (1.7). Hence any general member l of $|L_D|$ is nonsingular by Bertini's theorem. The natural homomorphism of $H^0(X,L)$ into $H^0(D,L_D)$ is surjective so that the curve l is just a complete intersection $D\cap D'$ for some $D'\in|L|$. By (1.7.4), l is connected, hence it is irreducible. By (2.3), l itself is also the unique irreducible component of $(D\cap D')_{\mathrm{red}}$ ($=D\cap D'$ in this case) with $Ll=1$. Hence l is a nonsingular rational curve with $Ll=1$.

Let D'' and D''' be arbitrary members of $|L|$ with $D''\neq D'''$ and let $l':=D''\cap D'''$ be the complete intersection. Then by (2.3) and by $\mathrm{Bs}|L|=\emptyset$, we have $\mathrm{cl}(l')=n_1\mathrm{cl}(A_1)+\cdots+n_a\mathrm{cl}(A_a)$ for some $n_i>0$ and A_j irreducible, where $LA_1=1$, $LA_j=0$ $(2\leqq j\leqq a)$. Hence $n_1=Ll'=Ll=1$. This shows in view of the criterion of multiplicity one [**2**, Prop. 4.6] (see also [**4**, Prop. 2.2]) that l' is reduced at a generic point of A_1, hence reduced over a Zariski open dense subset of A_1. Then by the same argument as the first half of (2.7), $(I_{l'}/I_{l'}^2)\otimes\mathcal{O}_{A_1}\cong I_1/I_1^2\cong\mathcal{O}_{A_1}(-1)\oplus\mathcal{O}_{A_1}(-1)$, $l'\cong A_1$. Thus l' is also a nonsingular rational curve with $Ll'=1$. q. e. d.

(2.8) COMPLETION OF THE PROOF OF (1.1). Let X be a compact threefold as in (1.1). Then $l:=D\cap D'$ is a nonsingular rational curve for arbitrary D and $D'\in|L|$, $D\neq D'$ by (2.7). Hence by (1.7), we have $h^0(X, L)=h^0(D, L_D)+1=h^0(l, L_l)+2=4$. By (2.7), we have a morphism f of X onto $\boldsymbol{P}^3$ associated with the complete linear system $|L|$. Since $L^3=L\cdot D\cdot D'=Ll=1$ by (2.7), f is surjective and bimeromorphic. Let E be the exceptional set of f, that is, the divisor defined by $(\det(\mathrm{Jac} f))$ on X. Then E is a Cartier divisor whose image by f is zero or one dimensional. Hence $f_*E=0$. However since f_* induces an isomorphism of $\mathrm{Pic}\,X$ onto $\mathrm{Pic}\,\boldsymbol{P}^3$, E is empty. Hence f is unramified, so that f is an isomorphism of X onto $\boldsymbol{P}^3$. q. e. d.

(2.9) Here is another proof of (1.1) which makes use of (2.7) in full strength, making however less use of $\mathrm{Pic}X\cong\boldsymbol{Z}$. Let X be a compact threefold as in (1.1). In the same manner as in (2.8), we have $h^0(X, L)=4$ and a bimeromorphic morphism f of X onto $\boldsymbol{P}^3$. Suppose that f is not an isomorphism. Then there is an irreducible curve C on X such that $LC=0$. Take a point p of C. Then by (2.7) and $h^0(X, L)=4$, we can choose two distinct members D and D' of $|L|$ passing through p. Let $l=D\cap D'$ be the complete intersection. Then by (2.7), l is a nonsingular rational curve with $Ll=1$. Since $DC=D'C=LC=0$, C is contained in both D and D', hence it is contained in $l_{\mathrm{red}}=l$. Therefore $C=l$, which contradicts $LC\neq Ll$. Hence f is an isomorphism of X onto $\boldsymbol{P}^3$. q. e. d.

(2.10) REMARK. The assumption $\mathrm{Pic}X\cong\boldsymbol{Z}$ in (1.1) was made use of only in the proof of (1.3) and (1.5). It is conjectured that the following is true;

CONJECTURE. *If a compact threefold X has a complex line bundle L such that $L^3>0$, $K_X=-dL$ $(d\geqq 4)$, then X is isomorphic to $\boldsymbol{P}^3$.*

Fujita kindly pointed out that (1.1) is true in the category of algebraic varieties over an algebraically closed field of arbitrary characteristic by additionally assuming that $H^1(X, \mathcal{O}_X)=0$. He kindly gave necessary modifications in the proof of (2.7). We notice that we have an alternative proof similar to (but much simpler than) [18] which works in arbitrary characteristic.

§ 3. Main theorems.

(3.1) THEOREM. *A Moishezon threefold homeomorphic to $\boldsymbol{P}^3$ is isomorphic to $\boldsymbol{P}^3$ if the Kodaira dimension of it is less than three.*

PROOF. Let X be a Moishezon threefold homeomorphic to $\boldsymbol{P}^3$. Then the Hodge spectral sequence $E_1^{p,q}=H^q(X, \Omega_X^p)$ with abutment $H^n(X, \boldsymbol{C})$ degenerates at E_1 [22, p. 99]. Hence $h^q(X, \mathcal{O}_X)=0$ for $q>0$ and $\chi(X, \mathcal{O}_X)=1$ because $b_1=b_3=0$, $b_2=1$. Hence $\operatorname{Pic} X=H^2(X, \boldsymbol{Z})=H^2(\boldsymbol{P}^3, \boldsymbol{Z})=\boldsymbol{Z}$. Let L be a generator of $\operatorname{Pic} X$ with $L^3=1$. Then by [8, pp. 207-208] (see also [17, pp. 317-318]), $K_X=-4L$. Since X is Moishezon and $\operatorname{Pic} X=\boldsymbol{Z}$, we have either $\kappa(X, L)=3$ or $\kappa(X, -L)=3$. In the second case, the Kodaira dimension of X is 3, which contradicts the assumption. Hence $\kappa(X, L)=3$. (See (3.3) below.) Therefore by (1.1), X is isomorphic to $\boldsymbol{P}^3$. q. e. d.

(3.2) THEOREM. *An arbitrary complex analytic* (*global*) *deformation of $\boldsymbol{P}^3$ is isomorphic to $\boldsymbol{P}^3$.*

PROOF. Let X be an arbitrary complex analytic deformation of $\boldsymbol{P}^3$. Then by the upper semi-continuity, $h^0(X, -mK_X)$ behaves as a polynomial of degree 3 in m as m goes to infinity. Hence X is Moishezon and the Kodaira dimension of X is $-\infty$. Hence X is isomorphic to $\boldsymbol{P}^3$ by (3.1). q. e. d.

(3.3) REMARK. It seems very plausible that a Moishezon threefold homeomorphic to $\boldsymbol{P}^3$ has Kodaira dimension less than three. However $\kappa(X, L)\geqq 1$ is not a consequence of $\operatorname{Pic} X=\boldsymbol{Z}L$ with $L^3>0$. The assertion $\mu>0$ in [19, p. 403, line 19] is equivalent to $\kappa(X, L)\geqq 1$. This part requires a proof. Indeed, as the following example of Fujiki (or someone else?) shows, there is a Moishezon threefold X with $\operatorname{Pic} X=\boldsymbol{Z}L$ such that $L^3<0$, $K_X=-2L$, $\kappa(X, L)=3$. This also gives a counterexample to [21] Theorem 5.3. We shall show the example.

Let S be a nonsingular quadric surface in $\boldsymbol{P}^3$, f_i a fiber of two rulings via the isomorphism $S\cong \boldsymbol{P}^1\times \boldsymbol{P}^1$, C a nonsingular curve on S with $[C]=3f_1+kf_2\in \operatorname{Pic} S$ ($k\geqq 7$). Let $f: Y\to \boldsymbol{P}^3$ be the blowing up of $\boldsymbol{P}^3$ with C center, $E=f^{-1}(C)_{\mathrm{red}}$, $T=f^*S-E$, $e_i=f^{-1}(f_i)_{\mathrm{red}}$. Then T (resp. e_i) is isomorphic to S (resp. f_i). One sees readily that $N_{T/Y}=-e_1-(k-2)e_2$. Hence by the contraction theorem of Nakano-Fujiki, there is a contraction $g: Y\to X$ with X nonsingular, $g(T)$ a nonsingular rational curve, $g(e_2)$ a point. Let H be a hyperplane bundle of $\boldsymbol{P}^3$, $L=g_*f^*H$. Then we see $K_X=-2L$, $L^3=6-k$ (<0), $\kappa(X, L)=3$.

First we see

$$(f^*H)^2T=((f^*H)_T)^2=(e_1+e_2)^2=2\,,$$

$$(f^*H)T^2=(f^*H)_T[T]_T=(e_1+e_2)(-e_1-(k-2)e_2)=1-k\,,$$

$$T^3=([T]_T)^2=(-e_1-(k-2)e_2)^2=2k-4\,,$$

$$L^3=(f^*H+T)^3=6-k\,.$$

Next we shall show $K_X=-2L$. Since $\mathrm{Pic}\,X=\boldsymbol{Z}$, it suffices to prove $K_Xg_*f^*l=-2Lg_*f^*l$ for a line l in $\boldsymbol{P}^3$. We see

$$K_Xg_*f^*l=(g^*K_X)(f^*l)=(-6f^*H+2E)(f^*l)=-6\,,$$

$$Lg_*f^*l=(f^*H)(g^*g_*f^*l)=(f^*H)(f^*l+2e_2)=3\,.$$

Since H is a hyperplane bundle, $\kappa(X, L)=3$ is clear.

Bibliography

[1] A. Altman and S. Kleiman, Introduction to Grothendieck Duality, Lecture Notes in Math., **146**, Springer, 1970.

[2] A. Borel and A. Heafliger, La classe d'homologie fundamentale d'une espace analytique, Bull. Soc. Math. France, **89** (1961), 461-513.

[3] R. Draper, Intersection theory in analytic geometry, Math. Ann., **180** (1969), 175-204.

[4] T. Fujita, On the structure of polarized varieties with Δ-genera zero, J. Fac. Sci. Univ. Tokyo, **22** (1975), 103-115.

[5] W. Fulton, Intersection Theory, Springer, 1984.

[6] H. Hironaka, An example of Kaehlerian complex-analytic deformation of Kaehlerian complex structures, Ann. of Math., **75** (1962), 190-208.

[7] H. Hironaka, Flattening theorem in complex-analytic geometry, Amer. J. Math., **97** (1975), 503-547.

[8] F. Hirzebruch, Topological Methods in Algebraic Geometry, Springer, 1966.

[9] F. Hirzebruch and K. Kodaira, On the complex projective spaces, J. Math. Pure Appl., **36** (1957), 201-216.

[10] S. Iitaka, On D-dimensions of algebraic varieties, J. Math. Soc. Japan, **23** (1971), 356-373.

[11] S. Iitaka, Algebraic Geometry, Graduate Texts in Math., **76**, Springer, 1981.

[12] J. King, Global residues and intersections on a complex manifold, Trans. Amer. Math. Soc., **192** (1974), 163-199.

[13] K. Kleiman, Toward a numerical theory of ampleness, Ann. of Math., **84** (1966), 293-344.

[14] S. Kobayashi and T. Ochiai, Characterizations of complex projective spaces and hyperquadrics, J. Math. Kyoto Univ., **13** (1973), 31-47.

[15] B. G. Moishezon, On n-dimensional compact varieties with n algebraically independent meromorphic functions, I, II, III, Izv. Akad. SSSR Ser. Mat., **30** (1966), 133-174, 345-386, 621-656, (English transl.), AMS transl. Ser. 2, **63**, 51-177.

[16] S. Mori, Threefolds whose canonical bundles are not numerically effective, Ann. of Math., **116** (1982), 133-176.

[17] J. Morrow, A survey of some results on complex Kähler manifolds, Global Analysis, Univ. Tokyo Press, 1969, 315-324.
[18] I. Nakamura, Moishezon threefolds homeomorphic to a hyperquadric in $\boldsymbol{P}^4$, preprint.
[19] T. Peternell, A rigidity theorem for $\boldsymbol{P}_3(\boldsymbol{C})$, Manuscripta Math., **50** (1985), 397-428.
[20] T. Peternell, On the rigidity problem for the complex projective space, preprint.
[21] T. Peternell, Algebraic structures on certain 3-folds, Math. Ann., **274** (1986), 133-156.
[22] K. Ueno, Classification theory of algebraic varieties and compact complex spaces, Lectures Notes in Math., **439**, Springer, 1975.
[23] H. Tsuji, Every deformation of $\boldsymbol{P}^n$ is again $\boldsymbol{P}^n$, preprint.
[24] S T. Yau, On Calabi's conjecture and some new results in algebraic geometry, Proc. Nat. Acad. Sci. U. S. A., **74** (1977), 1798-1799.

Iku Nakamura
Department of Mathematics
Hokkaido University
Sapporo 060
Japan