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On the global strong solutions of coupled Klein-Gordon-Schrödinger equations

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1. Introduction.

In this paper we will consider the following system of equations in three space dimensions:

$$id\psi/dt-A_1\psi = -\psi\phi\,, \tag{1.1}$$

$$d^2\phi/dt^2+A_2\phi = |\psi|^2\,. \tag{1.2}$$

Where A_1 and A_2 denote positive selfadjoint elliptic operators of order 2 with Dirichlet-zero conditions over a bounded or unbounded domain $\Omega\subset\boldsymbol{R}^3$. If $A_1=-\Delta$ and $A_2=-\Delta+I$, where Δ denotes the spatial Laplacian, (1.1) and (1.2) are the so called Klein-Gordon-Schrödinger (K-G-S) equations with Yukawa coupling in which ψ describes complex scalar neucleon field and ϕ describes real scalar meson field.

The first study for the K-G-S equations was done by I. Fukuda and M. Tsutsumi [7]. They considered the initial boundary value problem for the K-G-S equations under the initial conditions $\psi(0, x)=\psi_0(x)\in H_0^{1,2}(\Omega)\cap H^{3,2}(\Omega)$, $\phi(0, x)=\phi_0(x)\in H_0^{1,2}(\Omega)\cap H^{2,2}(\Omega)$, $\phi_t(0, x)=\phi_1(x)\in H_0^{1,2}(\Omega)$ and the boundary conditions $\psi(t, x)=\phi(t, x)=0$ for $x\in\partial\Omega$ and $t\in\boldsymbol{R}$. Here Ω is a bounded domain in $\boldsymbol{R}^3$ and $\partial\Omega$ is a smooth boundary of Ω. By using Galerkin's method, they proved the existence of global strong solutions of the K-G-S equations under the above conditions. The initial condition on $\psi_0(x)$ is unnatural and should be changed into the natural condition such as $\psi_0(x)\in H_0^{1,2}(\Omega)\cap H^{2,2}(\Omega)$.

The second study was done by J. B. Baillon and J. M. Chadam [2]. They proved the existence of global strong solutions of the initial value problem of the K-G-S equations under the initial conditions $\psi_0(x)\in H^{2,2}(\boldsymbol{R}^3)$ and $\phi_0(x)\in H^{2,2}(\boldsymbol{R}^3)$ and $\phi_1(x)\in H^{1,2}(\boldsymbol{R}^3)$. They obtained the above result by using L^p-L^q estimates for the elementary solution of the linear Schrödinger equation. The L^p-L^q estimates are very useful methods to the initial value problem for the K-G-S equations (see, e.g., A. Bachelot [1]). But such L^p-L^q estimates are not obtained in the case of initial boundary value problem. Therefore it does not seem that their method is directly applicable to the initial boundary value problem (1.1) and (1.2).

Our purpose in this paper is to show the existence of global strong solutions to (1.1) and (1.2) which include the K-G-S equations, under the same initial conditions as [2] and the same boundary conditions as [7]. We will get the result by using estimates of the nonlinearity in fractional order Besov spaces developed by P. Brenner and W. von Wahl [4], nonlinear interpolation theorem obtained by W. von Wahl [10], [11], [12] and the inequality of H. Brezis and T. Gallouet [5] (see also H. Brezis and S. Wainger [6]).

We introduce the following standard notations. For a multiindex $\alpha=(\alpha_1, \alpha_2, \alpha_3)$ we set $|\alpha|=\sum_{\nu=1}^{3}\alpha_\nu$ and $D^\alpha=\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$, where $\partial_j=\partial/\partial x_j$. Let Ω be an open set of $\boldsymbol{R}^3$ with smooth compact boundary $\partial\Omega$ and $s\geqq 0$, $1\leqq p\leqq\infty$. For simplicity, we denote the space of complex valued functions and real valued functions by the same symbols. $H^{s,p}(\boldsymbol{R}^3)$, $H^{s,p}(\Omega)$ and $H_0^{s',p}(\Omega)$ $(s'>1/2)$ are the usual Sobolev spaces of fractional order s or s' of L^p functions. Let $1<p$, $q<\infty$ and $s\geqq 0$, $s=[s]+\sigma$, where $[s]$ denotes the largest integer smaller than or equal to s and $0<\sigma<1$. The Besov space $B_p^{s,q}(\boldsymbol{R}^3)$ consists of tempered distributions u such that

$$\|u\|_{s,q,p}=\|u\|_{L^p(\boldsymbol{R}^3)}+\left(\int_0^\infty t^{-\sigma q}\sup_{|k|\leq t}\sum_{|\alpha|\leq[s]}\|D^\alpha(u_k-u)\|_{L^p(\boldsymbol{R}^3)}^q\frac{dt}{t}\right)^{1/q}$$

is finite, where $u_k=u(x+k)$ (see [9]). The norms of $L^p(\Omega)$ and $H^{s,p}(\Omega)$ are denoted by $\|\cdot\|_p$ and $\|\cdot\|_{s,p}$, respectively. We simply denote the norms of $L^p(\boldsymbol{R}^3)$ and $H^{s,p}(\boldsymbol{R}^3)$ by the same symbols as those of $L^p(\Omega)$ and $H^{s,p}(\Omega)$, respectively. For any Banach space X, $C^k(I;X)$ denotes the space of k times continuously differentiable functions from I to X.

2. Initial boundary value problem for (1.1) and (1.2) in $L^2(\Omega)$.

Let $a_{i,j}^k(x)$ and $a^k(x)$ $(1\leqq i, j\leqq 3, k=1, 2)$ be infinitely differentiable real valued functions on $\boldsymbol{R}^3$ and every derivative of them is bounded in $\boldsymbol{R}^3$, moreover we assume

$$C|\xi|^2\leqq\sum_{i,j=1}^{3}a_{i,j}^k(x)\xi_i\xi_j\leqq C^{-1}|\xi|^2,\qquad a_{i,j}^k(x)=a_{j,i}^k(x),$$

where $\xi\in\boldsymbol{R}^3$ and $C>0$. We define the operators A_1 and A_2 by

(2.1) $$A_k u=-\sum_{i,j=1}^{3}\partial(a_{i,j}^k(x)(\partial u/\partial x_j))/\partial x_i+a^k(x)u$$
$$\text{for } u\in D(A_k)=H_0^{1,2}(\Omega)\cap H^{2,2}(\Omega),\qquad k=1,2.$$

Then A_1 and A_2 are selfadjoint operators in $L^2(\Omega)$. For our purpose it is no loss of generality to assume that $a^k(x)\geqq\lambda>0$ in $\bar{\Omega}$, and that

(2.2) $$C\|u\|_{1,2}^2\leqq(A_k u,\bar{u})\ \left(=\int_\Omega(A_k u)\bar{u}dx\right),\qquad k=1,2,$$

where $C>0$. Therefore, the fractional powers A_k are defined in the standard manner, and we have the relations (see [4])

(2.3) $\qquad H_0^{2\rho,2}(\Omega)\subset D(A_k^\rho)\subset H^{2\rho,2}(\Omega)\,,\qquad 0\leqq\rho\leqq 1\,,$

(2.4) $\qquad H^{2\rho,2}(\Omega)=D(A_k^\rho)\,,\qquad 0\leqq\rho<(1/4)\,,$

(2.5) $\qquad H_0^{1,2}(\Omega)\cap H^{2\rho,2}(\Omega)=D(A_k^\rho)\,,\qquad (1/2)\leqq\rho\leqq 1\,,$

(2.6) $\qquad C\|u\|_{2\rho,2}\leqq\|A_k^\rho u\|_2\leqq C^{-1}\|u\|_{2\rho,2},\qquad u\in D(A_k^\rho),\quad 0\leqq\rho\leqq 1,\quad \rho\neq 1/4\,.$

Here and in the sequel C is a positive constant and will change from line to line.

We will consider the initial boundary value problem for (1.1) and (1.2) with the operators A_1 and A_2 defined by (2.1) satisfying (2.2).

Our main result is the following

THEOREM. *Suppose that* $\psi_0\in D(A_1)$ *is complex,* $\phi_0\in D(A_2)$ *and* $\phi_1\in D(A_2^{1/2})$ *are real. Then there exists a strong solution of* (1.1) *and* (1.2) *such that*

$$\psi\in C(\boldsymbol{R};\,D(A_1))\cap C^1(\boldsymbol{R};\,L^2(\Omega))\,,$$

$$\phi\in C(\boldsymbol{R};\,D(A_2))\cap C^1(\boldsymbol{R};\,D(A_2^{1/2}))\cap C^2(\boldsymbol{R};\,L^2(\Omega))\,.$$

We now summarize some lemmas needed below to prove the Theorem. The following two lemmas are derived by direct calculations of the nonlinearity in fractional order Besov space. We follow P. Brenner and W. von Wahl [4].

LEMMA 1. *Let* $u\in H^{7/6,2}(\Omega)$. *Then we have*

(2.7) $$\||u|^2\|_{2/3,2}\leqq C\|u\|_{1,2}\|u\|_{7/6,2}\,.$$

PROOF. Let u be extended to the whole of $\boldsymbol{R}^3$ as a function of $H^{7/6,2}(\boldsymbol{R}^3)$ by means of the $H^{7/6,2}(\Omega)$-extension operator (see, e.g., [9]). We denote the extension by u and the resulting extension of $|u|^2$ to $\boldsymbol{R}^3$ also by $|u|^2$. If $\||u|^2\|_{H^{2/3,2}(\boldsymbol{R}^3)}$ can be estimated in the desired way our lemma is proved. By Theorem 6.4.4 in [3] and the definition of the Besov space we obtain

(2.8) $$\||u|^2\|_{2/3,2}\leqq C\||u|^2\|_2+C\Big(\int_0^\infty t^{-4/3}\sup_{|k|\leqq t}\||u_k|^2-|u|^2\|_2^2\frac{dt}{t}\Big)^{1/2}\,.$$

From Hölder's inequality we have

(2.9) $$\||u_k|^2-|u|^2\|_2\leqq C\|u\|_6\|u_k-u\|_3\,.$$

(2.8) and (2.9) give

(2.10) $$\begin{aligned}\||u|^2\|_{2/3}&\leqq C\|u\|_4^2+C\|u\|_6\Big(\int_0^\infty t^{-4/3}\sup_{|k|\leqq t}\|u_k-u\|_3^2\frac{dt}{t}\Big)^{1/2}\\&\leqq C\|u\|_4^2+C\|u\|_6\|u\|_{2/3,2,3}\,.\end{aligned}$$

By Theorems 6.4.4, 6.5.1 in [3] and (2.10) we have (2.1). This completes the proof.

LEMMA 2. *Let* $u\in H^{13/10,2}(\Omega)$ *and* $v\in H^{3/2,2}(\Omega)\cap L^{\infty}(\Omega)$. *Then we have*

$$\|u\cdot v\|_{13/10,2}\leqq C\|u\|_{13/10,2}(\|v\|_{3/2,2}+\|v\|_{\infty}). \tag{2.11}$$

PROOF. (2.11) is derived in the same way as in the proof of Lemma 1. Indeed we have

$$\|u\cdot v\|_{13/10,2}\leqq C\|u\cdot v\|_2+C\Big(\int_0^\infty t^{-3/5}\sup_{|k|\leqq t}\sum_{|\alpha|\leqq 1}\|D^\alpha(u_k\cdot v_k-u\cdot v)\|_2^2\frac{dt}{t}\Big)^{1/2}.$$

On the other hand we have for $|\alpha|\leqq 1$

$$\begin{aligned}\|D^\alpha(u_k\cdot v_k-u\cdot v)\|_2\leqq\ &\|(v_k-v)\cdot D^\alpha u_k\|_2+\|u_k\cdot D^\alpha(v_k-v)\|_2\\ &+\|(u_k-u)\cdot D^\alpha v_k\|_2+\|v\cdot D^\alpha(u_k-u)\|_2.\end{aligned} \tag{2.12}$$

By Hölder's inequality the right hand side of (2.12) is estimated by

$$\|v_k-v\|_{10}\|D^\alpha u_k\|_{5/2}+\|u_k\|_{15}\|D^\alpha(v_k-v)\|_{30/13}+\|u_k-u\|_6\|D^\alpha v\|_3+\|v\|_\infty\|D^\alpha(u_k-u)\|_2.$$

Therefore $\|u\cdot v\|_{13/10,2}$ is dominated by

$$\begin{aligned}&C\|u\cdot v\|_2+C\|u\|_{1,5/2}\Big(\int_0^\infty t^{-3/5}\sup_{|k|\leqq t}\|v_k-v\|_{10}^2\frac{dt}{t}\Big)^{1/2}\\ &+C\|u\|_{15}\Big(\int_0^\infty t^{-3/5}\sup_{|k|\leqq t}\sum_{|\alpha|\leqq 1}\|D^\alpha(v_k-v)\|_{30/13}^2\frac{dt}{t}\Big)^{1/2}\\ &+C\|v\|_{1,3}\Big(\int_0^\infty t^{-3/5}\sup_{|k|\leqq t}\|u_k-u\|_6^2\frac{dt}{t}\Big)^{1/2}\\ &+C\|v\|_\infty\Big(\int_0^\infty t^{-3/5}\sup_{|k|\leqq t}\sum_{|\alpha|\leqq 1}\|D^\alpha(u_k-u)\|_2^2\frac{dt}{t}\Big)^{1/2}\\ &\leqq C(\|u\|_2\|v\|_\infty+\|u\|_{1,5/2}\|v\|_{3/10,2,10}+\|u\|_{15}\|v\|_{13/10,2,30/13}\\ &\quad+\|u\|_{3/10,2,6}\|v\|_{1,3}+\|u\|_{13/10,2,2}\|v\|_\infty).\end{aligned} \tag{2.13}$$

(2.13) and Theorems 6.4.4, 6.5.1 in [3] imply

$$\begin{aligned}\|u\cdot v\|_{13/10,2}&\leqq C(\|u\|_2\|v\|_\infty+\|u\|_{13/10,2}\|v\|_{3/2,2}+\|u\|_{13/10,2}\|v\|_\infty)\\ &\leqq C\|u\|_{13/10,2}(\|v\|_{3/2,2}+\|v\|_\infty).\end{aligned}$$

This completes the proof.

LEMMA 3 (The Brezis and Gallouet inequality). *Let* $u\in H^{3/2,2}(\Omega)\cap H^{5/3,2}(\Omega)$. *Then we have*

$$\|u\|_\infty\leqq C(1+\|u\|_{3/2,2}\sqrt{\log(1+\|u\|_{5/3,2})}). \tag{2.14}$$

PROOF. Proof is obtained in the same way as is the proof of [5] (see also [6]). Let $R>0$ and $u(x)=\int \hat{u}(\xi)e^{ix\cdot\xi}d\xi$. Then

$$\|u\|_\infty \leqq \|\hat{u}\|_1 \leqq \int_{|\xi|<R}|\hat{u}(\xi)|d\xi+\int_{|\xi|\geqq R}|\hat{u}(\xi)|d\xi \tag{2.15}$$
$$\leqq \int_{|\xi|<R}(1+|\xi|)^{3/2}|\hat{u}(\xi)|(1+|\xi|)^{-3/2}d\xi$$
$$+\int_{|\xi|\geqq R}(1+|\xi|)^{5/3}|\hat{u}(\xi)|(1+|\xi|)^{-5/3}d\xi\,.$$

By the Schwartz inequality and (2.15) we have

$$\|u\|_\infty \leqq \|(1+|\xi|)^{3/2}\hat{u}\|_2\Big(\int_{|\xi|<R}(1+|\xi|)^{-3}d\xi\Big)^{1/2}$$
$$+\|(1+|\xi|)^{5/3}\hat{u}\|_2\Big(\int_{|\xi|\geqq R}(1+|\xi|)^{-10/3}d\xi\Big)^{1/2}$$
$$\leqq C\|u\|_{3/2,2}\sqrt{\log(1+R)}+C\|u\|_{5/3,2}(1+R)^{-1/6}\,.$$

Proof is completed by taking $R=\max(1, \|u\|_{5/3,2}^6)$.

We have to take into consideration boundary values to prove the main result, therefore we need the following nonlinear interpolation lemma which follows from [12] (see also [10], [11]).

LEMMA 4. *Let $u\in D(A_2^{1/2})$. Then we have*

$$\|A_2^{1/4}|u|^2\|_2 \leqq C\|A_2^{1/2}u\|_2^2\,. \tag{2.16}$$

PROOF. Let $u, v\in D(A_2^{3/8})$. Then we obtain by Sobolev's inequality and (2.6)

$$\|u\cdot v\|_2 \leqq \|u\|_4\|v\|_4 \leqq C\|u\|_{3/4,2}\|v\|_{3/4,2} \leqq C\|A_2^{3/8}u\|_2\|A_2^{3/8}v\|_2\,. \tag{2.17}$$

If $u, v\in D(A_2^{5/8})$, a formal calculation yields

$$\partial(u\cdot v)/\partial x_i = (\partial u/\partial x_i)\cdot v+u\cdot(\partial v/\partial x_i)\,,$$
$$\|\partial(u\cdot v)/\partial x_i\|_2 \leqq \|\nabla u\|_{12/5}\|v\|_{12}+\|u\|_{12}\|\nabla v\|_{12/5}\,.$$

This gives by Sobolev's inequality and (2.6)

$$\|\partial(u\cdot v)/\partial x_i\|_2 \leqq C\|u\|_{5/4,2}\|v\|_{5/4,2} \leqq C\|A_2^{5/8}u\|_2\|A_2^{5/8}v\|_2\,.$$

This estimate also justifies the preceding calculations and we get

$$\|u\cdot v\|_{1,2} \leqq C\|A_2^{5/8}u\|_2\|A_2^{5/8}v\|_2\,.$$

If $u, v \in D(A_2)$, then $u \cdot v$ is in $H_0^{1,2}(\Omega)$ as it easily follows from the boundedness of u, v. Thus approximating u, v in the norm $\|A_2^{5/8}\cdot\|_2$ by $u_m, v_m \in D(A_2)$, we see that $u_m \cdot v_m$ converges to $u \cdot v$ in $H_0^{1,2}(\Omega)$ if $m \to \infty$. Thus $u \cdot v \in H_0^{1,2}(\Omega)$ and

$$\|A_2^{1/2}(u\cdot v)\|_2 \leqq C\|A_2^{5/8}u\|_2\|A_2^{5/8}v\|_2 . \tag{2.18}$$

The preceding estimates (2.17) and (2.18) also show that $M(u, v)=u\cdot v$ is an analytic mapping from $D(A_2^{3/8})\times D(A_2^{3/8})$ into $L^2(\Omega)$ and from $D(A_2^{5/8})\times D(A_2^{5/8})$ into $D(A_2^{1/2})$, respectively, in the following sense:

1. On any ball $(\|A_2^{\rho}u\|_2^2+\|A_2^{\rho}v\|_2^2)^{1/2}\leqq R$ $(\rho=3/8, 5/8)$ the expressions $M(u, v)$ stay bounded in the following sense: We have $\|M(u, v)\|_2 \leqq w((\|A_2^{3/8}u\|_2^2+\|A_2^{3/8}v\|_2^2)^{1/2})$ if $\rho=3/8$ and $\|A_2^{1/2}M(u, v)\|_2 \leqq w((\|A_2^{5/8}u\|_2^2+\|A_2^{5/8}v\|_2^2)^{1/2})$ if $\rho=5/8$ with some monotone non-decreasing function w.

2. The mapping $\zeta \mapsto M(u+\zeta u', v+\zeta v')$ is holomorphic from $\boldsymbol{C}$ into $L^2(\Omega)$ $(D(A_2^{1/2})=H_0^{1,2}(\Omega))$ if $u, u', v, v' \in D(A_2^{3/8})$ $(D(A_2^{5/8}))$.

Then it follows by interpolation result in [**12**, Satz V.2, p. 213 and the remark on p. 214] that M is also an analytic mapping from $D(A_2^{1/2})\times D(A_2^{1/2})$ into $D(A_2^{1/4})$ fulfilling the estimate

$$\|A_2^{1/4}(u\cdot v)\|_2 \leqq w((\|A_2^{1/2}u\|_2^2+\|A_2^{1/2}v\|_2^2)^{1/2})$$

with the same w as above. Inserting for v the function $\bar{u}$ we arrive at the desired estimate. This completes the proof.

3. Proof of Theorem.

We consider the following integral equations

$$\psi(t) = (\exp -iA_1t)\psi_0 - i\int_0^t (\exp -iA_1(t-s))\psi(s)\phi(s)ds , \tag{3.1}$$

$$\phi(t) = (\cos A_2^{1/2}t)\phi_0 + (A_2^{-1/2}\sin A_2^{1/2}t)\phi_1 - \int_0^t (A_2^{-1/2}\sin A_2^{1/2}(t-s))|\psi(s)|^2 ds. \tag{3.2}$$

(3.1) and (3.2) are the integral equations corresponding to (1.1) and (1.2), respectively. By the result of I. E. Segal [8], there exists a strong solution of (3.1) and (3.2) in some time interval $[-T, T]$ such that

$$\psi \in C([-T, T]; D(A_1)) \cap C^1([-T, T]; L^2(\Omega)),$$
$$\phi \in C([-T, T]; D(A_2)) \cap C^1([-T, T]; D(A_2^{1/2})) \cap C^2([-T, T]; L^2(\Omega)).$$

It follows by a standard argument that the strong solution of (3.1) and (3.2) satisfies (1.1) and (1.2) in $L^2(\Omega)$. Therefore the Theorem is proved if the desired a prior estimates of the local strong solution of (3.1) and (3.2) are obtained.

First we give a priori estimates of $\|A_1^{1/2}\psi(t)\|_2$, $\|A_2^{1/2}\phi(t)\|_2$ and $\|\phi_t(t)\|_2$. From (1.1) we have

(3.3) $$\|\psi(t)\|_2 = \|\psi_0\|_2\,, \qquad \text{for any } t\in[-T, T]\,.$$

Using (1.1) and (1.2), we have

$$(A_1\psi, \bar{\psi}_t)+(A_1\bar{\psi}, \psi_t)+(A_2\phi, \phi_t)+(\phi_{tt}, \phi_t) = d(|\psi|^2, \phi)/dt, \qquad \text{for any } t\in[-T, T]\,,$$

from which we get

(3.4) $$d(Re(A_1\psi, \bar{\psi})+(A_2\phi, \phi)/2+(\phi_t, \phi_t))/dt = d(|\psi|^2, \phi)/dt\,, \qquad \text{for any } t\in[-T, T]\,.$$

By Hölder's inequality, Sobolev's inequality and (2.6) we have

(3.5) $$\begin{aligned}\int_\Omega|\psi|^2|\phi|dx &\leqq \|\phi\|_6\|\psi\|_{12/5}^2 \leqq C\|A_2^{1/2}\phi\|_2\|\psi\|_{1,2}^{1/4}\|\psi\|_2^{3/4}\\ &\leqq C\|A_2^{1/2}\phi\|_2\|A_1^{1/2}\psi\|_2^{1/4}\|\psi\|_2^{3/4}\,, \qquad \text{for any } t\in[-T, T]\,.\end{aligned}$$

By (2.2), (2.6), (3.3), (3.4) and (3.5) we have

(3.6) $$\|A_1^{1/2}\psi(t)\|_2,\ \|A_2^{1/2}\phi(t)\|_2,\ \|\phi_t(t)\|_2 \leqq C\,, \qquad \text{for any } t\in[-T, T]\,.$$

Next we show the desired a priori estimates. We apply (3.6), Lemma 4 and (2.6) to (3.2) to obtain

(3.7) $$\begin{aligned}\|A_2^{3/4}\phi(t)\|_2 &\leqq C+C\int_0^t\|A_2^{1/4}|\psi(s)|^2\|_2ds\\ &\leqq C+C\int_0^t\|A_2^{1/2}\psi(s)\|_2^2ds\\ &\leqq C+C\int_0^t\|A_1^{1/2}\psi(s)\|_2^2ds\\ &\leqq C(T)\,, \qquad \text{for any } t\in[-T, T]\,.\end{aligned}$$

Here and in what follows $C(\cdot)$ is a continuous monotonically non-decreasing function from the non-negative reals into itself. From (3.1), Lemma 2 and (2.6) we have

(3.8) $$\begin{aligned}\|A_1^{13/20}\psi(t)\|_2 &\leqq C+C\int_0^t\|A_1^{13/20}\psi(s)\phi(s)\|_2ds\\ &\leqq C+C\int_0^t\|\psi(s)\phi(s)\|_{13/10,2}ds\\ &\leqq C+C\int_0^t\|\psi(s)\|_{13/10,2}(\|\phi(s)\|_{3/2,2}+\|\phi(s)\|_\infty)ds, \qquad \text{for any } t\in[-T, T]\end{aligned}$$

By Lemma 3, (2.6), (3.7) and (3.8)

(3.9) $$\|A_1^{13/20}\psi(t)\|_2 \leqq C+C(T)\int_0^t \|A_1^{13/20}\psi(s)\|_2(1+\sqrt{\log(1+\|A_2^{5/6}\phi(s)\|_2)})ds\,, \quad \text{for any } t\in[-T, T].$$

From (3.2), Lemma 1, (3.6) and (2.6) we have

(3.10) $$\begin{aligned}\|A_2^{5/6}\phi(t)\|_2 &\leqq C+C\int_0^t \|A_2^{1/3}|\psi(s)|^2\|_2 ds\\ &\leqq C+C\int_0^t \||\psi(s)|^2\|_{2/3,2}ds\\ &\leqq C+C\int_0^t \|\psi(s)\|_{7/6,2}^2 ds\\ &\leqq C+C\int_0^t \|A_1^{13/20}\psi(s)\|_2 ds\,, \qquad \text{for any } t\in[-T, T].\end{aligned}$$

We denote by $f(t)$ the right hand side in (3.9) and by $g(t)$ the right hand side in (3.10). Simple calculation gives

(3.11) $$dG(t)/dt \leqq C(T)G(t)(1+\sqrt{\log(1+G(t))}), \qquad \text{for any } t\in[-T, T],$$

where $G(t)=f(t)+g(t)$. From (3.11) and Gronwall's inequality we easily get

$$G(t) \leqq C(T)\,, \qquad \text{for any } t\in[-T, T].$$

Thus we have

(3.12) $$\|A_1^{13/20}\psi(t)\|_2,\ \|A_2^{5/6}\phi(t)\|_2 \leqq C(T)\,, \qquad \text{for any } t\in[-T, T].$$

By (3.2), (2.6), Sobolev's inequality and (3.12) we have

(3.13) $$\begin{aligned}\|A_2\phi(t)\|_2 &\leqq C+C\int_0^t \||\psi(s)|^2\|_{1,2}ds\\ &\leqq C+C\int_0^t \|\psi(s)\|_{13/10,2}^2 ds\\ &\leqq C+C\int_0^t \|A_1^{13/20}\psi(s)\|_2^2 ds\\ &\leqq C(T)\,, \qquad \text{for any } t\in[-T, T].\end{aligned}$$

In the same way as in the proof of (3.13) we see

$$\|A_1\psi(t)\|_2 \leqq C+C(T)\int_0^t \|A_1\psi(s)\|_2 ds\,, \qquad \text{for any } t\in[-T, T].$$

Hence we have

(3.14) $$\|A_1\psi(t)\|_2 \leqq C(T)\,, \qquad \text{for any } t\in[-T, T].$$

Therefore, by (3.1), (3.2), Sobolev's inequality, (2.6), (3.13) and (3.14) we easily get

(3.15) $\|\phi_t(t)\|_2, \|\phi_{tt}(t)\|_2, \|A_2^{1/2}\phi_t(t)\|_2 \leqq C(T)$, for any $t \in [-T, T]$.

Theorem follows from (3.13), (3.14) and (3.15). This completes the proof.

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