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Tightness of probability measures in $D([0, T]; C)$ and $D([0, T]; D)$

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Introduction.

The tightness criterion by Kolmogorov-Chentsov (Chentsov [2]) plays an important role in various limit theorems of stochastic processes with discontinuities of at most the first kind. The criterion is stated as follows: A sequence of stochastic processes X_t^n, $t\in[0, T]$, $n=1, 2, \cdots$, right continuous with the left hand limits is tight if there are positive constants K, γ and α not depending on n such that

$$E[|X_t^n-X_{t_1}^n|^{\gamma}|X_{t_2}^n-X_t^n|^{\gamma}]\leqq K|t_2-t_1|^{1+\alpha}, \qquad 0\leqq t_1<t<t_2\leqq T,$$

$$E[|X_t^n|^{\gamma}]\leqq K, \qquad 0\leqq t\leqq T.$$

In this paper, we shall give analogous tightness criteria for sequences of stochastic processes X_t^n, $n=1, 2, \cdots$, taking values in $C=C(R^d; R^d)$ and $D=D([0, T]; C)$ with discontinuities of at most the first kind. Here C is the space of continuous maps from R^d into itself and D is the set of all maps from $[0, T]$ into C, right continuous with the left hand limits. A main object of considering such processes is to obtain tightness criteria for sequences of stochastic flows, which will be discussed in the latter half of this paper.

In case that X_t^n are C-valued processes, $X_t^n(x)$, $x\in R^d$ can be regarded as a continuous random field with values in R^d if t is fixed. Then Kolmogorov-Prohorov's tightness criterion for continuous random field is applicable: There are positive constants K, γ and β not depending on n such that

$$E[|X_t^n(x)-X_t^n(y)|^{\gamma}]\leqq K|x-y|^{d+\beta},$$

$$E[|X_t^n(x)|^{\gamma}]\leqq K.$$

See Totoki [9]. However, since we consider $X_t^n(x)$ as random fields with two parameters t and x, the tightness criterion should be given in the mixed form

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of the above two criteria. Our criterion is in fact related to Chentsov [3].

In Section 1, we discuss the tightness of C-valued processes, right continuous with the left hand limits. Since their laws are given as probability measures in $D=D([0, T]; C)$, the criterion will be given to a sequence of probability measures in D. (Theorem 1.1). In Section 2, we discuss the tightness of D-valued processes, right continuous with the left hand limits. The criterion will be given to a sequence of probability measures in $D([0, T]; D)$.

In the latter half of the paper, we shall apply these criteria to obtain a tightness condition for Lévy processes with values in C. The criterion is stated in terms of the means, covariances and characteristic measures of the Lévy processes, called the characteristics of the processes. (Theorem 3.1). We then discuss the tightness of stochastic flows generated by C-valued Lévy processes. Let X_t be a C-valued Lévy process with some additional regularity conditions. We consider the stochastic differential equation of the jump type $d\xi_t=dX_t(\xi_{t-})$. The solution defines a stochastic flow or a G_+-valued Lévy process, where G_+ is the semi-group consisting of continuous maps. See Fujiwara-Kunita [5]. Tightness condition of stochastic flows is given in terms of the characteristics of their generators. (Theorems 4.1 and 4.3).

The convergence problem of stochastic flows will be discussed in a separate paper. [10].

1. Tightness of C-valued processes.

Let $C=C(R^d; R^d)$ be the totality of continuous maps from d-dimensional Euclidean space R^d into itself. It is a Fréchet space by the compact uniform metric ρ. Let $D=D([0, T]; C)$ be the totality of maps $X; [0, T]\to C(R^d; R^d)$ such that $X_t\equiv X(t)$ is right continuous with the left hand limits with respect to the metric ρ and $X_T=\lim_{t\uparrow T} X_t$, where T is a fixed positive integer. For X, Y of D, we define the *Skorohod metric* $\boldsymbol{s}$ by

$$\boldsymbol{s}(X, Y)=\inf_{\lambda\in H}\sup_{t\in[0,T]}\{\rho(X_t, Y_{\lambda(t)})+|\lambda(t)-t|\},$$

where H is the set of all homeomorphisms on $[0, T]$. Then D is a complete separable space with respect to a metric equivalent to $\boldsymbol{s}$. We denote by $\mathcal{B}_D$ the topological Borel field of D.

Suppose we are given a sequence of probability measures $\{P_n, n=1, 2, \cdots\}$ on $(D, \mathcal{B}_D)$. It is called tight if for any $\eta>0$, there is a compact subset A of D such that $P_n(A)>1-\eta$ holds for all n. The object of this section is to give a tightness criterion, which is a combination of the well known Kolmogorov-Prohorov's tightness criterion for continuous processes or Totoki's theorem for continuous random fields ([9]) and Kolmogorov-Chentsov's criterion for discontinuous processes ([2]).

For each t, $X_t(x)$ denotes the value of the map $X_t \in C$ at the point $x \in R^d$. The following notations will be used.

$$\Delta_{(t_1, t_2)} X(x) = X_{t_2}(x) - X_{t_1}(x), \qquad \Delta_{(x, y)} X_t = X_t(y) - X_t(x),$$
$$\Delta_{(t_1, t_2), (x, y)} X = \Delta_{(t_1, t_2)} X(y) - \Delta_{(t_1, t_2)} X(x).$$

THEOREM 1.1. *Let $\{P_n\}$ be a sequence of probability measures on $(D, \mathcal{B}_D)$. Suppose that for each hypercube I with the center 0 there are positive constants K, α, β, γ with $\gamma \geqq 1$ and a positive non-decreasing function $\varepsilon(t)$, $t>0$ with $\lim_{t\downarrow 0} \varepsilon(t)=0$ such that*

$$(1.1) \qquad E_n[|\Delta_{(t_1, t), (x, y)} X|^\gamma |\Delta_{(t, t_2), (x', y')} X|^\gamma]$$
$$\leqq K|t_2 - t_1|^{1+\alpha} |x-y|^{d+\beta} |x'-y'|^{d+\beta}, \qquad x, y, x', y' \in I, \quad t_1 < t < t_2,$$

$$(1.2) \qquad E_n[|\Delta_{(t_1, t), (x, y)} X|^\gamma |\Delta_{(t, t_2)} X(0)|^\gamma] \leqq K|t_2 - t_1|^{1+\alpha} |x-y|^{d+\beta},$$
$$x, y \in I, \quad t_1 < t < t_2 \quad \text{or} \quad x, y \in I, \quad t_2 < t < t_1,$$

$$(1.3) \qquad E_n[|\Delta_{(t_1, t)} X(0)|^\gamma |X_{(t, t_2)} X(0)|^\gamma] \leqq K|t_2 - t_1|^{1+\alpha}, \qquad t_1 < t < t_2,$$

$$(1.4) \qquad E_n[|X_t(x) - X_t(y)|^\gamma] \leqq K|x-y|^{d+\beta}, \qquad x, y \in I, \quad t=0, T,$$

$$(1.5) \qquad E_n[|X_t(0)|^\gamma] \leqq K, \qquad t=0, T,$$

$$(1.6) \qquad E_n[|\Delta_{(0, \delta), (x, y)} X|^\gamma + |\Delta_{(T-\delta, T), (x, y)} X|^\gamma] \leqq \varepsilon(\delta) |x-y|^{d+\beta},$$
$$E_n[|\Delta_{(0, \delta)} X(0)|^\gamma + |\Delta_{(T-\delta, T)} X(0)|^\gamma] \leqq \varepsilon(\delta)$$

hold for all n, where E_n denotes the expectation based on P_n. Then $\{P_n\}$ is tight.

Before we proceed to the proof of the theorem, we shall give a rather abstract and intermediate criterion of the tightness. The following notations will be used in the statements:

$$\|\phi\|_I = \sup_{x\in I} |\phi(x)|, \qquad \|\phi\|_I^0 = \sup_{x, y \in I} |\phi(x) - \phi(y)|,$$

where I is a hypercube. The inequality $\|\phi\|_I \leqq \|\phi\|_I^0 + |\phi(0)|$ holds if 0 is the center of I.

PROPOSITION 1.1. *Let $\{P_n\}$ be a sequence of probability measures on $(D, \mathcal{B}_D)$. Suppose that for each hypercube I with the center 0, there are positive constants K, α, β, γ and a positive non-decreasing function $\varepsilon(t)$, $t>0$ with $\lim_{t\downarrow 0} \varepsilon(t)=0$ such that*

$$(1.7) \qquad E_n[\{\|X_t - X_{t_1}\|_I \|X_{t_2} - X_t\|_I\}^\gamma] \leqq K|t_2 - t_1|^{1+\alpha}, \qquad t_1 < t < t_2,$$

$$(1.8) \qquad E_n[\sup_t |X_t(x) - X_t(y)|^\gamma] \leqq K|x-y|^{d+\beta}, \qquad x, y \in I,$$

(1.9) $$E_n[\sup_t |X_t(0)|^\gamma] \leqq K,$$

(1.10) $$E_n[\|\Delta_{(0,\delta)}X\|_I^\gamma + \|\Delta_{(T-\delta,T)}X\|_I^\gamma] \leqq \varepsilon(\delta)$$

hold for all n. Then $\{P_n\}$ is tight.

For the proof of the above proposition, it is convenient to characterize a relatively compact subset of D. For each hypercube I and positive number δ, we define the modulus of the continuity of $X \in D$ as

$$w_I''(X;\delta) = \sup \|X_{t_1} - X_t\|_I \wedge \|X_t - X_{t_2}\|_I$$

where the supremum extends over all t_1, t, t_2 such that $t_1 < t < t_2$ and $t_2 - t_1 < \delta$. Also, we put

$$w_I(X;[\tau,\sigma]) = \sup_{t_1, t_2 \in [\tau,\sigma]} \|X_{t_2} - X_{t_1}\|_I .$$

LEMMA 1.1. *A subset A of D is relatively compact if the following two conditions are satisfied.*
(a) *There is a compact subset K of C such that $X_t \in K$ holds for all $X \in A$,*
(b) *For any hypercube I,*

$$\lim_{\delta\to 0} \sup_{X\in A} w_I''(X;\delta) = 0,$$
$$\lim_{\delta\to 0} \sup_{X\in A} w_I(X;[0,\delta]) = 0,$$
$$\lim_{\delta\to 0} \sup_{X\in A} w_I(X;[T-\delta, T]) = 0.$$

The lemma can be proved by modifying the appropriate theorem in Billingsley [**1**]. It is omitted. Now, in the proof of Proposition 1.1, Garsia's inequality giving a priori estimate for modulus of continuity of functions plays an important role. We shall give its special form.

LEMMA 1.2. *Let $\phi(x)$ be a continuous map from the hypercube I of R^d into R^d. Then for any $\gamma > 0$ and $\kappa > 2d$*

(1.11) $$|\phi(x)-\phi(y)| < \frac{8\kappa}{d^{\kappa/2\gamma}(\kappa-2d)} |x-y|^{(\kappa-2d)/\gamma} \left\{ \int_I \int_I \frac{|\phi(\tilde{x})-\phi(\tilde{y})|^\gamma}{|\tilde{x}-\tilde{y}|^\kappa} d\tilde{x} d\tilde{y} \right\}^{1/\gamma}.$$

PROOF. Garsia's lemma [6] is stated as follows: Let p and Φ be continuous, strictly increasing functions on $[0, \infty)$ such that $p(0) = \Phi(0) = 0$ and $\lim_{t\uparrow\infty} \Phi(t) = \infty$. Suppose

$$\int_I \int_I \Phi\left(\frac{|\phi(\tilde{x})-\phi(\tilde{y})|}{p(e(I))} \right) d\tilde{x} d\tilde{y} \leqq B,$$

where $e(I)$ denotes the common length of the edge of the hypercube. Then

$$|\phi(x)-\phi(y)| \leqq 8 \int_0^{|x-y|} \Phi^{-1}\left(\frac{B}{u^{2d}} \right) dp(u).$$

Now set $\Phi(t)=t^{\gamma}$ and $p(t)=t^{\kappa/\gamma}$. Then

$$\int_I\int_I \Phi\Big(\frac{|\phi(\tilde{x})-\phi(\tilde{y})|}{p(e(I))}\Big)d\tilde{x}d\tilde{y} \leqq \int_I\int_I \frac{|\phi(\tilde{x})-\phi(\tilde{y})|^{\gamma}}{d^{\kappa/2}|\tilde{x}-\tilde{y}|^{\kappa}}d\tilde{x}d\tilde{y} \equiv B,$$

and

$$\int_0^{|x-y|} \Phi^{-1}\Big(\frac{B}{u^{2d}}\Big)p(du) \leqq \Big(\frac{\kappa}{\kappa-2d}\Big)|x-y|^{(\kappa-2d)/\gamma}B^{1/\gamma},$$

so that (1.11) follows.

PROOF OF PROPOSITION 1.1. What we have to show is that given a positive η there is a subset A of D satisfying (a), (b) and $P_n(A)>1-\eta$ for all n. First we shall construct a subset $B=B(\eta)$ satisfying (a) and $P_n(B)>1-\eta/2$ for all n.

Define the modulus of continuity of $\phi\in C(I\,;R^d)$ by

$$w_I(\phi\,;\delta)=\sup_{x,\,y\in I,\,|x-y|<\delta}|\phi(x)-\phi(y)|.$$

By Garsia's inequality (1.11),

$$\sup_t w_I(X_t\,;\delta)^{\gamma} \leqq C_1\delta^{\kappa-2d}\int_I\int_I \frac{\sup_t|X_t(x)-X_t(y)|^{\gamma}}{|x-y|^{\kappa}}dxdy.$$

By (1.8) the expected values of the above double integral with respect to P_n, $n=1, 2, \cdots$ are all dominated by the same finite number $K\int_I\int_I|x-y|^{d+\beta-\kappa}dxdy$ if $\kappa\in(2d, 2d+\beta)$. Therefore we have

$$E_n[\sup_t w_I(X_t\,;\delta)^{\gamma}] \leqq C_2\delta^{\kappa-2d}$$

for all n. Now let η and ζ be arbitrary positive numbers. Then Chebischev's inequality yields that there is $\delta=\delta(\zeta, \eta)$ such that

$$(1.12) \qquad P_n[\sup_t w_I(X_t\,;\delta)\geqq\zeta] \leqq \frac{\eta}{4}, \qquad n\geqq 1.$$

By the similar consideration, we obtain from (1.8)

$$E_n[(\sup_t\|X_t\|_I^0)^{\gamma}] \leqq C_3, \qquad n\geqq 1.$$

The above and (1.9) imply

$$E_n[\sup_t\|X_t\|_I^{\gamma}] \leqq C_4, \qquad n\geqq 1.$$

Consequently, there is a positive constant a such that

$$(1.13) \qquad P_n[\sup_t\|X_t\|_I\geqq a] \leqq \frac{\eta}{4}, \qquad n\geqq 1.$$

The above two inequalities (1.12) and (1.13) lead to $P_n(B_I(\zeta, \eta))\geqq 1-\eta/2$ for all n, where

$$B_I(\zeta, \eta)=\{X\in D\;;\;\sup_t w_I(X_t\,;\delta)<\zeta,\ \sup_t\|X_t\|_I<a\}.$$

Let I_l, $l=1, 2, \cdots$ be a sequence of hypercubes with the center 0 and the length l. Set

$$B(\eta)=\bigcap_{l=1}^{\infty}\bigcap_{m=1}^{\infty}B_{I_l}\left(\frac{1}{m}, \frac{\eta}{2^{m+l}}\right).$$

Then $P_n(B(\eta))>1-\eta/2$ for all n. Consider now a subset of C:

$$K_I(\zeta, \eta)=\{\phi\in C\ ;\ w_I(\phi; \delta)<\zeta, \|\phi\|_I<a\}$$

and set

$$K(\eta)=\bigcap_{l=1}^{\infty}\bigcap_{m=1}^{\infty}K_{I_l}\left(\frac{1}{m}, \frac{\eta}{2^{m+l}}\right).$$

By Ascoli-Arzelà's theorem, $K(\eta)$ is relatively compact in C. Since $X_t\in K(\eta)$ holds for all t if $X\in B(\eta)$, the condition (a) is fulfilled.

We shall next construct a subset $C=C(\eta)$ of D satisfying (b) and $P_n(B)>1-\eta/2$ for all n. For any $\zeta>0$, there is $\delta=\delta(\zeta, \eta)$ such that

$$P_n[w_I''(X; \delta)\geqq\zeta]\leqq\frac{\eta}{4}$$

holds for all n by (1.7). See Billingsley [1], Theorem 15.6 and its proof. Also by (1.10) there is $\delta>0$ such that

$$P_n[\|X_\delta-X_0\|_I\geqq\zeta \text{ or } \|X_T-X_{T-\delta}\|_I\geqq\zeta]\leqq\frac{\eta}{4}, \qquad n\geqq1.$$

Let δ be the minimum of the above two δ's. Then the set

$$C_I(\zeta, \eta)=\{X\in D\ ;\ w_I''(X; \delta)<\zeta, \|X_\delta-X_0\|_I<\zeta, \|X_T-X_{T-\delta}\|_I<\zeta\}$$

satisfies $P_n(C_I(\zeta, \eta))>1-\eta/2$. Note that $w_I''(X; \delta)<\zeta$ and $\|X_\delta-X_0\|_I<\zeta$ implies $\|X_s-X_0\|_I<2\zeta$ for any s in $[0, \delta]$ and hence $w_I(X; [0, \delta])<4\zeta$. Similarly it holds $w_I(X; [T-\delta, T])<4\zeta$ if $X\in C_I(\zeta, \eta)$. Consequently, the set

$$C(\eta)=\bigcap_{l=1}^{\infty}\bigcap_{m=1}^{\infty}C_{I_l}\left(\frac{1}{m}, \frac{\eta}{2^{m+l}}\right)$$

satisfies the condition (b). The inequality $P_n(C(\eta))\geqq1-\eta/2$ holds for all n.

Now the set $A(\eta)=B(\eta)\cap C(\eta)$ satisfies conditions (a) and (b) and $P_n(A(\eta))\geqq 1-\eta$ for all n. This proves the proposition.

We can now complete the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. It is enough to prove (1.7)-(1.10) of Proposition 1.1. We first consider (1.7). By Garsia's inequality,

$$\|\Delta_{(t_1, t)}X\|_I^\gamma\leqq C_1\int_I\int_I\frac{|\Delta_{(t_1, t), (x, y)}X|^\gamma}{|x-y|^\kappa}dxdy.$$

The same inequality is valid to $\|\Delta_{(t, t_2)}X\|_I^\gamma$. Therefore (1.1) yields

$$E_n[\{\|\Delta_{(t_1,t)}X\|_I^\circ\|\Delta_{(t,t_2)}X\|_I^\circ\}^r]$$
$$\leqq C_1^2\int_I\int_I\int_I\int_I \frac{E_n[|\Delta_{(t_1,t),(x,y)}X|^r|\Delta_{(t,t_2),(x',y')}X|^r]}{|x-y|^\kappa|x'-y'|^\kappa}dxdydx'dy'$$
$$\leqq C_1^2K|t_2-t_1|^{1+\alpha}\int_I\int_I\int_I\int_I |x-y|^{d+\beta-\kappa}|x'-y'|^{d+\beta-\kappa}dxdydx'dy'$$
$$\leqq C_3|t_2-t_1|^{1+\alpha}.$$

The above 4-ple integral is finite if we choose κ from $(2d, 2d+\beta)$. Apply the similar argument to (1.2). Then we have

$$E_n[\{\|\Delta_{(t_1,t)}X\|_I^\circ|\Delta_{(t,t_2)}X(0)|\}^r]\leqq C_4|t_2-t_1|^{1+\alpha}, \qquad t_1<t<t_2 \text{ or } t_2<t<t_1.$$

These two inequalities and (1.3) imply (1.7).

For the proof of (1.8), we will make use of a real variable lemma due to Kôno [7]. Let $0=t_0^{(m)}<t_1^{(m)}<\cdots<t_{2^m}^{(m)}=T$ be a partition of $[0, T]$ such that $t_k^{(m)}=k2^{-m}T$. For an R^d-valued function $f(t)$, $t\in[0, T]$ right continuous with the left hand limits, set

$$\delta_{m,k}(f)=\min\{|f(t_k')-f(t_k)|,\ |f(t_{k+1})-f(t_k')|\}$$

where $t_k=t_k^{(m)}$ and $t_k'=(t_k+t_{k+1})/2$, and

$$\bar{\Delta}_m(f)=\max_{0\leq k\leq 2^m-1}\delta_{m,k}(f).$$

Then for any positive integers p and l,

$$|f(p2^{-l}T)-f(0)|\wedge|f(p2^{-l}T)-f(T)|\leqq\sum_{m=1}^\infty\bar{\Delta}_m(f).$$

See Lemma 3 and its proof in [7]. This implies in particular

$$\sup_t|f(t)|\leqq\sum_{m=1}^\infty\bar{\Delta}_m(f)+|f(0)|+|f(T)|.$$

Now, setting $f(t)=\Delta_{(x,y)}X_t$ and taking L^r-norm, we have

(1.14) $$E_n[\sup_t|X_t(x)-X_t(y)|^r]^{1/r}$$
$$\leqq\sum_{m=1}^\infty E_n[\bar{\Delta}_m(\Delta_{(x,y)}X)^r]^{1/r}+E_n[|\Delta_{(x,y)}X_0|^r]^{1/r}+E_n[|\Delta_{(x,y)}X_T|^r]^{1/r}$$

Note that

$$\delta_{m,k}(\Delta_{(x,y)}X)\leqq|\Delta_{(x,y)}X_{t_k}-\Delta_{(x,y)}X_{t_k'}|^{1/2}|\Delta_{(x,y)}X_{t_{k+1}}-\Delta_{(x,y)}X_{t_k'}|^{1/2},$$

and

$$\bar{\Delta}_m(\Delta_{(x,y)}X)^{2r}\leqq\sum_{k=0}^{2^m-1}\delta_{m,k}(\Delta_{(x,y)}X)^{2r}.$$

Then we have by (1.1)

$$E_n[\bar{\Delta}_m(\Delta_{(x,y)}X)^{2\gamma}] \leqq \sum_{k=0}^{2m-1} E_n[\delta_{m,k}(\Delta_{(x,y)}X)^{2\gamma}]$$
$$\leqq 2^m K(2^{-m}T)^{1+\alpha}|x-y|^{2(d+\beta)}$$
$$\leqq KT^{(1+\alpha)}|x-y|^{2(d+\beta)}2^{-m\alpha}.$$

Therefore by Schwarz's inequality,

$$\sum_{m=1}^{\infty} E_n[\bar{\Delta}_m(\Delta_{(x,y)}X)^{\gamma}]^{1/\gamma} \leqq \frac{2^{-\alpha/(2\gamma)}}{1-2^{-\alpha/(2\gamma)}}(KT^{(1+\alpha)})^{1/(2\gamma)}|x-y|^{(d+\beta)/\gamma}.$$

Substitute this and (1.4) to (1.14). Then we obtain (1.8). Inequality (1.9) can be shown similarly using (1.3) and (1.5). The property (1.10) follows from (1.6) immediately, using Garsia's inequality. The proof is complete.

As an application of Theorem 1.1, we shall give a criterion of the existence of the right continuous C-valued process with the left hand limits.

THEOREM 1.2. *Let $X_t(x)$, $0 \leqq t \leqq T$, $x \in R^d$ be an R^d-valued random field. Suppose that for any hypercube I of R^d, there are positive constants K, α, β, γ with $\gamma \geqq 1$ satisfying these properties:*

(1.15) $$E[|\Delta_{(t_1,t),(x,y)}X|^{\gamma}|\Delta_{(t,t_2),(x',y')}X|^{\gamma}]$$
$$\leqq K|t_2-t_1|^{1+\alpha}|x-y|^{d+\beta}|x'-y'|^{d+\beta}, \qquad x, y, x', y' \in I, \quad t_1<t<t_2,$$

(1.16) $$E[|\Delta_{(t_1,t),(x,y)}X|^{\gamma}|\Delta_{(t,t_2)}X(0)|^{\gamma}] \leqq K|t_2-t_1|^{1+\alpha}|x-y|^{d+\beta},$$
$$x, y \in I, \quad t_1<t<t_2 \quad \text{or} \quad x, y \in I, \quad t_2<t<t_1,$$

(1.17) $$E[|\Delta_{(t_1,t)}X(0)|^{\gamma}|X_{(t,t_2)}X(0)|^{\gamma}] \leqq K|t_2-t_1|^{1+\alpha}, \qquad t_1<t<t_2,$$

(1.18) $$E[|X_t(x)-X_t(y)|^{\gamma}] \leqq K|x-y|^{d+\beta}, \qquad x, y \in I, \quad t=0, T,$$

(1.19) $$E[|X_t(0)|^{\gamma}] \leqq K, \qquad t=0, T,$$

(1.20) $$\lim_{\delta \to 0+} E[|\Delta_{(t,t+\delta)}X(x)|^{\gamma}]=0, \qquad x \in I, \quad 0 \leqq t \leqq T.$$

Then there is a right continuous C-valued process $\tilde{X}_t(x)$ such that $\tilde{X}_t(x)=X_t(x)$ holds a. s. P for any x and t.

PROOF. Let $\delta_n=\{(j/2^n)T\,;\, j=1, 2, \cdots, 2^n\}$ be a sequence of the partitions of $[0, T]$. By (1.15)-(1.18), there is a constant K' such that

$$E[\sup_{t\in\delta_n}|X_t(x)-X_t(y)|^{\gamma}] \leqq K'|x-y|^{d+\beta}, \qquad x, y \in I,$$
$$E[\sup_{t\in\delta_n}|X_t(0)|^{\gamma}] \leqq K'$$

hold for all n, The proof can be carried out similarly as in the proof of Theorem 1.1. Then by Totoki's theorem [**9**], X_t, $t \in \delta_n$ may be regarded as a C-valued process with the discrete parameter. For each n, we define a right continuous C-valued process X_t^n by setting $X_{(j/2^n)T}$ if $t\in[(j/2^n)T, ((j+1)/2^n)T)$, $j=0, 1, 2, \cdots$. Let P_n be the law of X_t^n defined on $(D, \mathscr{B}_D)$. We shall prove that the sequence $\{P_n\}$ is tight, assuming temporally that there is $\delta_0>0$ satisfying

$$(1.21)\qquad P(X_\delta(x)=X_0(x) \text{ and } X_{T-\delta}(x)=X_T(x))=1, \qquad x\in R^d, \quad 0\leqq\delta\leqq\delta_0.$$

The argument below is close to that of Theorem 1.1 and Proposition 1.1. Obviously, the law P_n satisfies (1.1)-(1.6) if we restrict t and δ to the points of δ_n. Then we can prove similarly as in the proof of Theorem 1.1 and Proposition 1.1, that for any $\eta>0$, $\zeta>0$ there are $\delta>0$, $a>0$ such that $P(B_I^n(\zeta, \eta))>1-\eta/4$ holds for all n, where $B_I^n(\zeta, \eta)$ is the set of all ω satisfying the following two conditions

$$\sup_t w_I(X_t^n\,;\delta)<\zeta, \qquad \sup_t \|X_t^n\|_I<a\,.$$

Also, there is $\delta\in[0, \delta_0]$ such that $P(C_I^n(\zeta, \eta))>1-\eta/4$ holds for all n, where $C_I^n(\zeta, \eta)$ is the set of all ω satisfying

$$w_I''(X^n\,;\delta)<\zeta, \qquad \|X_\delta^n-X_0^n\|_I<\zeta, \qquad \|X_T^n-X_{T-\delta}^n\|<\zeta.$$

These two properties conclude that $\{P_n\}$ is tight as before. Now let P_∞ be any limit point of $\{P_n\}$. Restricting the time parameter to the countable dense subset $\bigcup_n\delta_n$, the finite dimensional distribution of $\tilde{P}_\infty$ coincides with that of the process $X_t(x)$ with respect to P. Since $X_t(x)$ is right continuous in t in propability by (1.20), any finite dimensional distribution of $\tilde{P}_\infty$ coincides with that of $X_t(x)$. Therefore $\tilde{P}_\infty$ is the law of (X_t, P). This implies in particular that for almost all ω, $\lim_{t_k\downarrow t,\, t_k\in\cup_n\delta_n}X_{t_k}(\omega)$ exists for all t and the limit $\tilde{X}_t(\omega)$ is a right continuous C-valued process with the left hand limits. This $\tilde{X}_t$ is the desired modification of $X_t(x)$. Finally we can remove the restriction (1.21) by the change of time. See the proof of Theorem 15.7 in Billengsley [**1**].

2. Tightness of C-valued processes with double time parameters.

In this section, we shall discuss the tightness of C-valued processes with double time parameters $(s, t)\in[0, S]\times[0, T]$. Such a process appears, for example, as a solution of a stochastic differential equation or a stochastic flow, where the time s stands for the initial time of the solution. The tightness criterion given in this section will be applied to get a tightness criterion for stochastic flows in Section 4.

Let $\tilde{W}$ be the totality of maps; $[0, S]\to D=D([0, T]\,;C)$ such that X_s is right continuous with the left hand limits with respect to the Skorohod topology $\boldsymbol{s}$. We define the another Skorohod metric $\tilde{\boldsymbol{s}}$ on $\tilde{W}$ by

$$\tilde{\boldsymbol{s}}(X, Y)=\inf_{\lambda\in\tilde{\Pi}}\sup_s\{\boldsymbol{s}(X_s, Y_{\lambda(s)})+|\lambda(s)-s|\},$$

where $\tilde{H}$ is the set of all homeomorphisms of $[0, S]$. We denote by $X_{s,t}(x)$ the restriction of X_s to the point $(t, x)\in[0, T]\times R^d$. These notations will be used in the following.

$$\Delta_{(s_1,s_2)}X_t(x)=X_{s_2,t}(x)-X_{s_1,t}(x), \qquad \Delta_{(t_1,t_2)}X_s(x)=X_{s,t_2}(x)-X_{s,t_1}(x),$$
$$\Delta_{(s_1,s_2),(x,y)}X_t=\Delta_{(s_1,s_2)}X_t(y)-\Delta_{(s_1,s_2)}X_t(x),$$
$$\Delta_{(t_1,t_2),(x,y)}X_s=\Delta_{(t_1,t_2)}X_s(y)-\Delta_{(t_1,t_2)}X_s(x),$$
$$\Delta_{(s_1,s_2),(t_1,t_2),(x,y)}X=\Delta_{(t_1,t_2),(x,y)}X_{s_2}-\Delta_{(t_1,t_2),(x,y)}X_{s_1}.$$

THEOREM 2.1. *Let $\{\tilde{P}_n\}$ be a sequence of probability measures on $(\tilde{W}, \mathcal{B}_{\tilde{W}})$. Suppose that for any hypercube I with the center 0 there are positive constants K, α, β, γ with $\gamma\geqq 1$ satisfying the following* (2.1)-(2.6) *for any $s_1<s<s_2$, $t_1<t<t_2$, $t_3<t'<t_4$ and x_j, $y_j\in I$, $j=1, 2, 3, 4$.*

$$\text{(2.1)} \qquad \tilde{E}_n[A_{1i_1}A_{2i_2}A_{3i_3}A_{4i_4}]$$
$$\leqq K a_{1i_1}a_{2i_2}a_{3i_3}a_{4i_4}|s_2-s_1|^{1+\alpha}|t_2-t_1|^{1+\alpha}|t_4-t_3|^{1+\alpha} \quad \text{if } (t_1, t_2)\cap(t_3, t_4)=\emptyset,$$
$$\leqq K a_{1i_1}a_{2i_2}a_{3i_3}a_{4i_4}|s_2-s_1|^{1+\alpha}|t_2-t_1|^{1+\alpha} \quad \text{if } (t_1, t_2)\subset(t_3, t_4),$$

where

$$A_{11}=|\Delta_{(s_1,s),(t_1,t),(x_1,y_1)}X|^\gamma, \qquad A_{12}=|\Delta_{(s_1,s),(t_1,t)}X(0)|^\gamma,$$
$$A_{21}=|\Delta_{(s_1,s),(t,t_2),(x_2,y_2)}X|^\gamma, \qquad A_{22}=|\Delta_{(s_1,s),(t,t_2)}X(0)|^\gamma,$$
$$A_{31}=|\Delta_{(s,s_2),(t_3,t'),(x_3,y_3)}X|^\gamma, \qquad A_{32}=|\Delta_{(s,s_2),(t_3,t')}X(0)|^\gamma,$$
$$A_{41}=|\Delta_{(s,s_2),(t',t_4),(x_4,y_4)}X|^\gamma, \qquad A_{42}=|\Delta_{(s,s_2),(t',t_4)}X(0)|^\gamma,$$

and $a_{j1}=|x_j-y_j|^{d+\beta}$, $a_{j2}=1$, $j=1, 2, 3, 4$.

$$\text{(2.2)} \qquad \tilde{E}_n[B_{1i_1}^2A_{3i_3}A_{4i_4}]\leqq K b_{1i_1}^2 a_{3i_3}a_{4i_4}|s_2-s_1|^{1+\alpha}|t_4-t_3|^{1+\alpha},$$
$$\tilde{E}_n[B_{3i_3}^2A_{1i_1}A_{2i_2}]\leqq K b_{3i_3}^2 a_{1i_1}a_{2i_2}|s_2-s_1|^{1+\alpha}|t_2-t_1|^{1+\alpha},$$

where

$$B_{11}=|\Delta_{(s_1,s),(x_1,y_1)}X_t|^\gamma, \qquad B_{12}=|\Delta_{(s_1,s)}X_t(0)|^\gamma,$$
$$B_{31}=|\Delta_{(s,s_2),(x_3,y_3)}X_t|^\gamma, \qquad B_{32}=|\Delta_{(s,s_2)}X_t(0)|^\gamma,$$

and $b_{j1}=|x_j-y_j|^{d+\beta}$, $b_{j2}=1$, $j=1, 3$.

$$\text{(2.3)} \qquad \tilde{E}_n[B_{1i_1}B_{3i_3}]\leqq K b_{1i_1}b_{3i_3}|s_2-s_1|^{1+\alpha},$$
$$\text{(2.4)} \qquad \tilde{E}_n[C_{1i_1}C_{2i_2}]\leqq K c_{1i_1}c_{2i_2}|t_2-t_1|^{1+\alpha},$$

where

$$C_{11}=|\Delta_{(t_1,t),(x_1,y_1)}X_s|^\gamma, \qquad C_{12}=|\Delta_{(t_1,t)}X_s(0)|^\gamma,$$
$$C_{21}=|\Delta_{(t,t_2),(x_2,y_2)}X_s|^\gamma, \qquad C_{22}=|\Delta_{(t,t_2)}X_s(0)|^\gamma,$$

and $c_{j1}=|x_j-y_j|^{d+\beta}$, $c_{j2}=1$, $j=1, 2,$

$$\tilde{E}_n[|X_{s,t}(x)-X_{s,t}(y)|^{\gamma}]\leqq K|x-y|^{(d+\beta)}, \tag{2.5}$$

$$\tilde{E}_n[|X_{s,t}(0)|^{\gamma}]\leqq K. \tag{2.6}$$

Suppose further that there is a positive non-decreasing function $\varepsilon(t)$, $t>0$ *with* $\lim_{t\downarrow 0}\varepsilon(t)=0$ *satisfying the following* (2.7)-(2.9) *for any* $t_1<t<t_2$ *and* x_j, $y_j\in I$.

$$\tilde{E}_n[A_{1i_1}A_{2i_2}]\leqq\varepsilon(\delta)^2 a_{1i_1}a_{2i_2}|t_2-t_1|^{1+\alpha}, \quad \text{if } (s_1, s)=(0, \delta) \text{ or } (S-\delta, S), \tag{2.7}$$

$$\tilde{E}_n[B_{1i}]\leqq\varepsilon(\delta)b_{1i} \quad \text{if } (s_1, s)=(0, \delta) \text{ or } (S-\delta, S), \tag{2.8}$$

$$\tilde{E}_n[C_{1i}]\leqq\varepsilon(\delta)c_{1i} \quad \text{if } (t_1, t)=(0, \delta) \text{ or } (T-\delta, T). \tag{2.9}$$

Then $\{\tilde{P}_n\}$ *is tight.*

Before we proceed to the proof of the theorem, we shall give an intermediate criterion for the tightness.

PROPOSITION 2.1. *Let* $\{\tilde{P}_n\}$ *be a sequence of probability measures on* $(\tilde{W}, \mathcal{B}_{\tilde{W}})$ *such that for any hypercube* I *there are positive constants* K, α, β, γ *and a positive non-decreasing function* $\varepsilon(t)$, $t>0$ *with* $\lim_{t\downarrow 0}\varepsilon(t)=0$ *such that*

$$\tilde{E}_n[\{\sup_t\|\Delta_{(s_1,s)}X_t\|_I\cdot\sup_t\|\Delta_{(s,s_2)}X_t\|_I\}^{2\gamma}]\leqq K|s_2-s_1|^{1+\alpha}, \quad s_1<s<s_2, \tag{2.10}$$

$$\tilde{E}_n[\|\Delta_{(t_1,t)}X_s\|_I^{\gamma}\|\Delta_{(t,t_2)}X_s\|_I^{\gamma}]\leqq K|t_2-t_1|^{1+\alpha}, \quad t_1<t<t_2, \tag{2.11}$$

$$\tilde{E}_n[\sup_t|X_{s,t}(x)-X_{s,t}(y)|^{\gamma}]\leqq K|x-y|^{(d+\beta)}, \quad x, y\in I, \quad 0\leqq s\leqq S, \tag{2.12}$$

$$\tilde{E}_n[\sup_t|X_{s,t}(0)|^{\gamma}]\leqq K, \quad 0\leqq s\leqq S, \tag{2.13}$$

$$\tilde{E}_n[\sup_t\|\Delta_{(0,\delta)}X_t\|_I^{\gamma}+\sup_t\|\Delta_{(S-\delta,S)}X_t\|_I^{\gamma}]\leqq\varepsilon(\delta), \tag{2.14}$$

$$\tilde{E}_n[\|\Delta_{(0,\delta)}X_s\|_I^{\gamma}+\|\Delta_{(T-\delta,T)}X_s\|_I^{\gamma}]\leqq\varepsilon(\delta), \quad 0\leqq s\leqq T. \tag{2.15}$$

Then $\{\tilde{P}_n\}$ *is tight.*

For the proof of the proposition it is necessary to characterize a relatively compact subset of $\tilde{W}$. We shall introduce modulus of continuities

$$\tilde{w}_I''(X;\delta)=\sup\sup_t\|\Delta_{(s_1,s)}X_t\|_I\wedge\sup_t\|\Delta_{(s,s_2)}X_t\|_I, \tag{2.16}$$

where the supremum extends over all $s_1<s<s_2$ and $s_2-s_1<\delta$. Set.

$$\tilde{w}_I(X;[\tau,\sigma])=\sup_{\tau\leqq s_1<s\leqq\sigma}\sup_t\|\Delta_{(s_1,s)}X_t\|_I. \tag{2.17}$$

LEMMA 2.1. *A subset* $\tilde{A}$ *of* $\tilde{W}$ *is relatively compact if these two conditions are satisfied.*

(a′) *For each rational* s *of* $[0, S]$, *there is a compact subset* $K(s)$ *of* D *such that* $X_s\in K(s)$ *for all* $X\in\tilde{A}$.

(b′) *For any hypercube* I, *it holds*

$$\lim_{\delta\to 0}\sup_{X\in\tilde{A}} \tilde{w}''_I(X;\delta)=0, \tag{2.18}$$

$$\lim_{\delta\to 0}\{\sup_{X\in\tilde{A}} \tilde{w}_I(X;[0,\delta])+\sup_{X\in\tilde{A}}\tilde{w}_I(X;[S-\delta,S])\}=0. \tag{2.19}$$

The proof is omitted, since it is analogous to that of Lemma 1.1.

PROOF OF PROPOSITION 2.1. It is sufficient to prove two properties (a′) and (b′) of Lemma 2.1. We first consider (a′). The following argument is close to that of Proposition 1.1. Let ζ and η be any positive numbers. From (2.12), there is $\delta=\delta(\zeta,\eta)$ such that

$$P_n[\sup_t w_I(X_{s,t};\delta)\geqq\zeta]\leqq\frac{\eta}{8},\qquad \forall s\in[0,S]. \tag{2.20}$$

From (2.12) and (2.13), there is $a=a(\zeta,\eta)$ such that

$$\tilde{P}_n[\sup_t \|X_{s,t}\|_I\geqq a]\leqq\frac{\eta}{8},\qquad \forall s\in[0,S]. \tag{2.21}$$

From (2.11), there is $\delta=\delta(\zeta,\eta)$ such that

$$\tilde{P}_n[w''_I(X_s;\delta)\geqq\zeta]\leqq\frac{\eta}{8},\qquad \forall s\in[0,S]. \tag{2.22}$$

From (2.15), there is $\delta=\delta(\zeta,\eta)$ such that

$$\tilde{P}_n[\|\varDelta_{(0,\delta)}X_s\|_I\geqq\zeta \text{ or } \|\varDelta_{(T-\delta,T)}X_s\|_I\geqq\zeta]\leqq\frac{\eta}{8}. \tag{2.23}$$

Now let δ be the minimum of the above three δ's. Consider the subset of D:

$$\tilde{K}_I(\zeta,\eta)=\{\phi\in D\ ;\ \sup_t w_I(\phi_t;\delta)<\zeta,\ \sup_t\|\phi_t\|_I<a,\ w''_I(\phi;\delta)<\zeta,\ \|\varDelta_{(0,\delta)}\phi\|_I<\zeta,\ \|\varDelta_{(T-\delta,T)}\phi\|_I<\zeta\}. \tag{2.24}$$

Let I_l, $l=1,2,\cdots$ be a sequence of hypercubes with the center 0 and the length l. Set

$$\tilde{K}(\eta)=\bigcap_l\bigcap_m\tilde{K}_{I_l}\Big(\frac{1}{m},\frac{\eta}{2^{m+l}}\Big).$$

Then $\tilde{K}(\eta)$ satisfies conditions (a) and (b) of Lemma 1.1 and hence it is relatively compact in D as is shown in the proof of Proposition 1.1. It holds $\tilde{P}_n(X_s\in\tilde{K}(\eta))>1-\eta/2$ for all n by (2.20)-(2.23). Let $\{s_k,\ k=1,2,\cdots\}$ be the set of all rational numbers in $[0,S]$ and let η_k, $k=1,2,\cdots$ be positive numbers such that $\sum_k\eta_k=\eta$. Set $\tilde{K}(s_k)=\tilde{K}(\eta_k)$. Then it holds $\tilde{P}_n(X_{s_k}\in\tilde{K}(s_k)$ for all $k)\geqq 1-\eta/2$. Consequently, the set $\tilde{B}=\{X;X_{s_k}\in\tilde{K}(s_k)$ for all $k\}$ satisfies condition (a′) and $\tilde{P}_n(\tilde{B})\geqq 1-\eta/2$ for all n.

We shall next consider (b′). Let ζ and η be any positive numbers. From (2.10) there is a positive $\delta=\delta(\zeta,\eta)$ such that

$$\tilde{P}_n[\tilde{w}''_I(X;\delta)\geqq\zeta]\leqq\frac{\eta}{4}.$$

From (2.14), there is a positive $\delta=\delta(\zeta, \eta)$ such that

$$\tilde{P}_n[\sup_t \|\varDelta_{(0,\delta)}X_t\|_I \geqq \zeta \text{ or } \sup_t \|\varDelta_{(S-\delta,S)}X_t\|_I \geqq \zeta] \leqq \frac{\eta}{4}.$$

Let δ be the minimum of the above two δ's. Set

$$\tilde{C}_I(\zeta, \eta)=\{X\in\tilde{W} \ ; \ \tilde{w}_I''(X;\delta)<\zeta, \ \sup_t \|\varDelta_{(0,\delta)}X_t\|_I<\zeta \text{ and } \sup_t \|\varDelta_{(S-\delta,S)}X_t\|_I<\zeta\}$$

and

$$\tilde{C}(\eta)=\bigcap_{l=1}^{\infty}\bigcap_{m=1}^{\infty}\tilde{C}_{I_l}\Big(\frac{1}{m}, \frac{\eta}{2^{m+l}}\Big).$$

Then it holds $\tilde{P}_n(\tilde{C}(\eta))>1-\eta/2$. The set $\tilde{C}(\eta)$ clearly satisfies (b′). Consequently, the set $\tilde{A}(\eta)=\tilde{B}(\eta)\cap\tilde{C}(\eta)$ is relatively compact by Lemma 2.1 and satisfies $\tilde{P}_n(\tilde{A}(\eta))>1-\eta$, proving that $\{\tilde{P}_n\}$ is tight. The proof is complete.

PROOF OF THEOREM 2.1. It is enough to prove (2.10)-(2.15). We first consider (2.10). Making use of Garsia's inequality, (2.1) yields similarly as in the proof of Theorem 1.1,

$$\begin{aligned}(2.25)\quad &\tilde{E}_n[\{\|\varDelta_{(s_1,s),(t_1,t)}X\|_I\|\varDelta_{(s_1,s),(t,t_2)}X\|_I\|\varDelta_{(s,s_2),(t_3,t')}X\|_I\|\varDelta_{(s,s_2),(t',t_4)}X\|_I\}^\gamma]\\ &\leqq C_1|s_2-s_1|^{1+\alpha}|t_2-t_1|^{1+\alpha}|t_4-t_3|^{1+\alpha} \qquad \text{if } (t_1, t_2)\cap(t_3, t_4)=\emptyset,\\ &\leqq C_2|s_2-s_1|^{1+\alpha}|t_2-t_1|^{1+\alpha} \qquad \text{if } (t_1, t_2)\subset(t_3, t_4).\end{aligned}$$

Next, (2.3) implies similarly as the above

$$(2.26)\qquad \tilde{E}_n[\|\varDelta_{(s_1,s)}X_t\|_I^\gamma\|\varDelta_{(s,s_2)}X_t\|_I^\gamma]\leqq C_3|s_2-s_1|^{1+\alpha}.$$

Now let $0=t_0^{(m)}<\cdots<t_{2^m}^{(m)}=T$ be a partition such that $t_k^{(m)}=k2^{-m}T$. For a C-valued function f, set

$$\delta_{m,k}(f)=\min\{\|f(t_k)-f(t_k')\|_I, \ \|f(t_{k+1})-f(t_k')\|_I\},$$
$$\bar{\varDelta}_m(f)=\max_k \delta_{m,k}(f),$$

where $t_k=t_k^{(m)}$ and $t_k'=(t_k+t_{k+1})/2$. Then it holds

$$\sup_t\|f(t)\|_I\leqq\sum_{m=1}^{\infty}\bar{\varDelta}_m(f)+\|f(0)\|_I+\|f(T)\|_I, \qquad f\in D([0, T]; C).$$

Setting $f(t)=\varDelta_{(s_1,s)}X$ or $\varDelta_{(s,s_2)}X$, we have

$$\begin{aligned}(2.27)\quad &\sup_t\|\varDelta_{(s_1,s)}X_t\|_I\cdot\sup_t\|\varDelta_{(s,s_2)}X_t\|_I\\ &\leqq\sum_{m,m'}\bar{\varDelta}_m(\varDelta_{(s_1,s)}X)\bar{\varDelta}_{m'}(\varDelta_{(s,s_2)}X)\\ &\quad+(\sum_m\bar{\varDelta}_m(\varDelta_{(s,s_2)}X))(\|\varDelta_{(s_1,s)}X_0\|_I+\|\varDelta_{(s_1,s)}X_T\|_I)\\ &\quad+(\sum_{m'}\bar{\varDelta}_{m'}(\varDelta_{(s_1,s)}X))(\|\varDelta_{(s,s_2)}X_0\|_I+\|\varDelta_{(s,s_2)}X_T\|_I).\end{aligned}$$

We shall prove

$$\tilde{E}_n[\{\sum_{m,m'}\bar{\Delta}_m(\Delta_{(s_1,s)}X)\bar{\Delta}_{m'}(\Delta_{(s\ s_2)}X)\}^{2\gamma}]^{1/(2\gamma)}\leqq C_4|s_2-s_1|^{(1+\alpha)/(2\gamma)}. \tag{2.28}$$

The left hand side is dominated by

$$\sum_{m,m'}\{\sum_{k,k'}a^{(n)}_{m,k,m',k'}\}^{1/(2\gamma)},$$

where

$$a^{(n)}_{m,k,m',k'}=\tilde{E}_n[\delta_{m,k}(\Delta_{(s_1,s)}X)^{2\gamma}\delta_{m',k'}(\Delta_{(s,s_2)}X)^{2\gamma}].$$

By (2.25),

$$\begin{aligned}a^{(n)}_{m,k,m',k'}&\leqq C_1|s_2-s_1|^{1+\alpha}2^{-m(1+\alpha)}2^{-m'(1+\alpha)}, &&\text{if } (t^{(m)}_k, t^{(m)}_{k+1})\cap(t^{(m')}_{k'}, t^{(m')}_{k'+1})=\emptyset\\ &\leqq C_2|s_2-s_1|^{1+\alpha}2^{-m(1+\alpha)}, &&\text{if } m\geqq m' \text{ and } (t^{(m)}_k, t^{(m)}_{k+1})\subset(t^{(m')}_{k'}, t^{(m')}_{k'+1}).\end{aligned}$$

Therefore if $m\geqq m'$ we have

$$\sum_{k,k'}a^{(n)}_{m,k,m',k'}\leqq C_5|s_2-s_1|^{1+\alpha}(2^{-m\alpha}2^{-m'\alpha}+2^{-m\alpha}),$$

so that

$$\sum_{m=1}^{\infty}\sum_{m'=1}^{m}(\sum_{k,k'}a^{(n)}_{m,k,m',k'})^{1/(2\gamma)}\leqq C_6|s_2-s_1|^{(1+\alpha)/(2\gamma)}.$$

We have similarly

$$\sum_{m'=1}^{\infty}\sum_{m=1}^{m'}(\sum_{k,k'}a^{(n)}_{m,k,m',k'})^{1/(2\gamma)}\leqq C_7|s_2-s_1|^{(1+\alpha)/(2\gamma)}.$$

These two inequalities imply (2.28).

We shall next prove

$$\tilde{E}_n[(\sum_m\bar{\Delta}_m(\Delta_{(s,s_2)}X))^{2\gamma}\|\Delta_{(s_1,s)}X_t\|_I^{2\gamma}]^{1/(2\gamma)}\leqq C_8|s_2-s_1|^{(1+\alpha)/(2\gamma)},\qquad t=0 \text{ or } T, \tag{2.29}$$

$$\tilde{E}_n[(\sum_m\bar{\Delta}_m(\Delta_{(s_1,s)}X))^{2\gamma}\|\Delta_{(s,s_2)}X_t\|_I^{2\gamma}]^{1/(2\gamma)}\leqq C_9|s_2-s_1|^{(1+\alpha)/(2\gamma)},\qquad t=0 \text{ or } T. \tag{2.30}$$

The left hand side of (2.29) is dominated by

$$\sum_m\{\sum_k\tilde{E}_n[\delta_{m,k}(\Delta_{(s,s_2)}X)^{2\gamma}\|\Delta_{(s_1,s)}X_t\|_I^{2\gamma}]\}^{1/(2\gamma)}. \tag{2.31}$$

We have from (2.2),

$$\begin{aligned}&\tilde{E}_n[\|\Delta_{(s,s_2),(t_3,t')}X\|_I^{\gamma}\|\Delta_{(s,s_2),(t',t_4)}X\|_I^{\gamma}\|\Delta_{(s_1,s)}X_t\|_I^{2\gamma}]\\ &\leqq C_{10}|s_2-s_1|^{1+\alpha}|t_4-t_3|^{1+\alpha},\end{aligned}$$

making use of Garsia's inequality. This implies

$$\tilde{E}_n[\delta_{m,k}(\Delta_{(s,s_2)}X)^{2\gamma}\|\Delta_{(s_1,s)}X_t\|_I^{2\gamma}]\leqq C_{11}|s_2-s_1|^{1+\alpha}2^{-m(1+\alpha)}.$$

Then (2.31) is dominated by $C_{12}|s_2-s_1|^{(1+\alpha)/2\gamma}$ and (2.29) follows. Inequality (2.30) can be proved similarly. Then (2.28)-(2.30) prove (2.10) in view of (2.27).

Inequality (2.11) follows from (2.4) immediately using Garsia's inequality.

Inequality (2.12) follows from (2.4) and (2.5) as is shown in the proof of Theorem 1.1. Similarly (2.13) follows from (2.4) and (2.6). From (2.7) and (2.8), we have similarly

$$\tilde{E}_n[\sup_t \|\Delta_{(0,\delta)}X_t\|_I^\gamma + \sup_t \|\Delta_{(S-\delta,S)}X_t\|_I^\gamma] \leqq C_{13}\varepsilon(\delta).$$

This proves (2.14). From (2.9) we get (2.15) immediately. The proof is complete.

As an application of Theorem 2.1, we shall show that the random field $X_{s,t}(x)$, $(s, t)\in[0, S]\times[0, T]$, $x\in R^d$ satisfying the moment inequalities like (2.1)-(2.6) (replacing $\tilde{E}_n$ by E) has a modification $\tilde{X}_{s,t}$, which is a C-valued process right continuous with the left limits both in s and t. In order to state the property of the modification more precisely, we shall introduce a subspace $\overline{W}$ of $\tilde{W}$ equipped with a topology stronger than the Skorohod topology $\tilde{\mathfrak{s}}$ introduced before.

Let $D=D([0, T]; C)$. For X, $Y\in D$ we introduce the uniform topology $\bar{\rho}(X, Y)=\sup_t \rho(X_t, Y_t)$, where ρ is the compact uniform metric on $C=C(R^d; R^d)$. Let $\overline{W}$ be the totality of maps $X; [0, S]\to D$ which is right continuous with the left limits relative to $\bar{\rho}$. We define another Skorohod metric $\bar{\mathfrak{s}}$ on $\overline{W}$ by

$$\bar{\mathfrak{s}}(X, Y)=\inf_{\lambda\in\tilde{H}}\sup_s\{\bar{\rho}(X_{\lambda(s)}, Y_s)+|\lambda(s)-s|\},$$

where $\tilde{H}$ is the totality of homeomorphisms of $[0, S]$. Then $\overline{W}\subset\tilde{W}$ and $\bar{\mathfrak{s}}(X, Y)\geqq\tilde{\mathfrak{s}}(X, Y)$ if $X, Y\in\overline{W}$. Hence the $\bar{\mathfrak{s}}$-topology is stronger than $\tilde{\mathfrak{s}}$-topology. Now let $\pi_t; D\to C$ be the projection defined by $\pi_t(\phi)=\phi(t)$, $\phi\in D$. Then π_t is a continuous map from $(D, \bar{\rho})$ into (C, ρ). Hence if $X_s\in\overline{W}$, then $X_{s,t}\equiv\pi_t(X_s)$ is right continuous with the left limits not only in t but also in s with respect to ρ.

THEOREM 2.2. *Let $X_{s,t}(x)$, $(s, t)\in[0, S]\times[0, T]$, $x\in R^d$ be an R^d-valued random field such that for any hypercube I with the center* 0, *there are positive constants K, α, β, γ with $\gamma\geqq 1$ satisfying* (2.1)-(2.6) *for any $s_1<s<s_2$, $t_1<t<t_2$, $t_3<t'<t_4$ and x_i, $y_i\in I$, $i=1, \cdots, 4$, where $\tilde{E}_n$ is replaced by E. Suppose further that*

$$\lim_{\substack{\delta\to 0+\\ \delta'\to 0+}} E[|X_{s+\delta,t+\delta'}(x)-X_{s,t}(x)|^\gamma]=0, \qquad \forall s, t, x. \tag{2.32}$$

Then there is a random field $\tilde{X}_{s,t}(x)$ such that for almost all ω, $X_{\cdot,\cdot}(\cdot, \omega)$ is an element of $\overline{W}$ and $X_{s,t}(x)=\tilde{X}_{s,t}(x)$ holds a.s. P for any s, t, x.

PROOF. Let us temporally fix s and consider $X_{s,t}(x)$ as a random field with parameters $t\in[0, T]$ and $x\in R^d$. Then it satisfies conditions (1.15)-(1.20) of Theorem 1.2, because of (2.4), (2.5), (2.6) and (2.32). Consequently, in view of Theorem 1.2 there is a C-valued process with time parameter t denoting $\hat{X}_{s,t}$ which is right continuous with the left limit relative to t and satisfies $\hat{X}_{s,t}(x)=X_{s,t}(x)$ a.s. P for any s, t, x. Then similarly as in the proof of Proposition

2.1 we can prove that the process $\hat{X}_{s,t}$ satisfies (2.10)-(2.15) replacing $\tilde{E}_n$ by E.

Let now Q be the totality of positive numbers rS, where r's are dyadic rationals in $[0, 1]$. Let $f(s)$, $s\in Q$ be a $D=D([0, T]; C)$-valued function. Given a partition $0=s_0^{(m)}<\cdots<s_{2^m}^{(m)}=S$ where $s_k^{(m)}=k2^{-m}S$, we define

$$\tilde{\delta}^I_{m,k}(f)=\lim\{|||f(s_k)-f(s'_k)|||_I,\ |||f(s_{k+1})-f(s'_k)|||_I\},$$
$$\tilde{\Delta}^I_m(f)=\max_k \tilde{\delta}^I_{m,k}(f),$$

where I is a hypercube and $|||\phi|||_I=\sup_{x\in I, t\in[0,T]}|\phi_t(x)|$ and $s_k=s_k^{(n)}$, $s'_k=(s_{k+1}+s_k)/2$. Then $f(t)$, $t\in Q$ has the discontinuities of at most the first kind if $\sum_{m=1}^{\infty}\tilde{\Delta}^I_m(f)<\infty$ holds for any hypercube I. See Kôno [7], Lemma 3. Now setting $f(s)=\hat{X}_{s,\cdot}(\cdot)$, $s\in Q$, we have by (2.10)

$$E[\tilde{\Delta}^I_m(\hat{X})^{2\gamma}]\leqq\sum_k E[\tilde{\delta}^I_{m,k}(\hat{X})^{2\gamma}]\leqq K\cdot 2^m\cdot 2^{-m(1+\alpha)}\leqq K2^{-m\alpha}$$

and

$$E\left[\left|\sum_{m=1}^{\infty}\tilde{\Delta}^I_m(\hat{X})\right|^{2\gamma}\right]^{1/(2\gamma)}\leqq K^{1/(2\gamma)}\sum_{m=1}^{\infty}2^{-m\alpha/(2\gamma)}<\infty .$$

Consequently we have $\sum_{m=1}^{\infty}\tilde{\Delta}^I_m(\hat{X})<\infty$ a.s. for any hypercube I, proving that $\hat{X}_{s,\cdot}(\cdot)$ has the discontinuities of at most the first kind with respect to the topology $\bar{\rho}$. Define now $\tilde{X}_s=\lim_{s'\in Q, s'\downarrow s}\hat{X}_{s'}$ for any $s\in[0, S]$. Then $\tilde{X}_s$ is a D-valued process right continuous with the left limits with respect to the metric $\bar{\rho}$ of D. Hence $\tilde{X}_s(\omega)$ is an element of $\overline{W}$ for almost all ω. On the other hand, note that $\hat{X}_{s,t}(x)$ is continuous in s in probability for each t and x in view of (2.32). Then it turns out that $\tilde{X}_{s,t}(x)=X_{s,t}(x)$ holds a.s. P for each s, t and x. Hence this $\tilde{X}_{s,t}(x)$ is the desired modification of $X_{s,t}(x)$.

REMARK. Owing to the above theorem, probability measures $\tilde{P}_n$ of Theorem 2.1 are supported by the subset $\overline{W}$, i.e., $\tilde{P}_n(\overline{W})=1$ holds. However, the tightness of $\{\tilde{P}_n\}$ with respect to the $\bar{\mathfrak{s}}$-topology is false in general.

Now, the solution of a stochastic differential equation $X_{s,t}$ (denoted by $\xi_{s,t}$ in Section 4) has the time parameter $0\leqq s\leqq t\leqq T$, where s stands for the initial time and $X_{s,t}$ denotes the state of the solution at time t, $t\geqq s$. Defining $X_{s,t}=X_{s,s}$ for $t\leqq s$, we may consider that $X_{s,t}$ has the parameter in $[0, T]\times[0, T]$. Then the law of $X_{s,t}$ is a probability measure on $\widetilde{W}=D([0, T]; D)$, $D=D([0, T]; C)$ supported by the closed subset $\hat{W}=\{X\in\widetilde{W}; X_{s,t}=X_{s,t\vee s}$ for any $s, t\in[0, T]\}$ of $\widetilde{W}$. Let $\mathcal{B}_{\hat{W}}$ be the topological Borel field of $\hat{W}$. Then the law of $X_{s,t}$ is defined as a probability measure on $(\hat{W}, \mathcal{B}_{\hat{W}})$.

The next theorem gives a tightness criterion for a sequence of C-valued processes with parameters $0\leqq s\leqq t\leqq T$.

THEOREM 2.3. *Let $\tilde{P}_n$, $n=1, 2, \cdots$ be a sequence of probability measures on $(\hat{W}, \mathcal{B}_{\hat{W}})$ such that for any hypercube I with the center* 0 *there are positive con-*

stants K, α, β, γ *with* $\gamma\geqq 1$ *satisfying* (2.1)-(2.6) *for any* x_i, $y_i\in I$, $i=1, \cdots, 4$ *and* $0\leqq s_1<s<s_2\leqq t_1<t<t_2\leqq T$ *and* $0\leqq s_1<s<s_2\leqq t_3<t'\leqq t_4$. *Suppose further there is* $\varepsilon(t)>0$, $t>0$ *with* $\lim_{t\to 0}\varepsilon(t)=0$ *satisfying* (2.7)-(2.9) *for any* x_i, $y_i\in I$ *and* $t_1<t<t_2$. *Then* $\{\tilde{P}_n\}$ *is tight.*

PROOF. Set for any s, t, $t_1<t_2$, $s_1<s_2$

$$\hat{\Delta}_{(t_1,t_2)}X_s=\Delta_{(t_1,t_2)}X_{s\wedge t_1},\quad \hat{\Delta}_{(s_1,s_2)}X_t=\Delta_{(s_1,s_2)}X_{t\vee s_2},$$
$$\hat{\Delta}_{(s_1,s_2),(t_1,t_2)}X=\Delta_{(s_1,s_2),(t_1\vee s_2,t_2\vee s_2)}X.$$

Then similarly as in the proof of Proposition 2.1, we can show that the above $\hat{\Delta}_{(t_1,t_2)}X_s$ etc. satisfy (2.10)-(2.17) of Proposition 2.1. Then $\tilde{P}_n$, $n=1, 2, \cdots$ are tight. Indeed, in the proof of Proposition 2.1, replace $\Delta_{(t_1,t)}X$ etc. by $\hat{\Delta}_{(t_1,t)}X$ etc. Then we see that the whole argument is applicable to the present case without any essential change.

Finally we give a criterion of the existence of the C-valued process with two parameters $0\leqq s\leqq t\leqq T$.

THEOREM 2.4. *Let* $X_{s,t}(x)$, $0\leqq s\leqq t\leqq T$, $x\in R^d$ *be an* R^d*-valued random field such that for any hypercube* I *there are positive constants* K, α, β, γ *with* $\gamma\geqq 1$ *satisfying* (2.1)-(2.6) *for any* x_i, $y_i\in I$, $i=1, \cdots, 4$ *and* $0\leqq s_1<s<s_2\leqq t_1<t<t_2\leqq T$, *where* $\tilde{E}_n$ *is replaced by* E. *Suppose further* (2.32) *holds. Then there is a C-valued process* $\tilde{X}_{s,t}$ *right continuous with the left limits in both s and t such that* $X_{s,t}(x)=\tilde{X}_{s,t}(x)$ *holds a.s. P for any s, t and x.*

3. Tightness of C-valued Lévy processes.

Let $X_t=X_t(\omega)$, $t\in[0, T]$ be a C-valued process defined on a probability space $(\Omega, \mathcal{F}, P)$, right continuous with left limits a.s. It is called a *C-valued Lévy process* if it is continuous in probability and has the independent increments; $X_{t_{i+1}}-X_{t_i}$, $i=0, \cdots, n-1$ are independent for any $0\leqq t_1\leqq\cdots\leqq t_n\leqq T$. In particular, if X_t is continuous in t, it is called a C-valued Brownian motion. In the followings, we always assume that X_t is stationary, i.e., the law of X_t-X_s depends on $t-s$, $X_0=0$ and $E[|X_t(x)|^2]$ is finite for all t, x.

Given a C-valued Lévy process X_t, we define the Poisson random measure $N((0, t], A)$ associated with X_t by

$$N((0,t],A)=\#\{s;\Delta X_s\in A\},\qquad \Delta X_s=X_s-X_{s-}, \tag{3.1}$$

where A is a Borel set of C excluding 0. The intensity measure ν' is defined by $\nu'((0, t], A)=E[N((0, t], A)]$. Since X_t is stationary, ν' is the product measure $dt\otimes\nu(df)$. The measure ν is called the *characteristic measure* of X_t.

Let $x_1, \cdots, x_N$ be N points in R^d and consider the N-point motion $X_t(\underline{x})=$

$(X_t(x_1), \cdots, X_t(x_N))$. Its characteristic function $E[\exp i \sum_{k=1}^N (\alpha_k, X_t(x_k))]$ is represented by Lévy-Khinchin's formula:

$$\exp t\Big[i \sum_{k=1}^N (\alpha_k, b(x_k)) \quad \frac{1}{2} \sum_{k,l=1}^N \alpha_k a(x_k, x_l)\alpha_l$$
$$+\int_C \Big\{\exp\Big(i\sum_{k=1}^N (\alpha_k, f(x_k))\Big) - 1 - i\sum_{k=1}^N (\alpha_k, f(x_k))\Big\}\nu(df)\Big],$$

where

(3.2) $b(x)$ is an R^d-valued function,

(3.3) $a(x, y)$ is a $d\times d$-matrix valued function such that $a^{kl}(x, y)=a^{lk}(y, x)$ for any $k, l=1, \cdots, d$ and $x, y\in R^d$, and $\sum_{i,j=1}^N \alpha_i a(x_i, x_j)\alpha_j \geqq 0$ for any $x_i, \alpha_j \in R^d$, $i, j=1, \cdots, N$.

(3.4) ν is a σ-finite measure on C such that $\nu(\{0\})=0$ and $\int_C |f(x)|^2 \nu(df)<\infty$ for any $x\in R^d$.

Hence the law of a C-valued process is uniquely determined by the triple (a, b, ν). It is called the *characteristics* of the C-valued Lévy process.

The following proposition is shown in Fujiwara-Kunita [**5**].

PROPOSITION 3.1. *Let (a, b, ν) be a triple satisfying* (3.1), (3.2) *and* (3.3). *Suppose that there are positive constants L and ε satisfying the followings:*

$$|\mathrm{Trace}\, a(x, y)| \leqq L(1+|x|)(1+|y|), \tag{3.5}$$

$$|\mathrm{Trace}(a(x, x)-2a(x, y)+a(y, y))| \leqq L|x-y|^2, \tag{3.6}$$

$$|b(x)| \leqq L(1+|x|), \tag{3.7}$$

$$|b(x)-b(y)| \leqq L|x-y|, \tag{3.8}$$

$$\int_C |f(x)|^\gamma \nu(df) \leqq L(1+|x|)^\gamma, \tag{3.9}$$

$$\int_C |f(x)-f(y)|^\gamma \nu(df) \leqq L|x-y|^\gamma, \tag{3.10}$$

for all $x, y\in R^d$ and $\gamma\in[2, 2\vee d+\varepsilon)$. Then there is a C-valued Lévy process X_t with characteristics (a, b, ν). Furthermore, it satisfies

$$E[\sup_{0\leq s\leq t} |X_s(x)-X_s(y)|^\gamma] \leqq Kt|x-y|^\gamma, \tag{3.11}$$

$$E[\sup_{0\leq s\leq t} |X_s(x)|^\gamma] \leqq Kt(1+|x|)^\gamma, \tag{3.12}$$

for all $x, y\in R^d$, $t\in[0, T]$ and $\gamma\in[2, 2\vee d+\varepsilon)$, where K is a positive number depending only on L and ε.

Conversely let X_t be a C-valued Lévy process satisfying (3.11)-(3.12) *for all*

$x, y \in R^d$, $t \in [0, T]$ *and* $\gamma \in [2, 2\vee d+\varepsilon)$. *Then the associated characteristics satisfies* (3.5)-(3.10) *for all* $x, y \in R^d$ *and* $\gamma \in [2, 2\vee d+\varepsilon)$.

We shall give a tightness criterion for the sequence of C-valued Lévy process in terms of its characteristics.

THEOREM 3.1. *Let* X_t^n, $n=1, 2, \cdots$ *be a sequence of C-valued Lévy processes with characteristics* (a_n, b_n, ν_n), *respectively. Suppose that there are positive constants* L *and* ε *not depending on* n *such that* (3.5)-(3.10) *are satisfied for all* $x, y \in R^d$, $\gamma \in [2, 2\vee d+\varepsilon)$ *and* n. *Then* $\{X_t^n\}$ *is tight.*

PROOF. It is enough to prove conditions (1.1)-(1.6) of Theorem 1.1. We can calculate these moments easily making use of inequalities (3.11) and (3.12). Indeed, since the law of $\Delta_{(t_1, t),(x, y)}X^n$ coincides with that of $X_{t-t_1}^n(y)-X_{t-t_1}^n(x)$, we have from (3.11)

$$E[|\Delta_{(t_1, t),(x, y)}X^n|^\gamma] \leqq K(t-t_1)|x-y|^\gamma .$$

Note that $\Delta_{(t_1, t),(x, y)}X^n$ and $\Delta_{(t, t_2),(x', y')}X^n$ are independent if $t_1<t<t_2$. Then we get

$$E[|\Delta_{(t_1, t),(x, y)}X^n|^\gamma |\Delta_{(t, t_2),(x', y')}X^n|^\gamma] \\ \leqq K^2(t_2-t)(t-t_1)|x-y|^\gamma |x'-y'|^\gamma .$$

This proves (1.1), taking γ from $(2\vee d, 2\vee d+\varepsilon)$. Inequalities (1.2) and (1.3) can be shown similarly using (3.12). Inequalities (1.4) and (1.5) are obvious from (3.11) and (3.12). Inequality (1.6) is immediate from (3.11) and (3.12).

4. Tightness of G_+-valued Lévy processes.

In this section we shall discuss the tightness and the regularity of stochastic flows generated by C-valued Lévy processes. Let X_t be a C-valued Lévy process satisfying (3.11) and (3.12) for any $\gamma \in [2, 2\vee d+\varepsilon)$ where $\varepsilon>0$. Let $s<t$ and $\mathcal{F}_{s,t}$ be the least sub σ-field of $\mathcal{F}$ for which X_u-X_v, $s \leqq u \leqq v \leqq t$ are measurable. Suppose that f_t is a right continuous R^d-valued process with the left hand limits, adapted to $\mathcal{F}_{s,t}$. The stochastic integral of f_t by $dX_t(x)$ is defined by

$$\int_s^t dX_r(f_{r-}) = \lim_{|\Delta|\to 0} \sum_{i=0}^{n-1} X_{t_{i+1}}(f_{t_i}) - X_{t_i}(f_{t_i}), \tag{4.1}$$

where Δ are partitions $\Delta=\{s=t_0<t_1<\cdots<t_n=t\}$. As to the existence and some basic properties of the stochastic integral, we refer to Fujiwara-Kunita [5].

Let us consider the stochastic differential equation defined by X_t. Given $s \in [0, T]$ and $x \in R^d$, an R^d-valued $\mathcal{F}_{s,t}$-adapted process ξ_t, right continuous with the left hand limits is called the solution of equation

$$d\xi_t = dX_t(\xi_{t-})$$

if it satisfies

$$\xi_t = x + \int_s^t dX_r(\xi_{r-}). \tag{4.2}$$

It is known that the equation has a unique solution, which we denote by $\xi_{s,t}(x)$. The solution admits the following modification.

PROPOSITION 4.1 ([5]). *Suppose that X_t is a stationary C-valued Lévy process satisfying* (3.11) *and* (3.12) *for any $\gamma \in [2, 2\vee d+\varepsilon)$, where $\varepsilon > 0$. Then the solution $\xi_{s,t}$ admits the following properties:*

(i) *$\xi_{s,t}$ is stationary, i.e., the law of $\xi_{s,t}$ depends on $t-s$.*

(ii) *For each s, $\xi_{s,t}$, $t \in [s, T]$ is a right continuous C-valued process with the left limits a.s.*

(iii) *For each $s<t$, $\xi_{s,u} = \xi_{t,u} \circ \xi_{s,t}$ holds for all $t<u$ a.s. Furthermore, $\xi_{t_i, t_{i+1}}$, $i=0, \cdots, n-1$ are independent for any $0 \leqq t_1 < t_2 < \cdots < t_n \leqq T$.*

(iv) *There is a positive constant M depending only on L and ε in* (3.5)-(3.10) *such that*

$$|E[\xi_{s,t}(x) - x]| \leqq M(t-s)(1+|x|), \tag{4.3}$$

$$E[\sup_{s \leq r \leq t} |\xi_{s,r}(x) - x - (\xi_{s,r}(y) - y)|^\gamma] \leqq M(t-s)|x-y|^\gamma, \tag{4.4}$$

$$E[\sup_{s \leq r \leq t} |\xi_{s,r}(x) - x|^\gamma] \leqq M(t-s)(1+|x|)^\gamma \tag{4.5}$$

holds for all $x, y \in R^d$, $s, t \in [0, T]$ and $\gamma \in [2, 2\vee d+\varepsilon)$.

(v) *The following limits exist:*

$$A^{ij}(x, y) = \lim_{h \to 0+} \frac{1}{h} E[(\xi^i_{s,s+h}(x) - x^i)(\xi^j_{s,s+h}(x) - x^j)], \tag{4.6}$$

$$b^i(x) = \lim_{h \to 0+} \frac{1}{h} E[\xi^i_{s,s+h}(x) - x^i]. \tag{4.7}$$

Conversely, let $\xi_{s,t}(x)$, $0 \leqq s \leqq t \leqq T$, $x \in R^d$ be an R^d-valued random field, right continuous in t, satisfying (i)-(v). *Then there is a unique C-valued Lévy process X_t satisfying* (3.11) *and* (3.12) *for any $\gamma \in [2, 2\vee d+\varepsilon)$ such that $\xi_{s,t}$ is the solution of* (4.2) *for each s.*

Now the space C may be considered as a topological semigroup if we define the product of $f, g \in C$ by the composition $f \circ g$ of the maps. We denote the semigroup by G_+. Then the solution $\xi_{s,t}$ defines a Lévy process in the semigroup G_+ because of properties (i)-(iii) of the proposition. The associated C-valued Lévy process X_t is the infinitesimal generator of $\xi_{s,t}$.

REMARKS. 1. $A^{ij}(x, y)$ of (4.6) is represented by

$$A^{ij}(x, y) = a^{ij}(x, y) + \int_C f^i(x) f^j(y) \nu(df).$$

2. From (4.4), there is a positive constant M such that

$$E[\sup_{s\leq r\leq t}|\xi_{s,r}(x)-\xi_{s,r}(y)|^{\gamma}]\leq M|x-y|^{\gamma} \tag{4.8}$$

holds for all $x, y\in R^{d}$ and $t\in[s, T]$.

Consider now a sequence of G_{+}-valued Lévy processes $\xi_{s,t}^{n}$ generated by C-valued Lévy processes X_{t}^{n}. We shall give a tightness criterion of $\xi_{s,t}^{n}$ by means of the characteristics of X_{t}^{n}.

THEOREM 4.1. *Suppose that there are positive constants L and ε, $\varepsilon>2\vee d$, such that the sequence of characteristics associated with C-valued Lévy processes X_{t}^{n} satisfies* (3.5)-(3.10) *for all $\gamma\in[2, 2\vee d+\varepsilon)$. Then for each s, the sequence of G_{+}-valued Lévy processes $\{\xi_{s,t}^{n}\}$ generated by $\{X_{t}^{n}\}$ is tight.*

PROOF. It is enough to prove that $\{\xi_{s,t}^{n}\}$ satisfies (1.1)-(1.6) when s is fixed. For simplicity, we shall write $\xi_{s,t}^{n}$ as ξ_{t}^{n}. Let $s<t_{1}<t<t_{2}$ and $\gamma\in[2, (2\vee d+\varepsilon)/2)$. Then,

$$E[|\varDelta_{(t,t_{2}),(x,y)}\xi^{n}|^{\gamma}|\mathcal{F}_{s,t}]=E[|\xi_{t,t_{2}}^{n}(\tilde{y})-\xi_{t,t_{2}}^{n}(\tilde{x})-(\tilde{y}-\tilde{x})|^{\gamma}]$$

where $\tilde{y}=\xi_{t}^{n}(y)$ and $\tilde{x}=\xi_{t}^{n}(x)$. The above is dominated by $M(t_{2}-t)|\tilde{y}-\tilde{x}|^{\gamma}$ in view of (4.4). Therefore,

$$\begin{aligned}&E[|\varDelta_{(t,t_{2}),(x,y)}\xi^{n}|^{\gamma}|\varDelta_{(t_{1},t),(x',y')}\xi^{n}|^{\gamma}]\\&\leq M(t_{2}-t)E[|\xi_{t}^{n}(y)-\xi_{t}^{n}(x)|^{\gamma}|\varDelta_{(t_{1},t),(x',y')}\xi^{n}|^{\gamma}]\\&\leq M(t_{2}-t)E[|\xi_{t}^{n}(y)-\xi_{t}^{n}(x)|^{2\gamma}]^{1/2}E[|\varDelta_{(t_{1},t),(x',y')}\xi^{n}|^{2\gamma}]^{1/2}.\end{aligned}$$

Apply now (4.8) and (4.4) to each member of the right hand side. Then the above is dominated by

$$M^{2}(t_{2}-t)(t-t_{1})^{1/2}|y-x|^{\gamma}|y'-x'|^{\gamma}.$$

Therefore (1.1) is satisfied if $\gamma\in(2\vee d, (2\vee d+\varepsilon)/2]$. We can prove (1.2) and (1.3) similarly. The properties (1.4)-(1.6) are obvious. Hence $\{\xi_{t}^{n}\}$ is tight.

We have so far discussed the problems in case that the initial time s of G_{+}-valued Lévy process is fixed. In the following, we shall discuss the regularity and the tightness problems of G_{+}-valued Lévy process regarding it as a two parameter process, applying Theorem 2.2.

THEOREM 4.2. *Suppose that the characteristics of the C-valued Lévy process X_{t} satisfies* (3.5)-(3.10) *for $\gamma\in[2, 3d+\varepsilon)$ for some $\varepsilon>0$. Then the G_{+}-valued process $\xi_{s,t}(x, \omega)$ has a modification such that it is a G_{+}-valued process, right continuous with the left limits in both s and t.*

It is enough to prove that $\xi_{s,t}$ satisfies (2.1)-(2.6) and (2.32). We shall only prove the special case of (2.1), since other inequalities can be shown more or less in the same way. Set

$$A_1=|\Delta_{(s_1,s),(t_1,t),(x_1,y_1)}\xi|^\gamma, \qquad A_2=|\Delta_{(s_1,s),(t,t_2),(x_2,y_2)}\xi|^\gamma,$$

$$A_3=|\Delta_{(s,s_2),(t_3,t'),(x_3,y_3)}\xi|^\gamma, \qquad A_4=|\Delta_{(s,s_2),(t',t_4),(x_4,y_4)}\xi|^\gamma$$

where $\gamma>3d$.

LEMMA 4.1. *For any hypercube I, there are positive constants K, α, β such that for any $s_1<s<s_2$ and $t_1<t<t_2\leqq t_3<t'<t_4$ the following inequality holds:*

$$E[A_1A_2A_3A_4]\leqq K(s_2-s_1)^{1+\alpha}(t_2-t_1)^{1+\alpha}(t_4-t_3)^{1+\alpha}\Big(\prod_{i=1}^{4}|x_i-y_i|\Big)^{d+\beta}. \tag{4.9}$$

PROOF. We begin with proving

$$\begin{aligned}E[A_4|\mathcal{F}_{0,t'}]\leqq C_1(t_4-t')\{&(|\xi_{s_2,t'}(x_4)-\xi_{s_2,t'}(y_4)|^\gamma+|\xi_{s,t'}(x_4)-\xi_{s,t'}(y_4)|^\gamma)\\ &\wedge(|\xi_{s_2,t'}(y_4)-\xi_{s,t'}(y_4)|^\gamma+|\xi_{s_2,t'}(x_4)-\xi_{s,t'}(x_4)|^\gamma)\}.\end{aligned} \tag{4.10}$$

The random variable A_4 is written as

$$A_4=|(\xi_{t',t_4}(\hat{y}_4)-\xi_{t',t_4}(\hat{x}_4)-\hat{y}_4+\hat{x}_4)-(\xi_{t',t_4}(\hat{\hat{y}}_4)-\xi_{t',t_4}(\hat{\hat{x}}_4)-\hat{\hat{y}}_4+\hat{\hat{x}}_4)|^\gamma,$$

where $\hat{y}_4=\xi_{s_2,t'}(y_4)$, $\hat{x}_4=\xi_{s_2,t'}(x_4)$, $\hat{\hat{y}}_4=\xi_{s,t'}(y_4)$ and $\hat{\hat{x}}_4=\xi_{s,t'}(x_4)$. Since ξ_{t',t_4} is independent of $\mathcal{F}_{0,t'}$, we have by (4.4)

$$\begin{aligned}E[A_4|\mathcal{F}_{0,t'}]&=E[|(\xi_{t',t_4}(\hat{y}_4)-\xi_{t',t_4}(\hat{x}_4)-\hat{y}_4+\hat{x}_4)\\ &\qquad-(\xi_{t',t_4}(\hat{\hat{y}}_4)-\xi_{t',t_4}(\hat{\hat{x}}_4)-\hat{\hat{y}}_4+\hat{\hat{x}}_4)|^\gamma]\\ &\leqq M(t_4-t')(|\hat{x}_4-\hat{y}_4|^\gamma+|\hat{\hat{x}}_4-\hat{\hat{y}}_4|^\gamma).\end{aligned} \tag{4.11}$$

Also, (4.11) is dominated by

$$M(t_4-t')(|\hat{x}_4-\hat{\hat{x}}_4|^\gamma+|\hat{y}_4-\hat{\hat{y}}_4|^\gamma).$$

Therefore (4.10) is satisfied.

We next prove that for any $p>1$ there is a positive constant $C_2=C(p)$ such that

$$\begin{aligned}&E[A_3A_4|\mathcal{F}_{0,t_3}]\\ &\leqq C_2(t_4-t_3)^{1+(1/p)}\{|\xi_{s_2,t_3}(x_3)-\xi_{s_2,t_3}(y_3)|^\gamma+|\xi_{s,t_3}(x_3)-\xi_{s,t_3}(y_3)|^\gamma\}\\ &\quad\times\{|\xi_{s_2,t_3}(x_4)-\xi_{s_2,t_3}(y_4)|^\gamma+|\xi_{s,t_3}(x_4)-\xi_{s,t_3}(y_4)|^\gamma\}\\ &\leqq C_2(t_4-t_3)^{1+(1/p)}\{|\xi_{s_2,t_3}(x_3)-\xi_{s,t_3}(x_3)|^\gamma+|\xi_{s_2,t_3}(y_3)-\xi_{s,t_3}(y_3)|^\gamma\}\\ &\quad\times\{|\xi_{s_2,t_3}(x_4)-\xi_{s,t_3}(x_4)|^\gamma+|\xi_{s_2,t_3}(y_4)-\xi_{s,t_3}(y_4)|^\gamma\}.\end{aligned} \tag{4.12}$$

Apply (4.10) and then Hölder's inequality. Then we have

$$\begin{aligned}E[A_3A_4|\mathcal{F}_{0,t_3}]&\leqq C_1(t_4-t')E[A_3^p|\mathcal{F}_{0,t_3}]^{1/p}\\ &\quad\times E[\{|\xi_{s_2,t'}(x_4)-\xi_{s_2,t'}(y_4)|^\gamma+|\xi_{s,t'}(x_4)-\xi_{s,t'}(y_4)|^\gamma\}^q|\mathcal{F}_{0,t_3}]^{1/q},\end{aligned} \tag{4.13}$$

where q is the conjugate of p. It holds similarly as (4.10),

$$E[A_3^p|\mathcal{F}_{0,t_3}]^{1/p}\leqq C_3(t'-t_3)^{1/p}\{|\xi_{s_2,t_3}(x_3)-\xi_{s_2,t_3}(y_3)|^\gamma+|\xi_{s,t_3}(x_3)-\xi_{s,t_3}(y_3)|^\gamma\}.$$

The last member of (4.13) is dominated by

$$M^{1/p}\{|\xi_{s_2,t_3}(x_4)-\xi_{s_2,t_3}(y_4)|^\gamma+|\xi_{s,t_3}(x_4)-\xi_{s,t_3}(y_4)|^\gamma\}.$$

Therefore the first inequality of (4.12) is verified. The second inequality can be shown similarly.

We shall next prove that for any $p>1$ and $p'>1$, there is a positive constant $C_4=C_4(p, p')$ such that

$$E[A_1A_2A_3A_4]\leqq C_4(t_4-t_3)^{1+(1/p)}(t_2-t_1)^{(1+(1/p))(1/p')}\Big(\prod_{i=1}^4|x_i-y_i|\Big)^\gamma. \tag{4.14}$$

Indeed, note that

$$\begin{aligned}E[A_1A_2A_3A_4]&=E\{A_1A_2E[A_3A_4|\mathcal{F}_{0,t_3}]\}\\&\leqq E[(A_1A_2)^{p'}]^{1/p'}E\{E[A_3A_4|\mathcal{F}_{0,t_3}]^{q'}\}^{1/q'},\end{aligned}$$

where q' is the conjugate of p'. Replacing A_3 and A_4 of (4.12) by $A_1^{p'}$ and $A_2^{p'}$, respectively and $\mathcal{F}_{0,t_3}$ by $\mathcal{F}_{0,t_1}$ and taking the expectation of the first inequality of (4.12), we obtain

$$\begin{aligned}E[A_1^{p'}A_2^{p'}]\leqq C_5(t_2-t_1)^{1+(1/p)}E\{(&|\xi_{s_1,t_1}(x_1)-\xi_{s_1,t_1}(y_1)|^{p'\gamma}+|\xi_{s,t_1}(x_1)-\xi_{s,t_1}(y_1)|^{p'\gamma})\\&\times(|\xi_{s_1,t_1}(x_2)-\xi_{s_1,t_1}(y_2)|^{p'\gamma}+|\xi_{s,t_1}(x_2)-\xi_{s,t_1}(y_2)|^{p'\gamma})\}.\end{aligned}$$

By (4.8), the above is dominated by $C_6(t_2-t_1)^{1+(1/p)}(|x_1-y_1||x_2-y_2|)^{p'\gamma}$. Furthermore we have similarly as the above

$$E\{E[A_3A_4|\mathcal{F}_{0,t_3}]^{q'}\}^{1/q'}\leqq C_7(t_4-t_3)^{1+(1/p)}(|x_3-y_3||x_4-y_4|)^\gamma.$$

These two estimates prove (4.14).

We shall next prove that for any p, $p'>1$, there is a positive constant C_8 such that

$$E[A_1A_2A_3A_4]\leqq C_8(t_4-t_3)^{(1+(1/p))(1/p')}(t_2-t_1)^{1/q'}(s_2-s_1)^{1+(1/p')}, \tag{4.15}$$

where q' is the conjugate of p'. Since $E[A_3A_4|\mathcal{F}_{0,t_3}]$ is $\mathcal{F}_{s,T}$-measurable we have

$$E[A_1A_2A_3A_4]=E\{E[A_1A_2|\mathcal{F}_{s,T}]E[A_3A_4|\mathcal{F}_{0,t_3}]\}.$$

Set

$$Z=|\varDelta_{(t_1,t),(x_1,x_1')}\xi_s-\varDelta_{(t_1,t),(y_1,y_1')}\xi_s|^\gamma|\varDelta_{(t,t_2),(x_2,x_2')}\xi_s-\varDelta_{(t,t_2),(y_2,y_2')}\xi_s|^\gamma.$$

Then

$$E[A_1A_2|\mathcal{F}_{s,T}]=\int Z\mu(dx_1'dx_2'dy_1'dy_2'),$$

where $\mu(dx_1', \cdots, dy_2')=P(\xi_{s_1,s}(x_1)\in dx_1', \cdots, \xi_{s_1,s}(y_2)\in dy_2')$. Therefore we have

(4.16) $$E[A_1A_2A_3A_4]=\int E\{E[A_3A_4|\mathcal{F}_{0,t_3}]Z\}\mu(dx_1', \cdots, dy_2')$$
$$\leqq\int E\{E[A_3A_4|\mathcal{F}_{0,t_3}]^{p'}\}^{1/p'}E[Z^{q'}]^{1/q'}\mu(dx_1', \cdots, dy_2').$$

We have by (4.12)

$$E\{E[A_3A_4|\mathcal{F}_{0,t_3}]^{p'}\}^{1/p'}$$
$$\leqq C_9(t_4-t_3)^{(1+(1/p))(1/p')}\{\sum_{i=3,4}E[|\xi_{s_2,t_3}(x_i)-\xi_{s,t_3}(x_i)|^{2\gamma p'}]^{1/p'}$$
$$+\sum_{i=3,4}E[|\xi_{s_2,t_3}(y_i)-\xi_{s,t_3}(y_i)|^{2\gamma p'}]^{1/p'}\}.$$

Setting $\mu(d\tilde{x}_i)=P(\xi_{s,s_2}(x_i)\in d\tilde{x}_i)$, we have

$$E[|\xi_{s_2,t_3}(x_i)-\xi_{s,t_3}(x_i)|^{2\gamma p'}]$$
$$=\int E[|\xi_{s_2,t_3}(x_i)-\xi_{s_2,t_3}(\tilde{x}_i)|^{2\gamma p'}]\mu(d\tilde{x}_i)$$
$$\leqq M\int|x_i-\tilde{x}_i|^{2\gamma p'}\mu(d\tilde{x}_i)$$
$$\leqq ME[|\xi_{s,s_2}(x_i)-x_i|^{2\gamma p'}]$$
$$\leqq M^2(s_2-s).$$

Therefore we have

(4.17) $$E[E[A_3A_4|\mathcal{F}_{0,t_3}]^{p'}]^{1/p'}\leqq C_{10}(t_4-t_3)^{(1+(1/p))(1/p')}(s_2-s_1)^{1/p'}.$$

On the other hand, we have from (4.4)

$$E[|\Delta_{(t_1,t),(x_1,x_1')}\xi_s|^{2\gamma q'}]\leqq C_{11}(t-t_1)|x_1-x_1'|^{2\gamma q'}$$

etc. Therefore, using (4.8), we get

(4.18) $$\int E[Z^{q'}]^{1/q'}\mu(dx_1', \cdots, dy_2')$$
$$\leqq C_{12}(t-t_1)^{1/q'}\{E[|\xi_{s_1,s}(x_1)-x_1|^{2\gamma}]+E[|\xi_{s_1,s}(y_1)-y_1|^{2\gamma}]$$
$$+E[|\xi_{s_1,s}(x_2)-x_2|^{2\gamma}]+E[|\xi_{s_1,s}(y_2)-y_2|^{2\gamma}]\}$$
$$\leqq C_{13}(t-t_1)^{1/q'}(s-s_1).$$

The inequality (4.15) follows from (4.16), (4.17) and (4.18).

We now proceed to the proof of (4.9). Let $0<\varepsilon<1$. Then from (4.14) and (4.15) we get

(4.19) $$E[A_1A_2A_3A_4]\leqq C_{14}(t_4-t_3)^{(1+(1/p))(1/p')\varepsilon+(1+(1/p))(1-\varepsilon)}(t_2-t_1)^{(1/q')\varepsilon+(1+(1/p))(1/p'')(1-\varepsilon)}$$
$$\times(s_2-s_1)^{(1+(1/p'))\varepsilon}(|x_1-y_1||x_2-y_2||x_3-y_3||x_4-y_4|)^{(1-\varepsilon)\gamma}.$$

Choose p, p', p'' greater than 1 such that

$$\frac{1}{1+p'}<\frac{2}{3}<\frac{1}{(p'-1)(p+1)}\wedge\left(1-\frac{p''p}{p'(p+1)}\right)$$

and set $\varepsilon=2/3$. Then we see that powers of (t_4-t_3), (t_2-t_1) and (s_2-s_1) in (4.19) are greater than 1. Hence (4.9) is established with $\gamma>3d$.

REMARK. By virtue of Theorem 4.2, the G_+-valued Lévy process has the multiplicative property $\xi_{s,u}=\xi_{t,u}\circ\xi_{s,t}$ for all $s<t<u$ a.s.

Finally, we shall give a tightness criterion for the sequence of two parameter G_+-valued processes $\xi^n_{s,t}$.

THEOREM 4.3. *Suppose that there are positive constants L and ε such that the sequence of characteristics associated with C-valued Lévy processes X^n_t satisfies* (3.5)-(3.10) *for all $\gamma\in[2, 3d+\varepsilon)$. Then, the sequence of two parameter G_+-valued processes $\{\xi^n_{s,t}\}$ generated by $\{X^n_t\}$ is tight.*

The proof is immediate from Theorem 2.3 and the proof of Theorem 4.2.

As an example, we shall consider the tightness of solutions of SDE which is widely studied in the literatures. Let $(U, \mathcal{B}_U)$ be a measurable space and μ be a σ-finite measure on it. Let $\sigma(x)=(\sigma^{ik}(x))$; $R^d\to R^d\otimes R^d$, $b(x)$; $R^d\to R^d$ and $f(x, u)$; $R^d\times U\to R^d$ be measurable functions such that there are positive constants L and ε satisfying

(4.20) $$|\sigma(x)|+|b(x)|\leqq L(1+|x|),$$

(4.21) $$|\sigma(x)-\sigma(y)|+|b(x)-b(y)|\leqq L|x-y|,$$

(4.22) $$\int_U |f(x, u)|^\gamma \mu(du)\leqq L(1+|x|)^\gamma,$$

(4.23) $$\int_U |f(x, u)-f(y, u)|^\gamma \mu(du)\leqq L|x-y|^\gamma,$$

for any $\gamma\in[2, 3d+\varepsilon)$. Let $B_t=(B^1_t, \cdots, B^r_t)$ be a standard Brownian motion and $N_p(dt, du)$ be a stationary Poisson random measure on $[0, T]\times U$ with the characteristic measure μ. Set $\tilde{N}_p(dt, du)=N_p(dt, du)-dt\mu(du)$ and

(4.24) $$X_t(x)=\sum_{k=1}^r \sigma^{\cdot k}(x)B^k_t+b(x)t+\int_U f(x, u)\tilde{N}_p((0, t], du).$$

It is a C-valued Lévy process with characteristics (a, b, ν), where

(4.25) $$a^{ij}(x, y)=\sum_{k=1}^r \sigma^{ik}(x)\sigma^{jk}(y),$$

(4.26) $$\nu(A)=\mu\{u\,;\, f(\cdot, u)\in A\},\qquad A\in\mathcal{B}_C.$$

The characteristics (a, b, ν) satisfies (3.5)-(3.10). Therefore the solution of SDE

(4.27) $$d\xi_t=\sum_k \sigma^{\cdot k}(\xi_t)dB^k_t+b(\xi_t)dt+f(\xi_{t-}, u)\tilde{N}_p(dt, du)$$

defines a G_+-valued Lévy process.

Now let $(\sigma_n(x), b_n(x), f_n(x, u))$, $n=1, 2, \cdots$ be a sequence of functions such that there are positive constants L and ε satisfying (4.20)-(4.23) for all n. Let X_t^n, $n=1, 2, \cdots$ be a sequence of C-valued Lévy processes defined by (4.24) using $(\sigma_n(x), b_n(x), f_n(x, u))$, and let $\xi_{s,t}^{(n)}$ be the G_+-valued Lévy processes generated by X_t^n. Then $\{\xi_{s,t}^n\}$ is tight.

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