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Torus fibrations over the 2-sphere with the simplest singular fibers

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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§ 1. Introduction.

By a *torus fibration* $f: M \rightarrow B$ is roughly meant a certain singular fibration of an oriented smooth 4-manifold M over an oriented surface B with general fiber the 2-torus. (For a precise definition, see Definition 2.1.) Some special types of such fibrations have been studied by Thornton [11] and Zieschang [13] as a generalization of Seifert fibered spaces (into higher dimensions not necessarily 4 in their articles), and other special types by Harer [1] and Moishezon [9] as a smooth analog of Lefschetz' pencils or Kodaira's elliptic fiber spaces [4]. (General fibers of Harer's pencils need not be tori.) The author gave a general formulation of torus fibrations [6].

Among the possible types of singular fibers that torus fibrations can admit, the simplest one would be of type I_1^+ or I_1^-. A singular fiber of type I_1^+ (resp. I_1^-) consists of a smoothly immersed 2-sphere with a single transverse self-intersection of sign $+1$ (resp. -1).

In this paper we will deal with torus fibrations over the 2-sphere whose singular fibers are of type I_1^+ or I_1^-. Our goal will be to classify the (not necessarily fiber preserving) diffeomorphism types of the total spaces of such torus fibrations. The following is our main result.

THEOREM 1.1. *Let $f: M \rightarrow S^2$ be a torus fibration over the 2-sphere each of whose singular fibers is of type I_1^+ or I_1^-. Suppose that the signature of M is not zero. Then M is 1-connected, and the diffeomorphism type of M is determined by the euler number $e(M)$ and the signature $\sigma(M)$.*

REMARK. Assume that each singular fiber of a torus fibration $f: M \rightarrow S^2$ is of type I_1^+ or I_1^-, and that there are k_+ singular fibers of type I_1^+ and k_- singular fibers of type I_1^-. Then $e(M)$ and $\sigma(M)$ are given by $e(M)=k_+ + k_-$, $\sigma(M)$

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$=(-2/3)(k_+-k_-)$, [**1**], [**6**, II]. Thus by Theorem 1.1, the diffeomorphism type of M is determined by two numbers k_+, k_- provided $k_+\neq k_-$.

Our theorem is an extension of Kas' theorem [**3**] which is given an alternative, differential topological proof by Moishezon [**9**]. Let us state it in a modified way:

THEOREM (Kas). *Let V be an elliptic surface over $\boldsymbol{CP}_1$ with no multiple fibers, with at least one singular fiber and with no exceptional curve contained in a fiber. Then V is* 1*-connected, and the diffeomorphism type of V is determined by the euler number $e(V)$.*

It is known ([**3**], [**9**]) that the fibering structure of such an elliptic surface V can be deformed so that the resulting fibering has singular fibers only of type I_1^+. Thus the Kas-Moishezon theorem is considered as the diffeomorphism classification of the total spaces of torus fibrations over S^2 in which every singular fiber is of type I_1^+, while our theorem allows two types of singular fibers I_1^+ and I_1^-.

The euler number of an elliptic surface V as in Kas' theorem is known to be positive and divisible by 12 ([**9**]). Let V_k denote an elliptic surface with $e(V_k)=12k$. By Kas' theorem, the diffeomorphism type of V_k is well-defined. For example, it is known that $V_1\cong \boldsymbol{CP}_2\#9\overline{\boldsymbol{CP}_2}$ and $V_2\cong$Kummer manifold. ($\overline{V}$ denotes the manifold V with orientation reversed.) Also the signature $\sigma(V_k)$ is known to be equal to $-8k$ (cf. [**6**, II]).

With the above notation, our result is stated more precisely as follows:

THEOREM 1.1$'$. *Let $f: M\to S^2$ be as in Theorem* 1.1. *Then M is diffeomorphic to $V_k\# l(S^2\times S^2)$ or $\overline{V}_k\# l(S^2\times S^2)$ according as $\sigma(M)<0$ or $\sigma(M)>0$, where the integers k and l are related to $\sigma(M)$ and $e(M)$ by $|\sigma(M)|=8k$ and $|e(M)|=12k+2l$.*

Let $f_i: M_i\to B_i$, $i=1, 2$, be torus fibrations over closed surfaces. Following Moishezon [**9**, Definition 7, p. 174], we define the *direct sum* $f_1\oplus f_2: M_1\oplus M_2\to B_1\# B_2$ as follows: Let D_i be a 2-disk in B_i such that $f_i^{-1}(D_i)$ contains no singular fibers. Let $\tilde{\varphi}: \partial(M_1-\mathrm{Int}\, f_1^{-1}(D_1))\to\partial(M_2-\mathrm{Int}\, f_2^{-1}(D_2))$ be an orientation reversing and fiber preserving diffeomorphism which induces an orientation reversing diffeomorphism $\varphi: \partial(B_1-\mathrm{Int}\, D_1)\to\partial(B_2-\mathrm{Int}\, D_2)$. Glue $M_1-\mathrm{Int}\, f_1^{-1}(D_1)$ and $M_2-\mathrm{Int}\, f_2^{-1}(D_2)$ via $\tilde{\varphi}$ to obtain a manifold denoted by $M_1\oplus M_2$. We get a torus fibration $f_1\oplus f_2: M_1\oplus M_2\to B_1\# B_2$ by setting $f_1\oplus f_2|(M_i-\mathrm{Int}\, f_i^{-1}(D_i))=f_i|(M_i-\mathrm{Int}\, f_i^{-1}(D_i))$, for $i=1, 2$. The diffeomorphism type of $M_1\oplus M_2$ possibly depends on $\tilde{\varphi}$.

Now let $f_a: V_a\to S^2$ and $f_b: V_b\to S^2$ be elliptic surfaces as in Kas' theorem with $e(V_a)=12a$, $e(V_b)=12b$. By the Kas-Moishezon theorem, we see that the diffeomorphism type of $V_a\oplus V_b$ is independent of the pasting diffeomorphism $\tilde{\varphi}$ and is the same as that of V_{a+b}, because $e(V_a\oplus V_b)=12(a+b)$, [**9**].

If we reverse the orientation of V_b, we obtain a torus fibration $\bar{f}_b: \overline{V}_b\to S^2$

whose singular fibers are of type I_1^-.

COROLLARY TO THEOREM 1.1′. *Suppose that $a>b$. The diffeomorphism type of $V_a \oplus \bar{V}_b$ is independent of the pasting diffeomorphism, and is the same as that of $V_{a-b} \# 12b(S^2 \times S^2)$.*

PROOF. By Novikov additivity of the signature, $\sigma(V_a \oplus \bar{V}_b) = -8(a-b)$. Also we have $e(V_a \oplus \bar{V}_b) = 12(a+b) = 12(a-b)+24b$. Thus the corollary follows from Theorem 1.1′. $\square$

In this paper we always assume that the signature of the total spaces is not zero. However, what happens if it vanishes?

Let $f: M \to S^2$ be a torus fibration whose singular fibers are of type I_1^+ or I_1^-. Suppose that $\sigma(M)=0$. Then by Theorems 3.7 and 4.1 below (and by noting that a singular fiber of type I_1^+ or I_1^- contributes $-2/3$ or $2/3$ to $\sigma(M)$, [**1**], [**6**, II]), we can deform the fibering structure of $f: M \to S^2$ so that in the resulting fibration all the singular fibers are "twin". (For the definition of a twin singular fiber, see Definition 2.3.) Iwase [**2**] studies torus fibrations of this kind. He proves the following:

THEOREM (Iwase). *Suppose that $e(M) \neq 0$, then the diffeomorphism type of the total space of a torus fibration $M \to S^2$ whose singular fibers are twin (and are not multiple in the sense of* § 2*) is determined by the 4 data: the fundamental group $\pi_1(M)$, the euler number $e(M)$, the second Stiefel-Whitney class $w_2(M)$ and the type of the intersection form on $H_2(M; \boldsymbol{Z})$ (even or odd).*

For the proof, we refer the reader to [**2**].

Throughout the paper, all manifolds will be smooth and oriented. All diffeomorphisms will preserve orientations, unless otherwise stated.

Main results of this paper were announced in [**7**].

§ 2. Definitions.

Torus fibrations defined below will be *good* in the sense that their singular fibers have only normal crossings. For a more general definition, see [**6**].

A *proper* map $f: M \to B$ between manifolds is a map such that the preimage of each compact subset of B is compact and $f^{-1}(\partial B) = \partial M$.

DEFINITION 2.1. Let M and B be manifolds of dimension 4 and 2, respectively. Let $f: M \to B$ be a proper, surjective and smooth map. We call $f: M \to B$ a (*good*) *torus fibration* if it satisfies the following conditions:

(i) near each point $p \in \operatorname{Int} M$ (resp. $f(p) \in \operatorname{Int} B$), there exist local complex coordinates z_1, z_2 with $z_1(p)=z_2(p)=0$ (resp. local complex coordinate ξ with $\xi(f(p))=0$), so that f is locally written as $\xi = f(z_1, z_2) = z_1^m z_2^n$ or $(\bar{z}_1)^m z_2^n$, where m, n are non-negative integers with $m+n \geqq 1$, and $\bar{z}_1$ is the complex conjugate of z_1;

(ii) there exists a set Γ of isolated points of $\operatorname{Int} B$ so that $f|f^{-1}(B-\Gamma)$: $f^{-1}(B-\Gamma)\to B-\Gamma$ is a smooth T^2-bundle over $B-\Gamma$.

We call f, M and B, the *projection*, the *total space* and the *base space*, respectively. Given a (good) torus fibration $f: M\to B$, those points p of $\operatorname{Int} M$ at which $m+n\geqq 2$ make a nowhere dense subset Σ. We may assume that $f(\Sigma)=\Gamma$. We call Γ the *set of critical values*. The fiber $F_x=f^{-1}(x)$ is a *general* or *singular* fiber according as $x\in B-\Gamma$ or $x\in\Gamma$.

A singular fiber has a finite number of normal crossings. The complement $F_x-\{\text{normal crossings}\}$ is divided into a finite number of connected components. The closure of each component is called an *irreducible component* of F_x. Irreducible components are smoothly immersed surfaces, and F_x is the union of them:

$$F_x=\Theta_1\cup\cdots\cup\Theta_s\,.$$

Each irreducible component is naturally oriented. Thus it represents a homology class $[\Theta_i]$ in $H_2(f^{-1}(D_x);\boldsymbol{Z})$, where D_x ($\subset\operatorname{Int} B$) denotes a small 2-disk centered at x such that $D_x\cap\Gamma=\{x\}$. $H_2(f^{-1}(D_x);\boldsymbol{Z})$ is a free abelian group with basis $[\Theta_1],\cdots,[\Theta_s]$, with which the homology class $[F_y]$ of a nearby general fiber F_y ($y\in D_x-\{x\}$) is written as

$$[F_y]=m_1[\Theta_1]+\cdots+m_s[\Theta_s]\,,\qquad m_i\geqq 1\,.$$

The formal sum $\sum m_i\Theta_i$ is called the *divisor* of the singular fiber F_x. F_x is said to be *simple* or *multiple* according as $\gcd(m_1,\cdots,m_s)=1$ or >1.

Let F_0 be a general fiber over a base point $x_0\in B-\Gamma$. Let $l:[0,1]\to B-\Gamma$ be a loop based at x_0. As is easily shown, there exists a map $h: F_0\times[0,1]\to M-f^{-1}(\Gamma)$ such that

(i) $f(h(p,t))=l(t)$ for all $(p,t)\in F_0\times[0,1]$;

(ii) the map $h_t: F_0\to F_t$ defined by $h_t(p)=h(p,t)$ is a homeomorphism, where $F_t=f^{-1}(l(t))$;

(iii) $h_0=$identity of F_0.

The isotopy class of $h_1: F_0\to F_1=F_0$ is determined by x_0 together with the homotopy class $[l]$. h_1 induces an automorphism

$$(h_1)_*: H_1(F_0;\boldsymbol{Z})\longrightarrow H_1(F_0;\boldsymbol{Z})\,.$$

Fix an ordered basis (μ,λ) of $H_1(F_0;\boldsymbol{Z})$ so that it is compatible with the orientation of F_0. Then $(h_1)_*$ is represented by a matrix A called the *monodromy matrix*. This gives a map

$$\rho:\ \pi_1(B-\Gamma,x_0)\longrightarrow SL(2,\boldsymbol{Z})\,.$$

Recalling that the product $l\cdot l'$ of loops is the loop which goes first round l

and then l', we easily see that to make ρ a homomorphism we must adopt the following rule when assigning $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to $(h_1)_*$:

$$(h_1)_*(\mu)=a\mu+b\lambda, \qquad (h_1)_*(\lambda)=c\mu+d\lambda.$$

This rule is equivalent to considering that the monodromy acts on $H_1(F_0;\boldsymbol{Z})$ from the *right*. This convention coincides with the one in Moishezon [**9**] but is different from the one in [**4**] or [**7**]. For this reason, monodromy matrices here will be the transposed matrices of those in [**4**], [**7**].

A different basis (μ', λ') gives a different homomorphism $\rho': \pi_1(B-\Gamma, x_0) \to SL(2, \boldsymbol{Z})$. ρ' is related to ρ by $\rho'=C^{-1}\cdot\rho\cdot C$, C being a matrix in $SL(2, \boldsymbol{Z})$. The conjugacy class of the matrix $\rho([l])$ is called the *monodromy* associated with $[l]$.

Let x be a point of Γ, D_x a small disk in $\operatorname{Int} B$ such that $D_x \cap \Gamma=\{x\}$. Let x' be a point on ∂D_x. Then D_x is considered as a loop based at x'. (The direction of ∂D_x is determined by the orientation of D_x.) The monodromy associated with the loop ∂D_x is called the *local monodromy* of the singular fiber F_x.

For a classification of singular fibers and their local monodromies, see [**6**], [**7**].

To this paper only three types of singular fibers are relevant. They are I_1^+, I_1^- and Tw (twin). (These three types belong to the same class $\tilde{A}$ in the notation of [**7**].)

DEFINITION 2.2. A singular fiber is of *type* I_1^+ (resp. *type* I_1^-) if it is a simple singular fiber consisting of a smooth immersed 2-sphere (in the total space) which intersects itself transversely at one point, where the sign of the intersection is $+1$ (resp. -1). (Fig. 2.1).

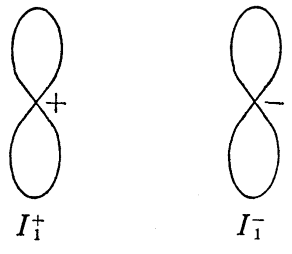

Figure 2.1.

The local monodromy of a singular fiber of type I_1^+ (resp. I_1^-) is represented by $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ (resp. $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$), [**9**], [**7**]. This is classically known as the Picard-Lefschetz formula.

DEFINITION 2.3. A singular fiber is of *type Tw* if it consists of two smoothly embedded 2-spheres R, S intersecting each other transversely at two points p_+, p_-. The sign of intersection at p_+ (resp. p_-) is $+1$ (resp. -1). The divisor is $mR+nS$. (Fig. 2.2).

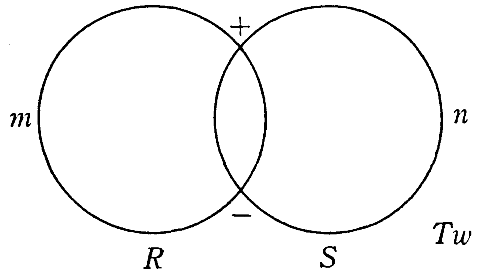

Figure 2.2.

In what follows all singular fibers of type Tw will have divisor $R+S$ (i.e., $m=n=1$) or $R+nS$.

Montesinos [**10**] first studied two 2-spheres in S^4 which intersect each other transversely at two points. Following him, we will call a singular fiber of type Tw a *twin singular fiber*.

If F_x is a twin singular fiber, the intersection numbers $R\cdot R$, $R\cdot S$, $S\cdot S$ are zero (cf. [**7**]). Therefore the neighborhood $f^{-1}(D_x)$ is obtained by plumbing $D^2\times S^2$ and $S^2\times D^2$ according to the graph $\bullet\!\overset{+}{\underset{-}{\bigcirc}}\!\bullet$. The boundary $\partial(f^{-1}(D_x))$ is diffeomorphic to $T^3=S^1\times S^1\times S^1$ ([**10**]), and the local monodromy is trivial $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

It is proved that the 4-sphere S^4 can be fibered as a torus fibration $S^4\to S^2$ with a single singular fiber of type Tw, [**6**].

§ 3. Elementary transformations.

In this section, we will extend the theorems of Livne and Moishezon [**9**] on elementary transformations of monodromies so that they may cover torus fibrations with $I_1^{\pm}$-singular fibers.

Let $f: M\to D^2$ be a torus fibration over the 2-disk each singular fiber of which is of type I_1^+ or I_1^-. We assume in this section that the monodromy around the boundary ∂D^2 is trivial.

Let $\Gamma=\{x_1, x_2, \cdots, x_\nu\}$ ($\subset \mathrm{Int}\, D^2$) be the set of critical values of f. Let D_i ($\subset \mathrm{Int}\, D^2$) be a small 2-disk centered at x_i such that $D_i\cap\Gamma=\{x_i\}$. We assume that $D_i\cap D_j=\emptyset$ for $i\neq j$. Take a base point $x_0\in D^2-\bigcup_{i=1}^{\nu}\mathrm{Int}(D_i)$ and points $x_1', x_2', \cdots, x_\nu'$ on $\partial D_1, \partial D_2, \cdots, \partial D_\nu$, respectively. Let $\gamma_1, \gamma_2, \cdots, \gamma_\nu: [0, 1]\to D^2-\bigcup_{i=1}^{\nu}\mathrm{Int}\, D_i$ be paths joining x_0 and $x_1', x_2', \cdots, x_\nu'$ as shown in Fig. 3.1.

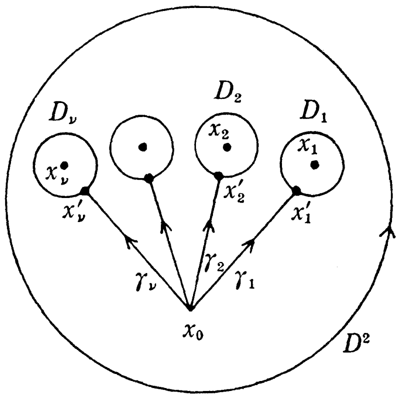

Figure 3.1.

Let l_i denote the loop $\gamma_i\cdot(\partial D_i)\cdot\gamma_i^{-1}$, $i=1, \cdots, \nu$, based at x_0. Throughout the argument we fix an ordered basis (μ, λ) of $H_1(F_0; \boldsymbol{Z})$, where $F_0=f^{-1}(x_0)$. The basis gives the monodromy homomorphism $\rho: \pi_1(D^2-\Gamma, x_0)\to SL(2, \boldsymbol{Z})$. The monodromy matrix $\rho([l_i])$ is denoted by B_i, $i=1, \cdots, \nu$.

Because of the triviality assumption on the monodromy around ∂D^2, we have

$$B_1B_2\cdots B_\nu=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \tag{3.1}$$

Since the singular fiber $F_i=f^{-1}(x_i)$ is of type I_1^+ or I_1^-, we see that

$$B_i \textit{ is conjugate to } \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \textit{ or } \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}, \quad \textit{for each } i=1, 2, \cdots, \nu. \tag{3.2}$$

Following Moishezon [9, pp. 177-178], we now study the effect of rechoosing the paths $\gamma_1, \gamma_2, \cdots, \gamma_\nu$ on the monodromy matrices $B_1, B_2, \cdots, B_\nu$. To examine this, fix an integer j, $1\leqq j\leqq \nu-1$. Let $\gamma'_1, \gamma'_2, \cdots, \gamma'_\nu$ be the new paths defined by $\gamma'_i=\gamma_i$ $(i\neq j, j+1)$, $\gamma'_j=\gamma_{j+1}$ and $\gamma'_{j+1}\simeq l_{j+1}^{-1}\cdot\gamma_j$, see Fig. 3.2.

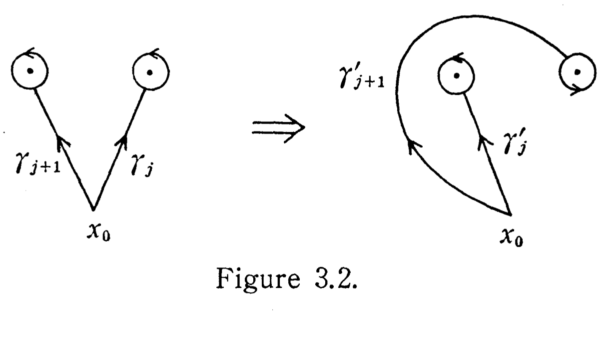

Figure 3.2.

Let $l'_1, l'_2, \cdots, l'_\nu$ denote the corresponding loops: $l'_i=\gamma'_i\cdot(\partial D_i)\cdot(\gamma'_i)^{-1}$, $i=1, 2, \cdots, \nu$. Then the new ν-tuple of the monodromy matrices $(B'_1, B'_2, \cdots, B'_\nu)$ corresponding to $l'_1, l'_2, \cdots, l'_\nu$ is given by

$$B'_i = B_i \qquad (i \neq j,\ j+1)\,,$$
$$B'_j = B_{j+1}\,,$$
$$B'_{j+1} = B_{j+1}^{-1} B_j B_{j+1}\,.$$

Clearly, the matrices $B'_1, B'_2, \cdots, B'_\nu$ satisfy the same conditions (3.1), (3.2) as $B_1, B_2, \cdots, B_\nu$ do.

Similarly, let $\gamma''_1, \gamma''_2, \cdots, \gamma''_\nu$ be the paths defined by $\gamma''_i = \gamma_i$ $(i \neq j,\ j+1)$, $\gamma''_j \simeq l_j \cdot \gamma_{j+1}$, $\gamma''_{j+1} = \gamma_j$, see Fig. 3.3.

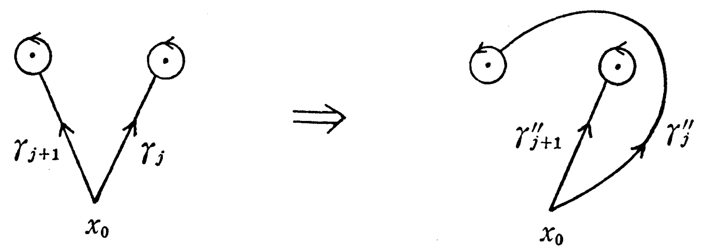

Figure 3.3.

Let $l''_1, l''_2, \cdots, l''_\nu$ denote the corresponding loops: $l''_i = \gamma''_i \cdot (\partial D_i) \cdot (\gamma''_i)^{-1}$, $i = 1, 2, \cdots, \nu$. Then the new ν-tuple of the monodromy matrices $(B''_1, B''_2, \cdots, B''_\nu)$ is given by

$$B''_i = B_i \qquad (i \neq j,\ j+1)\,,$$
$$B''_j = B_j B_{j+1} B_j^{-1},$$
$$B''_{j+1} = B_j\,.$$

Again $B''_1, B''_2, \cdots, B''_\nu$ satisfy the conditions (3.1), (3.2). These observations motivate the following definition:

DEFINITION 3.1 ([**9**], p. 223). Let G be a group. Let $\mathcal{S}_\nu$ be the set of ν-tuples $(g_1, g_2, \cdots, g_\nu)$ of elements of G such that $g_1 g_2 \cdots g_\nu = 1$. Let j be an integer with $1 \leqq j \leqq \nu - 1$.

The *j-th elementary transformation* $R_j : \mathcal{S}_\nu \to \mathcal{S}_\nu$ is a map defined by

$$R_j(g_1, \cdots, g_{j-1}, g_j, g_{j+1}, g_{j+2}, \cdots, g_\nu)$$
$$= (g_1, \cdots, g_{j-1}, g_{j+1}, g_{j+1}^{-1} g_j g_{j+1}, g_{j+2}, \cdots, g_\nu)\,.$$

The *j-th inverse transformation* $R_j^{-1} : \mathcal{S}_\nu \to \mathcal{S}_\nu$ is defined by

$$R_j^{-1}(g_1, \cdots, g_{j-1}, g_j, g_{j+1}, g_{j+2}, \cdots, g_\nu)$$
$$= (g_1, \cdots, g_{j-1}, g_j g_{j+1} g_j^{-1}, g_j, g_{j+2}, \cdots, g_\nu)\,.$$

Both R_j and R_j^{-1} are often called *elementary transformations*.

Using the assumption $g_1 g_2 \cdots g_\nu = 1$, one can easily see that the cyclic permutation $(g_1, g_2, \cdots, g_\nu) \to (g_2, \cdots, g_\nu, g_1)$ is a product of elementary transformations $(= R_{\nu-1} \cdots R_2 R_1)$.

The following is the main result of this section. Let X, Y always denote the matrices $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$, $\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$ respectively.

THEOREM 3.2. *Let $(B_1, B_2, \cdots, B_\nu)$ be a ν-tuple of matrices in $SL(2, \boldsymbol{Z})$ satisfying the conditions* (3.1), (3.2). *Then ν is even, and by successive application of elementary transformations, we can change $(B_1, B_2, \cdots, B_\nu)$ into a ν-tuple in one of the following normal forms,* (1) *or* (2):

(1) $(W_1, W_1^{-1}, \cdots, W_l, W_l^{-1}, X, Y, X, Y, \cdots, X, Y)$,

(2) $(W_1, W_1^{-1}, \cdots, W_l, W_l^{-1}, Y^{-1}, X^{-1}, Y^{-1}, X^{-1}, \cdots, Y^{-1}, X^{-1})$,

where $0 \leqq l \leqq \nu/2$ and $W_i \in SL(2, \boldsymbol{Z})$, for $i=1, \cdots, l$.

Note that $\nu-2l$ is divisible by 12, because $XY=\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}$ has order 6 in $SL(2, \boldsymbol{Z})$.

Theorem 3.2 generalizes Lemma 8 of [**9**, p. 179]. Our proof is globally the same as the one given in [**9**, pp. 180-188, pp. 223-230]. However, it differs in details. So we will give the full proof below.

Let A and B denote the matrices $\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 2\\ -1 & -1\end{bmatrix}$, respectively. We have $A^3=B^2=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$. A and B generate the group $SL(2, \boldsymbol{Z})$. Note that $X=ABA$, $Y=BA^2$ and Y is conjugate to $X: Y=A^{-1}XA$.

Now we pass to the modular group $PSL(2, \boldsymbol{Z})=SL(2, \boldsymbol{Z})\Big/\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\right\}$. Let $\pi: SL(2, \boldsymbol{Z}) \to PSL(2, \boldsymbol{Z})$ be the quotient map. The images $\pi(A)$, $\pi(B)$, $\pi(X)$ and $\pi(Y)$ will be denoted by the corresponding lowercase letters a, b, x, y respectively. Clearly, we have $x=aba$, $y=ba^2$.

$PSL(2, \boldsymbol{Z})$ has the presentation

$$PSL(2, \boldsymbol{Z})=\langle a, b \mid a^3=b^2=1\rangle.$$

In the proof below, we will always assume this presentation for $PSL(2, \boldsymbol{Z})$. Each element g of $PSL(2, \boldsymbol{Z})$ is expressed as a product $t_1t_2\cdots t_r$, where $t_i=a$, a^2 or b. Moreover, unless $g=1$, the expression $g=t_1t_2\cdots t_r$ is unique, provided that for each $i=1, \cdots, r-1$ the set of two adjacent elements $\{t_i, t_{i+1}\}$ coincides with the set $\{a, b\}$ or $\{a^2, b\}$. Such a product $t_1t_2\cdots t_r$ is said to be reduced, and r is called the *length* of the reduced product or of the element g which the product represents. The length of g is denoted by $l(g)$. For examples, $l(x)=l(aba)=3$, $l(y)=l(ba^2)=2$. We *define* $l(1)=0$.

It is easy to see that, if g is conjugate to x $(=aba)$ and $l(g)\leqq 3$, then $g=a^2b$, aba or ba^2. Also if g is conjugate to x^{-1} $(=a^2ba^2)$ and $l(g)\leqq 3$, then $g=ba$, a^2ba^2 or ab. We denote these six elements as follows (cf. [**9**, p. 180]):

$$s_0=a^2b\,,\quad s_1=aba\ (=x)\,,\quad s_2=ba^2\ (=y)$$
$$s_0^{-1}=ba\,,\quad s_1^{-1}=a^2ba^2,\qquad s_2^{-1}=ab\,.$$

Let g be conjugate to x or x^{-1}. Following [**9**], we say that g is *short* if it is one of the elements s_0, s_1, s_2, s_0^{-1}, s_1^{-1}, s_2^{-1}; otherwise it is *long*. If g is long, then g is expressed by a reduced product of the form $Q^{-1}a^\delta ba^\delta Q$, where $\delta=1$ or 2 and Q $(\neq 1)$ is a reduced product which begins with b. Thus $l(g)\geqq 5$.

LEMMA 3.3. *Let $g_1, g_2, \cdots, g_\nu$ be conjugate to x or x^{-1}. Assume that $g_1g_2\cdots g_\nu=1$. Then there exists at least one i $(1\leqq i\leqq \nu-1)$ for which $l(g_ig_{i+1})<\max(l(g_i), l(g_{i+1}))$.*

The proof of Lemma 3.3 is the same as that of Assertion on p. 225 of [**9**], so we omit it. The idea is to show that if we had $l(g_ig_{i+1})\geqq l(g_i)$, $l(g_{i+1})$ for each $i=1, \cdots, \nu-1$, then $g_1g_2\cdots g_\nu=1$ would not hold.

The next theorem is a slight extension of Livne's theorem ([**9**, Appendix II]).

THEOREM 3.4. *Let $g_1, g_2, \cdots, g_\nu$ be conjugate to x or x^{-1}, and assume that $g_1g_2\cdots g_\nu=1$. Then by successive application of elementary transformations, the ν-tuple $(g_1, g_2, \cdots, g_\nu)$ can be transformed into a ν-tuple $(h_1, h_2, \cdots, h_\nu)$ such that either every h_i is short or $h_ih_{i+1}=1$ for at least one i.*

PROOF. For completeness we will repeat the proof in [**9**] with necessary modifications. The proof proceeds by induction on the *total length* $l(g_1, \cdots, g_\nu)=\sum_{i=1}^\nu l(g_i)$. By Lemma 3.3, there exists an i such that $l(g_ig_{i+1})<\max(l(g_i), l(g_{i+1}))$. We will fix such an i for a while. There are three cases to be considered: Cases A.1, A.2, B.

Case A.1: g_i is long and $l(g_i)\geqq l(g_{i+1})$.

In this case we have $\max(l(g_i), l(g_{i+1}))=l(g_i)$, thus $l(g_ig_{i+1})<l(g_i)$. First of all, we prove the following

ASSERTION A.1.1. *$l(g_i)>l(g_{i+1})$ unless $g_ig_{i+1}=1$.*

It will suffice to show that if $l(g_i)=l(g_{i+1})$, then $g_ig_{i+1}=1$. Assume that $l(g_i)=l(g_{i+1})$. Since g_i is long, so is g_{i+1}. Express g_i and g_{i+1} as reduced products: $g_i=Q_i^{-1}a^\delta ba^\delta Q_i$, $g_{i+1}=Q_{i+1}^{-1}a^\varepsilon ba^\varepsilon Q_{i+1}$, δ, $\varepsilon=1$ or 2. Since $l(g_i)=l(g_{i+1})$, we have $l(Q_i)=l(Q_{i+1})$ and $g_ig_{i+1}=Q_i^{-1}a^\delta ba^\delta Q_iQ_{i+1}^{-1}a^\varepsilon ba^\varepsilon Q_{i+1}$. If the product $Q_iQ_{i+1}^{-1}$ does not cancel out, then we would have $l(g_ig_{i+1})>l(Q_i)+6+l(Q_{i+1})>l(g_i)$, a contradiction. Therefore, $Q_iQ_{i+1}^{-1}=1$ and $g_ig_{i+1}=Q_i^{-1}a^\delta ba^{\delta+\varepsilon}ba^\varepsilon Q_{i+1}$. If $\delta+\varepsilon\neq 3$, then we would have $l(g_ig_{i+1})=l(Q_i)+5+l(Q_{i+1})>l(g_i)$, a contradiction. Therefore, $\delta+\varepsilon=3$, and we have $g_ig_{i+1}=1$ as claimed.

ASSERTION A.1.2. *If $l(g_i)>l(g_{i+1})$, then $l(g_{i+1}^{-1}g_ig_{i+1})<l(g_i)$.*

To prove this assertion, we must consider two cases according as g_{i+1} is short or long. First, suppose that g_{i+1} is short, namely, $g_{i+1}\in\{s_0, s_1, s_2, s_0^{-1}, s_1^{-1}, s_2^{-1}\}$. By the assumption of Case A.1, g_i is long. Express g_i as a reduced product: $g_i=Q_i^{-1}a^\varepsilon ba^\varepsilon Q_i$, where $\varepsilon=1$ or 2 and $Q_i\neq 1$.

If $g_{i+1}=s_0=a^2b$, then Q_i must be of the form $b\cdots ba$, because $l(g_ig_{i+1})<l(g_i)$. Then, $g_{i+1}^{-1}g_ig_{i+1}=ba(Q_i^{-1}a^\varepsilon ba^\varepsilon Q_i)a^2b=ba(a^2b\cdots ba^\varepsilon ba^\varepsilon b\cdots ba)a^2b$, and we have $l(g_{i+1}^{-1}g_ig_{i+1})\leqq l(g_i)-4<l(g_i)$ as asserted.

If $g_{i+1}=s_1=aba$, then Q_i must be of the form $b\cdots ba^2$, because $l(g_ig_{i+1})<l(g_i)$. Then, $g_{i+1}^{-1}g_ig_{i+1}=a^2ba^2(ab\cdots ba^\varepsilon ba^\varepsilon b\cdots ba^2)aba$, and we have $l(g_{i+1}^{-1}g_ig_{i+1})\leqq l(g_i)-2<l(g_i)$ as asserted. The other cases when $g_{i+1}=s_2$, s_0^{-1}, s_1^{-1}, s_2^{-1} are treated similarly.

Secondly suppose that g_{i+1} is long. Express g_{i+1} as a reduced product: $g_{i+1}=Q_{i+1}^{-1}a^\delta ba^\delta Q_{i+1}$, where $\delta=1$ or 2 and $Q_{i+1}\neq 1$. By the assumption of Assertion A.1.2, we have $l(g_i)>l(g_{i+1})$, thus $l(Q_i)>l(Q_{i+1})$. If Q_{i+1}^{-1} were not canceled out by part of Q_i in the product $Q_iQ_{i+1}^{-1}$, we would have $l(g_ig_{i+1})>l(Q_i)+3+(l(Q_i)-l(Q_{i+1}))+3+l(Q_{i+1})>l(g_i)$, because $g_ig_{i+1}=Q_i^{-1}a^\varepsilon ba^\varepsilon Q_iQ_{i+1}^{-1}a^\delta ba^\delta Q_{i+1}$. This is a contradiction. Thus we have $Q_i=Q_i'Q_{i+1}$, where $Q_i'\neq 1$, and $g_ig_{i+1}=Q_i^{-1}a^\varepsilon ba^\varepsilon Q_i'a^\delta ba^\delta Q_{i+1}$. If this were a reduced product, we would have $l(g_ig_{i+1})=l(Q_i)+3+(l(Q_i)-l(Q_{i+1}))+3+l(Q_{i+1})>l(g_i)$, a contradiction. Thus $g_ig_{i+1}=Q_i^{-1}a^\varepsilon ba^\varepsilon Q_i'a^\delta ba^\delta Q_{i+1}$ is not a reduced product. This implies that Q_i' is of the form $Q_i'=b\cdots ba^{3-\delta}$. Then $g_{i+1}^{-1}g_ig_{i+1}=Q_{i+1}^{-1}a^{3-\delta}ba^{3-\delta}(Q_i')^{-1}a^\varepsilon ba^\varepsilon Q_i'a^\delta ba^\delta Q_{i+1}$ and we have $l(g_{i+1}^{-1}g_ig_{i+1})\leqq l(Q_{i+1})+3+(l(Q_i)-l(Q_{i+1}))+3+(l(Q_i)-l(Q_{i+1}))+3+l(Q_{i+1})-8=2l(Q_i)+1<l(g_i)$ as asserted. This proves Assertion A.1.2.

By Assertions A.1.1, A.1.2, we can conclude in Case A.1 that the i-th elementary transformation R_i reduces the total length of the ν-tuple $(g_1, g_2, \cdots, g_\nu)$, unless $g_ig_{i+1}=1$.

Case A.2: g_{i+1} is long and $l(g_i)\leqq l(g_{i+1})$.

In this case, one can prove the following:

ASSERTION A.2.1. *$l(g_i)<l(g_{i+1})$ unless $g_ig_{i+1}=1$.*

ASSERTION A.2.2. *If $l(g_i)<l(g_{i+1})$, then $l(g_ig_{i+1}g_i^{-1})<l(g_{i+1})$.*

The proofs of these assertions are similar to those of previous assertions. Combining Assertions A.2.1, A.2.2, we can conclude in Case A.2 that the inverse of the i-th elementary transformation, R_i^{-1}, reduces the total length of $(g_1, g_2, \cdots, g_\nu)$, unless $g_ig_{i+1}=1$.

Case B: Both g_i and g_{i+1} are short.

Since $l(g_ig_{+1})<\max(l(g_i), l(g_{i+1}))\leqq 3$, the ordered pair (g_i, g_{i+1}) must be one of the 12 pairs: (s_0, s_2), (s_0, s_0^{-1}), (s_1, s_0), (s_1, s_1^{-1}), (s_2, s_1), (s_2, s_2^{-1}), (s_0^{-1}, s_0), (s_0^{-1}, s_1^{-1}), (s_1^{-1}, s_1), (s_1^{-1}, s_2^{-1}), (s_2^{-1}, s_0^{-1}), (s_2^{-1}, s_2). In case $g_ig_{i+1}=1$, we are done. Thus we assume $g_ig_{i+1}\neq 1$. Then (g_i, g_{i+1}) is one of the 6 pairs: (s_0, s_2), (s_1, s_0), (s_2, s_1), (s_0^{-1}, s_1^{-1}), (s_1^{-1}, s_2^{-1}), (s_2^{-1}, s_0^{-1}). The first three are mutually transformed by elementary transformations. In fact, $(s_0, s_2)\mapsto(s_2, s_2^{-1}s_0s_2)=(s_2, s_1)\mapsto(s_1, s_1^{-1}s_2s_1)=(s_1, s_0)$. Similarly the second three are mutually transformed: $(s_0^{-1}, s_1^{-1})\mapsto(s_1^{-1}, s_1s_0^{-1}s_1^{-1})=(s_1^{-1}, s_2^{-1})\mapsto(s_2^{-1}, s_2s_1^{-1}s_2^{-1})=(s_2^{-1}, s_0^{-1})$.

If every g_j in $(g_1, \cdots, g_\nu)$ is short, then we are done. Therefore, we may assume that there exist j such that g_j is long. Moreover, after applying cyclic permutations, if necessary, we may assume that $j>i+1$. (Recall that any cyclic permutation is a product of elementary transformations.) Let j be the smallest in the set of indices $\{j \mid j>i+1$ and g_j is long$\}$. Let us denote the elements $g_i, g_{i+1}, \cdots, g_{j-1}$ in new notation $y_1, y_2, \cdots, y_u$. Of course, u is equal to $j-i$.

ASSERTION B.1. *For each v such that $2 \leqq v \leqq u$, one of the following three statements holds:*

(i) *by applying elementary transformations on the v-tuple $(y_1, \cdots, y_v)$, we can change y_v into any element of $\{s_0, s_1, s_2\}$ we want;*

(ii) *by applying elementary transformations on the v-tuple $(y_1, \cdots, y_v)$, we can change y_v into any element of $\{s_0^{-1}, s_1^{-1}, s_2^{-1}\}$ we want;*

(iii) *by applying elementary transformations on the v-tuple $(y_1, \cdots, y_v)$, we can transform it into a new v-tuple $(y'_1, \cdots, y'_v)$ such that $y'_l y'_{l+1}=1$ for at least one l $(1 \leqq l \leqq v-1)$.*

The proof proceeds by induction on v starting with $v=2$. If $v=2$, then $(y_1, y_2)=(g_i, g_{i+1})$, which can be transformed into either any of the 3 pairs (s_0, s_2), (s_1, s_0), (s_2, s_1) or any of the 3 pairs (s_0^{-1}, s_1^{-1}), (s_1^{-1}, s_2^{-1}), (s_2^{-1}, s_0^{-1}), as we remarked at the beginning of Case B. Therefore, (i) or (ii) holds. This proves the assertion for $v=2$.

Assume inductively that Assertion B.1 is proved for some v (with $v<u$). We will prove it for $v+1$. Note that y_{v+1} is short, because $y_{v+1}=g_{i+v}$ and $i+v<i+u=j$. First consider the case when $y_{v+1} \in \{s_0, s_1, s_2\}$. If (i) holds for v, then change y_v for s_1, s_2, or s_0 according as $y_{v+1}=s_0$, s_1 or s_2. Then (y_v, y_{v+1}) will become (s_1, s_0), (s_2, s_1) or (s_0, s_2), and these 3 pairs can be mutually transformed. Therefore (i) holds for $v+1$. If (ii) holds for v, then change y_v for s_0^{-1}, s_1^{-1} or s_2^{-1} according as $y_{v+1}=s_0$, s_1 or s_2. Then $y_v y_{v+1}=1$. Therefore (iii) holds for $v+1$. If (iii) holds for v, trivially (iii) holds for $v+1$.

Secondly consider the case when $y_{v+1} \in \{s_0^{-1}, s_1^{-1}, s_2^{-1}\}$. However, this case can be treated similarly as the first case.

Assertion B.1 is proved.

By Assertion B.1, one of the statements (i), (ii) or (iii) holds for the u-tuple $(y_1, \cdots, y_u)=(g_i, \cdots, g_{j-1})$. If (iii) holds for this u-tuple, then we are done. Thus we may assume that (i) or (ii) holds for the u-tuple $(g_i, g_{i+1}, \cdots, g_{j-1})$.

Recall that g_j is long, so that it is expressed as a reduced product: $g_j = Q_j^{-1} a^{\varepsilon} b a^{\varepsilon} Q_j$, where $\varepsilon=1$ or 2 and $Q_j \neq 1$. Obviously, Q_j^{-1} is equal to $a^2 b \cdots b$, $ab \cdots b$ or $b \cdots b$. This last case includes $Q_j^{-1}=b$ as a special case. Now by applying elementary transformations to $(g_i, g_{i+1}, \cdots, g_{j-1})$, change g_{j-1} into a new g'_{j-1} as the following table indicates:

Q_j^{-1}	$a^2b\cdots b$		$ab\cdots b$		$b\cdots b$	
the statement of Assertion B.1, (i) or (ii), which holds for the u-tuple $(y_1, \cdots, y_u)=(g_i, \cdots, g_{j-1})$	(i)	(ii)	(i)	(ii)	(i)	(ii)
g'_{j-1} which g_{j-1} is changed into	s_1	s_0^{-1}	s_2	s_1^{-1}	s_0	s_2^{-1}

For example, if $Q_j^{-1}=a^2b\cdots b$, and the statement (i) holds for $(g_i, \cdots, g_{j-1})$, change g_{j-1} into $g'_{j-1}=s_1=aba$. Then $g'_{j-1}g_j=(aba)(a^2b\cdots b)a^{\varepsilon}ba^{\varepsilon}(b\cdots ba)$ has shorter length than g_j, because of the cancellation $(aba)(a^2b\cdots b)=a\cdots b$, and we are led back to Case A.2. As is easily verified, we are similarly led to Case A.2 in all the remaining cases of the table above.

Thus in Case B, we find that at least one of the following three assertions holds:

(1) every g_j in $(g_1, \cdots, g_\nu)$ is short;

(2) by successive application of elementary transformations, the ν-tuple $(g_1, \cdots, g_\nu)$ can be transformed into a ν-tuple $(g'_1, \cdots, g'_\nu)$ in which $g'_l g'_{l+1}=1$ holds for at least one l $(1\leqq l\leqq \nu-1)$;

(3) the case is reduced to Case A.2.

Since Theorem 3.4 is obviously true for a ν-tuple with total length $\leqq 3$, we complete the proof of Theorem 3.4 by induction on the total length, combining the conclusions of Cases A.1, A.2 and B. □

Moishezon proved the following theorem. (See [**9**, pp. 180-187].)

MOISHEZON'S THEOREM. *Let $y_1, \cdots, y_\nu \in PSL(2, \boldsymbol{Z})$ be such that each y_i, $i=1, \cdots, \nu$, is equal to one of the elements s_0, s_1, s_2 and $y_1y_2\cdots y_\nu=1$. Then $\nu\equiv 0$* (mod 2) *and there exists a finite sequence of elementary transformations starting with some elementary transformation of $(y_1, y_2, \cdots, y_\nu)$ such that if $(z_1, z_2, \cdots, z_\nu)$ is the resulting ν-tuple, then for any $j=1, 2, \cdots, \nu/2$, $z_{2j-1}=s_1$, $z_{2j}=s_2$.*

We will extend Moishezon's theorem as follows:

THEOREM 3.5. *Let $y_1, \cdots, y_\nu \in PSL(2, \boldsymbol{Z})$ be such that each y_i, $i=1, \cdots, \nu$, is equal to one of the elements s_0, s_1, s_2, s_0^{-1}, s_1^{-1}, s_2^{-1} and $y_1y_2\cdots y_\nu=1$. Then $\nu\equiv 0$* (mod 2) *and there exists a finite sequence of elementary transformations starting with some elementary transformation of $(y_1, \cdots, y_\nu)$ such that if $(z_1, z_2, \cdots, z_\nu)$ is the resulting ν-tuple, then one of the three assertions holds for $(z_1, z_2, \cdots, z_\nu)$:*

(i) *for each $j=1, 2, \cdots, \nu/2$, $z_{2j-1}=s_1$, $z_{2j}=s_2$;*

(ii) *for each $j=1, 2, \cdots, \nu/2$, $z_{2j-1}=s_2^{-1}$, $z_{2j}=s_1^{-1}$;*

(iii) *for at least one i, $z_iz_{i+1}=1$.*

PROOF. *Case* I. For some $k\in\{0, 1, 2\}$, both s_k and s_k^{-1} are contained in $\{y_1, y_2, \cdots, y_\nu\}$. We may assume that $y_1=s_k$ and $y_i=s_k^{-1}$ $(\exists i>1)$. Consider the following sequence of elementary transformations:

$$(s_k, y_2, \cdots, y_{i-1}, s_k^{-1}, \cdots) \overset{R_{i-1}}{\longmapsto} (s_k, y_2, \cdots, s_k^{-1}, s_k y_{i-1} s_k^{-1}, \cdots)$$
$$\overset{R_{i-2}}{\longmapsto} \cdots\cdots \overset{R_2}{\longmapsto} (s_k, s_k^{-1}, s_k y_2 s_k^{-1}, \cdots, s_k y_{i-1} s_k^{-1}, \cdots).$$

Then assertion (iii) holds.

Case II. For each $k=0, 1, 2,$ either s_k or s_k^{-1} is not contained in $\{y_1, y_2, \cdots, y_\nu\}$.

There are 8 sub-cases to be considered according as

$$\bigcup_{i=1}^{\nu}\{y_i\}\subset\{s_0, s_1, s_2\}, \{s_0, s_1, s_2^{-1}\}, \{s_0, s_1^{-1}, s_2\}, \{s_0^{-1}, s_1, s_2\}, \{s_0, s_1^{-1}, s_2^{-1}\},$$
$$\{s_0^{-1}, s_1, s_2^{-1}\}, \{s_0^{-1}, s_1^{-1}, s_2\} \text{ or } \{s_0^{-1}, s_1^{-1}, s_2^{-1}\}.$$

Case II.1. $\bigcup_{i=1}^{\nu}\{y_i\}\subset\{s_0, s_1, s_2\}$.

This case is nothing but the situation of Moishezon's theorem.

Case II.2. $\bigcup_{i=1}^{\nu}\{y_i\}\subset\{s_0, s_1, s_2^{-1}\}$.

If $s_2^{-1}\notin\{y_1, \cdots, y_\nu\}$, the case is reduced to Case II.1. If $s_2^{-1}\in\{y_1, \cdots, y_\nu\}$, we may assume that $y_1=s_2^{-1}$ (after applying cyclic permutations). Suppose that there exists at least one $i>1$ for which $y_i=s_1$, $y_{i+1}=s_0$, then by the i-th inverse elementary transformation, R_i^{-1}, $(y_i, y_{i+1})=(s_1, s_0)$ is transformed into $(s_1 s_0 s_1^{-1}, s_1)$ $=(s_2, s_1)$. Therefore, we come to the situation in which both s_2^{-1} and s_2 are contained in the resulting ν-tuple. The case is reduced to Case I.

Suppose that there exists no i such that $y_i=s_1$, $y_{i+1}=s_0$. Then the ν-tuple $(y_1, \cdots, y_\nu)$ is of the form $(s_2^{-1}, \cdots, s_2^{-1}, s_0, \cdots, s_0, s_1, \cdots, s_1, s_2^{-1}, \cdots, s_2^{-1}, s_0, \cdots, s_0, s_1, \cdots, s_1, s_2^{-1}, \cdots, s_2^{-1}, s_0, \cdots, s_0, s_1, \cdots, s_1, \cdots)$. In this sequence, the subsequence $s_0, \cdots, s_0$, for instance, may be empty. However, the product $y_1\cdots y_\nu$ of the ν-tuple $(y_1, \cdots, y_\nu)$ of this form is not equal to 1. (Recall that $s_2^{-1}=ab$, $s_0=a^2b$, $s_1=aba$.) This contradicts the assumption $y_1 y_2\cdots y_\nu=1$.

Case II.3. $\bigcup_{i=1}^{\nu}\{y_i\}\subset\{s_0, s_1^{-1}, s_2\}$.

Case II.4. $\bigcup_{i=1}^{\nu}\{y_i\}\subset\{s_0^{-1}, s_1, s_2\}$.

Cases II.3, II.4 are treated similarly to Case II.2.

Case II.5. $\bigcup_{i=1}^{\nu}\{y_i\}\subset\{s_0, s_1^{-1}, s_2^{-1}\}$.

If $s_0\notin\{y_1, \cdots, y_\nu\}$, the case is reduced to Case II.8 below. If $s_0\in\{y_1, \cdots, y_\nu\}$, we may assume that $y_1=s_0$. Suppose that there exists at least one $i>1$ for which $y_i=s_1^{-1}$, $y_{i+1}=s_2^{-1}$, then by the elementary transformation R_i^{-1}, $(y_i, y_{i+1})=(s_1^{-1}, s_2^{-1})$ is transformed into $(s_1^{-1}s_2^{-1}s_1, s_1^{-1})=(s_0^{-1}, s_1^{-1})$. Therefore, the situation is altered so that both s_0 and s_0^{-1} are contained in the resulting ν-tuple. The case is reduced to Case I. So suppose that there exists no i such that $y_i=s_1^{-1}$, $y_{i+1}=s_2^{-1}$. Then the ν-tuple $(y_1, \cdots, y_\nu)$ is of the form $(s_0, \cdots, s_0, s_2^{-1}, \cdots, s_2^{-1}, s_1^{-1}, \cdots, s_1^{-1}, s_0, \cdots, s_0, s_2^{-1}, \cdots, s_2^{-1}, s_1^{-1}, \cdots, s_1^{-1}, \cdots)$. The product $y_1 \cdots y_\nu$ of the ν-tuple $(y_1, \cdots, y_\nu)$ of this form is not equal to 1. (Recall that $s_0=a^2b$, $s_2^{-1}=ab$, $s_1^{-1}=a^2ba^2$.) This contradicts the assumption $y_1 \cdots y_\nu=1$.

Case II.6. $\bigcup_{i=1}^{\nu}\{y_i\}\subset\{s_0^{-1}, s_1, s_2^{-1}\}$.

Case II.7. $\bigcup_{i=1}^{\nu}\{y_i\}\subset\{s_0^{-1}, s_1^{-1}, s_2\}$.

Cases II.6, II.7 are treated similarly to Case II.5.

Case II.8. $\bigcup_{i=1}^{\nu}\{y_i\}\subset\{s_0^{-1}, s_1^{-1}, s_2^{-1}\}$.

The situation of this case is "inverse" to that of Case II.1. So by the "inverse" of Moishezon's theorem, we can transform the ν-tuple $(y_1, \cdots, y_\nu)$ into $(z_1, \cdots, z_\nu)$ for which the assertion (ii) holds.

This completes the proof of Theorem 3.5. □

Combining Theorems 3.4, 3.5, we have the following

THEOREM 3.6. *Let $g_1, g_2, \cdots, g_\nu \in PSL(2, \boldsymbol{Z})$ be conjugates of x $(=s_1=aba)$ or x^{-1} $(=s_1^{-1}=a^2ba^2)$ satisfying $g_1g_2 \cdots g_\nu=1$. Then by successive application of elementary transformations, the ν-tuple $(g_1, g_2, \cdots, g_\nu)$ can be transformed into a ν-tuple in one of the two normal forms,* (1)′ *or* (2)′:

(1)′ $(w_1, w_1^{-1}, \cdots, w_l, w_l^{-1}, s_1, s_2, s_1, s_2, \cdots, s_1, s_2)$,

(2)′ $(w_1, w_1^{-1}, \cdots, w_l, w_l^{-1}, s_2^{-1}, s_1^{-1}, s_2^{-1}, s_1^{-1}, \cdots, s_2^{-1}, s_1^{-1})$,

where $0\leqq l\leqq \nu/2$ and $w_i\in PSL(2, \boldsymbol{Z})$ for each $i=1, \cdots, l$.

PROOF. The proof proceeds by induction on ν. If $\nu=1$, then $g_1\neq 1$ and the theorem is trivially true. Suppose that $\nu\geqq 2$. Then by Theorem 3.4, the ν-tuple $(g_1, \cdots, g_\nu)$ is transformed into $(h_1, \cdots, h_\nu)$ such that either every h_i is short or $h_ih_{i+1}=1$ for at least one i. If $h_ih_{i+1}=1$ for an i, then by cyclic permutation, we may assume that $h_1h_2=1$. The remaining $(\nu-2)$-tuple $(h_3, \cdots, h_\nu)$ satisfies the condition $h_3h_4 \cdots h_\nu=1$. Thus by the inductive hypothesis, $(h_3, \cdots, h_\nu)$ can be transformed into one of the two normal forms. If every h_i of $(h_1, \cdots, h_\nu)$ is short, then Theorem 3.5 applies. We can transform $(h_1, \cdots, h_\nu)$ into $(z_1, \cdots, z_\nu)$ such that either $(z_1, \cdots, z_\nu)=(s_1, s_2, \cdots, s_1, s_2)$, $(s_2^{-1}, s_1^{-1}, \cdots, s_2^{-1}, s_1^{-1})$ or there exists an i for which $z_iz_{i+1}=1$ holds. If $(z_1, \cdots, z_\nu)=(s_1, s_2, \cdots, s_1, s_2)$ or $(s_2^{-1}, s_1^{-1}, \cdots, s_2^{-1}, s_1^{-1})$, we are done. If there exists i such that $z_iz_{i+1}=1$, we may assume $z_1z_2=1$ by cyclic permutation. The remaining $(\nu-2)$-tuple $(z_3, \cdots, z_\nu)$ satisfies the condition $z_3z_4\cdots z_\nu=1$. Thus by the inductive hypothesis, $(z_3, \cdots, z_\nu)$ can be transformed into one of the normal forms. $\square$

Now we are in a position to prove Theorem 3.2. Let $(B_1, B_2, \cdots, B_\nu)$ be a ν-tuple in $SL(2, \boldsymbol{Z})$ such that $B_1B_2\cdots B_\nu=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and such that each B_i is conjugate to $X=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ or $X^{-1}=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$. Let $g_i=\pi(B_i)\in PSL(2, \boldsymbol{Z})$. Then by Theorem 3.6, the ν-tuple $(g_1, \cdots, g_\nu)$ can be transformed by a finite sequence of elementary transformations into a ν-tuple in one of the normal forms, (1)$'$ or (2)$'$. An elementary transformation in $PSL(2, \boldsymbol{Z})$ can be lifted to an elementary transformation in $SL(2, \boldsymbol{Z})$.

Therefore, the ν-tuple $(B_1, B_2, \cdots, B_\nu)$ can be transformed into a ν-tuple $(B_1', B_2', \cdots, B_\nu')$ such that $(\pi(B_1'), \pi(B_2'), \cdots, \pi(B_\nu'))$ is in one of the normal forms (1)$'$ or (2)$'$.

CLAIM 1. *If $\pi(B_i')\pi(B_{i+1}')=1\in PSL(2, \boldsymbol{Z})$ for some i, then* $B_i'B_{i+1}'=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

The abelianization of $PSL(2, \boldsymbol{Z})$ is a cyclic group of order 6 and x $(=aba)$ is taken as a generator of the cyclic group. Therefore, if $\pi(B_i')$ is conjugate to x (or x^{-1}), then $\pi(B_{i+1}')$ is conjugate to x^{-1} (or x). It follows that if B_i' is conjugate to X (or X^{-1}), then B_{i+1}' is conjugate to X^{-1} (or X). It will suffice to consider the case when $B_i'=A_i^{-1}XA_i$, $B_{i+1}'=A_{i+1}^{-1}X^{-1}A_{i+1}$. Then $B_i'B_{i+1}'=A_i^{-1}XA_iA_{i+1}^{-1}X^{-1}A_{i+1}$ belongs to the commutator subgroup $[SL(2, \boldsymbol{Z}), SL(2, \boldsymbol{Z})]$. By the assumption $\pi(B_i'B_{i+1}')=1$, we have $B_i'B_{i+1}'=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ or $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$. However, it is known that $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\notin[SL(2, \boldsymbol{Z}), SL(2, \boldsymbol{Z})]$. Therefore, $B_i'B_{i+1}'=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ as claimed.

CLAIM 2. *If $\pi(B_i')=s_1, s_2, s_1^{-1}$ or s_2^{-1} for some i, then $B_i'=X, Y, X^{-1}$ or Y^{-1}, accordingly.*

Note that $s_1=x=aba$ and that $\pi^{-1}(s_1)=\{X, -X\}$. Therefore, if $\pi(B_i')=s_1$, B_i' is equal to X or $-X$. But $B_i=-X$ is impossible, because $-X$ is not conjugate to $X^{\pm 1}$.

The other cases can be treated similarly. (Recall that $\pi(Y)=ba^2=s_2$ and that Y is conjugate to X.) Claim 2 is proved.

By Claims 1 and 2, we conclude that if $(B_1', B_2', \cdots, B_\nu')$ is a ν-tuple such that $(\pi(B_1'), \pi(B_2'), \cdots, \pi(B_\nu'))$ is in one of the normal forms (1)$'$ or (2)$'$ of Theorem 3.6, then $(B_1', B_2', \cdots, B_\nu')$ is in one of the normal forms (1) or (2) in the statement of Theorem 3.2. This completes the proof of Theorem 3.2. $\square$

A geometric implication of Theorem 3.2 is the following:

THEOREM 3.7. *Let $f: M\to D^2$ be a torus fibration over the 2-disk D^2 each of* *whose singular fibers is of type* I_1^+ *or* I_1^-. *Let* $\Gamma=\{x_1, \cdots, x_\nu\}$ *be the set of critical values. Suppose that the monodromy around the boundary* ∂D^2 *is trivial. Let* $x_0 \in D^2-\Gamma$ *be a base point,* (μ_0, λ_0) *an ordered basis of* $H_1(F_0; \boldsymbol{Z})$ *compatible with the orientation of* $F_0=f^{-1}(x_0)$. *Let* $\rho: \pi_1(D^2-\Gamma, x_0)\to SL(2, \boldsymbol{Z})$ *be the monodromy homomorphism determined by* (μ_0, λ_0).

Then ν *is even, and by relabeling the critical values* $x_1, \cdots, x_\nu$ *and by rechoosing the paths* $\gamma_1, \cdots, \gamma_\nu$ *as shown in Fig.* 3.1 *appropriately, we can make the* ν*-tuple of monodromy matrices* $(\rho(l_1), \cdots, \rho(l_\nu))$ *a* ν*-tuple in one of the normal forms,* (1) *or* (2):

(1) $(W_1, W_1^{-1}, \cdots, W_l, W_l^{-1}, X, Y, X, Y, \cdots, X, Y)$,

(2) $(W_1, W_1^{-1}, \cdots, W_l, W_l^{-1}, Y^{-1}, X^{-1}, Y^{-1}, X^{-1}, \cdots, Y^{-1}, X^{-1})$,

where W_i *belongs to* $SL(2, \boldsymbol{Z})$ *for each* $i=1, 2, \cdots, l$, *and* X, Y *denote* $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$, $\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$, *respectively.* (Recall that the order of XY in $SL(2, \boldsymbol{Z})$ is 6, and that $\nu-2l$ is divisible by 12.)

§4. Deformation of fibrations.

Let $f: M\to B$ be a torus fibration each singular fiber of which is of type I_1^+ or I_1^-. Suppose that there exists a 2-disk D in $\operatorname{Int} B$ which contains exactly two critical values x_α, x_β in its interior and that the monodromy around ∂D is trivial.

THEOREM 4.1. *We can deform the structure of torus fibration* $f|f^{-1}(D)$: $f^{-1}(D)\to D$, *without altering it in a neighborhood of* $\partial(f^{-1}(D))$, *so that the resulting torus fibration* $\tilde{f}: f^{-1}(D)\to D$ *has a single singular fiber of type Tw whose divisor is* $R+S$.

PROOF. (A rough idea was sketched in [7].) The singular fibers over x_α and x_β are denoted by F_α, F_β, respectively. Since the monodromy around ∂D is assumed to be trivial, one of them, say F_α, is of type I_1^+ and the other F_β is of type I_1^-.

Let D_α, D_β be mutually disjoint 2-disks in $\operatorname{Int} D$ defined by

$$D_\alpha=\{\xi_\alpha \mid |\xi_\alpha|\leqq\varepsilon\}, \qquad D_\beta=\{\xi_\beta \mid |\xi_\beta|\leqq\varepsilon\},$$

where ξ_α (resp. ξ_β) is a local complex coordinate in $\operatorname{Int} D$ near x_α (resp. x_β) which equals 0 at x_α (resp. x_β).

In the total space M, there are local complex coordinates z_α^1, z_α^2 (resp. z_β^1, z_β^2) near the self-intersection point of F_α (resp. F_β) with which the projection f is written locally as $\xi_\alpha=f(z_\alpha^1, z_\alpha^2)=z_\alpha^1 z_\alpha^2$ (resp. $\xi_\beta=f(z_\beta^1, z_\beta^2)=\bar{z}_\beta^1 z_\beta^2$). We define smooth 4-cells U_α, U_β with corners in M as follows:

$$U_\alpha = \{(z_\alpha^1, z_\alpha^2) \mid |z_\alpha^1 z_\alpha^2| \leqq \varepsilon, \ |z_\alpha^1| \leqq 1, \ |z_\alpha^2| \leqq 1\},$$

$$U_\beta = \{(z_\beta^1, z_\beta^2) \mid |\bar{z}_\beta^1 z_\beta^2| \leqq \varepsilon, \ |z_\beta^1| \leqq 1, \ |z_\beta^2| \leqq 1\}.$$

Furthermore, let us take a quadrilateral Q in $\operatorname{Int} D$ as shown in Fig. 4.1.

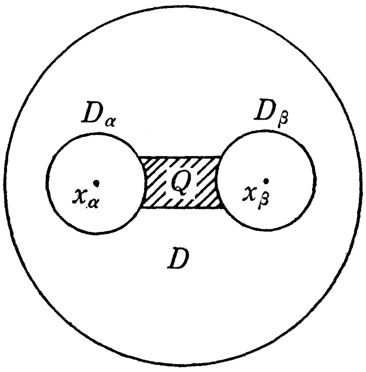

Figure 4.1.

Our proof below will split $f^{-1}(D_\alpha \cup Q \cup D_\beta)$ into two parts, say "upper" and "lower" parts, and will show that the upper part (resp. the lower part) is diffeomorphic to U_α (resp. U_β) via a fiber preserving diffeomorphism. (Here we are speaking of the fibering structures induced by the projection f.) Thus $f^{-1}(D_\alpha \cup Q \cup D_\beta)$, which is the union of the two parts, is diffeomorphic to $U_\alpha \cup U_\beta$, the manifold obtained by gluing U_α and U_β via an orientation reversing diffeomorphism $\tilde{\phi} : T_\alpha^1 \cup T_\alpha^2 \to T_\beta^2 \cup T_\beta^1$, where T_α^1, T_α^2, T_β^2, T_β^1 are certain solid tori embedded in the boundaries ∂U_α and ∂U_β. We will next deform the pasting diffeomorphism $\tilde{\phi}$ by isotopy so that the resulting diffeomorphism $\tilde{\phi}'$ will match the "section" of F_α with that of F_β. This process will correspond to the deformation of the fibration $f | f^{-1}(D_\alpha \cup Q \cup D_\beta) : f^{-1}(D_\alpha \cup Q \cup D_\beta) \to D_\alpha \cup Q \cup D_\beta$ which will give at the last stage a fibration over $D_\alpha \cup Q \cup D_\beta$ with a single singular fiber of type Tw.

Now we proceed into the details. Obviously, we have $f(U_\alpha) = D_\alpha$ and $f(U_\beta) = D_\beta$. We denote $\text{Closure}(f^{-1}(D_\alpha) - U_\alpha)$ and $\text{Closure}(f^{-1}(D_\beta) - U_\beta)$ by H_α and H_β, respectively. The intersection $U_\alpha \cap H_\alpha$ consists of the two solid tori T_α^1, T_α^2 mentioned above. In terms of local coordinates, they are given as follows:

$$T_\alpha^1 = \{(z_\alpha^1, z_\alpha^2) \mid |z_\alpha^1| = 1, \ |z_\alpha^2| \leqq \varepsilon\},$$

$$T_\alpha^2 = \{(z_\alpha^1, z_\alpha^2) \mid |z_\alpha^1| \leqq \varepsilon, \ |z_\alpha^2| = 1\}.$$

Similarly, the intersection $U_\beta \cap H_\beta$ consists of two solid tori denoted by T_β^1, T_β^2.

Since the singular fiber F_α is an immersed 2-sphere with a single self-intersection, $\text{Closure}(F_\alpha - F_\alpha \cap U_\alpha)$ is an annulus. Being a tubular neighborhood of this annulus in $M-\mathrm{Int}(U_\alpha)$, H_α can be identified with $D_\alpha\times S^1\times[0, 1]$ so that T_α^1 and T_α^2 are identified with $D_\alpha\times S^1\times\{0\}$ and $D_\beta\times S^1\times\{1\}$, respectively. Also the projection $f|H_\alpha: H_\alpha\to D_\alpha$ may be assumed to be the first projection $D_\alpha\times S^1\times[0, 1]\to D_\alpha$. Similarly H_β can be identified with $D_\beta\times S^1\times[0, 1]$ and the projection $f|H_\beta: H_\beta\to D_\beta$ with $D_\beta\times S^1\times[0, 1]\to D_\beta$.

Let J_α, J_β denote the arcs $D_\alpha\cap Q$, $D_\beta\cap Q$, and take a point x_α' (resp. x_β') in $\mathrm{Int}J_\alpha$ (resp. $\mathrm{Int}J_\beta$). We will choose an ordered basis $(\mu_\alpha, \lambda_\alpha)$ (resp. $(\mu_\beta, \lambda_\beta)$) of $H_1(f^{-1}(x_\alpha); \boldsymbol{Z})$ (resp. $H_1(f^{-1}(x_\beta); \boldsymbol{Z})$) as follows: Each fiber $f^{-1}(x)$ in $f^{-1}(D_\alpha)$ transversely intersects each of the solid tori T_α^1, T_α^2 in a circle. As μ_α, we take the suitably oriented circle $f^{-1}(x_\alpha')\cap T_\alpha^1$, and as λ_α, a simple closed curve in the fiber $f^{-1}(x_\alpha')$ intersecting the circle $f^{-1}(x_\alpha')\cap T_\alpha^1$ transversely in a point. If we choose an appropriate orientation for λ_α, the basis $(\mu_\alpha, \lambda_\alpha)$ gives the natural orientation of $f^{-1}(x_\alpha')$. The basis $(\mu_\beta, \lambda_\beta)$ is constructed similarly.

With these bases, the local monodromy matrices of the singular fibers F_α, F_β are computed as $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$, respectively. (See [**9**], [**7**].)

Q being a 2-cell in $\mathrm{Int}D$ (Fig. 4.1), the restricted T^2-bundle $f|f^{-1}(Q): f^{-1}(Q)\to Q$ is trivial. Thus there is a trivialization

$$\Phi: f^{-1}(Q)\longrightarrow Q\times T^2.$$

Our task below will to rechoose Φ so that it preserves the "mid levels" $U_\alpha\cap H_\alpha$, $U_\beta\cap H_\beta$ and interchanges the upper and lower parts of $f^{-1}(J_\alpha)$ and those of $f^{-1}(J_\beta)$ in a way soon clarified. Let us start with a given trivialization Φ.

Let $i_{\beta\alpha}: f^{-1}(x_\alpha')\to f^{-1}(x_\beta')$ be the diffeomorphism defined through the identity

$$p_2(\Phi(q))=p_2(\Phi(i_{\beta\alpha}(q))), \qquad \forall q\in f^{-1}(x_\alpha'),$$

where $p_2: Q\times T^2\to T^2$ is the second projection. The isotopy class of $i_{\beta\alpha}$ is specified by a matrix $B=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ such that

$$(i_{\beta\alpha})_*(\mu_\alpha)=a\mu_\beta+b\lambda_\beta, \qquad (i_{\beta\alpha})_*(\lambda_\alpha)=c\mu_\beta+d\lambda_\beta.$$

By the hypothesis, the monodromy around $\partial(D_\alpha\cup Q\cup D_\beta)$ is trivial. Thus we have

$$\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}B\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}B^{-1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

from which $B=\begin{bmatrix}\pm 1 & 0\\ c & \pm 1\end{bmatrix}$ follows. By replacing $(\mu_\beta, \lambda_\beta)$ by $(-\mu_\beta, -\lambda_\beta)$ if necessary, we may assume

$$(i_{\beta\alpha})_*(\mu_\alpha)=\mu_\beta, \qquad (i_{\beta\alpha})_*(\lambda_\alpha)=c\mu_\beta+\lambda_\beta.$$

This allows us to deform $i_{\beta\alpha}$ by isotopy so that the resulting diffeomorphism $i'_{\beta\alpha}$ satisfies

$$i'_{\beta\alpha}(f^{-1}(x'_\alpha)\cap T^1_\alpha)=f^{-1}_\beta(x'_\beta)\cap T^1_\beta, \qquad i'_{\beta\alpha}(f^{-1}(x'_\alpha)\cap T^2_\alpha)=f^{-1}(x'_\beta)\cap T^2_\beta .$$

Now "rotate" $i'_{\beta\alpha}$ in $f^{-1}(x'_\beta)$ through $180°$ along λ_β. The diffeomorphism $i''_{\beta\alpha}: f^{-1}(x'_\alpha)\to f^{-1}(x'_\beta)$ thus obtained will satisfy

$$i''_{\beta\alpha}(f^{-1}(x'_\alpha)\cap T^1_\alpha)=f^{-1}(x'_\beta)\cap T^2_\beta , \qquad i''_{\beta\alpha}(f^{-1}(x'_\alpha)\cap T^2_\alpha)=f^{-1}(x'_\beta)\cap T^1_\beta .$$

Therefore

$$i''_{\beta\alpha}(f^{-1}(x'_\alpha)\cap U_\alpha)=f^{-1}(x'_\beta)\cap H_\beta , \qquad i''_{\beta\alpha}(f^{-1}(x'_\alpha)\cap H_\alpha)=f^{-1}(x'_\beta)\cap U_\beta .$$

Via the use of these isotopies we can construct a desired trivialization $\Phi': f^{-1}(Q)\to Q\times T^2$ satisfying

$$\Phi'(f^{-1}(J_\alpha)\cap U_\alpha)=J_\alpha\times A^{(1)}, \qquad \Phi'(f^{-1}(J_\alpha)\cap H_\alpha)=J_\alpha\times A^{(2)},$$
$$\Phi'(f^{-1}(J_\beta)\cap U_\beta)=J_\beta\times A^{(2)}, \qquad \Phi'(f^{-1}(J_\beta)\cap H_\beta)=J_\beta\times A^{(1)},$$

where $A^{(1)}$, $A^{(2)}$ are certain annuli on T^2 with $T^2=A^{(1)}\cup A^{(2)}$, $A^{(1)}\cap A^{(2)}=\partial A^{(1)}$ $=\partial A^{(2)}$.

Note that the fibration $f|H_\beta: H_\beta\to D_\beta$ (with fiber the annulus) is isomorphic to the fibration $(f|f^{-1}(J_\beta)\cap H_\beta)\times\mathrm{id}: (f^{-1}(J_\beta)\times H_\beta)\times[0, 1]\to J_\beta\times[0, 1]$ with corners rounded. Therefore, there exist diffeomorphisms (if the corners are rounded):

$$\tilde{\varphi}_\alpha: U_\alpha\cup(\Phi')^{-1}(Q\times A^{(1)})\cup H_\beta \longrightarrow U_\alpha ,$$
$$\varphi_\alpha: D_\alpha\cup Q\cup D_\beta \longrightarrow D_\alpha$$

such that $(f|U_\alpha)\circ\tilde{\varphi}_\alpha=\varphi_\alpha\circ(f|U_\alpha\cup(\Phi')^{-1}(Q\times A^{(1)})\cup H_\beta)$.

Likewise there exist diffeomorphisms

$$\tilde{\varphi}_\beta: H_\alpha\cup(\Phi')^{-1}(Q\times A^{(2)})\cup U_\beta \longrightarrow U_\beta ,$$
$$\varphi_\beta: D_\alpha\cup Q\cup D_\beta \longrightarrow D_\beta$$

such that $(f|U_\beta)\circ\tilde{\varphi}_\beta=\varphi_\beta\circ(f|H_\alpha\cup(\Phi')^{-1}(Q\times A^{(2)})\cup U_\beta)$.

Let $\tilde{\phi}: T^1_\alpha\cup T^2_\alpha\to T^2_\beta\cup T^1_\beta$ be the orientation reversing diffeomorphism defined by $\tilde{\phi}=\tilde{\varphi}_\beta\circ(\tilde{\varphi}_\alpha|T^1_\alpha\cup T^2_\alpha)^{-1}$, and $U_\alpha\bigcup_{\tilde{\phi}}U_\beta$ the manifold obtained by gluing U_α and U_β via $\tilde{\phi}$. We define a projection $f': U_\alpha\bigcup_{\tilde{\phi}}U_\beta\to D_\beta$ by setting

$$f'(p)=\begin{cases} \phi\circ f(p) & p\in U_\alpha \\ f(p) & p\in U_\beta , \end{cases}$$

where $\phi=\varphi_\beta\circ\varphi_\alpha^{-1}: D_\alpha\to D_\beta$. As is easily verified, f' is well-defined.

We will see that $f' : U_\alpha \bigcup_{\tilde{\varphi}} U_\beta \to D_\beta$ is a torus fibration isomorphic to $f | f^{-1}(D_\alpha \cup Q \cup D_\beta) : f^{-1}(D_\alpha \cup Q \cup D_\beta) \to D_\alpha \cup Q \cup D_\beta$.

In fact, a diffeomorphism $\tilde{\varphi} : f^{-1}(D_\alpha \cup Q \cup D_\beta) \to U_\alpha \bigcup_{\tilde{\varphi}} U_\beta$ given by

$$\tilde{\varphi}(p) = \begin{cases} \tilde{\varphi}_\alpha(p) & p \in U_\alpha \cup (\Phi')^{-1}(Q \times A^{(1)}) \cup H_\beta\,, \\ \tilde{\varphi}_\beta(p) & p \in H_\alpha \cup (\Phi')^{-1}(Q \times A^{(2)}) \cup U_\beta \end{cases}$$

is well-defined and satisfies $f' \circ \tilde{\varphi} = \varphi_\beta \circ f$.

We now come to the last step of the proof. We will deform $\tilde{\phi}$ to get the desired torus fibration with a single singular fiber.

First note that the solid tori T_α^1, T_α^2, T_β^1, T_β^2 are foliated by circles as follows. The solid torus T_α^1 is foliated by the "sectional circles" $\{f^{-1}(x) \cap T_\alpha^1\}_{x \in D_\alpha}$, each of which is parametrized as $z_\alpha^1 = e^{i\theta}$, $z_\alpha^2 = \xi_\alpha e^{-i\theta}$, where $0 \leqq \theta \leqq 2\pi$ and the coordinate ξ_α corresponds to $x \in D_\alpha$. The solid tori T_α^2, T_β^1, T_β^2 are foliated similarly. (In T_β^1, the parametrization of a circle will be $z_\beta^1 = e^{i\theta}$, $z_\beta^2 = \xi_\beta e^{i\theta}$, because f is given there by $f(z_\beta^1, z_\beta^2) = \bar{z}_\beta^1 z_\beta^2$. Likewise for T_β^2.)

Call $f^{-1}(x_\alpha) \cap T_\alpha^1$, $f^{-1}(x_\alpha) \cap T_\alpha^2$, $f^{-1}(x_\beta) \cap T_\beta^1$, $f^{-1}(x_\beta) \cap T_\beta^2$ the *distinguished circles*. They are nothing but the sections of the singular fibers F_α, F_β.

Though $\tilde{\phi} : T_\alpha^1 \cup T_\alpha^2 \to T_\beta^2 \cup T_\beta^1$ preserves the leaves of the "sectional foliations", it does not preserve the distinguished circles. This is the point to be remedied.

In T_β^2, the distinguished circle of T_β^2 and the image of that of T_α^1 are situated as shown in Fig. 4.2.

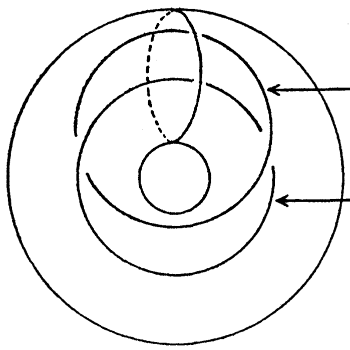

Figure 4.2.

The diffeomorphism $\tilde{\phi}_\alpha^1 := \tilde{\phi} | T_\alpha^1 : T_\alpha^1 \to T_\beta^2$ can be deformed via a leaf preserving isotopy $(\tilde{\phi}_\alpha^1)_t : T_\alpha^1 \to T_\beta^2$, $0 \leqq t \leqq 1$ so that the resulting diffeomorphism $(\tilde{\phi}_\alpha^1)' := (\tilde{\phi}_\alpha^1)_1$ maps $f^{-1}(x_\alpha) \cap T_\alpha^1$ (the distinguished circle of T_α^1) to $f^{-1}(x_\beta) \cap T_\beta^2$ (the distinguished circle of T_β^2). The isotopy $(\tilde{\phi}_\alpha^1)_t$ may be assumed not to alter $\tilde{\phi}_\alpha^1$ near the boundary ∂T_α^1.

Passing to the "base disks", the isotopy $(\tilde{\phi}_\alpha^1)_t$ induces an isotopy $(\psi)_t : D_\alpha \to D_\beta$

of $\phi: D_\alpha \to D_\beta$. Let ϕ' be the last stage of this isotopy $\phi' := (\phi)_1$. This isotopy $(\phi)_t$, in turn, induces a leaf preserving isotopy $(\tilde{\phi}_\alpha^2)_t : T_\alpha^2 \to T_\beta^1$ of $\tilde{\phi}_\alpha^2 := \tilde{\phi} | T_\alpha^2$, the last stage $(\tilde{\phi}_\alpha^2)' := (\tilde{\phi}_\alpha^2)_1$ of which maps $f^{-1}(x_\alpha) \cap T_\alpha^2$ (the distinguished circle of T_α^2) to $f^{-1}(x_\beta) \cap T_\beta^1$ (the distinguished circle of T_β^1). The isotopy $(\tilde{\phi})_t = (\tilde{\phi}_\alpha^1)_t \cup (\tilde{\phi}_\alpha^2)_t : T_\alpha^1 \cup T_\alpha^2 \to T_\beta^2 \cup T_\alpha^1$ of $\tilde{\phi}$ gives a family of manifolds $U_\alpha \bigcup_{(\tilde{\phi})_t} U_\beta$ equipped with the projection $f_t' : U_\alpha \bigcup_{(\tilde{\phi})_t} U_\beta \to D_\beta$, which is defined by

$$f_t'(p) = \begin{cases} (\phi)_t \circ f(p) & p \in U_\alpha\,, \\ f(p) & p \in U_\beta\,. \end{cases}$$

It is not difficult to see that $f_t' : U_\alpha \bigcup_{(\tilde{\phi})_t} U_\beta \to D_\beta$ is a torus fibration for each t.

Each manifold $U_\alpha \bigcup_{(\tilde{\phi})_t} U_\beta$ in the family is diffeomorphic to $U_\alpha \bigcup_{\tilde{\phi}} U_\beta$ via a diffeomorphism which is the identity near the boundary. Also, near the boundary, f_t' always restricts f'. Thus the family $(U_\alpha \bigcup_{(\tilde{\phi})_t} U_\beta, f_t')_{0 \leq t \leq 1}$ is considered as giving a deformation of $f' : U_\alpha \bigcup_{\tilde{\phi}} U_\beta \to D_\beta$. The last stage of this deformation is a torus fibration with a single singular fiber obtained by pasting $f^{-1}(x_\alpha) \cap U_\alpha$ (two disks intersecting transversely in a point with sign $+1$) and $f^{-1}(x_\beta) \cap U_\beta$ (two disks likewise intersecting with sign -1) along their boundaries (i.e. distinguished circles). This is a twin.

Pull back the above deformation to $f^{-1}(D_\alpha \cup Q \cup D_\beta)$ under $\tilde{\varphi} : f^{-1}(D_\alpha \cup Q \cup D_\beta) \to U_\alpha \bigcup_{\tilde{\phi}} U_\beta$, and extend the pulled back deformation by the identity to $f^{-1}(D)$. Then one obtains the desired deformation of $f | f^{-1}(D) : f^{-1}(D) \to D$. This completes the proof of Theorem 4.1. □

Reversing the whole process, we get the following:

THEOREM 4.2. *Let $f : N \to D$ be a torus fibration with a singular fiber of type Tw whose divisor is $R+S$. Then we can deform the fibration without altering it in a neighborhood of ∂N so that the resulting fibration $\tilde{f} : N \to D$ has exactly two singular fibers of types I_1^+ and I_1^-.*

This deformation was first observed in the torus fibration $S^4 \to S^2$ ([**6**], § 4).

§ 5. Fibered surgery.

In this section we will show that surgery on an irreducible component S of a singular fiber of type Tw (with divisor $R+nS$) will change the fiber to a general one and that conversely surgery along a simple closed curve in a general fiber will convert the fiber into a twin singular fiber. The two types of surgery are the inverse of each other. Such surgical operations in torus fibrations are not new. Iwase [2] has made use of them. Our contribution here is only to make the framing precise.

First of all we prepare a standard model of "fibered neighborhood" of a 2-sphere. (See [2], § 3.)

Let ε, δ be positive numbers with $0<2\varepsilon<\delta^n<\delta<1$ with n a fixed integer $\geqq 1$, and define manifolds (with corners) U, V, U_0, V_0 as follows:

$$U=\{(u_1, u_2)\in \boldsymbol{C}^2 \mid |u_1u_2^n|\leqq\varepsilon,\ |u_1|<2,\ |u_2|\leqq\delta\},$$

$$V=\{(v_1, v_2)\in \boldsymbol{C}^2 \mid |v_1v_2^n|\leqq\varepsilon,\ |v_1|<2,\ |v_2|\leqq\delta\},$$

$$U_0=\{(u_1, u_2)\in U \mid |u_1|>1/2\},$$

$$V_0=\{(v_1, v_2)\in V \mid |v_1|>1/2\}.$$

The map $\phi_{1,n}: U_0\to V_0$ given by

$$\phi_{1,n}(u_1, u_2)=(1/u_1,\ u_2|u_1|^{2/n})$$

is an orientation preserving diffeomorphism, via which we glue U and V to obtain a manifold $N_{1,n}=U\underset{\phi_{1,n}}{\cup}V$. The orientation of $N_{1,n}$ is chosen so as to be compatible with the orientations of U and V. Let D_ε denote the closed 2-disk $\{\xi\in\boldsymbol{C} \mid |\xi|\leqq\varepsilon\}$.

Map U (resp. V) to D_ε by the correspondence

$$f_U(u_1, u_2)=u_1u_2^n \qquad (\text{resp. } f_V(v_1, v_2)=\bar{v}_1v_2^n).$$

Since $f_V\phi_{1,n}(u_1, u_2)=f_U(u_1, u_2)$ for all $(u_1, u_2)\in U_0$, we get a well-defined map $f_{1,n}: N_{1,n}\to D_\varepsilon$ which equals f_U on U and f_V on V.

Let S_0 denote the 2-sphere $\{(u_1, u_2)\in U \mid u_2=0\}\cup\{(v_1, v_2)\in V \mid v_2=0\}$.

DEFINITION 5.1. We call $f_{1,n}: N_{1,n}\to D_\varepsilon$ the *standard fibered neighborhood* of the 2-sphere S_0 with multiplicity $(1, n)$. We will sometimes denote $N_{1,n}$ by $N_{1,n}(S_0)$.

The map $f_{1,n}$ is, in fact, the projection of a certain fibration, whose structure we study now. $N_{1,n}$ is a smooth manifold with corners. (Fig. 5.1). The corners appear along the boundaries of the two solid tori

$$T_U=\{(u_1, u_2)\in U \mid |u_1u_2^n|\leqq\varepsilon,\ |u_2|=\delta\}$$

$$T_V=\{(v_1, v_2)\in V \mid |v_1v_2^n|\leqq\varepsilon,\ |v_2|=\delta\}$$

contained in the boundary $\partial N_{1,n}$.

All the fibers $f_{1,n}^{-1}(\xi)$ are transverse to T_U and T_V. A general fiber $f_{1,n}^{-1}(\xi)$ $(\xi\neq 0)$ is an annulus, and the singular fiber $f_{1,n}^{-1}(0)$ consists of S_0 and two 2-disks $D_+=\{(u_1, u_2)\in U \mid u_1=0\}$ and $D_-=\{(v_1, v_2)\in V \mid v_1=0\}$.

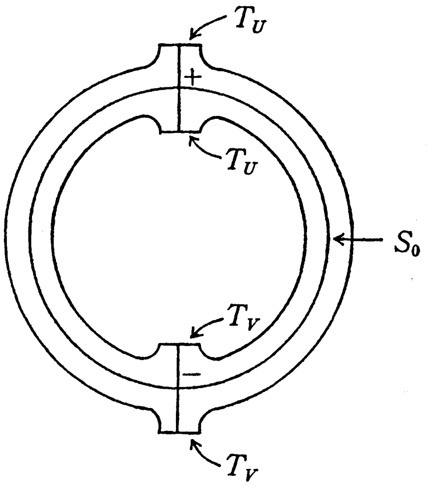

Figure 5.1.

LEMMA 5.2. *$N_{1,n}$ is homeomorphic to $S^2 \times D^2$.*

PROOF. Since $N_{1,n}$ is topologically a D^2-bundle over S^2, it suffices to show that the self-intersection number $[S_0]\cdot[S_0]$ of the zero-section vanishes. Giving the fibers $f_{1,n}^{-1}(\xi)$ the orientations determined by the ones of $N_{1,n}$ and D_ε, we consider them as representing relative homology classes in $H_2(N_{1,n}, T_U \cup T_V; \boldsymbol{Z})$. It is easy to see

$$[f_{1,n}^{-1}(\xi)]=[D_+]+n[S_0]+[D_-] \qquad (\xi \neq 0),$$

where [ ] denotes the relative homology class. Also the intersection numbers between the sphere S_0 and the disks $D_\pm$ are given as follows:

$$[D_+]\cdot[S_0]=1, \qquad [D_-]\cdot[S_0]=-1.$$

Since a general fiber $f_{1,n}^{-1}(\xi)$ and S_0 are disjoint,

$$[f_{1,n}^{-1}(\xi)]\cdot[S_0]=([D_+]+n[S_0]+[D_-])\cdot[S_0]=0.$$

This, together with $([D_+]+[D_-])\cdot[S_0]=1-1=0$, implies $[S_0]\cdot[S_0]=0$, completing the proof. □

A circle which is the intersection of a general fiber $f_{1,n}^{-1}(\xi)$ with T_U (resp. T_V) is parametrized as follows:

$$\{(\xi\delta^{-n}e^{-in\theta}, \delta e^{i\theta}) \in U\}_{0 \leq \theta \leq 2\pi} \qquad (\text{resp. } \{(\xi\delta^{-n}e^{in\theta}, \delta e^{i\theta}) \in V\}_{0 \leq \theta \leq 2\pi}).$$

See Fig. 5.2.

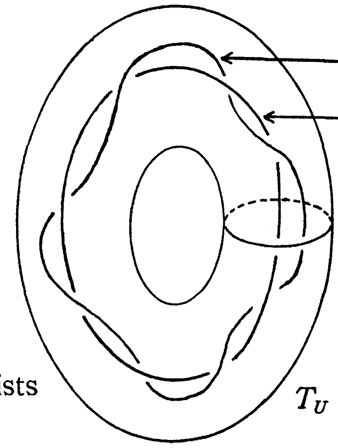

Figure 5.2.

Next we prepare another standard object denoted by $N(C_0)$, the *standard fibered neighborhood* of a circle C_0. It is simply a product $D_\varepsilon\times C_0\times[0, 1]$ equipped with the first projection $f_1: D_\varepsilon\times C_0\times[0, 1]\to D_\varepsilon$. All the fibers of f_1 are annuli. Clearly $N(C_0)$ is homeomorphic to $D^3\times S^1$.

Returning to $N_{1,n}(S_0)$, note that the boundary $\partial N_{1,n}(S_0)$ is the union

$$T_U\cup U^{\cdot}\cup V^{\cdot}\cup T_V,$$

where $U^{\cdot}=\{(u_1, u_2)\in U \mid |u_1u_2^n|=\varepsilon\}$ and $V^{\cdot}=\{(v_1, v_2) \mid |v_1v_2^n|=\varepsilon\}$. Obviously $T_U\cap U^{\cdot}=\{(u_1, u_2)\in U \mid |u_1|=\varepsilon\delta^{-n}, |u_2|=\delta\}$ and $T_V\cap V^{\cdot}=\{(v_1, v_2)\in V \mid |v_1|=\varepsilon\delta^{-n}, |v_2|=\delta\}$. If (u_1, u_2) (resp. (v_1, v_2)) belongs to $U^{\cdot}$ (resp. $V^{\cdot}$), then $2\geqq|u_1|\geqq\varepsilon\delta^{-n}$ (resp. $2\geqq|v_1|\geqq\varepsilon\delta^{-n}$).

Now define a map $h_{1,n}: \partial N_{1,n}(S_0)\to\partial N(C_0)$ by setting

$$h_{1,n}(u_1, \delta e^{i\theta})=(u_1\delta^n e^{in\theta}, e^{i\theta}, 1)\in D_\varepsilon\times C_0\times\{1\} \qquad \text{for } (u_1, \delta e^{i\theta})\in T_U,$$

$$\begin{aligned}h_{1,n}(u_1, u_2)&=(u_1u_2^n, \varepsilon^{-1/n}u_2|u_1|^{1/n}, (\varepsilon^{-1}\delta^n-|u_1|)/(\varepsilon^{-1}\delta^n-\varepsilon\delta^{-n}))\\ &\in\partial D_\varepsilon\times C_0\times[0, 1] \qquad \text{for } (u_1, u_2)\in U^{\cdot},\end{aligned}$$

$$\begin{aligned}h_{1,n}(v_1, v_2)&=(\bar{v}_1v_2^n, \varepsilon^{-1/n}v_2|v_1|^{1/n}, (\varepsilon^{-1}\delta^n-|v_1|^{-1})/(\varepsilon^{-1}\delta^n-\varepsilon\delta^{-n}))\\ &\in\partial D_\varepsilon\times C_0\times[0, 1] \qquad \text{for } (v_1, v_2)\in V^{\cdot},\end{aligned}$$

$$h_{1,n}(v_1, \delta e^{i\theta})=(\bar{v}_1\delta^n e^{in\theta}, e^{i\theta}, 0)\in D_\varepsilon\times C_0\times\{0\} \qquad \text{for } (v_1, \delta e^{i\theta})\in T_V.$$

The map $h_{1,n}$ is well-defined and a piecewise smooth homeomorphism. In fact, the first and the second (resp. the third and the fourth) expressions of $h_{1,n}$ coincide on $T_U\cap U^{\cdot}$ (resp. $T_V\cap V^{\cdot}$), and we have $h_{1,n}\phi_{1,n}(u_1, u_2)=h_{1,n}(u_1, u_2)$ on $U^{\cdot}$. Moreover, $h_{1,n}|T_U: T_U\to D_\varepsilon\times C_0\times\{1\}$, $h_{1,n}|U^{\cdot}\cup V^{\cdot}: U^{\cdot}\cup V^{\cdot}\to\partial D_\varepsilon\times C_0\times[0, 1]$ and $h_{1,n}|T_V: T_V\to D_\varepsilon\times C_0\times\{0\}$ are diffeomorphisms.

LEMMA 5.3. (i) *$h_{1,n}: \partial N_{1,n}(S_0)\to\partial N(C_0)$ is fiber preserving: that is*

$(f_1|\partial N(C_0))\circ h_{1,n}=f_{1,n}|\partial N_{1,n}(S_0)$.

(ii) *If we appropriately identify* $N_{1,n}(S_0)$ *with* $S^2\times D^2$, *and* $N(C_0)$ *with* $D^3\times S^1$ *respecting the projections to the factors so that*

$$\begin{pmatrix} D_\varepsilon\times C_0\times[0,1]\to C_0 \\ \downarrow \\ D_\varepsilon\times[0,1] \end{pmatrix} \cong \begin{pmatrix} D^3\times S^1\to S^1 \\ \downarrow \\ D^3 \end{pmatrix},$$

then $h_{1,n}: S^2\times\partial D^2\to\partial D^3\times S^1$ *can be written as the n-time rotation:*

$$h_{1,n}(x, e^{i\theta})=(\rho_n(e^{i\theta})(x), e^{i\theta}) \qquad (x, e^{i\theta})\in S^2\times\partial D^2,$$

where $\rho_n: S^1\to SO(3)$ *is the map defined by*

$$\rho_n(e^{i\theta})=\begin{pmatrix} \cos n\theta & -\sin n\theta & \\ \sin n\theta & \cos n\theta & \\ & & 1 \end{pmatrix}.$$

The statement (i) follows directly from the expressions of $h_{1,n}$. To see the second statement, Figure 5.2 would be helpful. The details are left to the reader.

Let F_0 be a singular fiber of type Tw of a torus fibration $f: M\to B$. Suppose the divisor of F_0 is $R+nS$. We want to perform "fibered surgery" on S. Recalling the definition of a singular fiber of type Tw (Definition 2.3), one can easily check that there exists an orientation preserving smooth embedding $\varphi_{1,n}: N_{1,n}(S_0)\to M$ such that

(i) $\varphi_{1,n}(S_0)=S$;

(ii) $\varphi_{1,n}(D_+\cup D_-)=R\cap\varphi(N_{1,n}(S_0))$;

(iii) $\varphi_{1,n}$ is fiber preserving: that is, identifying D_ε with an ε-disk in B centered at $x_0=f(F_0)$, we have $f\circ\varphi_{1,n}=f_{1,n}: N_{1,n}(S_0)\to D_\varepsilon$. (Fig. 5.3).

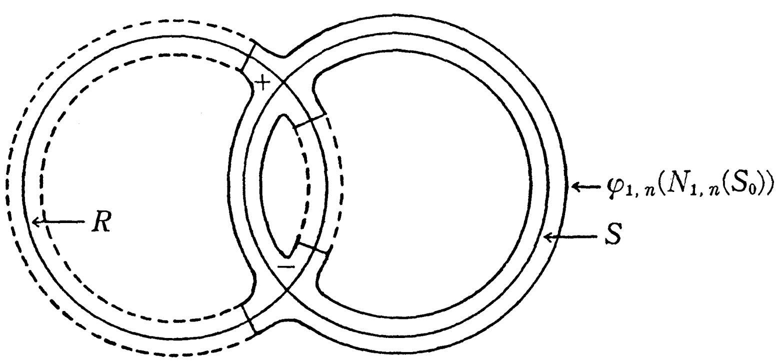

Figure 5.3.

Let M_0 denote Closure$(M-\varphi_{1,n}(N_{1,n}(S_0)))$. Glue M_0 and the standard fibered neighborhood $N(C_0)$ along their boundaries via the composed diffeomorphism

$h_{1,n}\circ\varphi_{1,n}^{-1}:\partial M_0\to\partial N_{1,n}(S_0)\to\partial N(C_0)$. Let $\chi_S(M)=M_0\cup N(C_0)$ denote the resulting manifold. Let $\chi_S(f):\chi_S(M)\to B$ denote the map which equals $f|M_0$ on M_0 and f_1 on $N(C_0)$. By the properties (i), (ii), (iii) of $\varphi_{1,n}$, it is clear that $\chi_S(f):\chi_S(M)\to B$ is a torus fibration in which the singular fiber F_0 has been replaced by $F_0'=(R-R\cap\varphi_{1,n}(N_{1,n}(S_0)))\cup(\{0\}\times C_0\times[0,1])$, which is a general fiber. This completes the description of *fibered surgery* on S. We summarize the above process into the following

THEOREM 5.4. *By performing fibered surgery on the irreducible component S of a twin singular fiber F_0 whose divisor is $R+nS$, we obtain a torus fibration $\chi_S(f):\chi_S(M)\to B$ of the surgered manifold in which F_0 is replaced by a general fiber F_0'.*

Conversely, let F_1 be a general fiber of a torus fibration $f:M\to B$. Let C be a smooth simple closed curve on F_1 which is not null-homotopic in F_1. Then there exists a smooth embedding $\varphi_1:N(C_0)\to M$ such that

(i) $\varphi_1(C_0)=\varphi_1(\{0\}\times C_0\times\{1/2\})=C$;

(ii) φ_1 is fiber preserving: that is, identifying D_ε with an ε-disk in B centered at $x_1=f(F_1)$, we have $f\circ\varphi_1=f_1:N(C_0)\to D_\varepsilon$.

Let M_1 denote Closure$(M-\varphi_1(N(C_0)))$, $\chi_C(M)$ the manifold obtained by gluing M_1 and $N_{1,n}(S_0)$ along their boundaries via the composed diffeomorphism $h_{1,n}^{-1}\circ\varphi_1^{-1}:\partial M_1\to\partial N(C_0)\to\partial N_{1,n}(S_0)$. Let $\chi_C(f):\chi_C(M)\to B$ denote a map which equals $f|M_1$ on M_1 and $f_{1,n}$ on $N_{1,n}(S_0)$. Then by the properties (i) (ii) of φ_1, $\chi_C(f):\chi_C(M)\to B$ is a torus fibration, in which the general fiber F_1 has been replaced by a twin singular fiber $F_1':=(F_1-F_1\cap\varphi_1(N(C_0)))\cup(D_+\cup S_0\cup D_-)$ whose divisor is $R+nS$. (Note that $R:=(F_1-F_1\cap\varphi_1(N(C_0)))\cup D_+\cup D_-$ and $S:=S_0$.) This completes the description of *fibered surgery of type* $(1, n)$ *on* C.

The embedding $\varphi_1:N(C_0)\to M$ is regarded as giving a normal framing of the simple closed curve C, which we call the *canonical framing* determined by the fibration f.

THEOREM 5.5. *By performing fibered surgery of type* $(1, n)$ *on an essential simple closed curve C in a general fiber F_1, we obtain a torus fibration $\chi_C(f):\chi_C(M)\to B$ of the surgered manifold in which F_1 is replaced by a twin singular fiber F_1' whose divisor is $R+nS$. The effect on the diffeomorphism type of M is the same as that of doing surgery on $C\subset M$ using a normal framing which twists n times with respect to the canonical framing determined by the fibration.*

The first assertion is obvious by the construction. The second assertion about the framing follows from Lemma 5.3 (ii). Note that, since $\pi_1(SO(3))=\boldsymbol{Z}/2$, the effect of the surgery on M depends only on C and on the parity of n.

THEOREM 5.6. *Let F_0 be a twin singular fiber whose divisor is $R+nS$ in a torus fibration $f:M\to B$. Perform fibered surgery on S to obtain a torus fibration*

$\chi_S(f):\chi_S(M)\to B$ in which F_0 is replaced by a general fiber F_0'. Let C be the simple closed curve in F_0' which is the image of $\{0\}\times C_0\times\{1/2\}$ $(\subset D_\varepsilon\times C_0\times[0,1]=N(C_0))$. Then, again by performing framed surgery of type $(1, n)$ on the curve C, we recover the original torus fibration $f:M\to B$.

The proof is straightforward.

§6. Proof of Theorem 1.1′.

The final ingredient we need in proving Theorem 1.1′ is Matumoto's extension theorem. Given three integers l, m, n, Matumoto [8] constructed a 4-manifold denoted by $W(l, m, n)$ as follows: Take a 3-torus $T^3=S^1\times S^1\times S^1$ and attach three 2-handles to $T^3\times[0,1]$ along the disjoint standard circles $S^1\times\{*\}\times\{*\}$, $\{*'\}\times S^1\times\{*'\}$, $\{*''\}\times\{*''\}\times S^1$ in $T^3\times\{1\}$ using framing numbers l, m, n, respectively. The resulting manifold is $W(l, m, n)$.

In what follows we need only $W(-1,-1,-1)$ which is denoted simply by W.

THEOREM (Matumoto [8, Theorem 1]). *Let $\alpha: T^3\times\{0\}\to T^3\times\{0\}$ be an orientation preserving diffeomorphism. Then there exists a diffeomorphism $\tilde{\alpha}:W\to W$ such that $\tilde{\alpha}|T^3\times\{0\}=\alpha$ and $\tilde{\alpha}|$(the other component of ∂W)$=$identity.*

Matumoto's next theorem states that W can be embedded in $(CP_2\#9\overline{CP}_2-\mathrm{Int}(D^2\times T^2))$. To state the result precisely, let us recall that $CP_2\#9\overline{CP}_2$ $(\cong V_1$ in the notation of §1) has the structure of an elliptic surface over S^2. Let $D^2\times T^2$ be a fibered neighborhood of a general fiber. Let N denote the closed complement $(CP_2\#9\overline{CP}_2-\mathrm{Int}(D^2\times T^2))$.

THEOREM (Matumoto [8, Proposition 5.1]). *W can be embedded in N so that $\partial N=W\cap\partial N=\partial W\cap\partial N=T^3\times\{0\}$.*

COROLLARY ([8, §7]). *Each orientation preserving diffeomorphism $\alpha:\partial N\to\partial N$ extends to a diffeomorphism $\tilde{\alpha}:N\to N$.*

These results of Matumoto imply a lemma of the Dehn type. Following Montesinos [**10**], we call a simple closed curve in T^3 a *canonical curve* if it is the image of the standard circle $S^1\times\{*\}\times\{*\}$ under an orientation preserving diffeomorphism $\alpha: T^3\to T^3$. For a canonical curve C, we define a *natural framing* to be the image (under α) of the product framing of $S^1\times\{*\}\times\{*\}$ in $S^1\times S^1\times S^1=T^3$. This terminology also applies to curves in a manifold diffeomorphic to T^3.

LEMMA 6.1. *Each canonical curve C in ∂N bounds a smoothly embedded disk Δ in N. Moreover we can take Δ so that if Δ' denotes a disk obtained by perturbing Δ slightly in such a way that $\partial\Delta'$ is pushed off $\partial\Delta$ in the direction of natural framing, then $\Delta\cdot\Delta'=-1$.*

PROOF. In the case when C is the standard circle $S^1\times\{*\}\times\{*\}\subset T^3\times\{0\}=\partial N$, the disk Δ can be taken as $\Delta=(S^1\times\{*\}\times\{*\})\times[0,1]\cup$(core of an attached 2-handle)$\subset W\subset N$. In the general case when $C=\alpha(S^1\times\{*\}\times\{*\})$, α being a diffeomorphism $\alpha:\partial N\to\partial N$, we have only to take the disk $\tilde{\alpha}(\Delta)$, where $\tilde{\alpha}:N\to N$ is an extension of α. $\square$

PROOF OF THEOREM 1.1′. Let $f:M\to S^2$ be a torus fibration, each singular fiber of which is of type I_1^+ or I_1^-. Suppose that $\sigma(M)\neq 0$. Let $\Gamma=\{x_1, x_2, \cdots, x_\nu\}$ be the set of critical values of f. Take a base point $x_0\in S^2-\Gamma$. Also take a disk D in S^2 which contains $\Gamma\cup\{x_0\}$ in $\operatorname{Int} D$. The restriction $f\,|\,f^{-1}(S^2-\operatorname{Int} D):f^{-1}(S^2-\operatorname{Int} D)\to S^2-\operatorname{Int} D$ is a T^2-bundle over a disk $S^2-\operatorname{Int} D$, hence is trivial. Therefore, the monodromy around ∂D is trivial.

By Theorem 3.7, we can permute $x_1, x_2, \cdots, x_\nu$ if necessary and choose paths $\gamma_1, \gamma_2, \cdots, \gamma_\nu$ in $\operatorname{Int} D$ joining x_0 to $x_1, x_2, \cdots, x_\nu$, respectively, so that the corresponding monodromy matrices (with respect to a certain basis (μ_0, λ_0) of $H_1(f^{-1}(x_0);\boldsymbol{Z})$) are in one of the two normal forms:

$$(W_1, W_1^{-1}, W_2, W_2^{-1}, \cdots, W_l, W_l^{-1}, X, Y, X, Y, \cdots, X, Y)$$

or

$$(W_1, W_1^{-1}, W_2, W_2^{-1}, \cdots, W_l, W_l^{-1}, Y^{-1}, X^{-1}, Y^{-1}, X^{-1}, \cdots, Y^{-1}, X^{-1}),$$

where $W_i\in SL(2,\boldsymbol{Z})$ and $X=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$, $Y=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$.

If $l\geqq 1$, then Theorem 4.1 applies. We can deform the fibration $f:M\to S^2$ so that the first two singular fibers $f^{-1}(x_1)$, $f^{-1}(x_2)$ (corresponding to W_1, W_1^{-1}) are fused together to make a single twin singular fiber. After repeating this process l times, we get a torus fibration $M\to S^2$ which contains l twin singular fibers $F_1, F_2, \cdots, F_l$ instead of the first $2l$ singular fibers $f^{-1}(x_1), f^{-1}(x_2), \cdots, f^{-1}(x_{2l-1}), f^{-1}(x_{2l})$. The divisor of each of these twin singular fibers is $R+S$ (Theorem 4.1).

Perform fibered surgery on an irreducible component of each twin singular fiber. Then all the twin singular fiber disappear (Theorem 5.4). Thus we obtain a new manifold $\chi(M)$ and a torus fibration $\chi(f):\chi(M)\to S^2$ all of whose singular fibers are of the same type (I_1^+ or I_1^-). (Note that X and Y represent the monodromy of I_1^+, and X^{-1}, Y^{-1} the monodromy of I_1^-.)

By [**9**, Theorem 9, p. 175], the euler number $e(\chi(M))$ of the manifold $\chi(M)$ is divisible by 12, and

$$\chi(M)\cong V_k \qquad \text{or} \qquad \chi(M)\cong \overline{V}_k$$

according as $\sigma(\chi(M))<0$ or $\sigma(\chi(M))>0$, where $k=e(\chi(M))/12$. (Note that $\sigma(\chi(M))=\sigma(M)\neq 0$.)

The original manifold M is recovered from $\chi(M)$ by performing fibered surgery of type (1, 1) on a simple closed curve in each of the l general fibers $F'_1, F'_2, \cdots, F'_l$ obtained from the twin singular fibers $F_1, F_2, \cdots, F_l$ (Theorem 5.6). The framing used for the surgery twists once with respect to the canonical framing determined by the fibration (Theorem 5.5).

Since $\chi(M)$ ($\cong V_k$ or $\bar{V}_k$) is 1-connected by Kas [**3**], surgery on a simple closed curve in $\chi(M)$ changes $\chi(M)$ into $\chi(M) \# S^2 \times S^2$ or $\chi(M) \# S^2 \tilde{\times} S^2$, where $S^2 \tilde{\times} S^2$ denotes the non-trivial S^2-bundle over S^2. We will see which is the case.

Consider the last 12 critical values of $\chi(f): \chi(M) \to S^2$, $x_{\nu-11}, x_{\nu-10}, \cdots, x_\nu$. The monodromy matrices of the corresponding singular fibers are

$$X,\ Y,\ X,\ Y,\ X,\ Y,\ X,\ Y,\ X,\ Y,\ X,\ Y,$$

or

$$Y^{-1},\ X^{-1},\ Y^{-1},\ X^{-1},\ Y^{-1},\ X^{-1},\ Y^{-1},\ X^{-1},\ Y^{-1},\ X^{-1},\ Y^{-1},\ X^{-1}.$$

Let D' be a 2-disk in S^2 such that $D' \cap \Gamma = \{x_{\nu-11}, x_{\nu-10}, \cdots, x_\nu\}$. The monodromy around $\partial D'$ is trivial, because either of $(XY)^6$ and $(Y^{-1}X^{-1})^6$ is trivial. Let N' denote $\chi(f)^{-1}(D')$, the part of $\chi(M)$ over D'.

We will show that N' is diffeomorphic to the manifold $N = (\boldsymbol{CP}_2 \# 9\overline{\boldsymbol{CP}}_2 - \mathrm{Int}(D^2 \times T^2))$ or $\bar{N}$ (with orientation reversed).

Matumoto proves his theorems using the fact that the fibration $N \cup D^2 \times T^2 \to S^2$ has two singular fibers of types II and II* (in Kodaira's notation [**4**]). We can deform the fibration $N \cup D^2 \times T^2 \to S^2$ by Moishezon's lemma [**9**, Lemma 6, p. 155] so that in the resulting fibration all the singular fibers are of type I_1^+. During the deformation, general fibers are moved by isotopy. Since the euler number $e(N \cup D^2 \times T^2) = e(\boldsymbol{CP}_2 \# 9\overline{\boldsymbol{CP}}_2)$ equals 12, the number of the singular fibers (of type I_1^+) is 12, and they can be arranged so that the corresponding monodromy matrices are $X, Y, X, Y, X, Y, X, Y, X, Y, X, Y$ ([**9**, Lemma 8, p. 179]). Thus by [**9**, Lemma 7a, p. 169], $N' = \chi(f)^{-1}(D')$ is diffeomorphic to N or $\bar{N}$.

Returning to our $\chi(M)$, let F'_i be a general fiber on which we perform fibered surgery of type (1, 1) along a simple closed curve C ($\subset F'_i$).

Let γ be an arc on S^2 joining the value $\chi(f)(F'_i)$ to a point $p \in \partial D'$ and missing all the other critical values of f. Also we assume $\gamma \cap \partial D' = \{p\}$. Move the simple closed curve C ($\subset F'_i$) along the path γ to obtain (as the trace of moving C) an annulus A embedded in $\chi(M)$. Let $C' = \partial A - C$. Then C' is a canonical curve in $\partial N'$. By Lemma 6.1, C' bounds a 2-disk Δ in N' with "relative" self-intersection number ± 1 with respect to the natural framing of C'.

Recall that when performing fibered surgery of type (1, 1) on C, we used a framing which turns once with respect to the canonical framing. Therefore, it is easily seen that if D_0 denotes the "attached" 2-disk along C by the surgery, the resulting 2-sphere $D_0 \cup A \cup \Delta$ has an even self-intersection number. This assures that the fibered surgery of type (1, 1) on C ($\subset F_i'$) changes $\chi(M)$ into $\chi(M) \# S^2 \times S^2$, not $\chi(M) \# S^2 \tilde{\times} S^2$ [12].

Repeating this process l times, we recover the original manifold M by Theorem 5.6, which is diffeomorphic to $\chi(M) \# l(S^2 \times S^2)$ by the above observation.

Therefore, M is diffeomorphic to $V_k \# l(S^2 \times S^2)$ or $\bar{V}_k \# l(S^2 \times S^2)$ according as $\sigma(M)<0$ or $\sigma(M)>0$. Obviously, we have $|\sigma(M)|=|\sigma(V_k)|=8k$ and $e(V_k)=12k+2l$. This completes the proof of Theorem 1.1′. □

A simple twin singular fiber whose divisor is $mR+nS$ is said to be *even* if $m+n\equiv 0$ (2) or *odd* if $m+n\equiv 1$ (2). The following theorem generalizes Theorem 1.1′.

THEOREM 6.2. *Let $f: M \to S^2$ be a torus fibration each of whose singular fibers is of type I_1^+, I_1^- or (simple) Tw. Suppose that $\sigma(M)\neq 0$. Then the diffeomorphism type of M is as follows:*

(i) *if $f: M \to S^2$ does not contain an odd twin singular fiber, then $M \cong V_k \# l(S^2 \times S^2)$ or $M \cong \bar{V}_k \# l(S^2 \times S^2)$ according as $\sigma(M)<0$ or $\sigma(M)>0$;*

(ii) *if $f: M \to S^2$ contains an odd twin singular fiber, then $M \cong k'\boldsymbol{CP}_2 \# l'\overline{\boldsymbol{CP}_2}$.*

OUTLINE OF PROOF. By [7], any simple twin singular fiber which is odd (resp. even) can be replaced without changing the diffeomorphism type of M by a twin singular fiber whose divisor is $R+2S$ (resp. $R+S$). Performing fibered surgery on S, we can eliminate the twin singular fiber. The inverse fibered surgery performed when recovering the twin singular fiber is of type (1, 2) or (1, 1) according as the divisor of the eliminated twin singular fiber was $R+2S$ or $R+S$. Thus the recovered manifold is a connected sum with $S^2 \tilde{\times} S^2$ or $S^2 \times S^2$ according as the twin singular fiber is odd or even. Once $S^2 \tilde{\times} S^2$ ($=\boldsymbol{CP}_2 \# \overline{\boldsymbol{CP}_2}$) appears, the whole manifold decomposes into $k'\boldsymbol{CP}_2 \# l'\overline{\boldsymbol{CP}_2}$ by Mandelbaum's theorem [5] (applied to V_k or $\bar{V}_k$). Details will be left to the reader.

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