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On the elimination of Morin singularities

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Introduction.

In this paper we study the problem of finding a smooth map between smooth manifolds with nice Morin singularities in a given homotopy class. A geometric interpretation of Morin singularities of a smooth map $f: N \to P$ is as follows. Let $S^i(f)$ denote the set of all points x of N such that the kernel rank of df_x is i. For a certain map f, $S^i(f)$ becomes a submanifold of N and we may define $S^{i,j}(f)$ as the set $S^j(f|S^i(f))$ for $f|S^i(f) : S^i(f) \to P$ similarly. Let $n=\dim N$, $p=\dim P$ and $i=\max(1, n-p+1)$. Let I_r be the r-sequence $(i, 1, \cdots, 1)$. Then we may continue to define $S^{I_r}(f)$ as $S^1(f|S^{I_{r-1}})$ inductively. A point of $S^{i,0}(f)$ or $S^{I_r}(f)$ is called a Morin singularity of symbol $(i, 0)$ or I_r respectively. However this approach does not make it clear for what part of smooth maps f, $S^{I_r}(f)$ can be defined. For this we review the following important observation due to Boardman [2].

There exist a submanifold $\Sigma^{i,0}(N, P)$ and a series of submanifolds; $\Sigma^{I_1}(N, P) \supset \Sigma^{I_2}(N, P) \supset \cdots \supset \Sigma^{I_r}(N, P) \supset \cdots$ in the infinite jet space $J^\infty(N, P)$. The codimension of $\Sigma^{i,0}(N, P)$ is $i(p-n+i)$ and that of $\Sigma^{I_r}(N, P)$ is $n-p+r$ for $n \geqq p$ and $r(p-n+1)$ for $n<p$. He has shown that if a jet map $j^\infty f : N \to J^\infty(N, P)$ of f is transverse to all submanifolds $\Sigma^{i,0}(N, P)$ and $\Sigma^{I_r}(N, P)$, then $S^{i,0}(f)$ and $S^{I_r}(f)$ coincide with $(j^\infty f)^{-1}(\Sigma^{i,0}(N, P))$ and $(j^\infty f)^{-1}(\Sigma^{I_r}(N, P))$ respectively. Therefore for generic maps f we may consider $S^{i,0}(f)$ and $S^{I_r}(f)$.

For any integer $r \geqq 1$ we define a subset $\Omega_r(N, P)$ of $J^\infty(N, P)$ as the set of all jets z such that either z is of maximal rank or a point of $\Sigma^{i,0}(N, P)$ or $\Sigma^{I_2}(N, P) \setminus \Sigma^{I_{r+1}}(N, P)$. Then $\Omega_r(N, P)$ becomes an open subbundle of the fibre bundle $J^\infty(N, P)$ over N. The first result of this paper is the following.

THEOREM 1. *Let $p \geqq 2$. Then for any continuous section s of N into $\Omega_r(N, P)$, there exists a smooth map $g : N \to P$ such that $j^\infty g$ becomes a section of N into $\Omega_r(N, P)$ homotopic to s in $\Omega_r(N, P)$.*

Next we will study the problem of eliminating the Morin singularities $S^{I_r}(f)$ with $\operatorname{codim} S^{I_r}(f)=n$ from f admitting only Morin singularities. Theorem 1 reduces it to a problem of finding a continuous section of N into $\Omega_{r-1}(N, P)$ homotopic to $j^\infty f$. We will show that if $j^\infty f$ is transverse to $\Sigma^{I_r}(N, P)$ for a

connected and closed manifold N, then the number of points of $S^{I_r}(f)$ modulo 2 is the unique obstruction of finding the above section. We should note that this number is just the Thom polynomial of the topological closure $\overline{\Sigma^{I_r}(N, P)}$ for f (see the definition of [**10**]).

THEOREM 2. *Let* $r\geqq 2$, $p\geqq 2$ *and* $\operatorname{codim}\Sigma^{I_r}(N, P)=n$. *Let* N *and* P *be orientable manifolds. Then*

(1) *A smooth map* f *with* $j^\infty f(N)\subset\Omega_r(N, P)$ *is homotopic to a smooth map* g *such that* $j^\infty g(N)\subset\Omega_{r-1}(N, P)$ *and* $j^\infty f$ *and* $j^\infty g$ *are homotopic as continuous sections of* N *into* $\Omega_r(N, P)$ *if and only if the Thom polynomial of* $\overline{\Sigma^{I_r}(N, P)}$ *for* f *vanishes.*

(2) *In particular* f *is homotopic to such a smooth map* g *in the following cases;*

i) $n>p$ *and* $r\equiv 1 \pmod 4$

ii) $n>p$, $r\equiv 2, 3$ *or* $4 \pmod 4$ *and* $n-p\equiv 1 \pmod 2$ *and*

iii) $n\leqq p$ *and* $n+p+r+r(r+1)/2\equiv 0 \pmod 2$.

It will be shown by the Morse inequalities that the similar statement of Theorem 1 for $p=1$ is not true. If N is an open manifold, then Theorem 1 is a direct consequence of Gromov [**8**, Theorem 4.1.1] and if $n<p$, it is also a special case of du Plessis [**4**, Theorem B]. So the rest cases will be treated in this paper. The case $r=2$ of Theorem 1 should be compared with [**6**, Theorem 1.3] which will play an important role in a proof of Theorem 1 (Sections 2 and 3).

The case $n\geqq p$ and $p=2$ of Theorem 2 has been proved by Levine [**12**, Theorems 1 and 2] for $n>2$ and by Eliasberg [**5**, Corollary of Theorem 4.9] for $n=2$. Let $\Sigma^{I_r}(n, p)$ be a fibre of $\Sigma^{I_r}(N, P)$ over $N\times P$. To estimate the primary obstruction class of $j^\infty f$ to be deformed to a section of $\Omega_{r-1}(N, P)$ over N (see [**18**]) we will calculate the number of connected components of $\Sigma^{I_r}(n, p)$ and their orientability in Section 4. In Section 5 we will define the dual class of each connected component of $\Sigma^{I_r}(N, P)$ for $j^\infty f$ in $H^{\operatorname{codim}\Sigma^{I_r}(N,P)}(N; G)$ where G is either $\boldsymbol{Z}$ or $\boldsymbol{Z}/2$ depending on its orientability. It will be shown that these dual classes vanish except for at most one class and that their sum is equal to the Thom polynomial modulo 2. A proof of Theorem 2 in Section 6 is based on these facts. For the calculation of Thom polynomials of Morin singularities see, for example, [**7**, **11**, **15**, **17** and **19**].

§ 1. Morin singularities.

Let N and P be paracompact and Hausdorff C^∞ manifolds (simply smooth manifolds) of dimensions n and p respectively. Let $J^k(N, P)$ denote the k jet space ($0\leqq k\leqq\infty$). If $s\geqq t$, then we have the canonical projection $\pi_t^s : J^s(N, P)\to J^t(N, P)$. Let π_N^k and π_P^k be the projections of $J^k(N, P)$ onto N and P mapping

a k jet onto its source and target respectively. In this section we review the definition of Boardman submanifold $\Sigma^I(N, P)$ only for $I=(i, 0)$ and I_r of $J^k(N, P)$ in [2].

We begin with recalling the total tangent bundle $\boldsymbol{D}$ over $J^\infty(N, P)$ ([2, Definition 1.9]). This notion is related to the derivation of functions on $J^\infty(N, P)$. A function ϕ from an open set U of $J^\infty(N, P)$ into $\boldsymbol{R}$ is called smooth if there exists a smooth function ψ on some open subset of $J^k(N, P)$ with $\phi=\psi\circ\pi^\infty_k$. Any smooth section D of $\boldsymbol{D}$ over U determines a homomorphism between the module of smooth functions on U. That is, $D\phi$ is a smooth function on U with property $D(\phi_1+\phi_2)=D\phi_1+D\phi_2$. Any vector field d on an open set V of N defines a smooth section D of $\boldsymbol{D}$ on $(\pi^\infty_N)^{-1}(V)$ characterized by the following equality for any smooth map $f : V\to P$

$$(*) \qquad D\phi\circ J^\infty f=d(\phi\circ j^\infty f).$$

The total tangent bundle $\boldsymbol{D}$ is identified with $(\pi^\infty_N)^*TN$ by $(*)$. Hence any smooth section d of $(\pi^k_N)^*TN$ yields a smooth section of $\boldsymbol{D}$ denoted by $(\pi^\infty_k)^*d$. If d is a section of $(\pi^k_N)^*TN$ and ϕ is a function on $J^k(N, P)$, then $((\pi^\infty_k)^*d)(\phi\circ\pi^\infty_k)$ is of the form $(\phi\circ\pi^\infty_{k+1})$ for some function ϕ on $J^{k+1}(N, P)$. In the sequel we simply write $d(\phi)$ for ϕ. Most of the arguments in the definition of $\Sigma^I(N, P)$ in [2] are treated over $J^\infty(N, P)$. However we will work over $J^k(N, P)$ where k is not less than the length of symbol I, for we will need finiteness of the dimension of $J^k(N, P)$. This approach is guaranteed mainly by [2, Lemmas 1.12, 1.20 and 2.20] and commented in [2, p. 412, line 33].

Let $\boldsymbol{K}_0=(\pi^k_N)^*TN$ and $\boldsymbol{P}=(\pi^k_P)^*TP$. First we recall a homomorphism

$$d_1 : \boldsymbol{K}_0 \longrightarrow \boldsymbol{P} \qquad \text{over} \quad J^k(N, P).$$

Let z be any k jet of $J^k(N, P)$ with target y of P. Let m_y denote the ideal of smooth function germs vanishing at y. For any section d of $\boldsymbol{K}_0$ near z and a smooth function ϕ in a neighbourhood of y we obtain a smooth function $d\phi$ on a neighbourhood of z in $J^k(N, P)$. This defines a homomorphism $h_1 : \boldsymbol{K}_{0,z}\otimes m_y \to \boldsymbol{R}$ by mapping $(d(z), \phi)$ onto $d\phi(z)$ where $\boldsymbol{K}_{0,z}$ is a fibre of $\boldsymbol{K}_0$ over z. Since d annihilate m_y^2 at y, h_1 induces $h_1' : \boldsymbol{K}_{0,z}\otimes m_y/m_y^2\to\boldsymbol{R}$. By identifying m_y/m_y^2 with $\mathrm{Hom}(TP_y, \boldsymbol{R})$ h_1' yields a homomorphism $d_{1,z} : \boldsymbol{K}_{0,z}\to TP_y$ which is what we want to define. Let $\Sigma^i(N, P)$ denote the set of all k jets z such that the kernel rank $d_{1,z}$ is i. We define bundles $\boldsymbol{K}_1$ and $\boldsymbol{Q}_1$ over $\Sigma^i(N, P)$ as the kernel bundle $\mathrm{Ker}(d_1)$ and the cokernel bundle $\mathrm{Cok}(d_1)$ respectively. Let e be the canonical projection of $\boldsymbol{P}$ onto $\boldsymbol{Q}_1$ over $\Sigma^i(N, P)$.

Next we define $\Sigma^{(i,j)}(N, P)$ for $i=\max(1, n-p+1)$ and $j=0$ or 1. There has been defined a symmetric homomorphism $h_2 : \boldsymbol{K}_1\otimes\boldsymbol{K}_1\to\boldsymbol{P}$ over $\Sigma^i(N, P)$ in [2, Corollary 4.5] (Later we will see briefly the definition of h_2 together with

h_r, $r \geqq 3$). Then $e \circ h_2$ yields a homomorphism

$$d_2 : \boldsymbol{K}_1 \longrightarrow \mathrm{Hom}(\boldsymbol{K}_1, \boldsymbol{Q}_1) \quad \text{over} \quad \Sigma^i(N, P).$$

We define $\Sigma^{(i,j)}(N, P)$ to be the set of all k jets z of $\Sigma^i(N, P)$ such that the kernel rank of d_2 over z is j. If $j=1$, we put $\boldsymbol{K}_2=\mathrm{Ker}(d_2)$ and $\mathrm{Cok}(d_2)$ becomes the line bundle $\mathrm{Hom}(\boldsymbol{K}_2, \boldsymbol{Q}_1)$. The definition of $\Sigma^{I_r}(N, P)$ goes by induction on r as follows. Again we have a symmetric homomorphism ($t \geqq 2$)

$$h_{t+1} : \overset{t}{\otimes} \boldsymbol{K}_2 \otimes \boldsymbol{K}_1 \longrightarrow \boldsymbol{P} \quad \text{over} \quad \Sigma^{I_t}(N, P).$$

The composition

$$c_{t+1} : \overset{t+1}{\otimes} \boldsymbol{K}_2 \subset \overset{t}{\otimes} \boldsymbol{K}_2 \otimes \boldsymbol{K}_1 \longrightarrow \boldsymbol{P} \overset{e}{\longrightarrow} \boldsymbol{Q}_1$$

induces a homomorphism

$$d_{t+1} : \boldsymbol{K}_2 \longrightarrow \mathrm{Hom}(\overset{t}{\otimes} \boldsymbol{K}_2, \boldsymbol{Q}_1) \quad \text{over} \quad \Sigma^{I_t}(N, P).$$

Then $\Sigma^{I_{t+1}}(N, P)$ denotes the set of all k jets z of $\Sigma^{I_t}(N, P)$ such that d_{t+1} is a null homomorphism over z. Let $\Sigma^{(I_t,0)}(N, P)$ denote the set $\Sigma^{I_t}(N, P) \setminus \Sigma^{I_{t+1}}(N, P)$ over which d_{t+1} is an isomorphism.

Here we see what a map h_{t+1} is (h_2 will be defined similarly). See the details in [2, Theorem 4.1]). Extend the vector bundles $\boldsymbol{K}_i$ to $\overline{\boldsymbol{K}_i}$ over a small neighbourhood of $\Sigma^{I_t}(N, P)$ ($i=1$ and 2) and take any smooth sections D_1 of $\overline{\boldsymbol{K}_1}$ and $D_2, \cdots, D_{t+1}$ of $\boldsymbol{K}_2$. For a germ ϕ near y of P we obtain a smooth function $D_{t+1}(\cdots D_2(D_1\phi))$ on a neighbourhood of $\Sigma^{I_t}(N, P)$. Furthermore it follows that if either ϕ is a germ of m_y^2 or one of $\{D_i\}$'s vanishes on z, then $D_{t+1}(\cdots D_2(D_1\phi))$ vanishes on z. Therefore we obtain a map $\overset{t}{\otimes} \boldsymbol{K}_{2,z} \otimes \boldsymbol{K}_{1,z} \otimes m_y/m_y^2 \to \boldsymbol{R}$ where $\boldsymbol{K}_{i,z}$ is a fibre of $\boldsymbol{K}_i$ over z. By identifying m_y/m_y^2 with $\mathrm{Hom}(TP_y, \boldsymbol{R})$ we have

$$h_{t+1,z} : \overset{t}{\otimes} \boldsymbol{K}_{2,z} \otimes \boldsymbol{K}_{1,z} \longrightarrow TP_y.$$

Since the operation of $\{D_i\}$'s on a function is a derivation, h_{t+1} becomes symmetric.

The next important fact is that d_{t+1} is extendable to a surjective homomorphism ([2, (7.6)])

$$d_{t+1} : T(\Sigma^{I_{t-1}}(N, P)) \longrightarrow \mathrm{Hom}(\overset{t}{\otimes} \boldsymbol{K}_2, \boldsymbol{Q}_1) \quad \text{over} \quad \Sigma^{I_t}(N, P).$$

The kernel bundle of d_{t+1} over $\Sigma^{I_t}(N, P)$ is equal to $T(\Sigma^{I_t}(N, P))$. This means that the normal bundle of $\Sigma^{I_t}(N, P)$ in $\Sigma^{I_{t-1}}(N, P)$ is given by $\mathrm{Hom}(\overset{t}{\otimes} \boldsymbol{K}_2, \boldsymbol{Q}_1)$. By [2, (7.7)] we have that $\boldsymbol{K}_1 \cap T(\Sigma^{I_{t-1}}(N, P)) = \boldsymbol{K}_t$ over $\Sigma^{I_t}(N, P)$.

REMARK 1.1. Although we have reviewed the definition of $\Sigma^{I_t}(N, P)$ over $J^k(N, P)$, more careful arguments show that we can actually construct a submani-

fold $\Sigma^{I^t}(N, P)'$ in $J^t(N, P)$ together with the bundles $\boldsymbol{K}_1'$, $\boldsymbol{Q}_1'$ over $\Sigma^i(N, P)'$, $\boldsymbol{K}_2'$ over $\Sigma^{I^2}(N, P)'$ and the homomorphisms d_{t+1}' over $(\pi_t^{t+1})^{-1}(\Sigma^{I^t}(N, P)')$ so that d_{t+1} comes from d_{t+1}' by π_{t+1}^k and consequently $\Sigma^{I^t}(N, P)$ coincides with $(\pi_t^k)^{-1}(\Sigma^{I^t}(N, P)')$. See [**2**, Lemma 2.20, 3.6 and 3.10].

§ 2. A generalization of a theorem of Eliasberg.

For a jet z of $\Sigma^{(i,0)}(N, P)$ we have a nonsingular homomorphism

$$(e\circ h_2)_z : \boldsymbol{K}_{1,z}\otimes\boldsymbol{K}_{1,z} \longrightarrow \boldsymbol{Q}_{1,z}.$$

For each orientation of $\boldsymbol{Q}_{1,z}$ we can consider the index s of $(e\circ h_2)_z$. We define the semi index of z as $\min(s, i-s)$. Let $\Sigma_s^{(i,0)}(N, P)$ denote the set of all jets z of $\Sigma^{(i,0)}(N, P)$ such that the semi index of z is s.

We take a sequence of submanifolds $N\supset N_1\supset N_2\supset\cdots\supset N_r$ and an open set U of N as follows. Every N_j is a closed subset in N with codim $N_j=n-p+j$. $N_1\setminus N_2$ is a disjoint union of $N_{1,s}$, $s=0, \cdots, [i/2]$. There exists a smooth map g of a neighbourhood of $N\setminus U$ into P. Let $C^\infty_{\Omega_r}(N, P; \{N_t\}, g)$ denote the space of all smooth maps $f: N\to P$ for $n\geqq p$ such that

(C-1) f coincides with g on a neighbourhood of $N\setminus U$,

(C-2) $(j^kf)^{-1}(\Sigma_s^{(i,0)}(N, P))=N_{1,s}$, $(j^kf)^{-1}(\Sigma^{I^t}(N, P))=N_t$ for $1\leqq t\leqq r$ and $(j^kf)^{-1}(\Sigma^{I^t}(N, P))=\emptyset$ for $t>r$,

(C-3) f has no other type of singularities.

Let $\mathrm{Hom}_{\Omega_r}(TN, TP; \{N_t\}, g)$ denote the space of all homomorphisms h of TN into TP such that

(H-1) h coincides with dg on a neighbourhood of $N\setminus U$,

(H-2) h has a neighbourhood V of N_1 where there exists a smooth map f_h in $C^\infty_{\Omega_r}(V, P; \{V\cap N_t\}, g|V)$ with $df_h=f|V$,

(H-3) h is of maximal rank outside of N_1.

THEOREM 2.1. *Let N_t, g and h be as above. Assume that N is connected, $N_{1,s}\cap U$, nonempty for $0\leqq s\leqq[i/2]$ and $n\geqq p\geqq 2$. Then for any homomorphism h there exists a smooth map f of $C^\infty_{\Omega_r}(N, P; \{N_t\}, g)$ such that df and h are homotopic in* $\mathrm{Hom}_{\Omega_r}(TN, TP; \{N_t\}, g)$.

PROOF. The case $r=1$ of the theorem is [**6**, Theorem 4.7]. We use it to prove the case $r\geqq 2$. For h we have a smooth map $f_h: V\to P$ in (H-2). Take a sufficiently small tubular disk neighbourhood W of N_2 in V so that $(N_{1,s}\cap U)\setminus W$ is nonempty. Then g is extendable to a smooth map $\bar{g}$ on a neighbourhood of $(N\setminus U)\cup W$

$$\bar{g}=\begin{cases} g & \text{on a neighbourhood of } N\setminus U \\ f_h & \text{on } W \end{cases}$$

by the fact that $h=dg$ on a neighbourhood of $N\setminus U$ and $df_h=h|V$. Now we apply the theorem of Eliasberg [6, Theorem 4.7] for $(N\setminus N_2, P; N_1\setminus N_2, \bar{g})$. Then we obtain a smooth map f' of $C^\infty_{\bar{\Omega}_1}(N\setminus N_2, P; N_1\setminus N_2, \bar{g})$ such that df' is homotopic to $h|(N\setminus N_2)$. Extend f' to a smooth map

$$f=\begin{cases} f' & \text{on } N\setminus N_2 \\ f_h & \text{on } W. \end{cases}$$

Then f has the required properties. Q.E.D.

§ 3. Proof of Theorem 1.

First we shall prove the relative form of Theorem 1 for the special case that N is an open set of $\boldsymbol{R}^n$ and $P=\boldsymbol{R}^p$.

PROPOSITION 3.1. *Let $n\geqq p\geqq 2$ and N be an open submanifold of $\boldsymbol{R}^n$. Let s be a smooth section of $\Omega_r(N, \boldsymbol{R}^p)$ over N for which there exists an open set U in N and a smooth map g of a neighbourhood of $N\setminus U$ into $\boldsymbol{R}^p$ such that $j^k g=s$ on $N\setminus U$ and that $j^k g$ is transverse to every submanifold $\Sigma^{I\iota}(N, \boldsymbol{R}^p)$ on $N\setminus U$. Then there exists a smooth map f such that $j^k f(N)\subset\Omega_r(N, \boldsymbol{R}^p)$ and $j^k f$ is homotopic to s relative to $N\setminus U$ as sections of $\Omega_r(N, \boldsymbol{R}^p)$ over N.*

We will need the following lemma.

LEMMA 3.2. *For a section s given in Proposition 3.1 there exists a homotopy s_λ $(0\leqq\lambda\leqq 1)$ of sections of $\Omega_r(N, \boldsymbol{R}^p)$ over N such that*

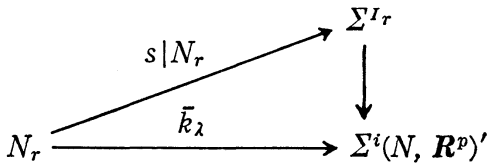
(1) *$s_0=s$ and $s_\lambda|N\setminus U=s|N\setminus U$ for any λ,*

(2) *s_1 is transverse to every $\Sigma^{I\iota}(N, \boldsymbol{R}^p)$,*

(3) *there is a smooth map $\bar{g}$ of a neighbourhood V of $(s_1)^{-1}(\Sigma^i(N, \boldsymbol{R}^p))$ and $N\setminus U$ into $\boldsymbol{R}^p$ with $\bar{g}|(N\setminus U)=g|(N\setminus U)$ and $j^k\bar{g}|V=s_1|V$.*

PROOF. We prove the lemma by induction on r. In the proof we simply write Σ^{Ir} for $\Sigma^{Ir}(N, \boldsymbol{R}^p)$ and N_r for $s^{-1}(\Sigma^{Ir})$. First we show that s is homotopic to a section represented by a smooth map on a neighbourhood of N_r and $N\setminus U$. We may suppose that $N_r\cap U$ is nonempty and that s is transverse to Σ^{Ir}. We use the same notations in Section 1. Let d_1'' denote the homomorphism of TN into $T\boldsymbol{R}^p$ induced from d_1 by s. It follows from Section 1 that $TN_r\subset(s|N_r)^*(T(\Sigma^{Ir}))$ and that $\boldsymbol{K}_1\cap T(\Sigma^{Ir})=\{0\}$ since $s^{-1}(\Sigma^{Ir+1})$ is empty. This means that $d_1''|TN_r$ is an injective homomorphism. By Hirsch's immersion theorem [9] we have a homotopy of monomorphisms $k_\lambda: TN_r\to T\boldsymbol{R}^p$ covering a homotopy relative to $(N\setminus U)\cap N_r$, $i_\lambda: N_r\to\boldsymbol{R}^p$ such that $k_0=d_1''|TN_r$ and i_1 is an immersion with $k_1=d(i_1)$. Extend k_λ to a homotopy $\bar{k}_\lambda: TN|N_r\to T\boldsymbol{R}^p$ of homomorphisms of rank $p-1$ for any λ so that $\bar{k}_0=d_1''$ over N_r. By using $\bar{k}_\lambda$ we can deform s to s_1 in $\Omega_r(N, \boldsymbol{R}^p)$ so that $d_1''|TN_r$ induced from d_1 by s_1 coincides with $d(i_1)$.

In fact we may apply the covering homotopy property of the fibre bundle $\pi_1^k|\Sigma^{I_r}:\Sigma^{I_r}\to\Sigma^i(N,\boldsymbol{R}^p)'$ (see Remark 1.1) to the following;



where $\bar{k}_\lambda$ is identified with a map sending a point x of N_r into $(\bar{k}_\lambda)|(TN)_x$. Let $s'_\lambda : N_r\to\Sigma^{I_r}$ be the homotopy over $\bar{k}_\lambda$ with $s'_0=s|N_r$. Since s is transverse to Σ^{I_r} we can extend s'_λ to a homotopy $s_\lambda : N\to\Omega_r(N,\boldsymbol{R}^p)$ relative to $N\setminus U$ so that $(s_\lambda)^{-1}(\Sigma^{I_r})=N_r$ and s_λ is transverse to Σ^{I_r} for any λ.

Let $(s_1)^*\boldsymbol{K}_i=K_i$, $(s_1)^*\boldsymbol{Q}_1=Q_1$ and d''_{t+1} be induced from d_{t+1} by s_1. Then the normal bundle of N_t in N is $\mathrm{Hom}(K_1\oplus(\bigoplus_{u=2}^{t}\overset{u}{\otimes}K_2),\ Q_1)$ and $d''_{t+1} : TN_{t-1}\to\mathrm{Hom}(\overset{t}{\otimes}K_2,\ Q_1)$ over N_t is a surjective homomorphism by Section 1. Therefore we obtain line bundles $L_2,\cdots,L_r$ in $TN|N_r$ such that L_t is mapped isomorphically onto $\mathrm{Hom}(\overset{t}{\otimes}K_2,\ Q_1)$ by d''_{t+1}. Then $K_1, L_2,\cdots,L_r$ are linearly independent in $TN|N_r$ and span $TN|N_r$ together with TN_r. Here we fix a diffeomorphism h of a neighbourhood of the zero section of $K_1|N_r\oplus\bigoplus_{t=2}^{r}L_t$ on a sufficiently small neighbourhood $U(N_r)$ of N_r in N. Next we consider a system of local coordinates of P near the image of N_r. We take a metric of TP. Since $d''_1|TN$ is of rank $p-1$ over N_r we have the orthogonal line bundle Q'_1 in $(i_1)^*TP$. Let $j : Q'_1\to(i_1)^*TP$ be the inclusion. Then $d''_1|(L_2\oplus\cdots\oplus L_r)\oplus j$ is an injective homomorphism. So we have an immersion i_P of a neighbourhood of the zero section of $L_2\oplus\cdots\oplus L_r\oplus Q'_1$ into P such that $i_P|N_r=i_1$.

Now we construct a smooth map f_s of a small neighbourhood V of N_r in $U(N_r)$ into P such that

(1) $(s_1)^*\boldsymbol{K}_i=(j^kf_s)^*\boldsymbol{K}_i=K_i$ ($i=1$ and 2) and $(s_1)^*\boldsymbol{Q}_1=(j^kf_s)^*\boldsymbol{Q}_1=Q_1$ over N_r,

(2) $d''_{t+1}|(TN_{t-1}|N_r)$ and the induced homomorphism of $TN_{t-1}|N_r$ into $\mathrm{Hom}(\overset{t}{\otimes}K_2,\ Q_1)$ from d_{t+1} by j^kf_s coincide over N_r for $1\leqq t\leqq r$.

In fact, let x be any point of N_r with $s_1(x)\in\Sigma^{I_r}_s(N,\boldsymbol{R}^p)$. We take a system of local coordinates $(t_1,\cdots,t_{p-r},k,k_1,\cdots,k_{i-1},l_2,\cdots,l_r)$ near x in N so that

(a) $(t_1,\cdots,t_{p-r})$ is a system of local coordinates of N_r near x,

(b) $k_1,\cdots,k_{i-1}$ are local coordinates coming from K_1/K_2 by h for which $d''_2(x)$ corresponds to the quadratic form $-\sum_{t=1}^{s}k_t^2+\sum_{t=s+1}^{i-1}k_t^2$,

(c) k comes from K_2 and l_t from L_t by h.

Since $i_1 : N_r\to P$ is an immersion, we can take $(t_1,\cdots,t_{p-r},l_2,\cdots,l_r,t_p)$ as a system of local coordinates of P near $i_1(x)$ where t_p comes from Q'_1 by i_P. Then f_s is given by the following normal form of a smooth map with a Morin singu-

larity of symbol I_r in [**14**] in a small neighbourhood of x.

$$(*)\quad (t_1, \cdots, t_{p-r}, k, k_1, \cdots, k_{i-1}, l_2, \cdots, l_r) \longrightarrow$$
$$(t_1, \cdots, t_{p-r}, l_2, \cdots, l_r,$$
$$(1/2)(-\sum_{1}^{s} k_t^2 + \sum_{s+1}^{i-1} k_t^2) + \sum_{t=2}^{r} (1/(t-1)!) l_t k^{t-1} + (1/(r+1)!) k^{r+1}).$$

We must note the compatibility of f_s and g. By [**14**] we can express g near every point of $(N \setminus U) \cap N_r$ as in (*) together with the properties (a), (b) and (c). After expressing g in this way we extend the coordinates to those in (a), (b) and (c) to construct f_s.

Now we construct a homotopy s_λ of $s_1 | N_r$ and $j^k f | N_r$ by using the structure of the vector bundle of $J^k(N, \boldsymbol{R}^p)$. Let

$$s_\lambda = (2-\lambda) s_1 | N_r + (\lambda - 1) j^k f | N_r \qquad (1 \leqq \lambda \leqq 2).$$

It follows from (1) and (2) that s_λ gives a homotopy of N_r into Σ^{I_r} and that s_λ induces the homotopy of bundle maps between normal bundles of N_r and Σ^{I_r}. Hence s_λ is extendable to a homotopy $\bar{s}_\lambda$ of V into a tubular neighbourhood of Σ^{I_r} relative to $V \cap (N \setminus U)$ so that $\bar{s}_\lambda$ is transverse to Σ^{I_r} and $(\bar{s}_\lambda)^{-1}(\Sigma^{I_r}) = N_r$. Then we can extend $\bar{s}_\lambda$ to a homotopy $\bar{s}_\lambda$ of N into $\Omega_r(N, P)$ relative to $N \setminus U$ so that $\bar{s}_2$ is transverse to Σ^{I_r} with $(\bar{s}_2)^{-1}(\Sigma^{I_r}) = N_r$ which is what we want.

Now we can prove the lemma by induction on r. The case of $r=1$ follows from the above result. For the case $r>1$, we use the above homotopy $\bar{s}_\lambda$ $(0 \leqq \lambda \leqq 2)$ and the inductive hypothesis of the case $r-1$ for $\bar{s}_2$. Since $\bar{s}_2$ is already represented by a smooth map on a neighbourhood of $N \setminus U$ and V, we can construct a homotopy $\bar{s}_\lambda$ $(2 \leqq \lambda \leqq 3)$ of N into $\Omega_r(N, P)$ relative to $N \setminus U$ with properties requested. Q.E.D.

PROOF OF PROPOSITION 3.1. Let U' be an open set with $\bar{U}' \subset U$ such that g is defined on $N \setminus U'$. By Lemma 3.2 we may suppose that the given section s has the properties (2) and (3) for U' of Lemma 3.2. Furthermore we may deform s so that $s^{-1}(\Sigma_u^{(i,0)}(N, \boldsymbol{R}^p) \cap U)$ is nonempty for any u. Then the section $\pi_1^k \circ s$ of $\mathrm{Hom}(TN, TP)$ over N becomes an element of $\mathrm{Hom}_{\Omega_r}(TN, TP; \{N_t\}, g)$. It follows from Theorem 2.1 that there exists a smooth map f of $C^\infty_{\Omega_r}(N, P; \{N_t\}, g)$ so that $\pi_1^k \circ s$ and df are homotopic in $\mathrm{Hom}_{\Omega_r}(TN, TP; \{N_t\}, g)$ by a homotopy s_λ with $s_0 = \pi_1^k \circ s$ and $s_1 = df$. By the definition every s_λ is realized by a smooth map $f_{(s_\lambda)}$ in a sufficiently small neighbourhood V of N_1 with Properties (H-1, 2 and 3). Since a fibre of π_1^k is contractible, there exists a lift $\bar{s}_\lambda$ of N into $J^k(N, P)$ covering s_λ with $s_0 = s$, $\bar{s}_1 = j^k f$ and $\bar{s}_\lambda | V = j^k f_{(s_\lambda)}$. Since s_λ is of maximal rank outside of N_1, it follows that $\bar{s}_\lambda$ is a homotopy of sections of $\Omega_r(N, \boldsymbol{R}^p)$ over N. This is what we want to prove. Q.E.D.

PROOF OF THEOREM 1. Let $\{V_\alpha\}$ be an open covering of P, each of which

is diffeomorphic to $\boldsymbol{R}^p$. Since N is compact we may take a finite covering U_1, $\cdots$, U_m of N such that every U_t is diffeomorphic to an open set of $\boldsymbol{R}^n$ and $\bar{U}_t$ is mapped into some V_α, say V_t by $\pi^k_P \circ s$. We show the following assertion (A_q) by induction on q.

(A_q) There exists a smooth map f_q of a neighbourhood W_q of $\bigcup_{i=1}^q \bar{U}_i$ into P such that $j^k f_q$ is homotopic to $s|W_q$ in $\Omega_r(W_q, P)$.

Let W_1 be a small neighbourhood of $\bar{U}_1$ in $(\pi^k_P \circ s)^{-1}(V_1)$. Then A_1 follows from Proposition 3.1 since V_1 is diffeomorphic to $\boldsymbol{R}^p$. We prove A_{q+1} under the inductive assumption of A_q. Take an open neighbourhood U of $\bigcup_{i=1}^q \bar{U}_i$ such that $\bar{U} \subset W_q$ and a neighbourhood O_{q+1} of $\bar{U}_{q+1}$ in $(\pi^k_P \circ s)^{-1}(V_{q+1})$. We apply Proposition 3.1 to a section $s|O_{q+1}$ and a smooth map $f_q|O_{q+1} \cap W_q$. Then we may extend f_q to a smooth map $f'_{q+1} : O_{q+1} \to V_{q+1}$ so that $j^k f'_{q+1}$ is homotopic to $s|O_{q+1}$ relative to $O_{q+1} \cap \bar{U}$. Then we can define $f_{q+1} : U \cup O_{q+1} \to P$ by

$$f_{q+1} = \begin{cases} f'_{q+1} & \text{on } O_{q+1} \\ f_q & \text{on } U. \end{cases}$$

If we put $W_{q+1} = U \cup O_{q+1}$, then f_{q+1} is a required smooth map. Q.E.D.

REMARK 3.3. By the similar proof of Theorem 1 we can show the relative form of Theorem 1 as Proposition 3.1.

§ 4. Topological properties of $\Sigma^{I_r}(n, p)$.

Let $\Sigma^{I_r}(n, p)$ (or simply Σ^{I_r}) be the fibre of $\Sigma^{I_r}(\boldsymbol{R}^n, \boldsymbol{R}^p)$ over the origin $(0, 0)$. In order to study the number of connected components of $\Sigma^{I_r}(n, p)$ we use the Boardman's construction of Remark 1.1 for $J^k(\boldsymbol{R}^n, \boldsymbol{R}^p)$ restricted on the fibre $J^k(n, p)$ over $(0, 0)$. By fixing bases of $\boldsymbol{R}^n$ and $\boldsymbol{R}^p$ we obtain a canonical identification

$$h : J^{t+1}(n, p) \longrightarrow J^t(n, p) \times \mathrm{Hom}(S^{t+1}\boldsymbol{R}^n, \boldsymbol{R}^p)$$

mapping a jet z to $\pi^{t+1}_t(z)$ and $t+1$ derivations of z. Let $\Sigma^{I_{t'}}$ be the fibre of $\Sigma^{I_t}(\boldsymbol{R}^n, \boldsymbol{R}^p)'$ over the origins. Let $K_1 = \boldsymbol{K}'_1|\Sigma^{i'}$, $Q_1 = \boldsymbol{Q}'_1|\Sigma^{i'}$ and $K_2 = \boldsymbol{K}'_2|\Sigma^{I_{2'}}$. We identify $(\pi^t_{\boldsymbol{R}^u})^* T_0 \boldsymbol{R}^u$ with $J^t(n, p) \times \boldsymbol{R}^u$ for $u = n$ or p. By the construction of Section 1 and Remark 1.1 we have inclusions

$$i_2 : \overset{2}{\bigcirc} K_1 \longrightarrow \Sigma^{i'} \times S^2\boldsymbol{R}^n \qquad \text{over} \quad \Sigma^{i'}$$

$$i_{t+1} : \overset{t+1}{\bigcirc} (\pi^t_2)^* K_2 \longrightarrow \Sigma^{I_{t'}} \times S^{t+1}\boldsymbol{R}^n \qquad \text{over} \quad \Sigma^{I_{t'}} \quad (t \geqq 2)$$

where $\overset{u}{\bigcirc} K_i$ denote the u symmetric product of K_i and the projection

$$e : \Sigma^{I_{t'}} \times \boldsymbol{R}^p \longrightarrow (\pi^t_1)^* Q_1 \qquad \text{over} \quad \Sigma^{I_{t'}}.$$

Then we can define the following homomorphisms by using i_t and e $(t\geqq 2)$

$$k_2 : \Sigma^{i'}\times \mathrm{Hom}(S^2\boldsymbol{R}^n, \boldsymbol{R}^p) \longrightarrow \mathrm{Hom}(\overset{2}{\bigcirc}K_1, Q_1) \quad \text{over} \quad \Sigma^{i'}$$

$$k_{t+1} : \Sigma^{I_t'}\times \mathrm{Hom}(S^{t+1}\boldsymbol{R}^n, \boldsymbol{R}^p) \longrightarrow \mathrm{Hom}(\overset{t+1}{\bigcirc}(\pi_2^t)^*K_2, (\pi_2^t)^*Q_1) \quad \text{over} \quad \Sigma^{I_t'}.$$

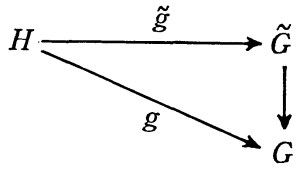
It follows from the definition of d'_{t+1} that $\Sigma^{I_2'}$ is the set of all 2 jets z in $\Sigma^{i'}\times \mathrm{Hom}(S^2\boldsymbol{R}^n, \boldsymbol{R}^p)$ for which $k_2(z)$ is a quadratic form of rank $i-1$ and $\Sigma^{I_{t+1}'}(t\geqq 2)$ is the set of all $t+1$ jets z in $\Sigma^{I_t'}\times \mathrm{Hom}(S^{t+1}\boldsymbol{R}^n, \boldsymbol{R}^p)$ for which $k_{t+1}(z)$ is a null homomorphism. Now we calculate the number of connected components of $\Sigma^{I_r'}$, that is, of Σ^{I_r}.

Let γ be the canonical i dimensional vector bundle over the grassmann manifold $G_{i,n-i}$ of all i spaces in $\boldsymbol{R}^n$. Then γ becomes a subbundle of the trivial n bundle θ^n over $G_{i,n-i}$. Let Σ denote the connected subspace of all homomorphisms of maximal rank in $\mathrm{Hom}(\theta^n/\gamma, \theta^p)$ over $G_{i,n-i}$ $(i=\max(1, n-p+1))$. Let $K_{i,z}$ and $Q_{1,z}$ be fibres determined by z. Then $\Sigma^{i'}$ is canonically identified with Σ by mapping a 1 jet z to a homomorphism $\bar{d}_{1,z} : \boldsymbol{R}^n/K_{1,z}\to\boldsymbol{R}^p$ induced from d'_1 over $K_{1,z}\in G_{i,n-i}$.

Next we see $\Sigma^{I_r'}$ $(r\geqq 2)$. For a jet z of $\Sigma^{I_r'}$ we have the symmetric quadratic form of rank $i-1$, $k_2(\pi_2^r(z))$ over $(\pi_1^r)(z)$. If we fix an orientation of Q_1, we can define an index of $k_2(\pi_2^r(z))$, say s. So the number $\min(s, i-s)$ is well defined for z which we call the semi index of z. Let $\Sigma_s^{I_r}(n, p)$ $(=\Sigma_s^{I_r})$ denote the set of all jets z with the semi index s of Σ^{I_r}. We show that $\Sigma_s^{I_r}$ is connected. Let H be the subset of $\mathrm{Hom}(\overset{2}{\bigcirc}K_2, Q_1)$ consisting of all quadratic forms of rank $i-1$ with semi index s. Let $\tilde{\gamma}$ be the canonical i bundle over the oriented grassmann manifold $\tilde{G}_{i,n-i}$. Let $G_{1,i-1}(\gamma)$ (resp. $G_{i,n-i}(\tilde{\gamma})$) be the associated grassmann bundle of γ (resp. $\tilde{\gamma}$). Let G (resp. $\tilde{G}$) denote the fibre product of Σ and $G_{1,i-1}(\gamma)$ (resp. $G_{1,i-1}(\tilde{\gamma})$) over $G_{i,n-i}$. For z of $\Sigma_s^{I_r'}$ we define a map

$$g : H \longrightarrow G$$

by $g(k_2(z))=(K_{1,z}, K_{2,z}, \bar{d}_{1,z})$. If $n>p$ and $s\neq(i-1)/2$, then we can define a map $\tilde{g}$ so that the following diagram commutes



where the vertical map comes from the covering map. In fact, for $z\in\Sigma_s^{I_2'}$ we can choose orientation of $Q_{1,z}$ so that the index of $k_2(z)$ is s. This orientation of $Q_{1,z}$ determines an orientation of $K_{1,z}$ denoted by $o(K_{1,z})$. So we define $\tilde{g}$ by $\tilde{g}(z)=(K_{1,z}, o(K_{1,z}), K_{2,z}, \bar{d}_{1,z})$. Let $\tilde{K}_1$ and $\tilde{K}_2$ be the induced bundles of dimen-

sions i and 1 over G or $\tilde{G}$ of γ and the canonical line bundle ever $G_{1,i-1}(\gamma)$ respectively. Let $\tilde{Q}_1$ be the induced bundle of Q_1 by the projection of G or $\tilde{G}$ onto $\Sigma^{i'}$. Then we define a map

$$u : \mathrm{Hom}(\tilde{K}_1 \bigcirc \tilde{K}_1, \tilde{Q}_1) \longrightarrow \mathrm{Hom}(\tilde{K}_2 \bigcirc \tilde{K}_1, \tilde{Q}_1)$$

induced from the inclusion $\tilde{K}_2 \bigcirc \tilde{K}_1 \subset \tilde{K}_1 \bigcirc \tilde{K}_1$ and an injection

$$i_H : H \longrightarrow \mathrm{Hom}((\tilde{K}_1/\tilde{K}_2) \bigcirc (\tilde{K}_1/\tilde{K}_2), \tilde{Q}_1) \cong \mathrm{Ker}(u)$$

so that i_H maps $k_2(z)$ onto a nonsingular quadratic form induced from $k_2(z)$ over $g(k_2(z))$ or $\tilde{g}(k_2(z))$.

Then the image of H is the space of all nonsingular quadratic forms of given index s which becomes connected. Hence $\Sigma^{I_{2'}}_s$ is connected for $n>p$ and $s\neq(i-1)/2$. Other cases for $\Sigma^{I_{2'}}_s$ follows similarly. The connectedness of $\Sigma^{I_r}_s$ $(r\geqq 3)$ follows by induction on r since $\Sigma^{I}{}^{t+1'}_{t+1}$ is the inverse image of the zero section of $\mathrm{Hom}(\overset{t+1}{\bigcirc}(\pi^t_2)^*K_2, (\pi^t_2)^*Q_1)$ by k_{t+1}. Therefore we have

PROPOSITION 4.1. (1) *If $n\leqq p$, then $\Sigma^{I}{}^{r}(n, p)$ is connected. If $n>p$, then $\Sigma^{I_r}_s(n, p)$ is connected.*

(2) *If $n>p$, then $\Sigma^{I_r}_s(n, p)\setminus\Sigma^{I_{r+1}}_s(n, p)$ has two connected components $(r\geqq 2)$.*

The next question is to see whether $\Sigma^{I_r}_s(n, p)$ is orientable or not.

PROPOSITION 4.2. *(Case 1 ; $n\leqq p$) $\Sigma^{I}{}^{r}(n, p)$ is orientable if and only if either $n+p+r+r(r+1)/2\equiv 0$ or $n=i=1$.*

(Case 2 ; $n>p$) $\Sigma^{I_r}_s(n, p)$ is orientable in the following cases

(i) *$s\neq(i-1)/2$ and $r(r+1)/2\equiv 1 \pmod 2$*

(ii) *$s=(i-1)/2$, $r\equiv 1 \pmod 2$ and $r(r+1)/2\equiv 1 \pmod 2$*

(iii) *$s=(i-1)/2$, $n=i$ and $r(r+1)/2\equiv 1 \pmod 2$.*

Otherwise $\Sigma^{I_r}_s(n, p)$ is nonorientable.

PROOF. In the proof we write Σ^I for $\Sigma^{I}{}^{r'}$. For the proof we will calculate the first Stiefel-Whitney class of Σ^I. Let p_r be the projection of Σ^I_s onto G $(r\geqq 2)$. Since H is an open set of $\mathrm{Ker}(u)$, we obtain that the tangent bundle of Σ^I is isomorphic to the Whitney sum of $(p_r)^*TG$, $(p_r)^*(\mathrm{Ker}(u))$ and $\oplus^r_{t=3}(\pi^r_t)^*(\mathrm{Ker}(k_t))\,|\,\Sigma^I$. Hence we have

$$W_1(\Sigma^I)=(p_r)^*(W_1(G)+W_1(\mathrm{Ker}(u)))+(\pi^r_t)^*(\sum_{t=3}^{r} W_1(\mathrm{Ker}(k_t))).$$

$$\begin{aligned} W_1(\mathrm{Ker}(u)) &= W_1(\mathrm{Hom}(\tilde{K}_2 \bigcirc \tilde{K}_1, \tilde{Q}_1)) \\ &= W_1(\tilde{K}_2 \bigcirc \tilde{K}_1)+iW_1(\tilde{Q}_1) \\ &= iW_1(\tilde{K}_2)+W_1(\tilde{K}_1)+iW_1(\tilde{K}_1) \\ &= iW_1(\tilde{K}_2)+(i+1)W_1(\tilde{K}_1), \end{aligned}$$

$$W_1(\sum_{t=3}^{r}\mathrm{Ker}(k_t))=\sum_{t=3}^{r}(W_1(\bigcirc^{t}\tilde{K}_2)+W_1(\tilde{Q}_1))$$
$$=\sum_{t=3}^{r}(tW_1(\tilde{K}_2)+W_1(\tilde{Q}_1))$$
$$=(r(r+1)/2-3)W_1(\tilde{K}_2)+(r-2)W_1(\tilde{Q}_1).$$

It follows from the standard topological properties of grassmann manifolds that $W_1(G)=(n+p+1)W_1(\tilde{K}_1)+iW_1(\tilde{K}_2)$. Therefore we have

$$W_1(\Sigma^I)=p_r^*\{(n+p+r+i)W_1(\tilde{K}_1)+(r(r+1)/2-3)W_1(\tilde{K}_2)\}.$$

If $n>p$, $s\neq(i-1)/2$ $(i>1)$, then it follows that $p_r^*(W_1(\tilde{Q}_1))=0$. Since the fibre of $\tilde{g}$ is connected, $p_r^*: H^1(\tilde{G}\ ;\ \boldsymbol{Z}/2)\to H^1(\Sigma_s^I\ ;\ \boldsymbol{Z}/2)$ is injective. Hence $W_1(\Sigma^I)$ is zero if and only if $r(r+1)/2\equiv 1 \pmod 2$.

If $n>p$, $s=(i-1)/2$ $(i>1)$, then Σ_s^I is orientable if and only if (i) either $r\equiv 1 \pmod 2$ or $n=i$ and (ii) $r(r+1)/2\equiv 1 \pmod 2$ by the similar argument. If $n\leqq p$, then $i=1$ and $\tilde{K}_1=\tilde{K}_2$. Hence $\Sigma^I(n, p)$ is orientable if and only if either $n+p+r+r(r+1)/2\equiv 0 \pmod 2$ or $n=i=1$. Q.E.D.

Let $\overline{\Sigma^{I_r}(n, p)}$ denote the topological closure of $\Sigma^{I_r}(n, p)$ in $J^k(n, p)$. It follows that both of $\overline{\Sigma^{I_r}(n, p)}$ and $\overline{\Sigma^{I_r}(n, p)}\setminus\Sigma^{I_r}(n, p)$ are algebraic sets in a Euclidean space $J^k(n, p)$. Let $d_{I_r}(n, p)$ be the dimension of $\overline{\Sigma^{I_r}(n, p)}$ as an algebraic set.

PROPOSITION 4.3. *Let $r\geqq 2$. The dimension of $\overline{\Sigma^{I_r}(n, p)}\setminus\Sigma^{I_r}(n, p)$ is smaller than $d_{I_r}(n, p)-1$.*

For a while we identify $J^k(n, p)$ with $\mathrm{Hom}(\oplus_{i=1}^{k}S^i\boldsymbol{R}^n, \boldsymbol{R}^p)$ by fixing a basis of $\boldsymbol{R}^n$ and $\boldsymbol{R}^p$. Then we write $z\in J^k(n, p)$ as $(z_1, \cdots, z_k)$ for homomorphisms $z_i: S^i\boldsymbol{R}^n\to\boldsymbol{R}^p$. For subspaces $L_2\subset L_1\subset\boldsymbol{R}^n$ with $L_1\subset\mathrm{Ker}(z_1)$ we define

$$h_z : \boldsymbol{R}^n/L_1\oplus L_2\bigcirc L_1\oplus(\bigoplus_{t=3}^{r}S^tL_2)\longrightarrow\boldsymbol{R}^p$$

by restricting z_i $(i\geqq 2)$ and $\bar{z}_1 : \boldsymbol{R}^n/L_1\to\boldsymbol{R}^p$ induced from z_1.

LEMMA 4.4. *An element z of $\Sigma^{i+s}(n, p)$ belongs to $\overline{\Sigma^{I_r}(n, p)}$ if and only if* (i) *$s\geqq 0$ and* (ii) *there exist an i dimensional subspace L_1 in $\mathrm{Ker}(d_{1,z})$ and 1 dimensional subspace L_2 in L_1 such that the kernel rank of the homomorphism*

$$h_z : \boldsymbol{R}^n/L_1\oplus L_2\bigcirc L_1\oplus(\bigoplus_{t=3}^{r}S^tL_2)\longrightarrow\boldsymbol{R}^p$$

is not less than $i+r-2$.

PROOF. Let z be an element of $\overline{\Sigma^{I_r}(n, p)}$. Then there exists a sequence $\{z^j\}$ in $\Sigma^{I_r}(n, p)$ which converges to z. $K_{1,zj}$ and $K_{2,zj}$ denote the subspaces of dimensions i and 1 in $\boldsymbol{R}^n$ determined by z^j respectively. Then $\{(K_{1,zj}, K_{2,zj})\}$ gives a sequence in $G_{1,i-1}(\gamma)$ which converges to an element, say (L_1, L_2). It is

clear that $L_1 \subset K_{1,z}$. Hence we can define a homomorphism h_z induced from z. Since

$$h_{zj} : \boldsymbol{R}^n / K_{1,zj} \oplus K_{2,zj} \bigcirc K_{1,zj} \oplus (\bigoplus_{t=3}^{r} S^t K_{2,zj}) \longrightarrow \boldsymbol{R}^p$$

converges to h_z and the kernel rank of h_{zj} is $i+r-2$, it follows that the kernel rank of h_z is not less than $i+r-2$.

Conversely we assume the conditions (i) and (ii) for z. Then it follows from the fact $rk(z_1) \leqq n-i$ that we may take a subspace L of dimension $n-i$ which contains the image of h_z and a sequence of homomorphisms $\{a_1^j\}$ of $\boldsymbol{R}^n$ into L of rank $n-i$ converging to z_1 such that $\mathrm{Ker}(a_1^j) = L_1$. Let $z^j = (a_1^j, z_2, \cdots, z_k)$ in $J^k(n, p)$. Then the kernel rank of h_{zj} is $i+r-2$ by the definition of z^j. This is equivalent to say that h_{zj} induces a zero homomorphism of $L_2 \bigcirc L_1 \oplus (\oplus_{t=3}^{r} S^t L_2)$ into $\boldsymbol{R}^p / L$. This means that $\dim \mathrm{Ker}(d_{t,zj}) = 1$ by Section 1 for $r \geqq t \geqq 2$. Hence z^j becomes a sequence of Σ^{I_r}. Clearly $\{z^j\}$ converges to z.

Q.E.D.

PROOF OF PROPOSITION 4.3. Let γ be the $i+s$ dimensional vector bundle over $\Sigma^{i+s}(n, p)$ induced from the canonical $i+s$ dimensional vector bundle over $G_{i+s,n-i-s}$. Let $G_{i,s}(\gamma)$ be its associated grassmann bundle over which we have the canonical i dimensional vector bundle denoted by γ_i. Let γ_2 be the canonical line bundle over $G_{1,i-1}(\gamma_i)$ and γ_1 the induced bundle of γ_i over $G_{1,i-1}(\gamma_i)$. Obviously $G_{1,i-1}(\gamma_i)$ consists of all triples (z, L_1, L_2) where $z \in \Sigma^{i+s}(n, p)$, L_1 is an i dimensional subspace of $K_{1,z}$ and L_2 is a 1 dimensional subspace of L_1. The projection of $G_{1,i-1}(\gamma_i)$ onto $\Sigma^{i+s}(n, p)$ is denoted by p. We consider the following vector bundle over $G_{1,i-1}(\gamma_i)$,

$$\mathrm{Hom}(\theta^n / \gamma_1 \oplus \gamma_2 \bigcirc \gamma_1 \oplus (\bigoplus_{t=3}^{r} \bigcirc^{t} \gamma_2),\ \theta^p).$$

Then we can define a smooth section s by mapping (z, L_1, L_2) to a homomorphism of h_z. It is clear that s is transverse to every manifold S^h of all homomorphisms of kernel rank h. It follows from Lemma 4.4 that $\overline{\Sigma^{I_r}(n, p)} \cap \Sigma^{i+s}(n, p)$ is equal to $p(s^{-1}(\overline{S^{i+r-2}}))$. Now we estimate the codimension of $\overline{\Sigma^{I_r}(n, p)} \cap \Sigma^{i+s}(n, p)$ in the case of either $n \geqq p$, $r > 2$ and $s > 0$ or $n < p$ as follows:

$$\begin{aligned}
&\mathrm{codim}(\overline{\Sigma^{I_r}(n, p)} \cap \Sigma^{i+s}(n, p)) - d_{I_r} \\
&\quad \geqq \mathrm{codim}(\Sigma^{i+s}(n, p)) + \mathrm{codim}(S^{i+r-2}) - \dim G_{1,i-1} - \dim G_{i,s} - d_{I_r} \\
&\quad \geqq (i+s)(p-n+i+s) + (i+r-2)(p-n+i) - is - (i-1) - d_{I_r} \\
&\quad = s(p-n+i) + s^2 + is + (i+r-2)(p-n+i) - is - (i-1) \\
&\quad = (s+i+r-2)(p-n+i) + s^2 - (i-1)
\end{aligned}$$

$$\geqq \begin{cases} s+s^2+r-1 & \text{if } n \geqq p \\ (s+r-1)(p-n+1)+s^2 & \text{if } n<p. \end{cases}$$

If $n \geqq p$, $r=2$ and $s=0$, then we have $\operatorname{codim}\Sigma^{(i,h)} \geqq \operatorname{codim}\Sigma^{(i,1)}+2$ for $h \geqq 2$. This proves the proposition. Q.E.D.

§ 5. Dual classes.

In the rest of the paper I means I_r. In this section we assume that N is a connected and closed manifold. Let c_I be $\operatorname{codim}\Sigma^I(N, P)$. We will define the dual class $c^I_{s,u}$ of $\overline{\Sigma^I_s(N, P)}$ for a section $u: N \to \Omega_r(N, P)$ in $H^{c_I}(N; \boldsymbol{Z})$ in case TN, $(\pi^k_P \circ s)^* TP$ and Σ^I_s are orientable. In the other cases $c^I_{s,u}$ can be definable in $H^{c_I}(N; \boldsymbol{Z}/2)$ similarly and we can also follow the method defining the dual class of $\overline{\Sigma^{I_2}(n, p)}$ in [**15**]. So we omit it. It will be seen that this dual class is the primary obstruction class of u to be homotopic to a section of $\Omega_r(N, P) \setminus \Sigma^I_s(N, P)$ over N. By fixing a structure of a vector bundle on $J^k(N, P)$ (see, for example, [**3**, Chapter 2]) we take a metric of $J^k(N, P)$. Note that $\Sigma^I_s(N, P)$ is invariant under the coordinate changes of this bundle structure. Let J^k_u be the induced vector bundle $(\pi^k_0 \circ u)^* J^k(N, P)$ and $S^I_{s,u}$, $(\pi^k_0 \circ u)^*(\Sigma^I_s(N, P))$. Let D_u (resp. S_u) denote the associated disk (resp. sphere) bundle of J^k_u and D (resp. S), the unit disk (resp. sphere) of $J^k(n, p)$ for a while. Then $\overline{\Sigma^I} \cap D$ is a cone of $\overline{\Sigma^I} \cap S$. By [**13**, Theorem 1] we can triangulate an algebraic set $\overline{\Sigma^I_s} \cap S$ so that $(\overline{\Sigma^I_s} \setminus \Sigma^I_s) \cap S$ becomes its subcomplex.

LEMMA 5.1. *Let P be a finite simplicial complex of dimension p and Q its subcomplex such that $P \setminus Q$ is a connected topological manifold and* $\dim Q \leqq p-2$. *Then we have the following*

(1) *If $P \setminus Q$ is orientable, then $H_{p+1}(CP, P; \boldsymbol{Z}) \cong \boldsymbol{Z}$ and $H^{p+1}(CP, P; \boldsymbol{Z}) \cong \boldsymbol{Z}$.*

(2) *If $P \setminus Q$ is nonorientable, then $H_{p+1}(CP, P; \boldsymbol{Z}) = \{0\}$ and $H^{p+1}(CP, P; \boldsymbol{Z}) = \boldsymbol{Z}/2$.*

The proof of the lemma will be elementary. It follows from Proposition 4.3 and Lemma 5.1 that

$$H_{d_I}(\overline{\Sigma^I_s} \cap D, \overline{\Sigma^I_s} \cap S\ ;\ \boldsymbol{Z}) \cong \boldsymbol{Z}.$$

Therefore we have

$$H_{d_I+n}(\overline{S^I_{s,u}} \cap D_u, \overline{S^I_{s,u}} \cap S_u)\ ;\ \boldsymbol{Z}) \cong \boldsymbol{Z}.$$

Let a generator of this homology group or its image in $H_{d_I}(D_u, S_u\ ;\ \boldsymbol{Z})$ be denoted by $[\overline{S^I_{s,u}}]$. By the Poincaré duality isomorphism $H_*(D_u\ ;\ \boldsymbol{Z}) \cong H^*(N\ ;\ \boldsymbol{Z})$, $[\overline{S^I_{s,u}}]$ is mapped onto an element of $H^{c_I}(N; \boldsymbol{Z})$ denoted by $c^I_{s,u}$. We call $c^I_{s,u}$ the dual class of $\overline{\Sigma^I_s(N, P)}$ for u. The sum of all $c^I_{s,u}$, $s=0, \cdots, [(i-1)/2]$, is

called the dual class of $\overline{\Sigma^I(N, P)}$ for u and denoted by c_u^I. If u is a jet section $j^k f$ of a smooth map f, this class modulo 2 is called the Thom polynomial of $\overline{\Sigma^I(N, P)}$ for f in [**10**].

Let Ω_r denote the fibre of $\Omega_r(\boldsymbol{R}^n, \boldsymbol{R}^p)$ over the origin 0×0. Let $c(u)$ denote the primary obstruction class of u to be homotopic to a section of $\Omega_{r-1}(N, P)$ over N ([**18**]). Then $c(u)$ is an element of $H^{c_I}(N\ ;\ \pi_{c_I}(\Omega_r, \Omega_{r-1}))$. Since $\pi_i(\Omega_r, \Omega_{r-1})$ vanishes for $i<c_I$ and Ω_{r-1} is simply connected except for the case $n=p$ and $I=(1, 1)$, we have

$$\pi_{c_I}(\Omega_r, \Omega_{r-1})\cong H_{c_I}(\Omega_r, \Omega_{r-1}\ ;\ \boldsymbol{Z})$$

by the Hurewicz isomorphism theorem except for the above case. (If this pair is c_I simple, this exception is unnecessary. In fact, for $n=p=r=2$, $\Omega_1(2, 2)$ is simply connected.) On the other hand we have by Alexander duality theorem

$$\begin{aligned} H_{c_I}(\Omega_r, \Omega_{r-1}\ ;\ \boldsymbol{Z}) &\cong H_{c_I}(\Omega_r\cap S, \Omega_{r-1}\cap S) \\ &\cong H^{d_I-1}((\overline{\Sigma^I}\cup\Omega^c)\cap S, \Omega^c\cap S) \\ &\cong H^{d_I-1}(\overline{\Sigma^I}\cap S, (\overline{\Sigma^I}\setminus\Sigma^I)\cap S) \\ &\cong \bigoplus_{s=0}^{[(i-1)/2]} H^{d_I-1}(\overline{\Sigma_s^I}\cap S\ ;\ \boldsymbol{Z}) \end{aligned}$$

where Ω^c denotes the complement of Ω_r in $J^k(n, p)$. Hence we have the following proposition.

PROPOSITION 5.2. *Let $r\geqq 2$ and for the case $n=p$ let $r>2$ or $n=p=r=2$. We assume that if one of $\Sigma_s^I(n, p)$, $s=0, \cdots, [(i-1)/2]$ and N are orientable, then $(\pi_0^k\circ u)^*TP$ is orientable. Then the primary obstruction class $c(u)$ is equal to the direct sum of the dual classes $c_{s,u}^I$, $s=0, \cdots, [(i-1)/2]$ in $H^{c_I}(N\ ;\ \bigoplus_{s=0}^{[(i-1)/2]}G_s)$ where $G_s=\boldsymbol{Z}$ or $\boldsymbol{Z}/2$ depending on whether $\Sigma_s^I(n, p)$ is orientable or not.*

Let u be a smooth section of $\Omega_r(N, P)$ over N transverse to every $\Sigma^{It}(N, P)$. Let $N_{t,s}$ be $u^{-1}(\Sigma_s^{It}(N, P))$. Then $c_{s,u}^{It}$ is equal to the dual class of $N_{t,s}$ in N. We will use the following fact in a proof of Proposition 5.3. Consider a closed submanifold M such that $N\supset M\supset N_{r,s}$. Then the dual class of $N_{r,s}$ is a cup product of that of M in N and that of $N_{r,s}$ in M coming from the cohomology group of N. Here we give a table of orientability of Σ_s^I for the case $r\geqq 2$ and $n>p$ by Proposition 4.2.

r	$4k+2$	$4k+3$	$4k+4$	$4k+5$
Σ_s^I $(s\neq(i-1)/2)$	orientable	non-orientable	non-orientable	orientable
Σ_s^I $(s=(i-1)/2)$	non-orientable	non-orientable	non-orientable	orientable

PROPOSITION 5.3. *Under the assumption as Proposition* 5.2 *in addition to* $c_I=n$, $c^I_{s,u}$ *vanishes in the following cases:*

(i) $n>p$ *and* $s\neq(i-1)/2$,

(ii) $n>p$, $s=(i-1)/2$ *and* $r\equiv 1 \pmod 4$ *and*

(iii) $n\leqq p$ *and* $n+p+r+r(r+1)/2\equiv 0 \pmod 2$.

PROOF. First we show that $c^{It}_{s,u}$ vanishes for $n>p$, $s\neq(i-1)/2$ and $t\equiv 2 \pmod 4$. In this case it follows from Propositions 4.1 and 4.2 that Σ^{It}_s and Σ^{It-1}_s are orientable and that $\Sigma^{It-1}_s\setminus\Sigma^{It}_s$ has two connected components, say Σ^+ and Σ^-. Therefore both of manifolds $N_{t-1,s}$ and $N_{t,s}$ are orientable. Let N^+ denote the closure of $(u)^{-1}(\Sigma^+)$. Then it is clear that $N_{t,s}$ bounds N^+. Hence the dual class of $N_{t,s}$ vanishes in $H^{c(It)}(N\,;\,\boldsymbol{Z})$. Consider the triple $N\supset N_{t,s}\supset N_{r,s}$ for $t\equiv 2 \pmod 4$. Then it follows from the above remark and the above table that the dual class of $N_{r,s}$ for $n>p$ and $s\neq(i-1)/2$ vanishes in $H^{cI}(N, G)$.

It follows from [**16**] and [**1**, Corollary 5.3] that $\Sigma^i(n, p)$ is orientable if $n+p$ is even and that the dual class of c^i_u is an element of 2 torsion if i is odd. Hence if $n>p$, $s=(i-1)/2$ and $r\equiv 1 \pmod 4$, then $p+n$ is even and the dual class of $N_{r,s}$ is an element of 2 torsion of $H^{cI}(N\,;\,\boldsymbol{Z})\cong\boldsymbol{Z}$, that is, vanishes by considering $N\supset N_1\supset N_{r,s}$.

For $n\leqq p$ it follows from Proposition 4.2 that Σ^{I_2} is orientable for $n+p$ odd and Σ^1 is orientable for $n+p$ even. By [**1**, Corollary 5.3] c^1_u and $c^{I_2}_u$ are elements of 2 torsion in both cases. Therefore by the similar arguments c^I_u vanishes. Q.E.D.

§ 6. Proof of Theorem 2.

We have assumed in Theorem 2 that N and P are always orientable for simplicity. However this assumption can be weakened in the first part (1) of Theorem 2 as 'If $\Sigma^{Ir}(n, p)$ and N are orientable, then f^*TP is also orientable.' We prove Theorem 2 in this form.

PROOF OF THEOREM 2. By Theorem 1 a proof is reduced to the problem of finding a section of $\Omega_{r-1}(N, P)$ homotopic to j^kf in $\Omega_r(N, P)$. This is possible if and only if $c^I_{s,jkf}$ vanishes for $0\leqq s\leqq[(i-1)/2]$ by Proposition 5.2. However it vanishes except for the case $s=(i-1)/2$ by Proposition 5.3. Therefore this is equivalent to say that the Thom polynomial of $\overline{\Sigma^I(N, P)}$ for f vanishes. This is the first part of Theorem 2. Furthermore $c^I_{s\,jkf}$ for $s=(i-1)/2$ vanishes in both cases of (i) and (iii). In case (ii) we have always $s\neq(i-1)/2$ since $n-p$ is odd. This is the second part. Q.E.D.

References

[1] Y. Ando, On the elimination of certain Thom-Boardman singularities of order two, J. Math. Soc. Japan, **34** (1982), 241-268.
[2] J. M. Boardman, Singularities of differentiable maps, IHES Publ. Math., **33** (1967), 21-57.
[3] J. Damon, Thom polynomials for contact class singularities, thesis, Harvard Univ., 1972.
[4] A. A. du Plessis, Maps without certain singularities, Comment. Math. Helv., **50** (1975), 363-382.
[5] J. M. Eliasberg, On singularities of folding types, Math. USSR-Izv., **4** (1970), 1119-1134.
[6] J. M. Eliasberg, Surgery of singularities of smooth mappings, Math. USSR-Izv., **6** (1972), 1302-1326.
[7] T. Gaffney, The Thom polynomial of $\overline{\Sigma^{1,1,1,1}}$, Singularities, Proc. Sympos. Pure Math., Vol. 40, Part I, Amer. Math. Soc., 1983, 399-408.
[8] M. L. Gromov, Stable mappings of foliations into manifolds, Math. USSR-Izv., **3** (1969), 671-694.
[9] M. W. Hirsch, Immersions of manifolds, Trans. Amer. Math. Soc., **93** (1959), 242-276.
[10] A. Haefliger and A. Kosinski, Un théorème de Thom sur les singularités des applications différentiables, Séminaire H. Cartan E. N. S., 1956/57, Exposé N° 8.
[11] A. Lascoux, Calcul de certains polynômes de Thom, C. R. Acad. Sci. Paris, **278** (1974), 889-891.
[12] H. I. Levin, Elimination of cusps, Topology, **3** (1965), 263-296.
[13] S. Łojasiewicz, Triangulation of semianalytic sets, Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat., **18** (1959), 87-136.
[14] B. Morin, Formes canoniques des singularités d'une application différentiable, C. R. Acad. Sci. Paris, **260** (1965), 5662-5665, 6503-6506.
[15] F. Ronga, Le calcul des classes duales singularités de Boardman d'ordre deux, Comment. Math. Helv., **47** (1972), 15-35.
[16] F. Ronga, Le calcul de la classe de cohomologie entière duale a Σ^k, Lecture Notes in Math., **192**, Springer, 1971, 313-315.
[17] F. Sergeraert, Expression explicite de certains polynômes de Thom, C. R. Acad. Sci. Paris, **276** (1973), 1661-1663.
[18] N. Steenrod, The Topology of Fibre Bundles, Princeton Univ. Press, Princeton, 1951.
[19] R. Thom, Les singularités des applications différentiables, Ann. Inst. Fourier, **6** (1955-56), 43-87.

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