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On Boolean powers of the group Z and (ω, ω)-weak distributivity

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For a homomorphism from the group Z^N to a Boolean power $Z^{(B)}$, the first author introduced a property "Infinite linearity" in Section 2 of [2], where Z^N is the direct product of countable copies of the group Z of integers and B is a complete Boolean algebra. There it was proved that (ω, ω)-weak distributivity of B implied infinite linearity of every homomorphism from Z^N to $Z^{(B)}$. In this paper we show that the same thing holds for a countably complete Boolean algebra (ccBa) B. It is known that any ccBa B is a quotient of a certain countably additive field F of subsets of the Stone space of B by the ideal of subsets of first category. This quotient map induces a homomorphism π from $Z^{(F)}$ to $Z^{(B)}$, where the Boolean power $Z^{(F)}$ is isomorphic to the group consisting of all F-measurable functions from the Stone space to Z and π corresponds to the quotient homomorphism modulo first category. We show that infinite linearity of $h: Z^N \to Z^{(B)}$ is equivalent to the existence of a lifting homomorphism $\tilde{h}: Z^N \to Z^{(F)}$ of h, i.e., $h = \pi \cdot \tilde{h}$. Infinite linearity of h also implies the existence of lifting homomorphisms of other quotient homomorphisms onto $Z^{(B)}$ with a certain property. Finally we show (ω, ω)-weak distributivity of certain quotient Boolean algebras. According to them we get another proof and an improvement of a result of [6] concerning a lifting problem of homomorphisms.

Our notation and terminology are common with those of [2], so see [2] for undefined notations. All groups in this paper are abelian and homomorphisms are group theoretic ones.

1. Infinite linearity and lifting.

Differing from [2], we only concern proper sequences of countable length. First we restate a few definitions for a countable case and prove some properties of proper sequences of countable length of $Z^{(B)}$ for a countably complete Boolean algebra (ccBa) B. B always stands for a ccBa.

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DEFINITION 1. An element x of a Boolean power $\boldsymbol{Z}^{(\boldsymbol{B})}$ is a function from $\boldsymbol{Z}$ to $\boldsymbol{B}$ such that $\bigvee_{a\in\boldsymbol{Z}} x(a)=\mathbf{1}$ and $x(a)\wedge x(b)=\mathbf{0}$ for $a\neq b$. For $x,\ y\in\boldsymbol{Z}^{(\boldsymbol{B})}$, $x+y$ is the element of $\boldsymbol{Z}^{(\boldsymbol{B})}$ such that $x+y(a)=\bigvee_{a=b+c} x(b)\wedge y(c)$.

A sequence $(x_n : n\in N)$ is a proper sequence of $\boldsymbol{Z}^{(\boldsymbol{B})}$ if there exists a partition P of $\mathbf{1}$ such that $b\leqq x_n(0)$ for almost all n for each $b\in P$, i.e., $\bigvee P=\mathbf{1}$, $b\wedge c=\mathbf{0}$ for distinct $b,\ c\in P$ and $\forall b\in P(\exists m\forall n\geqq m(b\leqq x_n(0)))$.

PROPOSITION 1. *Let* $(x_n : n\in N)$ *be a sequence of elements of* $\boldsymbol{Z}^{(\boldsymbol{B})}$. $(x_n : n\in N)$ *is a proper sequence iff* $\bigvee_m \bigwedge_{n\geqq m} x_n(0)=\mathbf{1}$.

PROOF. Let $(x_n : n\in N)$ be a proper sequence and P a related partition of $\mathbf{1}$. Suppose that $\bigvee_m \bigwedge_{n\geqq m} x_n(0)\neq\mathbf{1}$, then $\mathbf{0}\neq b=-\bigvee_m \bigwedge_{n\geqq m} x_n(0)$. Since $\bigvee P=\mathbf{1}$, there exists a $c\in P$ such that $b\wedge c\neq\mathbf{0}$. There exists m_0 such that $b\wedge c\leqq \bigwedge_{n\geqq m_0} x_n(0)$, because $c\in P$. Now $\mathbf{0}\neq b\wedge c\leqq(-\bigvee_m \bigwedge_{n\geqq m} x_n(0))\wedge \bigwedge_{n\geqq m_0} x_n(0)=\mathbf{0}$ which is a contradiction.

For the other direction of the proof, we only need a pairwise disjoint refinement of $\{\bigwedge_{n\geqq m} x_n(0) : m\in N\}$ and it is easy to get it.

Let $\bar{\boldsymbol{B}}$ be the canonical completion of $\boldsymbol{B}$, i. e., $\bar{\boldsymbol{B}}$ is a complete Boolean algebra which includes $\boldsymbol{B}$ as a subalgebra and for any non-zero element b of $\bar{\boldsymbol{B}}$ there exists a non-zero element of $\boldsymbol{B}$ that is less than or equal to b.

We remark that $\boldsymbol{Z}^{(\boldsymbol{B})}$ is a subgroup of $\boldsymbol{Z}^{(\bar{\boldsymbol{B}})}$ naturally.

PROPOSITION 2. *Let* $(x_n : n\in N)$ *be a sequence of elements of* $\boldsymbol{Z}^{(\boldsymbol{B})}$. *The sequence* $(x_n : n\in N)$ *is a proper sequence of* $\boldsymbol{Z}^{(\boldsymbol{B})}$ *iff it is a proper sequence of* $\boldsymbol{Z}^{(\bar{\boldsymbol{B}})}$.

PROOF. Since the infinite sums are preserved under the canonical completion, the proposition is clear by Proposition 1.

We use the following notations as in [2]. $[\![x=\check{a}]\!]=x(a)$ for $x\in\boldsymbol{Z}^{(\boldsymbol{B})}$ and $a\in\boldsymbol{Z}$, and $[\![x=y]\!]=\bigvee_{a\in\boldsymbol{Z}}(x(a)\wedge y(a))$ for $x,\ y\in\boldsymbol{Z}^{(\boldsymbol{B})}$. This notation is convenient when we use a Boolean extension of the universe.

PROPOSITION 3. *Let* $(x_n : n\in N)$ *be a proper sequence of* $\boldsymbol{Z}^{(\boldsymbol{B})}$, *then there exists a unique* $y\in\boldsymbol{Z}^{(\boldsymbol{B})}$ *such that*

$$\bigwedge_{n\geqq m} x_n(0)\leqq\left[\!\!\left[\sum_{k=1}^{m-1} x_k=y\right]\!\!\right]\qquad \textit{for every}\quad m\in N.$$

PROOF. Let $c_1=\bigwedge_{n\geqq 1} x_n(0)$ and $c_{m+1}=\bigwedge_{n\geqq m+1} x_n(0)-\bigvee_{k=1}^{m} c_k$, then $\bigvee_{m\in N} c_m=\mathbf{1}$ and $c_m\wedge c_n=\mathbf{0}$ for $m\neq n$. By the countably completeness of $\boldsymbol{B}$ there exists a unique element $y\in\boldsymbol{Z}^{(\boldsymbol{B})}$ such that $c_{m+1}\leqq\left[\!\!\left[\sum_{k=1}^{m} x_k=y\right]\!\!\right]$ where $\sum_{k=1}^{0} x_k=0$.

DEFINITION 2. For a proper sequence $(x_n : n\in N)$ of $\boldsymbol{Z}^{(\boldsymbol{B})}$, $\sum_{n\in N} x_n$ is the element of $\boldsymbol{Z}^{(\boldsymbol{B})}$ given by Proposition 3.

DEFINITION 3. For a homomorphism $h:\boldsymbol{Z}^N\to\boldsymbol{Z}^{(\boldsymbol{B})}$ h is infinitely linear, if $(h(\boldsymbol{e}_n) : n\in N)$ is a proper sequence and $h(\sum_{n\in N} a_n\boldsymbol{e}_n)=\sum_{n\in N} a_n h(\boldsymbol{e}_n)$.

A ccBa $\boldsymbol{B}$ has the slender property, if every homomorphism from $\boldsymbol{Z}^N$ to $\boldsymbol{Z}^{(\boldsymbol{B})}$ is infinitely linear.

PROPOSITION 4. *Let $h : \boldsymbol{Z}^N\to\boldsymbol{Z}^{(\boldsymbol{B})}$ be a homomorphism. Then, the following three propositions are equivalent:*
(1) *h is infinitely linear;*
(2) *$(h(\boldsymbol{e}_n) : n\in N)$ is a proper sequence;*
(3) *$\bigvee_m \bigwedge_{n\geq m} [\![h(\boldsymbol{e}_n)=\check{0}]\!]=\boldsymbol{1}$ holds.*

This is clear by Proposition 6 of [2], Propositions 2 and 3.

PROPOSITION 5. *Let $(x_n : n\in N)$ be a proper sequence of $\boldsymbol{Z}^{(\boldsymbol{B})}$. Then, there exists a unique infinitely linear homomorphism $h : \boldsymbol{Z}^N\to\boldsymbol{Z}^{(\boldsymbol{B})}$ such that $h(\boldsymbol{e}_n)=x_n$ for $n\in N$.*

Let ω be the least infinite ordinal, i. e., the set $N\cup\{0\}$. A ccBa $\boldsymbol{B}$ satisfies the (ω, ω)-weak distributive law (we abbreviate it by (ω, ω)-WDL), if $\bigwedge_{m<\omega}\bigvee_{n<\omega} b_{mn}=\bigvee_{f\in{}^\omega\omega}\bigwedge_{m<\omega}\bigvee_{n\leq f(m)} b_{mn}$ holds for any $b_{mn}\in\boldsymbol{B}$ $(m, n<\omega)$.

THEOREM 1. *If a ccBa $\boldsymbol{B}$ satisfies (ω, ω)-WDL, then $\boldsymbol{B}$ has the slender property.*

PROOF. Let $h : \boldsymbol{Z}^N\to\boldsymbol{Z}^{(\boldsymbol{B})}$ be a homomorphism. Then, there exists an element $\bar{h}$ of the Boolean extension $V^{(\bar{\boldsymbol{B}})}$ such that $[\![\bar{h} : \check{\boldsymbol{Z}}^N\to\boldsymbol{Z}$ is a homomorphism$]\!]^{(\bar{\boldsymbol{B}})}=\boldsymbol{1}$ and $[\![\bar{h}(\check{x})=h(x)]\!]^{(\bar{\boldsymbol{B}})}=\boldsymbol{1}$ for each $x\in\boldsymbol{Z}^N$. Suppose that $\bigvee_m\bigwedge_{n\geq m}[\![h(\boldsymbol{e}_n)=\check{0}]\!]\neq\boldsymbol{1}$. Since $\bigwedge_{n\in N}\bigvee_{a\in\boldsymbol{Z}}[\![h(\boldsymbol{e}_n)=\check{a}]\!]=\boldsymbol{1}$, there exists a function $f : N\to N$ such that $\boldsymbol{0}\neq(-\bigvee_m\bigwedge_{n\geq m}[\![h(\boldsymbol{e}_n)=\check{0}]\!])\wedge\bigwedge_{n\in N}\bigvee_{|a|\leq f(n)}[\![h(\boldsymbol{e}_n)=\check{a}]\!]$. This implies that $\boldsymbol{0}\neq[\![\forall m\exists n\geqq m$ $(h(\boldsymbol{e}_n)\neq 0)$ and $\forall n\in N(|h(\boldsymbol{e}_n)|\leqq\check{f}(n))]\!]^{(\bar{\boldsymbol{B}})}$. Apply Lemma 4 of [2] to $\boldsymbol{Z}^{\check{N}}$ in $V^{(\bar{\boldsymbol{B}})}$, then we get a contradiction.

Next we show that infinite linearity is equivalent to the existence of a lifting homomorphism.

For a quotient of a Boolean algebra by its ideal, we refer the reader to [5]. An ideal $\boldsymbol{I}$ of a ccBa $\boldsymbol{B}$ is countably complete, if $\bigvee X\in\boldsymbol{I}$ for any countable subset X of $\boldsymbol{I}$. Let $\boldsymbol{B}/\boldsymbol{I}$ be the quotient of a ccBa $\boldsymbol{B}$ by its countably complete ideal $\boldsymbol{I}$ and $[\ ] : \boldsymbol{B}\to\boldsymbol{B}/\boldsymbol{I}$ the quotient map. Then, $\boldsymbol{B}/\boldsymbol{I}$ is a ccBa and $[\ ]$ preserves countable sums, i. e., for any countable subset X of $\boldsymbol{B}$ $[\bigvee X]=\bigvee_{x\in X}[x]$. Let $(\boldsymbol{Z}^{(\boldsymbol{B})})_{\boldsymbol{I}}$ be the subgroup of $\boldsymbol{Z}^{(\boldsymbol{B})}$ such that $x\in(\boldsymbol{Z}^{(\boldsymbol{B})})_{\boldsymbol{I}}$ iff $-x(0)\in\boldsymbol{I}$, and $\pi : Z^{(\boldsymbol{B})}$

$\to Z^{(B)}/(Z^{(B)})_I$ be the canonical homomorphism. Then, $Z^{(B)}/(Z^{(B)})_I$ is isomorphic to $Z^{(B/I)}$. Therefore, we identify them.

LEMMA 1. *If* $(x_n : n\in N)$ *is a proper sequence of* $Z^{(B)}$, *then* $(\pi(x_n): n\in N)$ *is a proper sequence of* $Z^{(B/I)}$ *and* $\pi(\sum_{n\in N} x_n)=\sum_{n\in N}\pi(x_n)$ *holds.*

PROOF. Since $\bigvee_m \bigwedge_{n\geq m} x_n(0)=\mathbf{1}$ and $[x_n(0)]\leqq \pi(x_n)(0)$, $\bigvee_m \bigwedge_{n\geq m}\pi(x_n)(0)=\mathbf{1}$ holds. There exists an infinitely linear homomorphism $h : Z^N\to Z^{(B)}$ such that $h(e_n)=x_n$ for $n\in N$ by Proposition 5. Since $(\pi\cdot h(e_n) : n\in N)$ is a proper sequence, $\pi(\sum_{n\in N} x_n)=\pi\cdot h(\sum_{n\in N} e_n)=\sum_{n\in N}\pi\cdot h(e_n)=\sum_{n\in N}\pi(x_n)$ by Proposition 4.

DEFINITION 4. For a homomorphism $h : Z^N\to Z^{(B/I)}$, $\tilde{h} : Z^N\to Z^{(B)}$ is a lifting homomorphism of h if $h=\pi\cdot\tilde{h}$.

THEOREM 2. *Let* B *be a ccBa and* I *a countably complete ideal of* B. *If a homomorphism* $h : Z^N\to Z^{(B/I)}$ *is infinitely linear, then there exists a lifting homomorphism* $\tilde{h} : Z^N\to Z^{(B)}$ *of* h. *In the case that* B *has the slender property, a homomorphism* $h : Z^N\to Z^{(B/I)}$ *is infinitely linear iff there exists a lifting homomorphism* $\tilde{h} : Z^N\to Z^{(B)}$ *of* h.

PROOF. Let $h : Z^N\to Z^{(B/I)}$ be an infinitely linear homomorphism and $h(e_n)=\pi(x_n)$ for $n\in N$. Then, $\bigvee_m \bigwedge_{n\geq m}[x_n(0)]=\bigvee_m \bigwedge_{n\geq m}\pi(x_n)(0)=\mathbf{1}$. Hence $-\bigvee_m \bigwedge_{n\geq m} x_n(0)$ $(=b)$ belongs to I. Let $x'_n(a)=x_n(a)-b$ for $a\neq 0$ and $x'_n(0)=x_n(0)\vee b$. Then, $\bigvee_m \bigwedge_{n\geq m} x'_n(0)=\mathbf{1}$, so $(x'_n : n\in N)$ is a proper sequence. Let $\tilde{h}(\sum_{n\in N} a_n e_n)=\sum_{n\in N} a_n x'_n$. Then $\pi\cdot\tilde{h}=h$ holds by infinite linearity of h and Lemma 1. The second proposition is clear by the first one and Lemma 1.

DEFINITION 5. For a ccBa B let F be the least countably additive field of subsets of the Stone space of B that contains all clopen subsets and I the ideal of B consisting of all subsets of first category that belong to F.

Then, F is a ccBa and I is countably complete. The group $Z^{(F)}$ is isomorphic to the group consisting of all F-measurable functions f from the Stone space to Z, i.e., $f^{-1}(a)\in F$ for $a\in Z$.

PROPOSITION 6 (Theorem 29.1 of [5]). *A ccBa* B *is isomorphic to the quotient algebra* F/I.

COROLLARY 1. *Let* $h : Z^N\to Z^{(B)}$ *be a homomorphism for a ccBa* B $(=F/I)$. *Then,* h *is infinitely linear iff there exists a lifting homomorphism* $h : Z^N\to Z^{(F)}$ *of* h.

PROOF. Since F is a field of sets, F clearly satisfies (ω, ω)-WDL and hence has the slender property. Now the corollary is clear from Theorem 2.

Next we think of the field F^* of all Borel subsets of the unit interval $[0, 1]$.

There are two typical countably complete ideals of $\boldsymbol{F}^*$. The one is the ideal $\boldsymbol{I}_m$ consisting of all Borel subsets of Lebesgue measure zero and the other is the ideal $\boldsymbol{I}_c$ consisting of all Borel subsets of first category. Just like a case of the Stone space, $\boldsymbol{Z}^{(\boldsymbol{F}^*)}$ is isomorphic to the group consisting of all Borel functions from [0, 1] to $\boldsymbol{Z}$. It is well-known that the complete Boolean algebra $\boldsymbol{F}^*/\boldsymbol{I}_m$ satisfies (ω, ω)-WDL [1]. By Theorems 1 and 2 any homomorphism from $\boldsymbol{Z}^N$ to $\boldsymbol{Z}^{(\boldsymbol{F}^*/\boldsymbol{I}_m)}$ has a lifting homomorphism. However, we do not know whether the same holds for the ideal $\boldsymbol{I}_c$. Equivalently, does the cBa $\boldsymbol{F}^*/\boldsymbol{I}_c$ have the slender property? Equivalently, $[\![\forall h:\check{\boldsymbol{Z}}^N\to\boldsymbol{Z}(\exists m\forall n\geqq m\, h(\boldsymbol{e}_n)=0)]\!]^{(\boldsymbol{B})}=\boldsymbol{1}$ where $\boldsymbol{B}=\boldsymbol{F}^*/\boldsymbol{I}_c$?

2. (ω, ω)-weak distributivity of certain Boolean algebras.

In the following κ is a cardinal of uncountable cofinality and I a set of cardinality greater than or equal to κ, where a cardinal is an initial ordinal and an ordinal is the set of all ordinals less than itself. The cofinality of κ is denoted by $\mathrm{cf}(\kappa)$. A cardinal is regular if its cofinality is equal to itself, and singular otherwise. The ideal consisting of all subsets of I which are of cardinality less than κ is denoted by $\boldsymbol{P}_\kappa(I)$. Since $\boldsymbol{P}_\kappa(I)$ is closed under countable sums the quotient Boolean algebra $\boldsymbol{P}(I)/\boldsymbol{P}_\kappa(I)$ is a ccBa. Distributivity scarcely holds for the canonical completion of $\boldsymbol{P}(I)/\boldsymbol{P}_\kappa(I)$ [4]. However, it isn't the case for $\boldsymbol{P}(I)/\boldsymbol{P}_\kappa(I)$ itself. We investigate the (ω, ω)-weak distributivity of $\boldsymbol{P}(I)/\boldsymbol{P}_\kappa(I)$ in this section.

Let $D(\kappa)$ be the assertion: $\boldsymbol{P}(\kappa)/\boldsymbol{P}_\kappa(\kappa)$ satisfies (ω, ω)-WDL. Then, the following two propositions are easily shown.

PROPOSITION 7. *The ccBa* $\boldsymbol{P}(I)/\boldsymbol{P}_\kappa(I)$ *satisfies* (ω, ω)*-WDL for any* I*, if* $D(\kappa)$ *holds.*

PROPOSITION 8. *If* $2^{\aleph_0}<\mathrm{cf}(\kappa)$*, then* $D(\kappa)$ *holds.*

DEFINITION 6. Let ${}^\omega\omega$ be the set of all functions from ω to ω. For $f, g\in{}^\omega\omega$ $f\leqq^* g$ holds if $f(n)\leqq g(n)$ for almost all n, i.e., $\exists m\forall n\geqq m(f(n)\leqq g(n))$.

LEMMA 2. *The assertion* $D(\kappa)$ *does not hold iff there exist subsets* X_{mn} *of* κ $(m, n<\omega)$ *such that* $\bigcap_m\bigcup_n X_{mn}=\kappa$ *and* $X_{mn}\cap X_{mn'}=\emptyset$ *for* $n\neq n'$ *and the cardinality of* $\bigcap_m \bigcup_{n\leqq g(m)} X_{mn}$ *is less than* κ *for any* $g\in{}^\omega\omega$.

Since κ is of uncountable cofinality, the proof can be done just as for a homogeneous complete Boolean algebra. Therefore, we omit it.

LEMMA 3. $D(\kappa)$ *implies* $D(\mathrm{cf}(\kappa))$.

PROOF. Use Lemma 2.

LEMMA 4. *Let κ be a cardinal satisfying one of the following conditions:* (1) *κ is regular;* (2) *κ is singular and $D(\mathrm{cf}(\kappa))$ holds. Then $D(\kappa)$ does not hold iff there exists a subset S of ${}^{\omega}\omega$ of cardinality κ such that the cardinality of $\{f : f\in S$ and $f\leqq^* g\}$ is less than κ for any $g\in{}^{\omega}\omega$.*

PROOF. Suppose that $D(\kappa)$ does not hold. Then, there exist X_{mn} $(m, n<\omega)$ that satisfy the conditions in Lemma 2. Let $S=\{f : \bigcap_m X_{mf(m)}\neq\emptyset\}$. Since the cardinality of $\bigcap_m X_{mf(m)}$ is less than κ for any $f\in{}^{\omega}\omega$, the cardinality of S must be κ when κ is regular. Now we deal with the case that κ is singular. Suppose that the cardinality of $\bigcap_m X_{mf(m)}$ $(f\in{}^{\omega}\omega)$ are not bounded below κ. There exists a subset T of S of cardinality $\mathrm{cf}(\kappa)$ such that for any subset T' of T of cardinality $\mathrm{cf}(\kappa)$ the cardinality of $\bigcup_{f\in T'}\bigcap_m X_{mf(m)}$ is κ. Since $D(\mathrm{cf}(\kappa))$ holds, there exists a $g\in{}^{\omega}\omega$ such that the cardinality of $\{f : f\in T$ and $f(n)\leqq g(n)$ for all $n\}$ is $\mathrm{cf}(\kappa)$. Then, the cardinality of $\bigcap_m \bigcup_{n\leqq g(m)} X_{mn}$ is κ, which is a contradiction. Hence, the cardinality of $\bigcap_m X_{mf(m)}$ $(f\in{}^{\omega}\omega)$ are bounded below κ. Therefore, in any case the cardinality of S is κ. Let $\{g_i : i<\omega\}$ be an enumeration of all functions g' such that $g'(n)=g(n)$ for almost all $n<\omega$. Since $\{f : f\in S$ and $f\leqq^* g\}=\bigcup_{i<\omega}\{f : f\in S$ and $f(n)\leqq g_i(n)$ for all $n\}$ and $\mathrm{cf}(\kappa)$ is uncountable, the cardinality of $\{f : f\in S$ and $f\leqq^* g\}$ is less than κ for every $g\in{}^{\omega}\omega$. The converse is obvious.

COROLLARY 2. *If $D(\mathrm{cf}(\kappa))$ holds and $2^{\aleph_0}<\kappa$, then $D(\kappa)$ holds.*

This is immediate from Lemma 4. By the way, S. Kamo has shown that the condition "$2^{\aleph_0}<\kappa$" in Corollary 2 cannot be dropped.

LEMMA 5. *There exists a sequence $(g_\alpha : \alpha<\kappa)$ that satisfies the following:*
(1) *κ is regular;*
(2) *$g_\alpha\in{}^{\omega}\omega$ and $g_\alpha\leqq^* g_\beta$ and not $g_\beta\leqq^* g_\alpha$ for $\alpha<\beta$;*
(3) *for any $f\in{}^{\omega}\omega$ there exists $\alpha<\kappa$ such that $g_\alpha\leqq^* f$ does not hold.*
In addition, for such a κ $D(\kappa)$ does not hold.

PROOF. By axiom of choice there exists a sequence $(f_\alpha : \alpha<\lambda)$ that satisfies the conditions (2) and (3). Let $\kappa=\mathrm{cf}(\lambda)$ and $(g_\alpha : \alpha<\kappa)$ be a cofinal subsequence of $(f_\alpha : \alpha<\lambda)$. Next let $S=\{g_\alpha : \alpha<\kappa\}$ and $X_{mn}=\{f : f\in S$ and $f(m)=n\}$. Then, $\bigcap_m\bigcup_n X_{mn}=S$ and the cardinality of $\bigcap_m \bigcup_{n\leqq f(m)} X_{mn}$ is less than κ for any $f\in{}^{\omega}\omega$. Hence, $D(\kappa)$ does not hold.

It is well-known that Martin's axiom implies the following assertion: For any subset $A\subseteqq{}^{\omega}\omega$ of cardinality less than $2^{\aleph_0}$, there exists an $f\in{}^{\omega}\omega$ such that $g\leqq^* f$ holds for every $g\in A$ [3].

LEMMA 6. (*Under Martin's axiom*) *For any $\kappa<2^{\aleph_0}$ $D(\kappa)$ holds.*

PROOF. Let $\bigcap_m\bigcup_n X_{mn}=\kappa$ and $X_{mn}\cap X_{mn'}=\emptyset$ for $n\neq n'$. For $\alpha<\kappa$ let $f_\alpha\in{}^{\omega}\omega$

be the function such that $f_\alpha(m)=n$ iff $\alpha\in X_{mn}$. By Martin's axiom there exists $g^*\in{}^\omega\omega$ such that $f\leqq^* g^*$ for all $\alpha<\kappa$. Since $\mathrm{cf}(\kappa)$ is uncountable and there are only countably many $g\in{}^\omega\omega$ such that $g(n)=g^*(n)$ for almost all n, there exists $g\in{}^\omega\omega$ such that the cardinality of $\bigcap_m \bigcup_{n\leq g(m)} X_{mn}$ is κ. By Lemma 2, $D(\kappa)$ holds.

THEOREM 3. *(Under Martin's axiom) For a cardinal κ of uncountable cofinality, $D(\kappa)$ holds iff* $\mathrm{cf}(\kappa)$ *is not equal to* $2^{\aleph_0}$.

PROOF. Since Martin's axiom implies that $2^{\aleph_0}$ is regular, there exists a sequence $(g_\alpha : \alpha<2^{\aleph_0})$ that satisfies the conditions of Lemma 5 by the above consequence of Martin's axiom. Hence, $D(2^{\aleph_0})$ does not hold. Now, the conclusion follows from Lemmas 3, 6 and Corollary 2.

Theorems 2 and 3 imply that $\boldsymbol{P}(I)/\boldsymbol{P}_\kappa(I)$ has the slender property for a κ whose cofinality is uncountable but not equal to $2^{\aleph_0}$ under Martin's axiom. On the other hand, B. Wald [6] showed the existence of a homomorphism h : $\boldsymbol{Z}^N\to\boldsymbol{Z}^{(P(I)/P_{2^{\aleph_0}}(I))}$ whose lift-homomorphism from $\boldsymbol{Z}^N$ to $\boldsymbol{Z}^{(P(I))}$ does not exist under Martin's axiom. Therefore,

COROLLARY 3. *(Under Martin's axiom) $\boldsymbol{P}(I)/\boldsymbol{P}_\kappa(I)$ has the slender property iff the cofinality of κ is not equal to $2^{\aleph_0}$ for a κ of uncountable cofinality.*

Our Corollary 3 improves Theorem (a) of [6]. Since Wald deals with a lifting problem under a little different setting, there is a problem in the case that the cofinality of κ is countable. In appendix we shall show the existence of a homomorphism which has no lifting homomorphism for a κ of countable cofinality.

Next we show that in a certain well-known Boolean extension of the universe $D(2^{\aleph_0})$ holds. Let $\boldsymbol{B}$ be the measure algebra over a product space ${}^\kappa 2$ with a product measure, where $\kappa=(2^{\aleph_0})^+$ (p. 250 of [3]).

PROPOSITION 9. *The assertion $D(2^{\aleph_0})$ holds in $V^{(\boldsymbol{B})}$.*

PROOF. We work in $V^{(\boldsymbol{B})}$. It is known that for any $f\in{}^\omega\omega$ there exists a $g\in{}^{\check\omega}\check\omega$ such that $f(m)\leqq g(m)$ for every $m<\omega$ [3]. The cardinality of ${}^{\check\omega}\check\omega$ is less than $2^{\aleph_0}$ $(=\check\kappa)$ and $2^{\aleph_0}$ is regular. If $\bigcap_m\bigcup_n X_{mn}=2^{\aleph_0}$, then $\bigcup_{g\in{}^{\check\omega}\check\omega}\bigcap_m \bigcup_{n\leq g(m)} X_{mn}=\bigcup_{g\in{}^\omega\omega}\bigcap_m \bigcup_{n\leq g(m)} X_{mn}=2^{\aleph_0}$. Hence, there exists a $g\in{}^\omega\omega$ such that the cardinality of $\bigcap_m \bigcup_{n\leq g(m)} X_{mn}$ is $2^{\aleph_0}$. Therefore, $D(2^{\aleph_0})$ holds in $V^{(B)}$ by Lemma 2.

Proposition 9 and Corollary 3 imply

COROLLARY 4. *It is independent of ZFC set theory that $\boldsymbol{P}(I)/\boldsymbol{P}_{2^{\aleph_0}}(I)$ has the slender property.*

Appendix.

Here we show that there exists a homomorphism from $\boldsymbol{Z}^N$ to $\boldsymbol{Z}^{(\boldsymbol{B})}/(\boldsymbol{Z}^{(\boldsymbol{B})})_{\boldsymbol{I}}$ which has no lifting homomorphism from $\boldsymbol{Z}^N$ to $\boldsymbol{Z}^{(\boldsymbol{B})}$ for a certain ideal $\boldsymbol{I}$ of a ccBa $\boldsymbol{B}$ with the slender property.

THEOREM 4. *Let $\boldsymbol{B}$ be a ccBa with the slender property and an ideal $\boldsymbol{I}=\bigcup_{n\in N}\boldsymbol{I}_n$, where $\boldsymbol{I}_n$ is a countably complete ideal for each $n\in N$. If $\boldsymbol{I}$ is not countably complete, then there exists a homomorphism from $\boldsymbol{Z}^N$ to $\boldsymbol{Z}^{(\boldsymbol{B})}/(\boldsymbol{Z}^{(\boldsymbol{B})})_{\boldsymbol{I}}$ which has no lifting homomorphism from $\boldsymbol{Z}^N$ to $\boldsymbol{Z}^{(\boldsymbol{B})}$.*

Without loss of generality we may assume that $\boldsymbol{I}_n\subseteqq\boldsymbol{I}_{n+1}$ and $\boldsymbol{I}_n\neq\boldsymbol{I}_{n+1}$ for each $n\in N$. Then, there exist b_n $(n\in N)$ such that $b_n\notin\boldsymbol{I}_n$ and $b_n\in\boldsymbol{I}_{n+1}$ and $b_m\wedge b_n=\boldsymbol{0}$ for $m\neq n$. Clearly $\bigvee_{n\in N}b_n\notin\boldsymbol{I}$. Let C be the subgroup of $\boldsymbol{Z}^{(\boldsymbol{B})}$ such that $x\in C$ iff $b_n\leqq x(a)$ for some $a\in\boldsymbol{Z}$ for each $n\in N$ and $-\bigvee_{n\in N}b_n\leqq x(0)$. Let $\pi:\boldsymbol{Z}^{(\boldsymbol{B})}\to\boldsymbol{Z}^{(\boldsymbol{B})}/(\boldsymbol{Z}^{(\boldsymbol{B})})_{\boldsymbol{I}}$ be the canonical homomorphism.

LEMMA 7. *If the image of $\pi\cdot h$ is included by the image of the restriction of π to C for a homomorphism $h:\boldsymbol{Z}^N\to\boldsymbol{Z}^{(\boldsymbol{B})}$, then there exists a homomorphism $h^*:\boldsymbol{Z}^N\to C$ such that $\pi\cdot h^*=\pi\cdot h$.*

PROOF. Since $\boldsymbol{B}$ has the slender property, there exist c_n $(n\in N)$ with the following properties (consider the set $\{\bigwedge_{k\leq m}h(\boldsymbol{e}_k)(a_k)\wedge\bigwedge_{n>m}h(\boldsymbol{e}_n)(0):m\in N$ and $a_k\in\boldsymbol{Z}$ $(k\leqq m)\}$):

(1) $\bigvee_{n\in N}c_n=\boldsymbol{1}$, $c_n\neq\boldsymbol{0}$ and $c_m\wedge c_n=\boldsymbol{0}$ for $m\neq n$;

(2) For any m, $k\in N$ there exists an integer a such that $c_m\leqq h(\boldsymbol{e}_k)(a)$;

(3) For distinct m, n there exist k, a and b such that $c_m\leqq h(\boldsymbol{e}_k)(a)$, $c_n\leqq h(\boldsymbol{e}_k)(b)$ and $a\neq b$.

Since $b_m\notin\boldsymbol{I}_m$ and $b_m=\bigvee_{n\in N}b_m\wedge c_n$, there exists a d_m such that $d_m\notin\boldsymbol{I}_m$ and $d_m=b_m\wedge c_n$ for some n. If $\bigvee_{m\in N}b_m-\bigvee_{m\in N}d_m\in\boldsymbol{I}$, then let h^* be the homomorphism from $\boldsymbol{Z}^N$ to C such that $d_m\leqq h(\boldsymbol{e}_n)(a)$ implies $b_m\leqq h^*(\boldsymbol{e}_n)(a)$ for every m, n and a. Now, we have gotten the desired homomorphism h^*. In the rest we show that $\bigvee_{m\in N}b_m-\bigvee_{m\in N}d_m\in\boldsymbol{I}$. Otherwise, there exists an ascending sequence $(m_k:k\in N)$ of natural numbers and d'_{m_k} such that $d'_{m_k}\wedge\bigvee_{m\in N}d_m=\boldsymbol{0}$ and $d'_{m_k}\notin\boldsymbol{I}_k$ and $d'_{m_k}=b_{m_k}\wedge c_n$ for some n by countable completeness of $\boldsymbol{I}_k$ $(k\in N)$. We remark the following three facts:

(1) For any k there exist n, a and b such that $a\neq b$ and $d_{m_k}\leqq h(\boldsymbol{e}_n)(a)$ and $d'_{m_k}=h(\boldsymbol{e}_n)(b)$;

(2) Since $\pi\cdot h(\boldsymbol{e}_n)(a)\in\pi(C)$ for every $n\in N$, $\{k:d_{m_k}\leqq h(\boldsymbol{e}_n)(a)$ and $d'_{m_k}\leqq h(\boldsymbol{e}_n)(b)$ for $a\neq b\}$ is finite for every n;

(3) For any k, $d_{m_k}\vee d'_{m_k}\leqq h(\boldsymbol{e}_n)(0)$ for almost all n.

We define natural numbers n_i, n'_i ($i \in N$) and a subsequence $(p_i : i \in N)$ of $(m_k : k \in N)$ by induction.

Step 1: Let $p_1 = m_1$ and n_1 be a natural number such that $d_{p_1} \leqq h(\boldsymbol{e}_{n_1})(a)$ and $d'_{p_1} \leqq h(\boldsymbol{e}_{n_1})(b)$ for some distinct a, b. Let $n'_1 \geqq n_1$ be a natural number such that $d_{p_1} \vee d'_{p_1} \leqq h(\boldsymbol{e}_j)(0)$ for any $j > n'_1$.

We assume that we have defined $p_1 < \cdots < p_k$, $n_1 \leqq n'_1 < n_2 \leqq \cdots \leqq n'_k$ in such a way that for any $i \leqq k$ and $j > n'_k$ $d_{p_i} \vee d'_{p_i} \leqq h(\boldsymbol{e}_j)(0)$.

Step $k+1$: Take $p_{k+1} > p_k$ so that for any $j \leqq n'_k$ and $m_i \geqq p_{k+1}$ there exists $u \in \boldsymbol{Z}$; $d_{m_i} \vee d'_{m_i} \leqq h(\boldsymbol{e}_j)(u)$. There exists n_{k+1} such that $d_{p_{k+1}} \leqq h(\boldsymbol{e}_{n_{k+1}})(a)$ and $d'_{p_{k+1}} \leqq h(\boldsymbol{e}_{n_{k+1}})(b)$ for some distinct a, b. Then $n_{k+1} \geqq n'_k$ and

$$d_{p_{k+1}} \leqq h\Big(\sum_{i=1}^{k+1} \boldsymbol{e}_{n_i}\Big)(a), \qquad d'_{p_{k+1}} \leqq h\Big(\sum_{i=1}^{k+1} \boldsymbol{e}_{n_i}\Big)(b)$$

for some distinct a, b. Let $n'_{k+1} \geqq n_{k+1}$ such that for any $i \geqq n'_{k+1}$ and $j \leqq k+1$ $d_{p_j} \vee d'_{p_j} \leqq h(\boldsymbol{e}_i)(0)$. Thus we can continue this construction.

Let $\boldsymbol{a} = \sum_{i \in N} \boldsymbol{e}_{n_i}$. By the assumption of the lemma there exists $b \in \boldsymbol{I}$ such that $b_n \wedge -b \leqq h(\boldsymbol{a})(u)$ for some u for any $n \in N$. Let k be a natural number such that $b \in \boldsymbol{I}_k \subseteqq \boldsymbol{I}_{p_k}$. Then, $\boldsymbol{0} \neq d_{p_k} \wedge -b \leqq h\Big(\sum_{i=1}^{k} \boldsymbol{e}_{n_i}\Big)(u) \leqq h(\boldsymbol{a})(u)$ and $\boldsymbol{0} \neq d'_{p_k} \wedge -b \leqq h\Big(\sum_{i=1}^{k} \boldsymbol{e}_{n_i}\Big)(v) \leqq h(\boldsymbol{a})(v)$ for distinct u and v, but this contradicts the fact that $b_{p_k} \wedge -b \leqq h(\boldsymbol{a})(u)$ for some u. Now the proof of Lemma 7 has been completed.

PROOF OF THEOREM 4. Let x be an element of C such that $x(n!) = b_n$ and $x(0) = -\bigvee_{n \in N} b_n$. Then $\pi(x) \neq 0$ and it is divisible in $\pi(C)$. Therefore, $\pi(C)$ includes a non-trivial divisible subgroup, so there exist $2^{2^{\aleph_0}}$-many homomorphisms from $\boldsymbol{Z}^N$ to $\pi(C)$. On the other hand there exist only $2^{\aleph_0}$-many homomorphisms from $\boldsymbol{Z}^N$ to C, because C is isomorphic to $\boldsymbol{Z}^N$. Hence, there exists a homomorphism from $\boldsymbol{Z}^N$ to $\boldsymbol{Z}^{(B)}/(\boldsymbol{Z}^{(B)})_I$ which has no lifting homomorphism by Lemma 7.

COROLLARY 5. *Let λ be a cardinal of countable cofinality. Then, there exists a homomorphism from $\boldsymbol{Z}^N$ to $\boldsymbol{Z}^\lambda/(\boldsymbol{Z}^\lambda)_{\boldsymbol{P}_\lambda(\lambda)}$ which has no lifting homomorphism from $\boldsymbol{Z}^N$ to $\boldsymbol{Z}^\lambda$.*

PROOF. If λ is the first infinite cardinal ω, the proof is obtained by the same argument as in the proof of Theorem 4. Otherwise, there exist regular infinite cardinals κ_n ($n \in N$) such that λ is the least upper bound of κ_n ($n \in N$). Since $\boldsymbol{P}_{\kappa_n}(\lambda)$ is countably complete for each $n \in N$ and $\boldsymbol{P}_\lambda(\lambda) = \bigcup_{n \in N} \boldsymbol{P}_{\kappa_n}(\lambda)$, the conclusion follows from Theorem 4.

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