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Fluctuations of Markovian systems in Kac's caricature of a Maxwellian gas

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0. Introduction.

Kac [5] exhibited with his caricature of a Maxwellian gas, that the spatially homogeneous solution of Boltzmann problem is obtained as a limit of empirical distributions induced from n molecules Markov processes (as $n\to\infty$) which are regulated by master equations associated with the collision operator in the Boltzmann equation. Kac [6] also considered a fluctuation problem: he gave a formal derivation for a convergence of fluctuations of the empirical distributions about the solution of Boltzmann problem and observed that a kind of Ornstein-Uhlenbeck process appears in the limit. To this problem McKean [8] gave a rigorous result for his model of a two speed Maxwellian gas and made also a heuristic argument for the model of a gas of hard balls. Recently H. Tanaka [11] treated the same problem for Kac's caricature and obtained a convergence result in an equilibrium case.

In this paper we shall study the fluctuation problem for Kac's caricature and prove, in nonequilibrium (as well as equilibrium) cases, that the family of distributions on $D[[0, \infty), \mathcal{S}_\delta']$ induced by fluctuation processes converges weakly to a distribution of a kind of time-inhomogeneous Ornstein-Uhlenbeck process on $\mathcal{S}_\delta'$, where $\mathcal{S}_\delta'$ is a Hilbert space of tempered distributions.

To get the convergence result we shall follow the martingale approach as exposed in Stroock-Varadhan's book [10] and as applied by Holley-Stroock [4] to handle a convergence in law of tempered distribution valued Markov processes. Guided by their schedule, we shall first prove the tightness of the fluctuation processes as $\mathcal{S}_\delta'$-valued processes (§4), then that any limiting law solves an associated martingale problem (§5), and finally a uniqueness of a solution of the martingale problem (§6). A similar approach to the present problem has been already adopted by H. Tanaka [11].

In §1 we shall review the Kac's model and his result. In §2 we shall in-

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troduce the fluctuation processes and derive an asymptotic form for their infinitesimal generator. In § 3 we shall prepare several lemmas which provide fundamental moment estimates for the fluctuation processes. Through §§ 4 to 6 the convergence of fluctuation processes will be formulated and proved. In § 7 we shall give an explicit form for transition densities of the limiting process, which will incidentally show that the process is Gaussian if so is its initial distribution. In the last section we shall give several examples of initial distributions of the velocity processes which satisfy hypotheses of the main theorem (Theorem 6.2) of this paper.

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1. Markov process for an n molecules gas and propagation of chaos.

Here is given a review of Kac's caricature and his result. A concise and complete exposition for them is found in McKean [7].

Consider a fictitious gas of n like molecules having one dimensional velocities which collide in pair at random times: after a collision velocities of collided pairs are transformed by the two dimensional rotation

$$x \longrightarrow x^* = x\cos\theta - y\sin\theta$$

$$y \longrightarrow y^* = x\sin\theta + y\cos\theta\,,$$

preserving the energy $(1/2)(x^{*2}+y^{*2})=(1/2)(x^2+y^2)$ (but not the momentum). A collision within a small arc $[\theta, \theta+d\theta)$, $-\pi\leqq\theta<\pi$, takes place during a time interval $[t, t+dt)$, $t\geqq 0$, with probability

$$\frac{1}{n}\bar{d}\theta\, dt\,, \qquad \left[\bar{d}\theta=\frac{d\theta}{2\pi}\right].$$

If we further impose the Markovian nature on the velocity process, we obtain a Markov process on $\boldsymbol{R}^n$ (n-dimensional Euclidean space) which is regulated by the infinitesimal generator G_n:

$$\text{(1.1)} \quad G_n w(x_1, \cdots, x_n)=\frac{1}{n}\sum_{1\leq i<j\leq n}\int_{-\pi}^{\pi}(w(x_1, \cdots, x_i^*, \cdots, x_j^*, \cdots, x_n)-w(x_1, \cdots, x_n))\bar{d}\theta$$

for $w\in C_b(\boldsymbol{R}^n)$ (the totality of bounded continuous functions on $\boldsymbol{R}^n$). We denote a right continuous version of this Markov process by $X^n(t)=(X_1^n(t), \cdots, X_n^n(t))$ and its initial distribution by μ_n. We shall consider a system of Markov processes $\{X^n\}_{n=1}^{\infty}$ and suppose that they are all defined on a common probability space $(\boldsymbol{P}, \boldsymbol{\Omega}, \mathcal{M})$. Let $u_n(t)=u_n(t, dx)$ denote the distribution on $\boldsymbol{R}^n$ induced by $X^n(t)$:

$$u_n(t,\ \cdot)=\boldsymbol{P}[X^n(t)\in\cdot\,].$$

Since G_n is a bounded operator on $C_b(\boldsymbol{R}^n)$,

$$\langle u_n(t),\ w\rangle=\sum_{p=0}^{\infty}\frac{t^p}{p\,!}\langle\mu_n,\ G_n^p w\rangle\,,\qquad w\in C_b(\boldsymbol{R}^n) \tag{1.2}$$

where $\langle u_n,\ w\rangle$ denotes an integral of w by a measure u_n (later we shall continue to use $\langle\eta,\ \phi\rangle$ to denote a value of a generalized function η at a testing function ϕ). We shall assume throughout this paper that μ_n is symmetric in the sense that

$$\langle\mu_n,\ \phi_1\otimes\cdots\otimes\phi_n\rangle=\langle\mu_n,\ \phi_{\sigma(1)}\otimes\cdots\otimes\phi_{\sigma(n)}\rangle$$

for any $\phi_1,\cdots,\phi_n\in C_b(\boldsymbol{R}^n)$ and any permutation σ of n letters. Here $w\otimes v(x_1,\cdots,x_k)=w(x_1,\cdots,x_j)v(x_{j+1},\cdots,x_k)$ for w a function of j variables and v of $k-j$ variables. Since G_n commutes with the permutation of coordinates, the symmetry of μ_n is inherited by $u_n(t)$. We denote by $u_{n|m}(t)$, $m\leqq n$, the m-dimensional marginal distribution of $u_n(t)$:

$$\langle u_{n|m}(t),\ w\rangle=\langle u_n(t),\ w\otimes\underbrace{1\otimes\cdots\otimes 1}_{n-m}\rangle\qquad w\in C_b(\boldsymbol{R}^m)$$

and call $\{\mu_n\}$ chaotic if there exists a probability measure μ on $\boldsymbol{R}^1$ such that $\mu_{n|m}$ converges weakly to $\mu^{m\otimes}=\mu\otimes\cdots\otimes\mu$ (m-fold direct product), i.e.

$$\lim_{n\uparrow\infty}\langle\mu_{n|m},\ w\rangle=\langle\mu^{m\otimes},\ w\rangle\qquad\text{for all}\quad w\in C_b(\boldsymbol{R}^m)\,. \tag{1.3}$$

Now the so-called "propagation of chaos" discovered by Kac is stated as follows. *If $\{\mu_n\}$ is chaotic, then so is $\{u_n(t)\}$ for every $t>0$ and $u(t)=\lim u_{n|1}(t)$ is a unique solution of Boltzmann problem:*

$$\frac{d}{dt}\langle u(t),\ \phi\rangle=\frac{1}{2}\int_{-\pi}^{\pi}\langle u(t)\otimes u(t),\ \varepsilon^\theta\phi\rangle\tilde{d}\theta \tag{1.4}$$

for $\phi\in C_b(\boldsymbol{R}^1)$ with $u(0)=\mu$. Here ε^θ maps $\phi\in C(\boldsymbol{R}^1)$ (a continuous function on $\boldsymbol{R}^1$) into $\varepsilon^\theta\phi\in C(\boldsymbol{R}^2)$ so that

$$\varepsilon^\theta\phi(x,\ y)=\phi(x^*)+\phi(y^*)-\phi(x)-\phi(y)\,.$$

REMARK 1.1. If $u(t,\ \cdot)$ has a continuous density $f(t,\ x)$, (1.4) implies

$$\frac{\partial}{\partial t}f(t,\ x)=\int_{-\infty}^{\infty}\int_{-\pi}^{\pi}(f(t,\ x^*)f(t,\ y^*)-f(t,\ x)f(t,\ y))\tilde{d}\theta\,dy\,,$$

which is a usual form of Boltzmann equation.

2. Fluctuation processes.

Let δ_x be a delta measure carrying a unit mass at x. The empirical distribution of velocities at time t is

$$\alpha_t^n \equiv \frac{1}{n} \sum_{k=1}^n \delta_{x_k^n(t)} . \tag{2.1}$$

α_t^n, $t \geqq 0$, is a Markov process taking values of probability measures on $\boldsymbol{R}^1$. It is not hard to see that the propagation of chaos stated in §1 is equivalent to the assertion that

(2.2) α_t^n *converges weakly to* $u(t)$ $(n \to \infty)$ *in probability,*

where $u(t)$ is a solution of Boltzmann problem. This is the first order approximation and regarded as a law of large numbers. The second order approximation is a convergence of the fluctuation of α_t^n about $u(t)$ which should be defined by

$$\eta_t^n \equiv \sqrt{n}(\alpha_t^n - u(t)) \tag{2.3}$$

on the analogy of the central limit theorem. The process η_t^n, $t \geqq 0$, is also a Markov process (but temporarily inhomogeneous) whose values at time t are signed measures on $\boldsymbol{R}^n$ of the form

$$\eta \equiv \sqrt{n}(\alpha - u(t)), \quad \alpha = \frac{1}{n} \sum_{k=1}^n \delta_{x_k} \qquad ((x_1, \cdots, x_n) \in \boldsymbol{R}^n). \tag{2.4}$$

Our objective in this paper is to prove a convergence of the distribution induced by $\eta_\cdot^n$. In the rest of this section we shall derive a form of the infinitesimal generator of $\eta_\cdot^n$ and its formal limit as $n \to \infty$ according to Tanaka [**11**].

Let $\mathcal{S}$ be the Schwartz space of real valued C^∞-functions on $\boldsymbol{R}^1$ which together with all their derivatives are rapidly decreasing. For $\phi \in \mathcal{S}$, $f \in C_0^\infty(\boldsymbol{R}^1)$ (a C^∞-function vanishing off a compact set) and η in (2.4) we set

$$\mathfrak{G}_t^n(\eta ; \phi, f) \equiv \lim_{h \downarrow 0} \frac{1}{h} \{\boldsymbol{E}[f(\langle \eta_{t+h}^n, \phi \rangle) | \eta_t^n = \eta] - f(\langle \eta, \phi \rangle)\} .$$

(This is regarded as the infinitesimal generator for the Markov process $\eta_\cdot^n$ operating on the function $\eta \rightsquigarrow f(\langle \eta, \phi \rangle)$.) If $\eta^{i,j,\theta}$ denotes a signed measure obtained from η by replacing x_i, x_j by x_i^*, x_j^* in (2.4), then $\mathfrak{G}_t^n(\eta ; \phi, f)$ is expressed as

$$\frac{1}{n} \sum_{i<j} \int_{-\pi}^{\pi} \{f(\langle \eta^{i,j,\theta}, \phi \rangle) - f(\langle \eta, \phi \rangle)\} \tilde{d}\theta - \sqrt{n} \frac{d}{dt} \langle u(t), \phi \rangle f'(\langle \eta, \phi \rangle) . \tag{2.5}$$

By noticing that $u(t)$ is a solution of Boltzmann problem,

$$\langle \eta^{i,j,\theta}, \phi \rangle = \langle \eta, \phi \rangle + \frac{1}{\sqrt{n}} \varepsilon^\theta \phi(x_i, x_j) ,$$

and $\varepsilon^\theta \phi(x, y) = \varepsilon^{-\theta} \phi(y, x)$, we can write (2.5) as

$$\frac{n}{2} \int \langle \alpha \dot{\otimes} \alpha, w^{\theta,n} \rangle \tilde{d}\theta - \frac{\sqrt{n}}{2} \int \langle u(t) \otimes u(t), \varepsilon^\theta \phi \rangle \tilde{d}\theta f'(\langle \eta, \phi \rangle)$$

where $w^{\theta,n} \in C_b(\boldsymbol{R}^2)$, being defined by

$$w^{\theta,n}(x,y)=f\Big(\langle\eta,\phi\rangle+\frac{1}{\sqrt{n}}\varepsilon^{\theta}\phi(x,y)\Big)-f(\langle\eta,\phi\rangle)$$

and

$$\alpha\dot{\otimes}\alpha=n^{-2}\sum_{i\neq j}\delta_{x_i}\otimes\delta_{x_j}. \tag{2.6}$$

By writing $\omega^{\theta}\phi(x)$ for $\varepsilon^{\theta}\phi(x,x)$ we set

$$\mathfrak{A}_t^n\eta(\phi)\equiv\int\langle\eta\otimes u(t),\varepsilon^{\theta}\phi\rangle\tilde{d}\theta+\frac{1}{2\sqrt{n}}\int(\langle\eta\otimes\eta,\varepsilon^{\theta}\phi\rangle-\langle\alpha,\omega^{\theta}\phi\rangle)\tilde{d}\theta$$

$$\begin{aligned}Q_t^n(\phi;\eta)&\equiv\frac{1}{2}\int\langle\alpha\dot{\otimes}\alpha,(\varepsilon^{\theta}\phi)^2\rangle\tilde{d}\theta\\&=\frac{1}{2}\int\langle u(t)\otimes u(t),(\varepsilon^{\theta}\phi)^2\rangle\tilde{d}\theta+\frac{1}{\sqrt{n}}\int\langle\eta\otimes u(t),(\varepsilon^{\theta}\phi)^2\rangle\tilde{d}\theta\\&\quad+\frac{1}{2n}\int(\langle\eta\otimes\eta,(\varepsilon^{\theta}\phi)^2\rangle-\langle\alpha,(\omega^{\theta}\phi)^2\rangle)\tilde{d}\theta.\end{aligned}$$

Then, by observing

$$n\langle\alpha\dot{\otimes}\alpha,\varepsilon^{\theta}\phi\rangle=\langle(\eta+\sqrt{n}u(t))^{2\otimes},\varepsilon^{\theta}\phi\rangle-\langle\alpha,\omega^{\theta}\phi\rangle,$$

we can easily deduce

$$\mathfrak{G}_t^n(\eta;\phi,f)=\mathfrak{A}_t^n\eta(\phi)f'(\langle\eta,\phi\rangle)+\frac{1}{2}Q_t^n(\phi;\eta)f''(\langle\eta,\phi\rangle)+R_n \tag{2.7}$$

with

$$|R_n|\leqq\frac{\|f'''\|_\infty}{12\sqrt{n}}\int\langle\alpha\otimes\alpha,|\varepsilon^{\theta}\phi|^3\rangle\tilde{d}\theta.$$

If we set

$$Q_t(\phi)=\frac{1}{2}\int\langle u(t)\otimes u(t),(\varepsilon^{\theta}\phi)^2\rangle\tilde{d}\theta \tag{2.8}$$

and let $n\to\infty$ in (2.7), we get a formal limit of $\mathfrak{G}_t^n$:

$$\int\langle\eta\otimes u(t),\varepsilon^{\theta}\phi\rangle\tilde{d}\theta f'(\langle\eta,\phi\rangle)+\frac{1}{2}Q_t(\phi)f''(\langle\eta,\phi\rangle) \tag{2.9}$$

which should regulate the limiting process. The limiting form (2.9) is consistent with what McKean [8] derived formally for a model of a gas of hard balls.

3. Expectations of functionals of η_t^n.

Let us write for $\lambda\in\boldsymbol{R}^1$

$$\chi_\lambda(x)=e^{i\lambda x}\qquad x\in\boldsymbol{R}^1$$

and introduce a condition for $\{\mu_n\}$:

(A.1) *there exists a* $\gamma \geqq 0$ *such that*

$$\sup_n \sup_{\lambda \in \boldsymbol{R}^1} \{\boldsymbol{E}[|\langle \eta_0^n, \chi_\lambda \rangle|^2](1+|\lambda|)^{-2\gamma}\} < \infty .$$

The next lemma, which will play a crucial role in the whole story of this paper, asserts that the finiteness of (A.1) propagates.

LEMMA 3.1. *If* (A.1) *holds, then there exists a nondecreasing function* K_t *such that for all* $t \geqq 0$

$$\sup_n \sup_{\lambda \in \boldsymbol{R}^1} \{\boldsymbol{E}[|\langle \eta_t^n, \chi_\lambda \rangle|^2](1+|\lambda|)^{-2\gamma}\} < K_t .$$

PROOF. Let $\phi \in C_b(\boldsymbol{R}^1)$ and be real valued and define

$$M_t = M_{t,\phi}^n = \langle \eta_t^n, \phi \rangle - \int_0^t \mathfrak{A}_s^n \eta_s^n(\phi) ds , \tag{3.1}$$

$$S_t = S_{t,\phi}^n = (M_{t,\phi}^n)^2 - \int_0^t Q_s^n(\phi ; \eta_s^n) ds . \tag{3.2}$$

Then both M_t and S_t are martingales, as easily deduced from general formulas concerning Markov processes which read as follows: if x_t is a (time-inhomogeneous) Markov process with an infinitesimal generator A_t, then, under some general conditions, the process $m_t \equiv F(x_t) - \int_0^t A_s F(x_s) ds$ and the process

$$(m_t)^2 - \int_0^t \{A_s F^2(x_s) - 2F(x_s) A_s F(x_s)\} ds$$

are martingales for each F which belongs to some reasonable class of functions on the state space of x_t.

It follows that

$$\begin{aligned} \boldsymbol{E}\langle \eta_t^n, \phi \rangle^2 &\leqq 2\boldsymbol{E}M_t^2 + 2\boldsymbol{E}\Big(\int_0^t \mathfrak{A}_s^n \eta_s^n(\phi) ds\Big)^2 \\ &\leqq 2\boldsymbol{E}M_0^2 + 2\int_0^t \boldsymbol{E}Q_s^n(\phi ; \eta_s^n) ds + 2t\int_0^t \boldsymbol{E}\{\mathfrak{A}_s^n \eta_s^n(\phi)\}^2 ds . \end{aligned}$$

Since

$$Q_s^n(\phi ; \eta_s^n) \leqq 8\|\phi\|_\infty^2 , \tag{3.3}$$

we have

$$\boldsymbol{E}\langle \eta_t^n, \phi \rangle^2 \leqq 16\|\phi\|_\infty t + 2\boldsymbol{E}\langle \eta_0^n, \phi \rangle^2 + 2t\int_0^t \boldsymbol{E}\{\mathfrak{A}_s^n \eta_s^n(\phi)\}^2 ds$$

and, by noticing that $\mathfrak{A}_s^n \eta(\phi)$ is linear in ϕ,

$$\boldsymbol{E}|\langle \eta_t^n, \chi_\lambda \rangle|^2 \leqq 32t + 2\boldsymbol{E}|\langle \eta_0^n, \chi_\lambda \rangle|^2 + 2t\int_0^t \boldsymbol{E}|\mathfrak{A}_s^n \eta_s^n(\chi_\lambda)|^2 ds .$$

To handle the last term on the right side above look at the definition of $\mathfrak{A}_t^n\eta(\phi)$. Since $|\langle\eta_t^n, \chi_\lambda\rangle|\leqq 2\sqrt{n}$ and

$$\langle\eta\otimes\eta,\ \varepsilon^\theta\chi_\lambda\rangle=\langle\eta\otimes\eta,\ \chi_{\lambda\cos\theta}\otimes\chi_{-\lambda\sin\theta}+\chi_{\lambda\sin\theta}\otimes\chi_{\lambda\cos\theta}\rangle\,,$$

we have for each θ

$$\sup_\lambda\{\boldsymbol{E}|\langle\eta_t^n\otimes\eta_t^n,\ \varepsilon^\theta\chi_\lambda\rangle|^2(1+|\lambda|)^{-2\gamma}\}\leqq 16n\cdot\sup_\lambda\{\boldsymbol{E}|\langle\eta_t^n,\ \chi_\lambda\rangle|^2(1+|\lambda|)^{-2\gamma}\}\,.$$

This and a similar bound for $\langle\eta_t^n\otimes u(t),\ \varepsilon^\theta\chi_\lambda\rangle$ yield

$$\begin{aligned}(3.4)\qquad &\sup_\lambda\{\boldsymbol{E}|\mathfrak{A}_s^n\eta_s^n(\chi_\lambda)|^2(1+|\lambda|)^{-2\gamma}\}\\ &\leqq(2\cdot 3^2+16)\sup_\lambda\{\boldsymbol{E}|\langle\eta_s^n,\ \chi_\lambda\rangle|^2(1+|\lambda|)^{-2\gamma}\}+\frac{16}{n}\,.\end{aligned}$$

Therefore if we set

$$y^n(t)=\sup_\lambda\{\boldsymbol{E}|\langle\eta_t^n,\ \chi_\lambda\rangle|^2(1+|\lambda|)^{-2\gamma}\}\,,$$

then

$$\begin{aligned}y^n(t)&\leqq 2y^n(0)+32t+32t^2+68t\int_0^t y^n(s)ds\\ &\leqq(2y^n(0)+32t+32t^2)\exp(68t^2)\,,\end{aligned}$$

which proves the lemma.

In the rest of this section we shall assume that (A. 1) holds and that ϕ can be expressed as a Fourier-Stieltjes transform:

$$(3.5)\qquad \phi(x)=\int_{-\infty}^{\infty}\chi_\lambda(x)\hat{\phi}(d\lambda)\,,$$

where $\hat{\phi}$ is some complex measure on $\boldsymbol{R}^1$. Let us introduce a norm

$$|\hat{\phi}|_\gamma=\int_{-\infty}^{\infty}(1+|\lambda|)^\gamma|\hat{\phi}|(d\lambda)\,,$$

where $|\hat{\phi}|(d\lambda)$ denotes the total variation measure of $\hat{\phi}$. $|\hat{\phi}|_\gamma$ will be assumed to be finite. Fixing a finite time T arbitrarily we shall denote by C_T any bounding constant which is independent of $0\leqq t<T$, ϕ and n.

LEMMA 3.2. $\sup_n\boldsymbol{E}[|\langle\eta_t^n,\ \phi\rangle|^2]\leqq C_T|\hat{\phi}|_\gamma^2$ *for* $0\leqq t<T$.

PROOF. Since $\langle\eta,\ \phi\rangle=\int\langle\eta,\ \chi_\lambda\rangle\hat{\phi}(d\lambda)$, we have, by applying Schwartz inequality,

$$\begin{aligned}\boldsymbol{E}|\langle\eta_t^n,\ \phi\rangle|^2&\leqq\left\{\int(\boldsymbol{E}|\langle\eta_t^n,\ \chi_\lambda\rangle|^2)^{1/2}|\hat{\phi}|(d\lambda)\right\}^2\\ &\leqq\sup_\lambda\{\boldsymbol{E}|\langle\eta,\ \chi_\lambda\rangle|^2(1+|\lambda|)^{-2\gamma}\}\cdot|\hat{\phi}|_\gamma^2\,.\end{aligned}$$

Thus the lemma follows from Lemma 3.1.

Noticing the inequality (3.4) and the relation

$$\mathfrak{A}_t^n\eta(\phi)=\int\mathfrak{A}_t^n\eta(\chi_\lambda)\hat{\phi}(d\lambda),$$

we get, in the same way as in the proof of Lemma 3.2, that for $0\leqq t<T$

$$\sup_n \boldsymbol{E}[|\mathfrak{A}_t^n\eta_t^n(\phi)|^2]\leqq C_T|\hat{\phi}|_\gamma^2. \tag{3.6}$$

LEMMA 3.3. $\sup_n \boldsymbol{E}[\sup_{0\leqq t\leqq T}|\langle\eta_t^n, \phi\rangle|^2]\leqq C_T|\hat{\phi}|_\gamma^2$.

PROOF. Let $M_{t,\phi}^n$ be as in (3.1). Since by a martingale inequality $\boldsymbol{E}\sup_{t\leqq T}|M_{t,\phi}^n|^2\leqq 4\boldsymbol{E}|M_{T,\phi}^n|^2$, we have

$$\begin{aligned}\boldsymbol{E}\sup_{t\leqq T}|\langle\eta_t^n, \phi\rangle|^2 &\leqq 2\boldsymbol{E}\sup_{t\leqq T}|M_{t,\phi}^n|^2+2\boldsymbol{E}\Big(\int_0^T|\mathfrak{A}_s^n\eta_s^n(\phi)|ds\Big)^2\\ &\leqq 16\boldsymbol{E}|\langle\eta_T^n, \phi\rangle|^2+18T\int_0^T\boldsymbol{E}|\mathfrak{A}_s^n\eta_s^n(\phi)|^2ds,\end{aligned}$$

which combined with Lemma 3.2 and the bound (3.6) proves the lemma.

By the same calculations as above we can easily obtain also the following inequalities:

$$\sup_n \boldsymbol{E}\Big[\sup_{0\leqq t\leqq T}\Big|\int\langle\eta_t^n\otimes\eta_t^n, \varepsilon^\theta\phi\rangle\tilde{d}\theta\Big|\Big]\leqq C_T|\hat{\phi}|_{2\gamma}; \tag{3.7}$$

$$\sup_n \boldsymbol{E}\Big[\sup_{0\leqq t\leqq T}\Big|\int\langle\eta_t^n\otimes u(t), \varepsilon^\theta\phi\rangle\tilde{d}\theta\Big|\Big]\leqq C_T|\hat{\phi}|_\gamma; \tag{3.8}$$

$$\sup_n \frac{1}{\sqrt{n}}\boldsymbol{E}\Big|\int\langle\eta_t^n\otimes\eta_t^n, (\varepsilon^\theta\phi)^2\rangle\tilde{d}\theta\Big|\leqq C_T|\hat{\phi}|_\gamma^2,\quad 0\leqq t<T; \tag{3.9}$$

$$\sup_n \boldsymbol{E}\Big|\int\langle\eta_t^n\otimes u(t), (\varepsilon^\theta\phi)^2\rangle\tilde{d}\theta\Big|\leqq C_T|\hat{\phi}|_\gamma^2,\qquad 0\leqq t<T. \tag{3.10}$$

4. Tightness of fluctuation processes.

To discuss the problem of the convergence of fluctuation processes η_t^n we shall consider them as $\mathcal{S}'$-valued processes, for in the limit the state space of the process is not confined in the space of signed measures, but extends to a larger subspace of $\mathcal{S}'$ as has been revealed by Tanaka [11]. To set up the framework we introduce the following notations.

$e_k(x)=(-1)^k(\sqrt{2\pi}k\,!)^{-1/2}\exp(x^2/4)(d^k/dx^k)\exp(-x^2/2)\qquad x\in\boldsymbol{R}^1$.

$\mathcal{P}=\{\phi=\sum a_k e_k \text{ (finite sum)}: a_k\in\boldsymbol{R}^1\}$.

$|||\phi|||_\delta=\Big(\sum a_k^2\Big(k+\dfrac{1}{2}\Big)^{2\delta}\Big)^{1/2}$ for $\phi\in\mathcal{P}$ (δ is any real number).

$\mathcal{S}_\delta$: the completion of the pre-Hilbert space $(\mathcal{P}, |||\cdot|||_\delta)$.

$\mathcal{S}_\delta'$: the dual space of $\mathcal{S}_\delta$; this space will be identified with $\mathcal{S}_{-\delta}$; the dual norm of $\mathcal{S}_\delta'$ can be expressed as

$$|||\eta|||^{(\delta)}=\left(\sum_{k=0}^{\infty}\langle\eta, e_k\rangle^2\left(k+\frac{1}{2}\right)^{-2\delta}\right)^{1/2} \qquad (\eta\in\mathcal{S}_\delta').$$

$\mathcal{S}=\bigcap_{m=0}^{\infty}\mathcal{S}_m$; $\mathcal{S}'=\bigcup_{m=0}^{\infty}\mathcal{S}_m'$.

$\boldsymbol{R}_+=[0, \infty)$.

$D[\boldsymbol{R}_+, \mathcal{S}']$: the space of right continuous functions of $\boldsymbol{R}_+$ into $\mathcal{S}'$ with left limits.

$D[\boldsymbol{R}_+, \mathcal{S}_\delta']$: the space similarly defined, but equipped with the Skorohod topology associated with the Hilbert norm $|||\cdot|||^{(\delta)}$.

$C[\boldsymbol{R}_+, \mathcal{S}_\delta']$: the space of continuous functions of $\boldsymbol{R}_+$ into $\mathcal{S}_\delta'$.

$\mathcal{F}_t$: the smallest σ-field in $D[\boldsymbol{R}_+, \mathcal{S}']$ with respect to that the mappings $\eta_\cdot\in D[\boldsymbol{R}_+, \mathcal{S}']\rightsquigarrow\langle\eta_s, \phi\rangle\in\boldsymbol{R}^1$ are measurable for all $0\leqq s\leqq t$ and $\phi\in\mathcal{S}$.

$\mathcal{F}$: the smallest σ-field containing all $\mathcal{F}_t$, $t\geqq 0$.

(We shall use the same symbols to denote the restrictions of $\mathcal{F}_t$ or $\mathcal{F}$ to the spaces $D[\boldsymbol{R}_+, \mathcal{S}_\delta']$ or $C[\boldsymbol{R}_+, \mathcal{S}_\delta']$.)

P^n; E^n: the probability measure on $(D[\boldsymbol{R}_+, \mathcal{S}'], \mathcal{F})$ induced by $(\eta_\cdot^n, \boldsymbol{P})$; and the associated expectation.

THEOREM 4.1. *Let $\{\mu_n\}$ satisfy* (A. 1) *in* § 3 *and let $\delta>\dfrac{3+2\gamma}{4}$. Then P^n, $n=1, 2, \cdots$ are all concentrated on $D[\boldsymbol{R}_+, \mathcal{S}_\delta']$ and their restrictions to the space $D[\boldsymbol{R}_+, \mathcal{S}_\delta']$ form a tight family of probability measures on it. Any limit measure of the family is concentrated on $C[\boldsymbol{R}_+, \mathcal{S}_\delta']$.*

LEMMA 4.1. $\sup_n \boldsymbol{E}[\sup_{0\leq t\leq T}|\langle\eta_t^n, e_k\rangle|^2]\leqq C_T\left(k+\dfrac{1}{2}\right)^{\gamma+1/2}$.

PROOF. The inequality of the lemma is immediate from Lemma 3.3 if we prove

$$|\hat{e}_k|_\gamma\leqq C_\gamma\sqrt{k+1/2}^{\gamma+1/2}, \qquad k=0, 1, 2, \cdots, \tag{4.1}$$

where C_γ is a constant depending on γ only.

Since

$$e_k(x)=(i\sqrt{\pi})^{-1}\int e_k(2\lambda)\chi_\lambda(x)d\lambda,$$

our task for the proof of (4.1) is to compute an upper bound for

$$J_{k,\gamma}\equiv\int|\lambda|^\gamma|e_k(\lambda)|d\lambda$$

for each $\gamma\geqq 0$. Let m be an integer such that $2(\gamma-m)<-1$. Then

$$J_{k,\gamma}\leqq\int_{|\lambda|\leq\varepsilon}|\lambda|^{\gamma}|e_k|d\lambda+\int_{|\lambda|>\varepsilon}|\lambda|^{\gamma-m}|\lambda^m e_k|d\lambda$$

$$\leqq\left\{\frac{2}{2\gamma+1}\varepsilon^{2\gamma+1}\right\}^{1/2}\|e_k\|+\left\{\frac{2}{2(m-\gamma)-1}\varepsilon^{2(\gamma-m)+1}\right\}^{1/2}\|\lambda^m e_k\|,$$

where $\|\cdot\|$ denotes the usual L^2-norm. Now applying the recurrence formula

$$\lambda e_k(\lambda)=\sqrt{k+1}\cdot e_{k+1}(\lambda)+\sqrt{k}e_{k-1}(\lambda)$$

and the norming condition $\|e_k\|=1$, and then substituting $\sqrt{k+1/2}$ for ε, we obtain

$$J_{k,\gamma}\leqq\text{const.}\sqrt{k+1/2}^{\gamma+1/2}$$

as desired. The proof of Lemma 4.1 is complete.

Lemma 4.1 proves the first assertion of Theorem 4.1. For the proof of the other two of Theorem 4.1 we shall follow Holley and Stroock [4] (pp. 767-768). To carry out this we need the following fact

$$\text{(4.2)}\qquad P^n\left[\sup_{t\geq 0}|\langle\eta_t,\phi\rangle-\lim_{s\uparrow t}\langle\eta_s,\phi\rangle|\leqq\frac{4}{\sqrt{n}}\|\phi\|_\infty\right]=1\qquad\text{for all}\quad n,$$

in addition to (3.3), (3.7), (3.8) and Lemma 4.1. (4.2) follows from the fact that the probability that more than two molecules change their velocities at the same time is zero.

PROOF OF THEOREM 4.1. In view of Lemma 4.1 it suffices to show that for each $T<\infty$, $\phi\in\mathcal{S}$ and $\varepsilon>0$

$$\text{(4.3)}\qquad \sup_{n\geq m}\boldsymbol{E}[\sup_{|t-s|<b}{}^{*}|\langle\eta_t^n,\phi\rangle-\langle\eta_s^n,\phi\rangle|>\varepsilon]\to 0\quad\text{as}\quad b\downarrow 0\quad\text{and}\quad m\to\infty,$$

where the star means that the variables under the supremum are taken from the interval $[0, T]$: because the tightness of $\{P^n\}$ is implied by that of the family of $\boldsymbol{R}^m$-valued processes $((\langle\eta_\cdot,e_k\rangle)_{k=1}^m;P^n)$ (for each $m=1, 2, \cdots$) coupled with the condition that

$$\lim_{N\uparrow\infty}\sup_n P^n\left[\sup_{0\leq t\leq T}\sum_{k=N}^{\infty}\langle\eta_t,e_k\rangle^2\left(k+\frac{1}{2}\right)^{-2\delta}>\varepsilon\right]=0$$

(for $\delta>(3+2\gamma)/4$), and because it follows from (4.3) that the family $\{((\langle\eta_\cdot,e_k\rangle)_{k=1}^m;P^n)\}_{n=1}^\infty$ is tight and any limiting process of it is a continuous process (cf. Billingsley [1] Theorem 15.5). From (4.2) and from the fact that the inequality

$$\sup_{|t-s|<b}{}^{*}|x_t-x_s|\leqq 2w_b''(x.)+\sup_t{}^{*}|x_t-x_{t-}|$$

holds for each right continuous function x_t having left limits and each $b>0$, where

$$w_b''(x.)=\sup{}^{*}\{|x_t-x_r|\wedge|x_r-x_s|:s\leqq r\leqq t,\ |t-s|\leqq b\}$$

(cf. Lemma 6.4 and its proof in Parthasarathy [**9**]), the relation (4.3) follows if we prove

(4.4) $$\sup_n \boldsymbol{P}(w_b''(\langle \eta_\cdot^n, \phi\rangle)>\varepsilon)\to 0 \qquad \text{as} \quad b\downarrow 0\,.$$

Let

$$\tau=\tau_N^n=\inf\{t\geqq 0:\ |\mathfrak{A}_t^n\eta_t^n(\phi)|>N\}$$

and

$$x_t^n=\langle \eta_{t\wedge\tau}^n, \phi\rangle\,.$$

Since by (3.7) and (3.8) $\sup_n \boldsymbol{P}(\tau_N^n>T)\to 0$ as $N\to\infty$, (4.4) follows if we prove

(4.5) $$\sup_n \boldsymbol{P}(w_b''(x_\cdot^n)>\varepsilon)\to 0 \qquad \text{as} \quad b\downarrow 0\,.$$

Since τ is a stopping time, $M_{t\wedge\tau,\phi}^n$ and $S_{t\wedge\tau,\phi}^n$ are martingales. Noticing (3.3) and recalling the definitions of $M_{t,\phi}^n$ and $S_{t,\phi}^n$ (see (3.1) and (3.2) for definitions), it is routine to see that

$$\sup_n \boldsymbol{E}[(x_t^n-x_r^n)^2(x_r^n-x_s^n)^2]\leqq \text{const.}\,(t-s)^2$$

for $0\leqq s\leqq r\leqq t\leqq T$, which implies (4.5) in view of Censov's criterion (cf. [**1**] Theorem 15.6). Thus the proof of Theorem 4.1 is complete.

5. Limiting processes and a martingale problem.

The process on $\mathcal{S}_\delta'$ under a limit measure of $\{P^n\}$ should be a Markov process being regulated by a generator as expressed in (2.9). Since the integrand $\langle \eta\otimes u(t), \varepsilon^\theta\phi\rangle$ appearing in (2.9) does not necessarily make sense for all $\eta\in\mathcal{S}_\delta'$, $\phi\in\mathcal{S}$, by noticing $\langle \eta_t, 1\rangle=0$ a.s. P^n we shall replace the integral in the first term of (2.9) by

(5.1) $$\mathfrak{A}_t\eta(\phi)\equiv 2\int_{-\infty}^{\infty}\int_{-\pi}^{\pi}\langle \eta, \phi^{\theta,y}\rangle \bar{d}\theta u(t, dy)-\langle \eta, \phi\rangle$$

where $\phi^{\theta,y}$ is a function of $x\in\boldsymbol{R}^1$ and defined by

$$\phi^{\theta,y}(x)=\phi(x\cos\theta-y\sin\theta)\,.$$

The meaning of the integral on the right-hand side of (5.1) is still indefinite. Its precise meaning will be given in Remark 5.2 below. We shall assert that a limit measure P of $\{P^n\}$ solves the following martingale problem:

(5.2) *for every* $f\in C_0^\infty(\boldsymbol{R}^1)$ *and* $\phi\in\mathcal{S}$

$$f(\langle \eta_t, \phi\rangle)-\int_0^t\left\{\mathfrak{A}_s\eta_s(\phi)f'(\langle \eta_s, \phi\rangle)+\frac{1}{2}Q_s(\phi)f''(\langle \eta_s, \phi\rangle)\right\}ds$$

is a $(P, \mathfrak{F}_t)$*-martingale,* where $Q_t(\phi)$ is defined in (2.8), and $\bar{\mathcal{F}}_t$ is the completion of $\mathcal{F}_t$ by P.

THEOREM 5.1. *Let* $\{\mu_n\}$ *satisfy* (A.1). *Then a limit point* P *of* $\{P^n\}$ *solves the martingale problem* (5.2) *and satisfies*

$$\sup_{0\leq t\leq T} E|\langle\eta_t, \phi\rangle| \leqq C_T|\hat{\phi}|_\gamma \quad \textit{for all} \quad \phi\in\mathcal{S}. \tag{5.3}$$

(*Here* E *denotes the integration with* dP.)

REMARK 5.1. From (5.3) it follows that for $\phi\in\mathcal{S}$

$$E|\langle\eta_t, \phi^{\theta,y}\rangle| \leqq C_T|\hat{\phi}|_\gamma \quad \text{for} \quad 0\leqq t\leqq T, \quad \theta\neq\pm\frac{\pi}{2}, \quad y\in\boldsymbol{R}^1 \tag{5.4}$$

in view of

$$\begin{aligned} |(\phi^{\theta,y})^\wedge|_\gamma &= \frac{1}{2\pi}\int\left|\int\chi_\lambda(-x)\phi(x\cos\theta - y\sin\theta)dx\right|(1+|\lambda|)^\gamma d\lambda \\ &= \frac{1}{2\pi}\int\left|\int\chi_{\lambda/\cos\theta}(-x)\chi_{\lambda\tan\theta}(y)\phi(x)\frac{dx}{\cos\theta}\right|(1+|\lambda|)^\gamma d\lambda \\ &\leqq \frac{1}{2\pi}\int\left|\int\chi_\lambda(-x)\phi(x)dx\right|(1+|\lambda|)^\gamma d\lambda \\ &= |\hat{\phi}|_\gamma. \qquad (\theta\neq\pm\pi/2,\ y\in\boldsymbol{R}^1.) \end{aligned} \tag{5.5}$$

REMARK 5.2. Since $\phi\in\mathcal{S}$ implies $\phi^{\theta,y}\in\mathcal{S}$ (for $\theta\neq\pm\pi/2$), $\langle\eta, \phi^{\theta,y}\rangle$ makes sense. Though $\langle\eta, \phi^{\theta,y}\rangle$ is not necessarily integrable with respect to $\tilde{d}\theta u(t, dy)$ for all $\eta\in\mathcal{S}_0'$, $\langle\eta_t, \phi^{\theta,y}\rangle$ is integrable for P-a.a. $\eta.$, provided (5.3) is valid, in view of Remark 5.1. If $\langle\eta_t, \phi^{\theta,y}\rangle$ is not integrable we assign the value zero to $\mathfrak{A}_t\eta_t(\phi)$ so that $\mathfrak{A}_t\eta_t(\phi)$ becomes $\bar{\mathcal{F}}_t$-progressively-measurable. Similarly let $\int_0^t\mathfrak{A}_s\eta_s(\phi)f'(\langle\eta_s, \phi\rangle)ds$ be zero, if $\mathfrak{A}_s\eta_s(\phi)$ is not integrable on $[0, t]$.

PROOF OF THEOREM 5.1. By (5.5) and Lemma 3.2

$$\sup_n E^n[|\langle\eta_t, \phi^{\theta,y}\rangle|^2] \leqq C_T|\hat{\phi}|_\gamma^2 \quad \left(\theta\neq\pm\frac{\pi}{2}\right), \tag{5.6}$$

which implies (5.4) and hence (5.3). Since

$$\mathfrak{A}_s\eta_s(\phi) = \int\langle\eta_s\otimes u(s), \varepsilon^\theta\phi\rangle\tilde{d}\theta \qquad \text{a.s. } P^n$$

and for $f\in C_0^\infty(\boldsymbol{R}^1)$

$$f(\langle\eta_t, \phi\rangle) - \int_0^t \mathfrak{G}_s^n(\eta_s; \phi, f)ds$$

is a $(P^n, \mathcal{F}_t^n)$-martingale, we see by (2.7), (3.7), (3.9) and (3.10) that

$$\begin{aligned} f(\langle\eta_t, \phi\rangle) - \int_0^t\left\{\mathfrak{A}_s\eta_s(\phi)f'(\langle\eta_s, \phi\rangle) + \frac{1}{2}Q_s(\phi)f''(\langle\eta_s, \phi\rangle)\right\}ds \\ = \text{a } (P^n, \mathcal{F}_t^n)\text{-martingale} + o(1), \end{aligned}$$

(where $\bar{\mathcal{F}}_t^n$ denotes the completion of $\mathcal{F}_t$ by P^n) with $\lim_{n\to\infty} E^n|o(1)|=0$. Therefore it suffices to show that if $s<t$, Φ is a $\mathcal{F}_s$-measurable bounded continuous function of $\eta.\in D[\boldsymbol{R}_+, \mathcal{S}_\delta']$ and $P^{n'}$ converges weakly to P, then

$$\lim_{n'} E^{n'}\left[\int_0^t \mathfrak{A}_r\eta_r(\phi)f'(\langle\eta_r, \phi\rangle)dr\cdot\Phi\right] \tag{5.7}$$
$$=E\left[\int_0^t \mathfrak{A}_r\eta_r(\phi)f'(\langle\eta_r, \phi\rangle)dr\cdot\Phi\right].$$

The expectation on the left-hand side equals

$$\int_0^t dr\iint \tilde{d}\theta u(r, dy)E^{n'}[(2\langle\eta_r, \phi^{\theta,y}\rangle-\langle\eta_r, \phi\rangle)f'(\langle\eta_r, \phi\rangle)\cdot\Phi],$$

and by (5.6) the expectation in the triple integral above converges to the corresponding one with respect to P, because the mapping $\eta.\rightsquigarrow\langle\eta_r, \phi\rangle$ is continuous a.s. P. Thus an application of the bounded convergence theorem with the help of (5.6) concludes (5.7).

6. A uniqueness result for the martingale problem and convergence of fluctuation processes.

THEOREM 6.1. *A probability measure P on $(C[\boldsymbol{R}_+, \mathcal{S}'], \mathcal{F})$ satisfying* (5.2) *and* (5.3) *is uniquely determined by $P|_{\mathcal{F}_0}$ (the restriction of P on $\mathcal{F}_0$).*

We prepare a lemma which is concerned with functionals ξ_t, $t\geqq 0$ defined by

$$\xi_t(\phi)\equiv\xi_t(\phi\,;\eta.)\equiv\langle\eta_t-\eta_0, \phi\rangle-\int_0^t \mathfrak{A}_s\eta_s(\phi)ds, \qquad \phi\in\mathcal{S} \tag{6.1}$$

where $\eta.\in C[\boldsymbol{R}_+, \mathcal{S}']$; we also define

$$\mathcal{F}_t^*\equiv\sigma\{\xi_s(\phi):0\leqq s\leqq t, \phi\in\mathcal{S}\}, \qquad \mathcal{F}^*=\sigma\{\mathcal{F}_t^*:t\geqq 0\}.$$

($\sigma\{\cdot\}$ denotes the σ-field generated by $\{\cdot\}$.)

LEMMA 6.1. *If P satisfies* (5.2) *and* (5.3), *then the family of random variables $\xi_t(\phi):t\geqq 0$, $\phi\in\mathcal{S}$ on $(C[\boldsymbol{R}_+, \mathcal{S}'], \mathcal{F}, P)$ satisfies the following conditions:*

i) *For each $\phi\in\mathcal{S}$, $\xi_t(\phi)$ is a continuous function of $t\geqq 0$ a.s. P and $\xi.(\phi)$ is $\bar{\mathcal{F}}_t$-adapted;*

ii) *For each $\phi\in\mathcal{S}$, $\xi_0(\phi)=0$ and*

$$E[\exp\{i\xi_t(\phi)\}\,|\,\mathcal{F}_s]=\exp\left\{i\xi_s(\phi)-\frac{1}{2}\int_s^t Q_r(\phi)dr\right\} \qquad a.s.\ P;$$

iii) *For each ϕ, $\psi\in\mathcal{S}$ and each a, $b\in\boldsymbol{R}^1$*

$$\xi_t(a\phi+b\psi)=a\xi_t(\phi)+b\xi_t(\psi) \qquad \text{for all } t\geqq 0, \ a.s.\ P.$$

PROOF. i) and iii) are clear from the definition of $\mathfrak{A}_t\eta(\phi)$ and (5.3). For the proof of ii) we follow Stroock-Varadhan [10].

Let

$$\tau_N=\sup\left\{t\geqq 0:\int_0^t|\mathfrak{A}_s\eta_s(\phi)|\,ds\leqq N\right\}.$$

By (5.3) (or rather by (5.4))

$$P(\tau_N<T)\leqq 3TC_T|\hat{\phi}|_\gamma\frac{1}{N},$$

and hence

$$P(\lim_{N\uparrow\infty}\tau_N=\infty)=1.$$

If we set

$$F(t)=\exp\left\{-\int_0^{t\wedge\tau_N}\left(i\mathfrak{A}_s\eta_s(\phi)-\frac{Q_s(\phi)}{2}\right)ds\right\}$$

$$M_t=\exp\{i\langle\eta_{t\wedge\tau_N},\phi\rangle\}-\int_0^{t\wedge\tau_N}\left(i\mathfrak{A}_s\eta_s(\phi)-\frac{Q_s(\phi)}{2}\right)\exp\{i\langle\eta_s,\phi\rangle\}\,ds,$$

then M_t is a $(P, \mathcal{F}_t)$-martingale and

$$M_tF(t)-\int_0^t M_s dF(s)=\exp\{i\langle\eta_{t\wedge\tau_N},\phi\rangle\}F(t). \tag{6.2}$$

Since

$$\left(\sup_{0\leq t\leq T}|M_t|\right)\left(\int_0^T|F'(t)|\,dt+|F(T)|\right)$$
$$\leqq\left(1+N+\frac{1}{2}\int_0^T Q_s(\phi)ds\right)^2\exp\left\{\frac{1}{2}\int_0^T Q_s(\phi)ds\right\},$$

the left-hand side of (6.2) is a $(P, \mathcal{F}_t)$-martingale (cf. Theorem 1.2.8 of [10]). By letting $N\to\infty$, it follows that

$$\exp\left\{i\left(\langle\eta_t,\phi\rangle-\int_0^t\mathfrak{A}_s\eta_s(\phi)ds\right)+\frac{1}{2}\int_0^t Q_s(\phi)ds\right\}$$

is an $\mathcal{F}_t$-martingale, proving ii).

PROOF OF THEOREM 6.1. Let $\Omega=C[\boldsymbol{R}_+, \mathcal{S}']$ and $P^{(1)}$ and $P^{(2)}$ be two probability measures on $(\Omega, \mathcal{F})$ which satisfy (5.2) and (5.3) and coincide on $\mathcal{F}_0$. Since the relation in ii) of Lemma 6.1 determines $P^{(j)}$ on $\mathcal{F}^*$ when conditioned on $\mathcal{F}_0$, the coincidence of $P^{(1)}$ and $P^{(2)}$ on $\mathcal{F}_0$ implies that on $\mathcal{F}^*\vee\mathcal{F}_0$ $(=\sigma\{\mathcal{F}^*, \mathcal{F}_0\})$:

$$P^{(1)}|_{\mathcal{F}^*\vee\mathcal{F}_0}=P^{(2)}|_{\mathcal{F}^*\vee\mathcal{F}_0}.$$

Denote this common distribution on $(\Omega, \mathcal{F}^*\vee\mathcal{F}_0)$ by P^*. By applying assumptions (5.2) and (5.3) and the inequality in (5.5) it can be easily verified that $P^{(1)}$ and $P^{(2)}$ are concentrated on $C[\boldsymbol{R}_+, \mathcal{S}'_\delta]$ for some $\delta>0$ (we can take $\delta=(3+2\gamma)/4+1$). Since the space $C[\boldsymbol{R}_+, \mathcal{S}'_\delta]$ equipped with the topology of uniform convergence on finite intervals is Polish and $\mathcal{F}$ restricted on it coincides with its topological Borel field, there exist regular conditional probability measures $Q_{\eta.}^{(i)}(\cdot)$ of $P^{(i)}$ $(i=1, 2)$ given $\mathcal{F}^* \vee \mathcal{F}_0$. Now we let $\tilde{\Omega}=\Omega\times\Omega$ and $\tilde{\mathcal{F}}=\mathcal{F}\times\mathcal{F}$ and define a probability measure $\tilde{P}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ via the relation

$$\tilde{P}(A\times B)=\int Q_{\eta.}^{(1)}(A)Q_{\eta.}^{(2)}(B)P^*(d\eta.)$$

for $A, B\in\mathcal{F}$. Clearly marginals of $\tilde{P}$ agree with $P^{(1)}$ and $P^{(2)}$, i.e. if Φ is an $\mathcal{F}$-measurable bounded function, then $\int\Phi(\eta_.^{(j)})d\tilde{P}=E^{(j)}\Phi$ $(j=1, 2)$. This implies

$$|E^{(1)}\Phi-E^{(2)}\Phi|\leqq\int|\Phi(\eta_.^{(1)})-\Phi(\eta_.^{(2)})|\,d\tilde{P}\,.$$

Therefore it suffices to prove $\tilde{P}[\eta_.^{(1)}=\eta_.^{(2)}]=1$ or equivalently

$$\tilde{E}[|\langle\eta_t^{(1)}, \phi\rangle-\langle\eta_t^{(2)}, \phi\rangle|]=0 \tag{6.3}$$

for all $t\geqq 0$ and $\phi\in\mathcal{S}$. Let $\zeta_t=\eta_t^{(1)}-\eta_t^{(2)}$ for $(\eta_.^{(1)}, \eta_.^{(2)})\in\tilde{\Omega}$. Then by definition of $\tilde{P}$

$$\tilde{P}\left[\langle\zeta_t, \phi\rangle=\int_0^t\mathfrak{A}_s\zeta_s(\phi)ds\right]=1 \qquad \text{for} \quad \phi\in\mathcal{S}$$

where $\mathfrak{A}_s\zeta_s(\phi)=\mathfrak{A}_s\eta_s^{(1)}(\phi)-\mathfrak{A}_s\eta_s^{(2)}(\phi)$. If we define the iteration $\mathfrak{A}_t\mathfrak{A}_s$ by

$$\begin{aligned}\mathfrak{A}_t\mathfrak{A}_s\eta(\phi)&=2\iint\mathfrak{A}_s\eta(\phi^{\theta,y})\bar{d}\theta u(t, dy)-\mathfrak{A}_s\eta(\phi)\\&=4\iint\bar{d}\theta u(t, dy)\iint\langle\eta, (\phi^{\theta,y})^{\theta',y'}\rangle\bar{d}\theta' u(s, dy')\\&\quad-2\iint\langle\eta, \phi^{\theta,y}\rangle\bar{d}\theta u(t, dy)-\mathfrak{A}_s\eta(\phi)\,,\end{aligned}$$

and define $\mathfrak{A}_{t_1}\mathfrak{A}_{t_2}\cdots\mathfrak{A}_{t_m}$ analogously, then

$$\begin{aligned}\tilde{E}|\langle\zeta_t, \phi\rangle|&\leqq\int_0^t\tilde{E}|\mathfrak{A}_s\zeta_s(\phi)|ds\leqq\cdots\\&\leqq\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{m-1}}\tilde{E}|\mathfrak{A}_{t_1}\cdots\mathfrak{A}_{t_m}\zeta_{t_m}(\phi)|\,dt_m\,.\end{aligned}$$

Since (5.4) implies

$$\tilde{E}|\mathfrak{A}_{t_1}\cdots\mathfrak{A}_{t_m}\zeta_{t_m}(\phi)|\leqq 2\cdot 3^m C_T|\hat{\phi}|_\gamma\,,$$

we conclude $\tilde{E}|\langle\zeta_t, \phi\rangle|=0$, proving (6.3). Thus Theorem 6.1 has been proved.

REMARK 6.1. Let P be as in Theorem 6.1. Then by Lemma 6.1 $\xi_t(\phi)$ defined in (6.1) is a $(P, \mathcal{F}_t)$-martingale with

$$E[\xi_t(\phi)^2]=\int_0^t Q_s(\phi)ds\,, \qquad \phi\in\mathcal{S}\,.$$

By (3.3), (4.1), the inequality $\|\phi\|_\infty \leqq |\hat{\phi}|_0$ and a martingale inequality, we have

$$E[\sup_{0\leq t\leq T}|\xi_t(e_k)|^2]\leqq C_T\sqrt{k+1/2}\,.$$

Therefore for P-a.e. $\eta_\cdot$, $(\xi_\cdot(e_k))_{k=0}^\infty$ can be identified with an element of $C[\boldsymbol{R}_+, \mathcal{S}_1']$, say $\xi_\cdot$, via the equality

$$\langle\xi_t, \phi\rangle=\sum_{k=0}^\infty \xi_t(e_k)\int e_k\phi dx\,, \qquad \phi\in\mathcal{S}_1$$

consistently in the sense that $\langle\xi_\cdot, \phi\rangle=\xi_\cdot(\phi)$ a.s. P for $\phi\in\mathcal{S}$. We can take an $\mathcal{F}_t$-adapted version for $\xi_\cdot$; then in view of the property ii) in Lemma 6.1 ξ_t may be called a *Q_t-Wiener process* on $\mathcal{S}_1'$. Now (6.1) becomes

$$\text{(6.5)} \qquad \langle\eta_t-\eta_0, \phi\rangle=\langle\xi_t, \phi\rangle+\int_0^t \mathfrak{A}_s\eta_s(\phi)ds \qquad \text{for all} \quad t\geqq 0 \quad \text{a.s. } P.$$

In the proof of Theorem 6.1 we have proved, by showing (6.3), the pathwise uniqueness for the stochastic integral equation (6.5) where ξ_t is understood as a given Q_t-Wiener process.

A combination of Theorems 4.1, 5.1 and 6.1 yields the next theorem, in which we shall assume the following condition:

(A.2) *for each $\phi\in\mathcal{S}$, the distribution on $\boldsymbol{R}^1$ induced by $(\langle\eta_0^n, \phi\rangle, \boldsymbol{P})$ converges weakly to a distribution, say F_ϕ, as $n\to\infty$.*

This condition coupled with (A.1) implies the weak convergence of the family of distributions on $\mathcal{S}_\delta'$ induced by $(\eta_0^n, \boldsymbol{P})$ if $\delta>(3+2\gamma)/4$.

THEOREM 6.2. *If* (A.2) *as well as* (A.1) *holds and $\delta>(3+2\gamma)/4$, then P^n converges weakly to P where P is concentrated on $C[\boldsymbol{R}_+, \mathcal{S}_\delta']$ and is a unique solution of the martingale problem* (5.2) *with the collateral condition* (5.3) *and the initial condition*

$$P(\langle\eta_0, \phi\rangle\in\cdot)=F_\phi \qquad \textit{for} \quad \phi\in\mathcal{S}\,.$$

7. Distribution of the limiting process.

In this section we shall assume the condition (A.1) and find an expression of a limit measure of $\{P^n\}$, guided by Theorem (1.4) of Holley-Stroock [**4**]. To handle the operator $\mathfrak{A}_t$ appearing in the drift term of the limiting process we introduce a Banach space $\mathcal{E}$: $\mathcal{E}$ consists of all real functions ϕ of the form (3.5) with absolutely continuous $\hat{\phi}$ and is normed by $\|\phi\|\equiv|\hat{\phi}|_\gamma$ with a constant γ appearing in (A.1); hence, if $\phi\in\mathcal{E}$, ϕ is a Fourier transform of a (at least summable) complex function on $\boldsymbol{R}^1$.

For each $t \geqq 0$, we define a linear operator $\mathfrak{A}_t$ on $\mathcal{E}$ by

$$\mathfrak{A}_t \phi(x) = 2\iint \phi^{\theta, y}(x) u(t, dy) \bar{d}\theta - \phi(x)$$

($\phi^{\theta, y}$ is defined in §5). Then

$$\|\mathfrak{A}_t \phi\| \leqq 3\|\phi\| \tag{7.1}$$

$$\|\mathfrak{A}_t \phi - \mathfrak{A}_s \phi\| \leqq 4(t-s)\|\phi\| \quad \text{for} \quad s \leqq t. \tag{7.2}$$

The first bound follows from (5.5) and the second also from (5.5) with the help of

$$\frac{1}{2}\frac{\partial}{\partial t}\mathfrak{A}_t \phi(x) = \iiiint \phi^{\theta, y^*}(x) u(t, dy) u(t, dz) \bar{d}\theta \bar{d}\sigma$$

$$-\iint \phi^{\theta, y}(x) u(t, dy) \bar{d}\theta$$

where

$$y^* = y\cos\sigma - z\sin\sigma.$$

In view of (7.1) and (7.2) we can construct without difficulty a strongly continuous semigroup of bounded operators $U(s, t): 0 \leqq s \leqq t$, on $\mathcal{E}$ such that

$$U(s, r)U(r, t) = U(s, t) \quad \text{for} \quad 0 \leqq s \leqq r \leqq t;$$

$$U(s, s) = \text{the identity operator};$$

$$\frac{\partial}{\partial s}U(s, t) = -\mathfrak{A}_s U(s, t); \qquad \frac{\partial}{\partial t}U(s, t) = U(s, t)\mathfrak{A}_t,$$

where the partial derivatives are taken in the uniform norm of bounded operators. Let P be a limit measure of $\{P^n\}$. We want to give a meaning to $\langle \eta_s, U(s, t)\phi \rangle$ for each $\phi \in \mathcal{S}$, each $0 \leqq s \leqq t$ and P-a.a. $\eta. \in D[\boldsymbol{R}_+, \mathcal{S}']$. Since $U(s, t)\phi$ belongs in general not to $\mathcal{S}$, but to $\mathcal{E}$, we introduce for each $t > 0$ and $\phi \in \mathcal{E}$ a real valued functional $z_t(\phi) = z_t(\phi; \eta.)$ of $\eta. \in D[\boldsymbol{R}_+, \mathcal{S}']$ which has the following properties:

$z_t(\phi)$ is $\mathcal{F}_t$-measurable;
if $\phi_k \in \mathcal{S}$ and $\lim_{k \uparrow \infty} \|\phi_k - \phi\| = 0$, then $\lim_{k \uparrow \infty} E|\langle \eta_t, \phi_k \rangle - z_t(\phi)| = 0$;
$P^n[\langle \eta_t, \phi \rangle = z_t(\phi)] = 1$ for $n = 1, 2, \cdots$.

If $\phi \in \mathcal{S}$ let $z_t(\phi) = \langle \eta_t, \phi \rangle$. For a general $\phi \in \mathcal{E}$, by taking $\rho_k \in \mathcal{S}$ such that

$$\lim_{m \uparrow \infty} \left\| \sum_{k=1}^{m} \rho_k - \phi \right\| = 0 \quad \text{and} \quad \|\rho_k\| \leqq 2^{-k},$$

set

$$z_t(\phi) = \begin{cases} \sum_{k=1}^{\infty} \langle \eta_t, \rho_k \rangle & \text{if} \quad \sum_{k=1}^{\infty} |\langle \eta_t, \rho_k \rangle| < \infty \\ 0 & \text{otherwise.} \end{cases}$$

By virtue of Lemma 3.2 $z_t(\phi)$ then satisfies the requirements above. Because of the second condition $z_t(\phi)$ is independent of the choice of $\{\rho_k\}$ up to P-null sets.

In the next theorem we write

$$\phi_t^s=U(s, t)$$

and

$$p(t, x)=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right) \qquad x\in \boldsymbol{R}^1 .$$

THEOREM 7.1. *Assume* (A.1) *to hold and let P be a limit measure of $\{P^n\}$ and $z_t(\cdot)$ be as above. Then for each $\phi\in\mathcal{E}$, $0\leqq s\leqq t$ and $x\in\boldsymbol{R}^1$*

$$E[z_t(\phi)\leqq x\,|\,\mathcal{F}_s]=\int_{-\infty}^{x} p\left(\int_s^t Q_r(\phi_t^r)dr,\ z_s(\phi_t^s)-y\right)dy\,, \qquad a.s.\ P. \tag{7.3}$$

PROOF. Fixing a positive time T arbitrarily, set $\Omega=D[[0, T], \boldsymbol{R}^1]$ which is endowed with the usual Skorohod topology. By approximation argument, it suffices to show (7.3) for elements of $\mathcal{S}$. Given a $\phi\in\mathcal{S}$, we consider the real valued process

$$y_t^n\equiv\langle\eta_t^n, \phi_T^t\rangle:\ \ 0\leqq t\leqq T$$

where $\eta_\cdot^n$ is a process on $(\boldsymbol{P}, \boldsymbol{\Omega}, \mathcal{M})$ defined in § 2. Clearly $y_\cdot^n\in\Omega$ a.s. $\boldsymbol{P}$. By using the identity $\partial\phi_T^t/\partial t=-\mathfrak{A}_t\phi_T^t$ and noticing that $\eta_\cdot^n$ has no fixed discontinuity and its total variations are at most $2\sqrt{n}$, it can be easily verified that for any $f\in C_0^\infty(\boldsymbol{R}^1)$

$$f(y_t^n)-\int_0^t\{\mathfrak{G}_r^n(\eta_r^n\,;\,\phi_T^r, f)-\langle\eta_r^n, \mathfrak{A}_r\phi_t^r\rangle f'(y_r^n)\}\,dr \tag{7.4}$$

is a martingale. By the same reasoning as in § 4 and §5, we have that

$$\mathfrak{G}_r^n(\eta_r^n\,;\,\phi_T^r, f)-\langle\eta_r^n, \mathfrak{A}_r\phi_T^r\rangle f'(y_r^n)=\frac{1}{2}Q_r(\phi_T^r)f''(y_r^n)+o(1) \tag{7.5}$$

with $\lim_{n\to\infty}\boldsymbol{E}[|o(1)|]=0$ (note that $\phi\in\mathcal{S}$ implies $|(\phi_t^s)^\wedge|_{2\gamma}<\infty$ when applying the estimate (3.7)), and that a family of Ω-valued random variables $\{y_\cdot^n\}$ is tight. Take $\delta>(3+2\gamma)/4$ and let $\tilde{P}^n$ be the measure on $D[[0, T], \mathcal{S}_\delta']\times\Omega$ induced by $(\eta_\cdot^n, y_\cdot^n)$. Then $\{\tilde{P}^n\}$ is tight in view of Theorem 4.1. Let $\tilde{P}$ be a weak limit of $\{\tilde{P}^n\}$ along $\{n'\}$. We note that $P^{n'}$ converges weakly to the $\eta_\cdot$-marginal of $\tilde{P}$. By (7.4) and (7.5)

$$f(y_t)-\frac{1}{2}\int_0^t Q_r(\phi_T^r)f''(y_r)dr$$

is a $(\tilde{P}, \mathcal{F}_t\times\mathcal{R}_t)$-martingale, where $\{\mathcal{R}_t\}_{t=0}^T$ is a usual increasing family of σ-fields attached to Ω. Therefore for each $0\leqq s\leqq t\leqq T$ we have (for $\tilde{P}$-a.a. $(\eta_\cdot, y_\cdot)$)

$$\tilde{E}[f(y_t)\,|\,\mathcal{F}_s\times\mathcal{R}_s]=\int f(y)p\left(\int_s^t Q_r(\phi_T^r)dr,\ y_s-y\right)dy\,.$$

We shall soon observe that for $0 \leqq s \leqq T$

$$\tilde{P}[y_T = z_T(\phi\,;\,\eta.) \text{ and } y_s = z_s(\phi_T^s\,;\,\eta.)] = 1\,. \tag{7.6}$$

If this is admitted,

$$\tilde{E}[f(z_T(\phi)) \,|\, \mathcal{F}_s \times \mathcal{R}_s] = \int f(y) p\Big(\int_s^T Q_r(\phi_T^r) dr,\; z_s(\phi_T^s) - y\Big) dy\,.$$

Since the right-hand side above is independent of $y.$, this equation yields (7.3) with $t = T$ and with P the $\eta.$-marginal of $\tilde{P}$.

The proof of (7.6) is carried out as follows. Let $\phi = \phi_T^s = \sum_{k=1}^{\infty} \rho_k$ and $\phi_m = \sum_{k=1}^{m} \rho_k$, then by using Lemma 3.2

$$\begin{aligned}
\tilde{E}\,|z_s(\phi) - y_s| &\leqq \tilde{E}\,|z_s(\phi) - \langle \eta_s, \phi_m \rangle| + \tilde{E}\,|\langle \eta_s, \phi_m \rangle - y_s| \\
&\leqq C_T \sum_{k>m} \|\rho_k\| + \lim \tilde{E}^{n'} |\langle \eta_s, \phi_m \rangle - \langle \eta_s, \phi \rangle| \\
&\leqq 2C_T \sum_{k>m} \|\rho_k\|\,,
\end{aligned}$$

which vanishes as $m \to \infty$. This proves (7.6), because $\phi_T^T = \phi$.

8. Examples of $\{\mu_n\}$ satisfying (A. 1) and (A. 2).

i) Let $\mu_n = \mu^{n\otimes}$. Then (A. 1) is trivial and (A. 2) follows from the central limit theorem for i. i. d. random variables.

ii) Consider $n!$ linear arrangements of n distinct objects and attribute the equal probabilities to all arrangements. Let $\varepsilon_k = 1$ or -1 according as the k-th object is arranged in the first $[n/2]$ places or in the rest ($[x]$ is the largest integer which does not exceed x). Let $\{\boldsymbol{a}_j\}_{j=1}^{\infty}$ and $\{\boldsymbol{b}_j\}_{j=1}^{\infty}$ be two sequences of i. i. d. random variables which are independent of each other and of $\{\varepsilon_k\}$. Now we set

$$X_j = \boldsymbol{a}_j \text{ or } \boldsymbol{b}_j \qquad \text{according as} \quad \varepsilon_j = 1 \text{ or } -1$$

for $j = 1, 2, \cdots, n$, and define μ_n as the distribution of $(X_1, \cdots, X_n) \in \boldsymbol{R}^n$. Clearly μ_n is symmetric and since the probabilities of $\{\varepsilon_1 = \varepsilon_2 = 1\}$, $\{\varepsilon_1 = 1, \varepsilon_2 = -1\}$, etc. are $1/4 + O(1/n)$, (A. 1) is satisfied with $\mu = \mu_{2n|1}$ $(n = 1, 2, \cdots)$ and $\gamma = 0$. For $\phi \in C_b(\boldsymbol{R}^1)$, $\langle \eta_0^n, \phi \rangle$ has the same distribution as

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{[n/2]} (\phi(\boldsymbol{a}_j) - A) + \frac{1}{\sqrt{n}} \sum_{j=1}^{n-[n/2]} (\phi(\boldsymbol{b}_j) - B)$$
$$+ \frac{1}{\sqrt{n}} \Big([n/2] A + (n - [n/2]) B - \frac{n}{2}(A + B) \Big),$$

where $A = \boldsymbol{E}\phi(\boldsymbol{a}_1)$ and $B = \boldsymbol{E}\phi(\boldsymbol{b}_1)$. Therefore its distribution converges to a

Gaussian distribution with mean zero and variance

$$V_\phi \equiv \frac{1}{2}\boldsymbol{E}[(\phi(\boldsymbol{a}_1)-A)^2+(\phi(\boldsymbol{b}_1)-B)^2].$$

It should be noted that if $A \neq B$ this value is smaller than

$$V_{\mu,\phi} \equiv \langle \mu, (\phi-\langle \mu, \phi\rangle)^2\rangle = V_\phi + \frac{1}{4}(A-B)^2.$$

iii) Let s be a positive number and c a positive integer. Let us consider a Polya's urn scheme. An urn initially contains the equal number $[ns]$ of black and red balls. At each trial a ball is drawn at random, it being replaced and c balls of the color drawn added. Suppose one makes n successive trials and let $\varepsilon_k=1$ or -1 according as the k-th trial turns black or red. Given a set of m distinct indices $k_1, \cdots, k_m$, the probability that the k_j-th trial turns black for $j=1, \cdots, m$ and the rest $(n-m)$ trials turn red equals

$$p_{n,m} \equiv \left(\frac{[ns]}{c}+m-1\right)_m \left(\frac{[ns]}{c}+n-m-1\right)_{n-m} \Big/ \left(2\frac{[ns]}{c}+n-1\right)_n$$

(cf. Feller **[3]** p. 110). Since this probability is independent of a combination of m indices, the distribution of $(\varepsilon_1, \cdots, \varepsilon_n)$ is symmetric. Let $\{\boldsymbol{a}_j\}$ and $\{\boldsymbol{b}_j\}$ be the same as in the previous example and define μ_n in the same way too, but by the present $\{\varepsilon_k\}$. Then μ_n is symmetric and (A. 1) is satisfied with $\mu=\mu_{n|1}$ ($n= 1, 2, \cdots$) and $\gamma=0$. By using Stirling's formula and an elementary relation that as $k\to\infty$ and $x-k\to\infty$

$$\log(x)_k = \int_{x-k}^{x} (\log y)dy + \frac{1}{2}\log\frac{x}{x-k} + O\left(\frac{1}{x-k}\right),$$

it can be verified that if $k=[(n+x\sqrt{n})/2]$ and $\nu_n=\sum_{j=1}^{n}(1+\varepsilon_j)/2$,

$$\boldsymbol{P}[\nu_n=k] = p_{n,k}\cdot\binom{n}{k} = \frac{2}{\sqrt{n}}\frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{x^2}{2v}\right)\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)$$

as $n\to\infty$, where $v=1+c/2s$. From this relation it is ready to see that the distribution of $\langle \eta_0^n, \phi\rangle$ converges to a Gaussian distribution with mean zero and variance $V_\phi+(A-B)^2v/4=V_{\mu,\phi}+(A-B)^2c/8s$ (where V_ϕ and $V_{\mu,\phi}$ are the same as in ii)).

iv) Let J be a countable subset of $\boldsymbol{R}^1$ and let $(\xi_k; k=1, 2, \cdots; \boldsymbol{P})$ be a Markov chain on J starting with any initial distribution. Let τ_j be the first passage time of ξ_k through $j\in J$. We assume

$$\boldsymbol{E}[|\tau_j|^2 \mid \xi_1=l] < \infty$$

for any pair $(j, l)\in J\times J$. Then it is known that there exists a unique invariant probability measure, say μ, of the Markov chain and that there exist a constant

C and a functional $V(\phi)$ such that for any $\phi\in C_b(\boldsymbol{R}^1)$

$$\text{(8.1)}\qquad \boldsymbol{E}\Big[\Big(\sum_{k=1}^{n}(\phi(\xi_k)-\langle\mu,\phi\rangle)\Big)^2\Big]\leqq Cn\|\phi\|_\infty^2.$$

$$\text{(8.2)}\qquad \lim_{n\uparrow\infty}\boldsymbol{P}\Big[\frac{1}{\sqrt{nV(\phi)}}\sum_{k=1}^{n}(\phi(\xi_k)-\langle\mu,\phi\rangle)\leqq x\Big]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\Big(-\frac{y^2}{2}\Big)dy$$

(cf. Chung [2], pp. 94, 97 and 98). Now we define μ_n via the relation that

$$\langle\mu_n, w\rangle=\frac{1}{n!}\sum_{\sigma}\boldsymbol{E}[w(\xi_{\sigma(1)},\cdots,\xi_{\sigma(n)})]$$

for $w\in C_b(\boldsymbol{R}^n)$, where σ runs over all permutations of n objects. Clearly μ_n is symmetric. Since $\langle\alpha^n,\phi\rangle$ is identical in law to $n^{-1}\Big\{\sum_{k=1}^{n}\phi(\xi_k)\Big\}$ as being clear from

$$\int f\Big(\sum_{k=1}^{n}\phi(x_k)\Big)\mu_n=\frac{1}{n!}\sum_{\sigma}\boldsymbol{E}f\Big(\sum_{k=1}^{n}\phi(\xi_{\sigma(k)})\Big)=\boldsymbol{E}f\Big(\sum_{k=1}^{n}\phi(\xi_k)\Big),$$

(A. 1) and (A. 2) follow from (8.1) and (8.2), respectively.

v) In Kac's paper [5] the initial distribution μ_n (and hence $u_n(t)$) is supposed to be confined in the $(n-1)$-dimensional sphere of radius $\sqrt{n}$, which is natural from a physical point of view that the average frequency of collision should depend the total energy of n molecules. The proposition given here demonstrates that there are many $\{\mu_n\}$'s which are confined in these spheres and satisfy the conditions (A. 1) (with $\gamma=1$) and (A. 2).

Let μ be a probability measure with

$$0<\sigma\equiv\Big(\int x^2\mu(dx)\Big)^{1/2}<\infty$$

and $Z^n=(Z_1^n,\cdots,Z_n^n)$, $n=1,2,\cdots$ be $\boldsymbol{R}^n$-valued random variables such that $Z^n\neq 0$ a. s. and set

$$\zeta^n=\sqrt{n}\Big\{\frac{1}{n}\sum_{k=1}^{n}\delta_{Z_k^n}-\mu\Big\}.$$

Let μ_n be the distribution of the $\boldsymbol{R}^n$-valued random variable X^n which is defined by

$$X^n=\frac{\sigma\sqrt{n}}{|Z^n|}Z^n,\qquad |Z^n|=\Big\{\sum_{k=1}^{n}(Z_k^n)^2\Big\}^{1/2}.$$

Note that μ_n is concentrated on the $(n-1)$-sphere of radius $\sigma\sqrt{n}$. Now we claim that: *If the following conditions are satisfied*

$$\text{(8.3)}\qquad \sup_{n}\sup_{\lambda\in\boldsymbol{R}^1}\{\boldsymbol{E}[|\langle\zeta^n,\chi_\lambda\rangle|^2](1+|\lambda|)^{-2}\}<\infty;$$

(8.4) *for each* $\phi\in\mathcal{S}$ *the distribution of* $\langle\zeta^n,\phi+x^2\rangle$ *weakly converges as* $n\rightarrow\infty$;

(8.5) $$\sup_n \boldsymbol{E}[|\langle\zeta^n, x^2\rangle|^2]<\infty,$$

then $\lim_{n\uparrow\infty}\mu_{n|1}=\mu$, $\{\mu_n\}$ *satisfies* (A. 1) ($\gamma=1$) *and* (A. 2) *and* F_ϕ *in* (A. 2) *agrees with the limiting distribution of* $\langle\zeta^n, \tilde{\phi}\rangle$ *where*

$$\tilde{\phi}(x)=\phi(x)-(2\sigma^2)^{-1}\left(\int\phi'(y)y\mu(dy)\right)\cdot x^2.$$

PROOF. From the identity

(8.6) $$Z_k^n-X_k^n=\frac{|Z^n|^2/n-\sigma^2}{(|Z^n|/\sqrt{n}+\sigma)|Z^n|/\sqrt{n}}Z_k^n=\frac{1}{\sqrt{n}}(L_n/K_n)Z_k^n$$

where

$$L_n=\langle\zeta^n, x^2\rangle \quad\text{and}\quad K_n=\left(\frac{|Z^n|}{\sqrt{n}}+\sigma\right)\frac{|Z^n|}{\sqrt{n}},$$

it follows that

(8.7) $$\phi(Z_k^n)-\phi(X_k^n)=\frac{1}{\sqrt{n}}(L_n/K_n)Z_k^n\phi'(Z_k^n)+R_{n,k}$$

with

$$|R_{n,k}|\leqq\frac{1}{2n}\{(L_n/K_n)Z_k^n\}^2\|\phi''\|_\infty.$$

Therefore

(8.8) $$\langle\zeta^n-\eta_0^n, \phi\rangle=\frac{1}{\sqrt{n}}\sum_{k=1}^n(\phi(Z_k^n)-\phi(X_k^n))$$
$$=(L_n/K_n)\frac{1}{n}\left\{\sum_{k=1}^n Z_k^n\phi'(Z_k^n)\right\}+R_n$$

with

$$|R_n|\leqq\frac{1}{2\sqrt{n}}(L_n/\sigma)^2\|\phi\|_\infty.$$

Now the assumptions (8.3) and (8.4) is applied to ensure

$$\langle\eta_0^n, \phi\rangle=\langle\zeta^n, \phi\rangle-\frac{1}{K_n}\langle\zeta^n, x^2\rangle\langle\mu, x\phi'\rangle+o(1),$$

where $o(1)\to 0$ in probability, which proves the assertion for (A. 2), because (8.4) implies

$$K_n\to 2\sigma^2 \quad\text{in probability.}$$

From the inequality

$$\frac{1}{n}\sum_{k=1}^n|Z_k^n|\leqq\frac{|Z^n|}{\sqrt{n}}\leqq K_n/\sigma$$

and from (8.6) and the first line of (8.8) it follows that

$$|\langle\zeta^n-\eta_0^n, \chi_\lambda\rangle|\leqq(K_n/L_n)\left\{\frac{1}{n}\sum_{k=1}^n|Z_k^n|\right\}\|\chi_\lambda'\|_\infty\leqq(L_n/\sigma)|\lambda|.$$

Therefore (8.3) and (8.5) implies (A.1) with $\gamma=1$. The proof is complete.

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