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Countable metacompactness and tree topologies

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1. Introduction.

Many properties of the tree topologies of ω_1-trees are related to the normal Moore space problem and to the Dowker space problem. Countable metacompactness is related to the latter. We refer the reader to Devlin and Shelah [1], Fleissner [2] and Rudin [3] for explanations about the background of this subject.

This work was motivated by Fleissner's paper [2]. In this paper we improve one of his results, answer a question raised there and prove some more facts concerning them.

It is known that:

(1) *a special Aronszajn tree is countably metacompact* (*cmc*),

(2) *a Souslin tree is almost Souslin.*

The following is due to P. Nyikos (see [2]):

(3) *An almost Souslin tree is cmc.*

By these facts, we see that both Souslin trees and special Aronszajn trees are cmc. Since a Souslin tree and a special Aronszajn tree are very different in nature (e.g. "Souslin" and "special Aronszajn" are incompatible properties), it may be natural to ask the following:

QUESTION 1. *Is every Aronszajn tree is cmc?*

But Fleissner [2] gave a counter example. For the purpose he assumed Jensen's combinatorial principle $\diamondsuit^+$, which is a consequence of the axiom of constructibility $V=L$. However more popular principle $\diamondsuit$ weaker than $\diamondsuit^+$ suffices for the task (see Section 2 for the definition of $\diamondsuit$, also for that of $\diamondsuit^*$ also used in this paper):

(4) *If* $\diamondsuit$ *holds, there is an Aronszajn tree which is not cmc* (Theorem 1).

Relating to countable metacompactness, we consider the property that every closed set is G_δ. We call an ω_1-tree with this property a *perfect* tree here. The following are easy facts:

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(5) *A special Aronszajn tree is perfect and a perfect tree is a cmc Aronszajn tree.*
If MA+$\neg$CH holds, the converse are also true, since every Aronszajn tree is then special. So the following are reasonable questions:

QUESTION 2 (Fleissner [**2**]). *Does "perfect" imply "special Aronszajn" in* ZFC?

QUESTION 3. *Does "cmc" imply "perfect" for Aronszajn trees in* ZFC?
The latter is easily answered negatively. For, the following hold:

(6) *Every Souslin tree is not perfect,*

(7) *If $\diamondsuit$ holds, there is a Souslin tree* (Jensen).
And hence by (2) and (3), if $\diamondsuit$ holds, we have a Souslin tree as a counter example of Question 3. To answer Question 2, we first observe the following:

(8) *Every perfect tree is $\boldsymbol{R}$-embeddable* (Theorem 2).
Hence by (5), every perfect tree is $\boldsymbol{R}$-embeddable and cmc. This gives rise to the following further:

QUESTION 4. *Does $\boldsymbol{R}$-embeddability characterize perfectness for a cmc tree in ZFC?*
As may be expected, the answer is negative. But one must fail if one attempts to construct as its counter example an almost Souslin tree which is $\boldsymbol{R}$-embeddable but not perfect, under the observation of the fact (3):

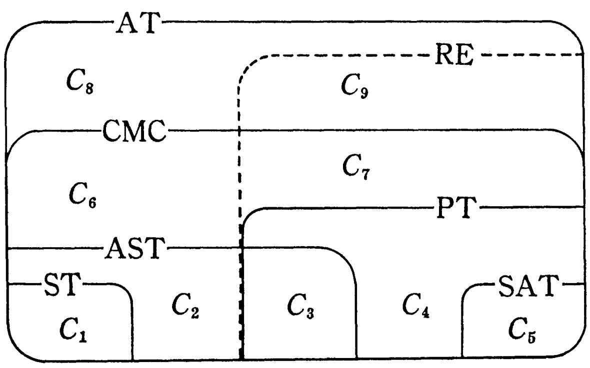
(9) *Every $\boldsymbol{R}$-embeddable almost Souslin tree is perfect* (Theorem 3).
This means that perfectness can be characterized by $\boldsymbol{R}$-embeddability for an almost Souslin tree (equivalently a collectionwise Hausdorff tree). This answers Question 2 negatively, since the following hold:

(10) *If $\diamondsuit^*$ holds, there is an $\boldsymbol{R}$-embeddable almost Souslin tree* (Devlin and Shelah [**1**]),

(11) *An almost Souslin tree is not a special Aronszajn tree.*
The following answers Question 4 negatively:

(12) *If $\diamondsuit^*$ holds, there is a cmc tree which is $\boldsymbol{R}$-embeddable but not perfect* (Theorem 4).

Our results are summarized in the following diagram:



where AT=the Aronszajn trees, RE=the $\boldsymbol{R}$-embeddable trees, CMC=the countably metacompact Aronszajn trees, PT=the perfect trees, AST=the almost Souslin trees, SAT=the special Aronszajn trees, ST=the Souslin trees. The Aronszajn trees are thus divided into nine categories in ZFC. If $V=L$ holds, these categories are all non-empty. More precisely (a) $\diamondsuit \Rightarrow C_1 \neq \emptyset$ (Jensen) & $C_2 \neq \emptyset$ (Devlin and Shelah [**1**]), (b) $C_5 \neq \emptyset$, (c) $\diamondsuit^* \Rightarrow C_3 \neq \emptyset$ (Devlin and Shelah [**1**]) & $C_7 \neq \emptyset$ (Theorem 4), (d) $[\diamondsuit \Rightarrow C_8 \neq \emptyset]$ & $[\diamondsuit \Rightarrow C_6 \neq \emptyset]$ & $[\diamondsuit^* \Rightarrow C_4 \neq \emptyset]$ follows easily from the others; e.g. $C_4 \neq \emptyset$ is immediate from $C_3 \neq \emptyset$ & $C_5 \neq \emptyset$, (e) $C_9 \neq \emptyset$ can be proven also under $\diamondsuit$, but we only give the proof of $\diamondsuit \Rightarrow C_8 \cup C_9 \neq \emptyset$ because this proof is much simpler and displays clearly the idea to destroy cmc property.

2. Preliminaries.

The cardinality of a set X is denoted by $|X|$. A subset C of ω_1 is *club* (closed and unbounded) iff $|C|=\omega_1$ and whenever D is a countable subset of C, then $\sup D \in C$. A subset D of ω_1 is *stationary* iff it meets every club set.

A *tree* $\mathcal{T}$ is a partially ordered set $(T, <_T)$ such that for every $t \in T$, the set $\hat{t}=\{s \in T: s <_T t\}$ is well ordered by $<_T$. The order type of $(\hat{t}, <_T)$ is denoted by $\mathrm{ht}(t)$, $\{t \in T: \mathrm{ht}(t)=\alpha\}$ is denoted by T_α. A *branch* of $\mathcal{T}$ is a maximal totally ordered subset of T. For C a set of ordinals, $T \restriction C=\{t \in T: \mathrm{ht}(t) \in C\}$. An *antichain* of $\mathcal{T}$ is a pairwise incomparable subset of T.

$\mathcal{T}$ is an *α-tree* iff (i) $\{\mathrm{ht}(t): t \in T\}=\alpha$, (ii) for all $\xi<\alpha$, $|T_\xi| \leqq \omega$, (iii) for every $t \in T$ and for every ξ, $\mathrm{ht}(t)<\xi<\alpha$, t has at least two successors of height ξ, and (iv) if $\mathrm{ht}(t)=\mathrm{ht}(s)$ is a limit ordinal, $t=s$ iff $\hat{t}=\hat{s}$. An *Aronszajn* tree is an ω_1-tree with no uncountable branch. A *Souslin* tree is an ω_1-tree with no uncountable antichain.

Let $\mathcal{T}$ be an ω_1-tree. The *tree topology* on $\mathcal{T}$ has a basis of all sets of the following forms:

$$[t, s)=\{u \in T: t \leqq u < s\}, \quad \text{where} \quad t \in T_0,\ s \in T\,,$$
$$(t, s)=\{u \in T: t < u < s\}, \quad \text{where} \quad t \in T,\ s \in T.$$

An ω_1-tree $\mathcal{T}$ is said to be *almost Souslin* iff whenever A is an antichain of $\mathcal{T}$ then $\{\mathrm{ht}(a): a \in A\}$ is not stationary. It is known that $\mathcal{T}$ is almost Souslin iff the tree topology of $\mathcal{T}$ is collectionwise Hausdorff. An ω_1-tree $\mathcal{T}$ is $\boldsymbol{R}$-embeddable iff there is a function $f: T \to \boldsymbol{R}$ such that whenever $x<y$ in $\mathcal{T}$, then $f(x)<f(y)$ in $\boldsymbol{R}$. We call such f an *$\boldsymbol{R}$-embedding*. A space is *countably metacompact* (*cmc*) iff every countable open cover has a point finite open refinement.

$\langle S_\alpha: \alpha<\omega_1\rangle$ is a *$\diamondsuit$-sequence* iff (i) $S_\alpha \subseteqq \alpha$, (ii) whenever X is a subset of ω_1, then the set $\{\alpha: X \cap \alpha=S_\alpha\}$ is stationary. $\langle F_\alpha: \alpha<\omega_1\rangle$ is a *$\diamondsuit^*$-sequence* iff (i) $F_\alpha \subseteqq \mathcal{P}(\alpha)$, (ii) $|F_\alpha| \leqq \omega$, (iii) whenever X is a subset of ω_1, then the set

$\{\alpha: X\cap\alpha\in F_\alpha\}$ contains a club set. $\diamondsuit$ is the assertion that a $\diamondsuit$-sequence exists. $\diamondsuit^*$ is the assertion that a $\diamondsuit^*$-sequence exists. Both are consequences of $V=L$.

LEMMA 2.1. *Let* $\{P_\alpha: \alpha<\omega_1\}$ *be a partition of* ω_1. *Let* $\langle S_\alpha: \alpha<\omega_1\rangle$ *be a* $\diamondsuit$-*sequence and* $\langle F_\alpha: \alpha<\omega_1\rangle$ *a* $\diamondsuit^*$-*sequence. Then whenever* X *is a subset of* ω_1 *such that* $X\cap P_\alpha$ *is at most countable for all* $\alpha<\omega_1$, *then the following hold:*

(i) $\{\alpha: X\cap\bigcup_{\xi<\alpha}P_\xi=S_\alpha\}$ *is stationary,*

(ii) $\{\alpha: X\cap\bigcup_{\xi<\alpha}P_\xi\in F_\alpha\}$ *contains a club set.*

The proof is left to the reader.

T^* stands for $\bigcup_{\alpha<\omega_1}{}^\alpha\omega$, the set of all functions f such that $\mathrm{dom}(f)\in\omega_1$ and $\mathrm{ran}(f)\subseteq\omega$. $\mathcal{T}^*=(T^*, <)$ is a tree by defining $t<s\leftrightarrow t\subset s$, i.e. s is a function extension of t. If $x\in T^*$, then $x^\frown\langle n\rangle$ stands for $x\cup\{\langle \mathrm{ht}(x), n\rangle\}$, a function from $\mathrm{ht}(x)+1$ to ω, an immediate successor of x in $\mathcal{T}^*$. If T is a subset of $T^*\restriction\alpha$ and $(\forall x\in T)(\forall y\in T^*\restriction\alpha)[y<x\rightarrow y\in T]$ and $(T, <)$ is an α-tree, then $\mathcal{T}=(T, <)$ is said to be an *α-subtree* of $(T^*\restriction\alpha, <)$. If $\mathcal{T}$ is an ω_1-subtree of $\mathcal{T}^*$, $T_\alpha\subseteq T^*_\alpha$ obviously.

LEMMA 2.2 ($\diamondsuit$). *There is a sequence* $\langle Z_\alpha: \alpha<\omega_1\rangle$ *such that*

(i) $Z_\alpha\subseteq T^*\restriction\alpha\times\omega$,

(ii) *whenever* $X\subseteq T^*\times\omega$ *and* $|X\cap(T^*_\alpha\times\omega)|\leqq\omega$ *for all* $\alpha<\omega_1$, *then* $\{\alpha: X\cap(T^*\restriction\alpha\times\omega)=Z_\alpha\}$ *is stationary.*

PROOF. $\diamondsuit$ implies CH. Hence $|T^*\times\omega|=|T^*|=|\bigcup_{\alpha<\omega_1}{}^\alpha\omega|=2^\omega=\omega_1$. Let f be a bijection: $T^*\times\omega\rightarrow\omega_1$. Fix a $\diamondsuit$-sequence $\langle S_\alpha: \alpha<\omega_1\rangle$ and put $Z_\alpha=f^{-1}(S_\alpha)\cap(T^*\restriction\alpha\times\omega)$ for $\alpha<\omega_1$. It is easy to check that by Lemma 2.1 $\langle Z_\alpha: \alpha<\omega_1\rangle$ satisfies the desired conditions (notice that $\bigcup_{\xi<\alpha}f(T^*_\xi\times\omega)=f(T^*\restriction\alpha\times\omega)$). Lemma 2.2 is thus proved.

LEMMA 2.3 ($\diamondsuit^*$). *There is a sequence* $\langle Y_\alpha: \alpha<\omega_1\rangle$ *such that*

(i) $(\forall X\in Y_\alpha)[X\subseteq T^*\restriction\alpha]$ & $|Y_\alpha|\leqq\omega$,

(ii) *whenever* $X\subseteq T^*$ *and* $|X\cap T^*_\alpha|\leqq\omega$ *for all* $\alpha<\omega_1$, *then* $\{\alpha: X\cap T^*\restriction\alpha\in Y_\alpha\}$ *contains a club set.*

PROOF. Similar to the above.

3. A not countably metacompact Aronszajn tree.

The following is proved in this section:

THEOREM 1 ($\diamondsuit$). *There is a not cmc Aronszajn tree.*

Let $\langle Z_\alpha: \alpha<\omega_1\rangle$ be the sequence written in Lemma 2.2.

Our goal is the same as Fleissner's: i.e. it is to define an Aronszajn tree $\mathcal{T}$ with antichain A and partition of A into $\{A_n: n\in\omega\}$ such that there are no functions f from T into ω that would satisfy

(P1) if $a\in A_n$, then $f(a)=n$,

(P2) for all $t\in T-T_0$, there is $s<t$ such that for all $u\in(s,t)$, $f(u)\geqq f(t)$.

Such a tree is not cmc (see Fleissner [2]). We construct an ω_1-subtree $\mathcal{T}$ of $\mathcal{T}^*$ as such a tree. We define $T_\alpha\subseteqq T^*_\alpha$ and $\{A_n\cap T_\alpha:\ n\in\omega\}$ by induction on $\alpha<\omega_1$. It is assumed that $T\restriction\alpha$ and $\{A_n\cap T\restriction\alpha:\ n\in\omega\}$ has been befined already. We denote $\{x\in T\restriction\alpha:\ \hat{x}\cup\{x\}\cap A=\emptyset\}$ by B. We further assume the following inductively at each stage α:

(1) $(T\restriction\alpha,<)$ is an α-subtree of $(T^*\restriction\alpha,<)$,

(2) $\{A_n\cap T\restriction\alpha:\ n\in\omega\}$ is a partition of an antichain $A\cap T\restriction\alpha$,

(3) Whenever $x\in B$ and $\mathrm{ht}(x)<\beta<\alpha$, then there is $y\in B$ such that $x<y\in T_\beta$.

I. CASE $\alpha=0$. Define $T_0=\{0\}$ and $A\cap T_0=\emptyset$.

II. CASE $\alpha=\beta+1$. Define $T_{\beta+1}=\{x^\frown\langle n\rangle:\ x\in T_\beta,\ n\in\omega\}$ and $A\cap T_{\beta+1}=\emptyset$.

III. CASE $\lim(\alpha)$. Fix an increasing sequence $\{\alpha_n:\ n\in\omega\}$ cofinal in α. We associate with each $x\in T\restriction\alpha$ a sequence $\{x_n:\ n\in\omega\}\subseteqq T\restriction\alpha$ such that (i) $x_0=x$, (ii) $x_n<x_{n+1}$ and $\alpha_n<\mathrm{ht}(x_{n+1})$, (iii) if $x_n\in B$ then $x_{n+1}\in B$. This is possible by (3). Put

$$t(x)=\bigcup\{x_n:\ n<\omega\}\in T^*_\alpha.$$

There are three cases to consider.

CASE 1. $f_\alpha=^{\mathrm{df}}Z_\alpha$ is a function from $T\restriction\alpha$ to ω and satisfies

(i) if $x\in A_n\cap T\restriction\alpha$, then $f_\alpha(x)=n$,

(ii) if $x\in T\restriction\alpha-T_0$, then $(\exists y<x)(\forall z\in(y,x))[f_\alpha(z)\geqq f_\alpha(x)]$. We divide this into two cases further.

SUBCASE 1.1. $(\forall n\in\omega)(\forall x\in B)(\exists y\in B)[y>x$ and $(\forall z\in B)[z\geqq y\rightarrow f_\alpha(z)\geqq n]]$. Take a sequence $\{u_n\in B:\ n\in\omega\}$ so that the following hold:

(i) $u_0=\emptyset\in T_0$,

(ii) $u_{2k}<u_{2k+1}\in B$ and $(\forall z\in B)[z\geqq u_{2k+1}\rightarrow f_\alpha(z)\geqq k]$,

(iii) $u_{2k+1}<u_{2k+2}\in B$ and $\mathrm{ht}(u_{2k+2})\geqq\alpha_k$.

Put $u=\bigcup_{n\in\omega}u_n\in T^*_\alpha$,

$$T_\alpha=\{u\}\cup\{t(x):\ x\in T\restriction\alpha\}\quad\text{and}\quad A\cap T_\alpha=\emptyset.$$

SUBCASE 1.2. Otherwise. Then we can take $m\in\omega$ and $w\in B$ such that $(\forall y\in B)[y>w\rightarrow(\exists z\in B)[z\geqq y\ \&\ f_\alpha(z)<m]]$. Take a sequence $\{v_n\in B:\ n\in\omega\}\subset T\restriction\alpha$ so that the following hold:

(i) $v_0=w\in B$,

(ii) $v_{2k}<v_{2k+1}\in B$ and $\mathrm{ht}(v_{2k+1})\geqq\alpha_k$,

(iii) $v_{2k+1}\leqq v_{2k+2}\in B$ and $f_\alpha(v_{2k+2})<m$.

Put $v=\bigcup\{v_n:\ n\in\omega\}\in T^*_\alpha$,

$$T_\alpha=\{v\}\cup\{t(x):\ x\in T\restriction\alpha\}\quad\text{and}\quad A_n\cap T_\alpha=\begin{cases}\{v\} & \text{if}\quad n=m\,,\\ \emptyset & \text{else}\,.\end{cases}$$

(To check that $(\forall x\in B)(\exists y\in T_\alpha)[x<y\ \&\ \hat{y}\cup\{y\}\cap A=\emptyset]$, observe that for every $x\in B$, there is $n\in\omega$ such that $x^\frown\langle n\rangle\notin\hat{v}$ and for such n, $y=t(x^\frown\langle n\rangle)$ satisfies $\hat{y}\cup\{y\}\cap A=\emptyset$.)

CASE 2. $Z=\{x\in T\restriction\alpha:\ (\exists n\in\omega)[<x,\ n>\in Z_\alpha]\}$ is a cofinal branch of $T\restriction\alpha$. Then with each $x\in T\restriction\alpha$, we associate $n_x\in\omega$ such that $x^\frown\langle n_x\rangle\notin Z$ and put

$$T_\alpha=\{t(x^\frown\langle n_x\rangle):\ x\in T\restriction\alpha\}\quad\text{and}\quad A\cap T_\alpha=\emptyset\,.$$

CASE 3. Otherwise. Put $T_\alpha=\{t(x):\ x\in T\restriction\alpha\}$ and $A\cap T_\alpha=\emptyset$.

Having defined $\mathcal{T}$, we prove that it is as required.

CLAIM 1. *$\mathcal{T}$ is an Aronszajn tree.*

This proof is left to the reader.

CLAIM 2. *$\mathcal{T}$ is not cmc.*

To prove this, suppose that there were a function $f:T\to\omega$ which satisfies (P1) and (P2). Let B' stand for $\{x\in T:\ \hat{x}\cup\{x\}\cap A=\emptyset\}$. Put

$$E=\{\alpha<\omega_1:\ \lim(\alpha)\quad\text{and}\quad f\restriction(T\restriction\alpha)=Z_\alpha\},$$

$$C=\{\alpha<\omega_1:\ (\forall x\in B'\cap T\restriction\alpha)(\forall n\in\omega)[(\exists y\in B')[x<y\ \&\ f(y)=n]\to(\exists y\in B'\cap T\restriction\alpha)[x<y\ \&\ f(y)=n]]\}.$$

E is stationary and C is club. Take $\alpha\in E\cap C$. Put $B=B'\cap T\restriction\alpha$. There are two cases to consider.

CASE 1. $(\forall n\in\omega)(\forall x\in B)(\exists y\in B)[y>x\ \&\ (\forall z\in B)[z\geqq y\to f(z)\geqq n]]$. Since $f\restriction(T\restriction\alpha)=f_\alpha$, T_α must have been defined in Subcase 1.1. Hence $u\in T_\alpha$. Put $m=f(u)$. Recall the definition of $u=\bigcup_{n\in\omega}u_n$. By the definition, $(\forall z\in B)[z\geqq u_{2m+3}\to f(z)\geqq m+1]$. Since $\alpha\in C$, $(\forall z\in B')[z\geqq u_{2m+3}\to f(z)\geqq m+1]$. $u\in B'$, since $A\cap T_\alpha=\emptyset$. Besides $u\geqq u_{2m+3}$. Hence $f(u)\geqq m+1$. This contradicts the definition of m.

CASE 2. Otherwise. T_α must have been defined in Subcase 1.2. Let m, w and v be the ones that were used in the definition of T_α. $v\in A_m$ by the definition of $A_m\cap T_\alpha$ and hence $f(v)=m$ by (P1). So, by (P2), there is $y<v$ such that $(\forall z\in(y,\ v))[f(z)\geqq m]$. But, for some k, $y<v_{2k+1}\leqq v_{2k+2}<v$ and $f(v_{2k+2})<m$. This is absurd. Claim 2 is thus proved. This completes the proof of Theorem 1.

4. Perfectness implies R-embeddability.

In this section, we prove

THEOREM 2. *If an ω_1-tree $\mathcal{T}$ is perfect, i.e. every closed set is G_δ, then $\mathcal{T}$*

*is **R**-embeddable.*

LEMMA 1. *If $T-T\restriction\mathrm{Lim}$ is the union of countably many antichains, then $\mathcal{T}$ is **R**-embeddable, where* $\mathrm{Lim}=\{\alpha<\omega_1:\ \lim(\alpha)\}$.

The proof is easy and omitted.

LEMMA 2. *If F is a closed set such that $F\subseteqq T-T\restriction C$ for some club set C, then F is the union of countably many antichains.*

To prove this, let F be a closed set and $F\subseteqq T-T\restriction C$ for a club set C. Let $\langle c(\alpha):\ \alpha<\omega_1\rangle$ be a monotone enumeration of C. Let $\langle x_n^\alpha:\ n\in\omega\rangle$ be an enumeration of $\{x\in T:\ c(\alpha)<\mathrm{ht}(x)<c(\alpha+1)\}$ for each $\alpha<\omega_1$. Put $B_n=\{x_n^\alpha:\ \alpha<\omega_1\}\cap F$ for each $n\in\omega$. Then for every $x\in B_n$, the set $\hat{x}\cap B_n$ is finite. So, if we put $B_n^m=\{x\in B_n:\ |\hat{x}\cap B_n|=m\}$ for each m and n, then every B_n^m is antichain and $F=\bigcup_{m,n\in\omega} B_n^m$. Lemma 2 is thus proved.

To prove the theorem, suppose $\mathcal{T}$ is perfect. Then, since $T-T\restriction\mathrm{Lim}$ is open, there is $\{F_n:\ n\in\omega\}$, a family of closed sets such that $T-T\restriction\mathrm{Lim}=\bigcup_{n\in\omega} F_n$. By Lemma 2, F_n is the union of countably many antichains and hence so is $T-T\restriction\mathrm{Lim}$. By Lemma 1, $\mathcal{T}$ is **R**-embeddable. Theorem 2 is thus proved.

5. A characterization of perfectness for an almost Souslin tree.

In this section, we prove

THEOREM 3. *If an almost Souslin tree $\mathcal{T}$ is **R**-embeddable, then $\mathcal{T}$ is perfect.*

LEMMA 1. *If $\mathcal{T}$ is **R**-embeddable and C is a club set, then there is an **R**-embedding $e:\ T\to\boldsymbol{R}$ such that $e(T-T\restriction C)\subseteqq\boldsymbol{Q}$.*

To prove this, take an **R**-embedding $f:\ T\to\boldsymbol{R}$. Let $\langle c_\alpha:\ \alpha<\omega_1\rangle$ be a monotone enumeration of C and let $\langle x_n^\alpha:\ n<\omega_1\rangle$ enumerate the elements of $\{x\in T:\ c_\alpha<\mathrm{ht}(x)<c_{\alpha+1}\}$. Fix α arbitrarily. For each x_n^α, define $r(x_n^\alpha)\in\boldsymbol{Q}$ by induction on $n\in\omega$ so that the following hold:

(i) $f(y)<r(x_n^\alpha)\leqq f(x_n^\alpha)$, where y is the element of $\hat{x}_n^\alpha$ such that $\mathrm{ht}(y)=c_\alpha$,

(ii) $x_m^\alpha<x_n^\alpha\to r(x_m^\alpha)<r(x_n^\alpha)$, for all $m,n\in\omega$. Then this embedding $r:\ (T-T\restriction C)\to\boldsymbol{Q}$ can be extended to an embedding $r:\ T\to\boldsymbol{R}$ naturally. Lemma 1 is thus proved.

As a corollary of this lemma, we obtain

LEMMA 2. *If $\mathcal{T}$ is **R**-embeddable, then for any club set C, $T-T\restriction C$ is the union of countably many antichains.*

Now, to prove Theorem 3, suppose that $\mathcal{T}$ is almost Souslin and a function f embeds T into $\boldsymbol{R}$. To show that $\mathcal{T}$ is perfect, take an open set $U\subset T$ arbitrarily. Put

$$I=\{u\in U:\ (\forall x<u)[[x,u)\nsubseteqq U]\}.$$

For each $u\in I$, put

$$V(u)=\{x\in U:\ [u,x]\subseteqq U\}.$$

Clearly $U=\bigcup_{u\in I} V(u)$ and $V(u)\cap V(v)=\emptyset$ for all $u, v\in I$ with $u\neq v$. Since U is open, $I\subseteq T-T\restriction\mathrm{Lim}$ and so $V(u)$ is open.

LEMMA 3. *$V(u)$ is F_σ for all $u\in I$.*

To prove this, put $A=\overline{V(u)}-V(u)$. Then A is clearly an antichain. Since $\mathcal{T}$ is almost Souslin, there is a club set $C\subset\omega_1$ such that $T\restriction C\cap A=\emptyset$. By Lemma 2, $V(u)\cap(T-T\restriction C)$ is the union of countably many antichains and hence F_σ. On the other hand, $V(u)\cap T\restriction C$ is a closed set. For, if $x\in\overline{V(u)\cap T\restriction C}$, then clearly $x\in V(u)\cap T\restriction C$ since $T\restriction C\cap A=\emptyset$. Thus $V(u)$ is F_σ. Lemma 3 is thus proved.

By Lemma 3, we obtain $\{F_n(u)\colon n\in\omega\}$ a family of closed sets such that $V(u)=\bigcup_{n\in\omega} F_n(u)$ for all $u\in I$. Since $I\subseteq T-T\restriction\mathrm{Lim}$, by Lemma 2, we can take $\{A_k\colon k\in\omega\}$ a disjoint family of antichains such that $I=\bigcup_{k\in\omega} A_k$. With each $u\in I$, we associate $k(u)\in\omega$ such that $u\in A_{k(u)}$. Put

$$B_n=\bigcup\{F_{n-k(u)}(u)\colon u\in I,\ k(u)\leqq n\}.$$

Clearly $U=\bigcup_{n\in\omega} B_n$. It remains to show that each B_n is closed. Suppose $x\notin B_n$. To show $(y, x]\cap B_n=\emptyset$ for some $y<x$, take $y'<x$ so that $(y', x)\cap\bigcup_{k\leqq n} A_k=\emptyset$. Suppose $(y', x]\cap B_n\neq\emptyset$. Take $z\in(y', x]\cap B_n$. Then $z\in F_{n-k(u)}(u)$ for some $u\in I$ with $k(u)\leqq n$. Take $y<x$ so that $(y, x]\subseteq(y', x]-F_{n-k(u)}(u)$. We show $(y, x]\cap B_n=\emptyset$. If there were $u'\in I$ with $k(u')\leqq n$ such that $(y, x]\cap F_{n-k(u')}(u')\neq\emptyset$, then $u'>u$ and so $u'>z$ $(>y')$ since $z\in V(u)$. This is absurd since $u'\in(y', x)\cap I$ implies $k(u')>n$ by the choice of y'. Theorem 3 is thus proved.

6. An $\boldsymbol{R}$-embeddable, not perfect, cmc tree.

In this section, we prove

THEOREM 4 ($\diamondsuit^*$). *There is a cmc tree which is $\boldsymbol{R}$-embeddable but not perfect.*

We construct an ω_1-subtree $\mathcal{T}$ of $\mathcal{T}^*$ with an initial segment U and an $\boldsymbol{R}$-embedding $r\colon T\to\boldsymbol{R}$. Let $\langle Y_\alpha\colon \alpha<\omega_1\rangle$ be a $\diamondsuit^*$-sequence in Lemma 2.3 and $\langle Z_\alpha\colon\alpha<\omega_1\rangle$ a $\diamondsuit$-sequence in Lemma 2.2. We define $T_\alpha\subseteq T^*_\alpha$, $U\cap T_\alpha$ and $r\restriction T_\alpha$ by induction on $\alpha<\omega_1$. We assume that $T\restriction\alpha$, $U\cap T\restriction\alpha$ and $r\restriction(T\restriction\alpha)$ have been defined. The letter q is used to denote an element of $\boldsymbol{Q}$. We write $x>^q_U y$ for $[x>y\ \&\ r(x)<q\ \&\ [y\in U\to x\in U]]$. We ensure the following at each stage α, where α' denotes $\alpha+1$:

(1) $(T\restriction\alpha', <)$ is an α'-subtree of $\mathcal{T}^*\restriction\alpha'$,

(2) $U\cap T\restriction\alpha'$ is an initial segment of $T\restriction\alpha'$,

(3) $r\colon T\restriction\alpha'\to\boldsymbol{R}$ is an $\boldsymbol{R}$-embedding,

(4) $x\in T\restriction\alpha\ \&\ \mathrm{ht}(x)<\beta\leqq\alpha\ \&\ r(x)<q\in\boldsymbol{Q}\to(\exists y\in T_\beta)[y>^q_U x]$.

I. CASE $\alpha=0$. $T_0=\{\emptyset\}$, $U\cap T_0=\{\emptyset\}$, $r(\emptyset)=0$.

II. CASE $\alpha=\beta+1$. $T_{\beta+1}=\{x^\frown\langle n\rangle: x\in T_\beta, n\in\omega\}$, $U\cap T_{\beta+1}=\{x^\frown\langle n\rangle: x\in U\cap T_\beta, n\in\omega\}$, $r(x^\frown\langle n\rangle)=q_n$, where $\langle q_n: n\in\omega\rangle$ is an enumeration of $\{q\in\boldsymbol{Q}: r(x)<q\}$.

III. CASE $\lim(\alpha)$. Fix a sequence $\{\alpha_n: n\in\omega\}$ such that $\sup_{n\in\omega}\alpha_n=\alpha$. Let $\langle V_n: n\in\omega\rangle$ be an enumeration with infinitely many iterations of $\{X\in\{\emptyset\}\cup Y_\alpha: X$ is an open set$\}$. First we associate $t(x, q)$ an element of T^*_α with each pair of $x\in T\restriction\alpha$ and $q>r(x)$ as follows: (i) Put $x_0=x$; (ii) Take $x_{2k+1}>^q_U x_{2k}$ so that $\mathrm{ht}(x_{2k+1})>\alpha_k$; (iii) Take $x_{2k+2}>^q_U x_{2k+1}$ so that if possible, $x_{2k+2}\notin V_k$; And put $t(x, q)=\bigcup_{k\in\omega} x_k$. Now we divide the case into two cases.

CASE 1. $F^\alpha_n=^{\mathrm{df}}\{x\in T\restriction\alpha: \langle x, n\rangle\in Z_\alpha\}$ is closed in $T\restriction\alpha$ for all $n\in\omega$ and $\bigcup_{n\in\omega} F^\alpha_n=U\cap T\restriction\alpha$ and $F^\alpha_m\subseteqq F^\alpha_n$ if $m\leqq n$.

SUBCASE 1.1. There are $x\in U\cap T\restriction\alpha$, $p\in(r(x), \infty)\cap\boldsymbol{Q}$ and $m\in\omega$ such that $(\forall y>^p_U x)(\exists z>^p_U y)[z\in F^\alpha_m]$. Let x, p and m be such ones. (i) Put $u_0=x$; (ii) Take $u_{3k+1}>^p_U u_{3k}$ so that $\mathrm{ht}(u_{3k+1})>\alpha_k$; (iii) Take $u_{3k+2}>^p_U u_{3k+1}$ so that if possible, $u_{3k+2}\notin V_k$; (iv) Take $u_{3k+3}>^p_U u_{3k+2}$ so that $u_{3k+3}\in F^\alpha_m$. Put $u_\alpha=\bigcup_{k\in\omega} u_k\in T^*_\alpha$. Define $T_\alpha=\{u_\alpha\}\cup\{t(x, q): x\in T\restriction\alpha, q>r(x)\}$, $r(u_\alpha)=\sup_{k\in\omega} r(u_k)\leqq p$, $r(t(x, q))=\sup_{k\in\omega} r(x_k)\leqq q$, $U\cap T_\alpha=\{t(x, q): x\in U\cap T\restriction\alpha, q>r(x)\}-\{u_\alpha\}$.

SUBCASE 1.2. Otherwise. Then for each $x\in U\cap T\restriction\alpha$, $q>r(x)$, $k\in\omega$, there is $y>^q_U x$ such that $(\forall z>^q_U y)[z\notin F^\alpha_k]$. (i) Put $v_0=\emptyset$ and $p(0)=1\in\boldsymbol{Q}$; (ii) Take $v_{3k+1}>^{p(k)}_U v_{3k}$ so that $\mathrm{ht}(v_{3k+1})>\alpha_k$; (iii) Take $v_{3k+2}>^{p(k)}_U v_{3k+1}$ so that if possible, $v_{3k+2}\notin V_k$; (iv) Take $p(k+1)\in\boldsymbol{Q}$ so that $r(v_{3k+2})<p(k+1)<p(k)$; (v) Take $v_{3k+3}>^{p(k+1)}_U v_{3k+2}$ so that $(\forall z>^{p(k+1)}_U v_{3k+3})[z\notin F^\alpha_k]$. Put $v_\alpha=\bigcup_{k\in\omega} v_k\in T^*_\alpha$ and define: $T_\alpha=\{v_\alpha\}\cup\{t(x, q): x\in T\restriction\alpha, q>r(x)\}$; $r(v_\alpha)=\sup_{k\in\omega} v_k$ $(<p(k)\leqq 1$ for all $k\in\omega)$; $r(t(x, q))=\sup_{k\in\omega} r(x_k)\leqq q$; $U\cap T_\alpha=\{v_\alpha\}\cup\{t(x, q): x\in U\cap T\restriction\alpha, q>r(x)\}$.

CASE 2. Otherwise. Define: $T_\alpha=\{t(x, q): x\in T\restriction\alpha, q>r(x)\}$; $r(t(x, q))=\sup_{k<\omega} r(x_k)\leqq q$; $U\cap T_\alpha=\{t(x, q): x\in U\cap T\restriction\alpha, q>r(x)\}$.

$\mathcal{T}$, $U\subset T$ and $r: T\to\boldsymbol{R}$ is thus defined. We prove that $\mathcal{T}$ is as required. $\mathcal{T}$ is $\boldsymbol{R}$-embeddable obviously.

LEMMA 1. *$\mathcal{T}$ is not perfect.*

To prove this, it suffices to show that U is not F_σ, since U is an initial segment of T and hence an open set. Suppose to the contrary that there is $\{F_n: n\in\omega\}$ a family of closed sets such that $U=\bigcup\{F_n: n\in\omega\}$ and $F_m\subseteqq F_n$ for $m\leqq n\in\omega$. Put $F=\bigcup_{n\in\omega} F_n\times\{n\}$ and $C=\{\alpha<\omega_1: (\forall x\in U\cap T\restriction\alpha)(\forall q>r(x))(\forall n\in\omega)[(\exists y)[x<^q_U y\in F_n]\to(\exists y\in T\restriction\alpha)[x<^q_U y\in F_n]]\}$. Since C is club and $F\subset T\times\omega$, there is a limit ordinal $\alpha\in C$ such that $F\cap(T\restriction\alpha\times\omega)=Z_\alpha$. Then $F^\alpha_n=\{x\in T\restriction\alpha: \langle x, n\rangle\in Z_\alpha\}=F_n\cap T\restriction\alpha$ is a closed set in $T\restriction\alpha$ for all $n\in\omega$. So, T_α must have been defined in Case 1 and hence contains u_α or v_α.

CASE 1. $u_\alpha\in T_\alpha$. Recall the definition of u_α and let x, p and m be the ones in the definition. Then $(x, u_\alpha)\cap F_m^\alpha$ is cofinal in $\hat{u}_\alpha$ and hence $u_\alpha \in F_m (\subset U)$ since $(x, u_\alpha)\cap F_m^\alpha \subset F_m$ and F_m is closed. But this is absurd since $u_\alpha \notin U$ by the definition of $U\cap T_\alpha$.

CASE 2. $v_\alpha \in T_\alpha$. By the definition of $U\cap T_\alpha$, $v_\alpha \in U=\bigcup_{n\in\omega} F_n$. Take n so that $v_\alpha \in F_n$. Recall the definition of v_α:

$$(\forall z\in T\restriction\alpha)[z>_U^{p(n+1)} v_{3n+3} \to z\notin F_n^\alpha = F_n\cap T\restriction\alpha].$$

Hence by $\alpha\in C$, $(\forall z)[z>_U^{p(n+1)} v_{3n+3}\to z\notin F_n]$. But $v_\alpha\in F_n\subset U$ and $r(v_\alpha)<p(n+1)$ and $v_\alpha > v_{3n+3}$. This is absurd. Lemma 1 is thus proved.

LEMMA 2. *$\mathcal{T}$ is countably metacompact.*

To prove this, suppose $\mathcal{U}=\{U_n : n\in\omega-\{0\}\}$ be a countable open cover of T. Take a club set C_n so that $C_n \subseteq \{\alpha<\omega_1 : U_n\cap T\restriction\alpha\in Y_\alpha\}$ for $n\in\omega-\{0\}$. Put:

$$C_0=\{\alpha<\omega_1 : (\forall x\in T\restriction\alpha)(\forall q>r(x)) [(\exists y)[y>_U^q x]\to(\exists y\in T\restriction\alpha)[y>_U^q x]]\};$$

$$C'=\{\alpha<\omega_1 : (\forall x\in T\restriction\alpha)(\forall q>r(x))(\forall n\in\omega) [(\exists y)[y>_U^q x \ \&\ y\notin U_n]\to(\exists y\in T\restriction\alpha)[y>_U^q x \ \&\ y\notin U_n]]\}.$$

Put $C=C'\cap\bigcap_{n\in\omega} C_n$, a club set. We define two point finite refinements $\mathcal{W}$ and $\mathcal{W}'$ of $\mathcal{U}$ satisfying $\bigcup\mathcal{W}\supseteq T\restriction C$ and $\bigcup\mathcal{W}'\supseteq T-T\restriction C$.

SUBLEMMA 2.1. *Let $\alpha\in C$. Then:*

(i) *If $u_\alpha\in T_\alpha$ and $u_\alpha\in U_k$, then $(\exists q\geqq r(u_\alpha))(\exists y<u_\alpha)(\forall z\in T\restriction\alpha)[z>_U^q y\to z\in U_k]$;*

(ii) *If $v_\alpha\in T_\alpha$ and $v_\alpha\in U_k$, then $(\exists q\geqq r(v_\alpha))(\exists y<v_\alpha)(\forall z\in T\restriction\alpha)[z>_U^q y\to z\in U_k]$;*

(iii) *If $t(x, q)\in U_k\cap T_\alpha$, then $(\exists q\geqq r(t(x, q)))(\exists y<t(x, q))(\forall z\in T\restriction\alpha)[z>_U^q y\to z\in U_k]$.*

We prove only (i) because (ii) and (iii) can be proved similarly. Suppose $u_\alpha\in T_\alpha$ and $u_\alpha\in U_k$. Then T_α has been defined in Subcase 1.1. Let x, p and m be the ones described there. Since U_k is open, we can take $y<u_\alpha$ satisfying $(y, u_\alpha]\subseteq U_k$. Since $\alpha\in C_k$, $(y, u_\alpha)\subseteq U_k\cap T\restriction\alpha\in Y_\alpha$. Since every open set belonging to Y_α appears in $\langle V_n : n\in\omega\rangle$ infinitely many times and $\langle u_k : k\in\omega\rangle$ is cofinal in $\hat{u}_\alpha$, there is i such that $V_i=U_k\cap T\restriction\alpha$ and $y<u_{3i+1}$. Since $u_{3i+2}\in U_k\cap T\restriction\alpha=V_i$, by choice of u_{3i+2}, $(\forall z\in T\restriction\alpha)[z>_U^p u_{3i+1}\to z\in V_i\subseteq U_k]$. This asserts (i), since $r(u_\alpha)\leqq p$ and $u_{3i+1}<u_\alpha$. Sublemma 2.1 is thus proved.

SUBLEMMA 2.2. *If $\alpha\in C$ and $t\in T_\alpha\cap U_k$, then*

$$(\exists q\geqq r(t))(\exists y<t)(\forall z\in T)[z>_U^q y\to z\in U_k].$$

This follows immediately from the previous lemma, since $\alpha\in C'$.

Now let $\alpha \in C$. For each $t \in T_\alpha$, putting $k(t)$=the least k such that $t \in U_k$, we take $q(t) \in \boldsymbol{Q}$ and $t^* \in T \upharpoonright \alpha$ so that:

$$r(t) \leqq q(t) \ \& \ (\exists y < t)(\forall z)[z >_U^{q(t)} y \to z \in U_{k(t)}];$$

$$t^* < t \ \& \ r(t) - r(t^*) < 1/k(t) \ \& \ (\forall z)[z >_U^{q(t)} t^* \to z \in U_{k(t)}].$$

For each $i \in \omega - \{0\}$, put $W_i = \bigcup \{(t^*, t] : t \in T \upharpoonright C \ \& \ k(t) = i\}$. Clearly $W_i \subseteqq U_i$ for each $i \in \omega - \{0\}$.

CLAIM. $\mathcal{W} = \{W_i : i \in \omega - \{0\}\}$ *is point finite.*

To the contrary, assume $\bigcap_{i \in I} W_i \neq \emptyset$ for some infinite subset $I \subset \omega - \{0\}$. Then we can take $z \in T$ and $\{t_i : i \in I\}$ such that $z \in (t_i^*, t_i]$ & $k(t_i) = i$ & $t_i \in T \upharpoonright C$ for all $i \in I$. We may assume $z \neq t_i$ for all $i \in I$ by taking I appropriately. Let i be the least element of I. Take $j \in I$ so that $r(t_i) - r(z) > 1/j$. Then $r(t_j) - r(z) < r(t_j) - r(t_j^*) < 1/k(t_j) = 1/j < r(t_i) - r(z)$. Hence $r(t_i^*) < r(z) < r(t_j) < r(t_i) \leqq q(t_i)$. Put $p = q(t_i)$ for simplicity. Since $t_i^* <_U^p z <_U^p t_j$, we have $t_i^* <_U^p t_j$. But this means $t_j \in U_i$, because by the definition of t_i^* $(\forall z)[z >_U^p t_i^* \to z \in U_i]$ holds. This contradicts that j is the least k satisfying $t_j \in U_k$. Claim is thus proved.

Clearly $\mathcal{W}$ covers $T \upharpoonright C$. Thus $\mathcal{W}$ is a point finite refinement of $\mathcal{U}$ which covers $T \upharpoonright C$.

Now we define $\mathcal{W}'$ a point finite refinement of $\mathcal{U}$ which covers $T - T \upharpoonright C$. Let $\langle c_\alpha : \alpha < \omega_1 \rangle$ be the monotone enumeration of $\{0\} \cup C$. Let $\langle t_n^\alpha : n \in \omega \rangle$ be an enumeration of $\{t \in T : c_\alpha < \mathrm{ht}(t) < c_{\alpha+1}\}$ for $\alpha < \omega_1$. Fix $\alpha < \omega_1$ arbitrarily. By induction on n, take $u_n^\alpha \in T \upharpoonright c_{\alpha+1} - T \upharpoonright c_\alpha$ so that: (i) if $t_n^\alpha \notin \bigcup_{i<n} (u_i^\alpha, t_i^\alpha]$, then $u_n^\alpha < t_n^\alpha$ & $(u_n^\alpha, t_n^\alpha] \cap \bigcup_{i<n} (u_i^\alpha, t_i^\alpha] = \emptyset$ & $(u_n^\alpha, t_n^\alpha] \subseteqq U_k$, where k is the least k such that $t_n^\alpha \in U_k$; (ii) if $t_n^\alpha \in \bigcup_{i<n} (u_i^\alpha, t_i^\alpha]$, then $u_n^\alpha = t_n^\alpha$, i.e. $(u_n^\alpha, t_n^\alpha] = \emptyset$. Then $\mathcal{W}' = \{(u_n^\alpha, t_n^\alpha] : n \in \omega, \alpha < \omega_1\}$ is clearly a point finite open refinement of $\mathcal{U}$ and covers $T - T \upharpoonright C$.

Now $\mathcal{W} \cup \mathcal{W}' \cup \{T_0\}$ is a point finite open refinement of $\mathcal{U}$ which covers whole T. Lemma 2 is thus proved. The proof of Theorem 4 is complete.

REMARK. An $\boldsymbol{R}$-embeddable, not perfect, cmc tree can not be almost Souslin by Theorem 3. We can however obtain an $\boldsymbol{R}$-embeddable, not perfect, cmc tree which is almost an almost Souslin tree in the sense that there is an antichain A such that whenever X is an antichain, $\{\mathrm{ht}(x) : x \in X - A\}$ is not stationary. Such a tree can be obtained by modifying slightly the definition of $t(x, q)$ in the proof of Theorem 4 and putting $A = \bar{U} - U$.

References

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