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Invariant subspaces of shift operators of arbitrary multiplicity

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Introduction.

Let $\mathfrak{K}$ be a separable complex Hilbert space and $\{\mathfrak{K}_n\}_{n\in Z}$ be the countable family of copies of $\mathfrak{K}$, where Z is the set of all integers. Let $\mathfrak{H}$ be the direct sum $\sum_{n\in Z}\oplus\mathfrak{K}_n$ of $\{\mathfrak{K}_n\}_{n\in Z}$. A unitary operator U on $\mathfrak{H}$ is called a *shift* operator if U maps $\mathfrak{K}_n$ onto $\mathfrak{K}_{n+1}$ for all n in Z. We denote by S the shift operator on $\mathfrak{H}$:

$$x=\sum_{n\in Z}\oplus\xi_n \longrightarrow y=\sum_{n\in Z}\oplus\eta_n ,$$

where $\eta_n=\xi_{n-1}$.

A study of invariant subspaces of the shift operator S was originated by the Beurling's paper [2], in which he completely described the structure of invariant subspaces of the unilateral shift operator of multiplicity one. The so-called Beurling's theorem for invariant subspaces was stated in Helson [4] and Halmos [3]. They demonstrated the theorem from geometric consideration in contrast to a function theoretic proof by Beurling. Halmos, in his paper [3], considered the case of countable (and finite) multiplicity and noticed that the study of shift-invariant subspaces might be useful for the case of general bounded linear operators on a Hilbert space. The works [6], [7] of McAsey, Muhly and Saitô were the first attempt to characterize invariant subspaces of a family of shift operators. But they seem to study non-self-adjoint algebras rather than invariant subspaces. Thus the underlying Hilbert space in their paper heavily depends on the structure of algebras.

Our purpose is to study the structure of invariant subspaces of a family $\mathcal{S}$ of shift operators on a given Hilbert space. In our discussion, we use geometric methods in the theory of operator algebras as in [6] and [7]. In the present paper, we give a necessary and sufficient condition for a family $\mathcal{S}$ under which every invariant subspaces is of Beurling type. The condition, of course, is

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deeply related to the underlying Hilbert space. In the case where $\mathcal{S}$ does not satisfy the condition, the structure of invariant subspaces of $\mathcal{S}$ seems to be complicated. However, for the case of multiplicity one, the author has succeeded in showing the structure of those spaces [**11**].

For a family $\mathcal{S}$ of shift operators, we denote by $W(\mathcal{S})$ the set of diagonal operators corresponding to the operators in $\mathcal{S}$. In this paper, the diagonal part of a shift operator U means the unitary operator W on $\mathfrak{H}$ such that $U=WS$. Since every shift operator is unitarily equivalent to S, we assume that $\mathcal{S}$ contains S, that is, $W(\mathcal{S})$ contains the identity operator I on $\mathfrak{H}$. Indeed, for an operator U in $\mathcal{S}$, $W=US^*$ is of the form $W=\sum_{n\in\boldsymbol{Z}}\oplus u_n$, where each u_n is a unitary operator on $\mathfrak{K}$. For this unitary operator W, we define a unitary operator $V=\sum_{n\in\boldsymbol{Z}}\oplus v_n$ as follows; $v_n=u_nu_{n-1}\cdots u_1$ $(n\geqq 1)$, $v_0=1$ and $v_n=u_{n+1}^*u_{n+2}^*\cdots u_0^*$ $(n\leqq -1)$. We put $\mathcal{S}'=V^*\mathcal{S}V$. Then $\mathcal{S}'$ contains S and it follows that the subspace $\mathfrak{M}$ is invariant under $\mathcal{S}$ if and only if $V^*\mathfrak{M}$ is invariant under $\mathcal{S}'$.

An invariant subspace $\mathfrak{M}$ is said to be simply invariant if the closed linear span $[\mathcal{S}\mathfrak{M}]$ of $\mathcal{S}\mathfrak{M}$ is a proper subspace of $\mathfrak{M}$. In § 1, we study the relation between the structure of simply invariant subspaces of $\mathcal{S}$ and the properties of $\mathcal{S}$. We prove that each pure simply invariant subspace $\mathfrak{M}$, i.e. $\bigcap_{n=0}^{\infty}[\mathcal{S}^n\mathfrak{M}]=\{0\}$, is of the form

$$\mathfrak{M}=\mathfrak{M}_0\oplus[\mathcal{S}\mathfrak{M}_0]\oplus[\mathcal{S}^2\mathfrak{M}_0]\oplus\cdots$$

under the condition that $W(\mathcal{S})$ is a group and $S^*W(\mathcal{S})S\subset W(\mathcal{S})$ (Proposition 1.7). In particular, in the case where $S^*W(\mathcal{S})S=W(\mathcal{S})$, we have $\mathfrak{M}_n=[\mathcal{S}^n\mathfrak{M}_0]=S^n[W(\mathcal{S})\mathfrak{M}_0]$. Obviously, each simply invariant subspace does not reduce $\mathcal{S}$. However, in general, there are many examples of non-reducing invariant subspaces $\mathfrak{M}$ such that $[\mathcal{S}\mathfrak{M}]=\mathfrak{M}$. At the end of this section, we prove that only reducing subspaces $\mathfrak{M}$ have the property $[\mathcal{S}\mathfrak{M}]=\mathfrak{M}$ under the condition that $W(\mathcal{S})^k$ is a group for some integer $k>0$ and $S^*W(\mathcal{S})S=W(\mathcal{S})$.

We now recall the Beurling's theorem [**4**, Lecture II, Theorem 3]. Namely, every simply invariant subspace $\mathfrak{M}$ of the shift S on $L^2(T)$ is of the form $\mathfrak{M}=M_uH^2(T)$, where M_u is the multiplication operator by a unitary function u in $L^\infty(T)$. We easily find that M_u commutes with S. In this context, we say that $\mathcal{S}$ has Property (B) if $\mathcal{S}$ satisfies the following condition: every pure simply invariant subspace $\mathfrak{M}$ of $\mathcal{S}$ is of the form $\mathfrak{M}=U\mathfrak{N}$, where $\mathfrak{N}$ is an invariant subspace of $\mathcal{S}$ such that $\mathfrak{N}_n$ is contained in $\mathfrak{K}_n$ for all $n\geqq 0$ and U is a partial isometry on $\mathfrak{H}$ which satisfies a suitable condition (Definition 2.3). In § 2, we seek a necessary and sufficient condition for $\mathcal{S}$ to have Property (B). We denote by $M(\mathcal{S})$ (resp. $D(\mathcal{S})$) the von Neumann algebra generated by $\mathcal{S}$ (resp. $W(\mathcal{S})$). We prove that if $\mathcal{S}$ has Property (B) then $M(\mathcal{S})$ must be the crossed product of a von Neumann algebra D_0 on $\mathfrak{K}$ by $\boldsymbol{Z}$ with respect to a spatial *-automorphism

α of D_0 (Proposition 2.7). Ultimately we prove that $\mathcal{S}$ has Property (B) if and only if α leaves the central finite projections in the commutant D_0' element-wise fixed (Theorem 2.12). Hence we find, in the case of tensor product, that for each group A of unitary operators on $\mathfrak{K}$, the family of shift operators $\mathcal{S}=s\otimes A$ on $L^2(T)\otimes\mathfrak{K}$ always has Property (B). At the end of this section, we show some results concerning the reducing subspaces generated by simply invariant subspaces.

§ 1. Decompositions of invariant subspaces.

Let $\mathfrak{M}$ be a subspace of the Hilbert space $\mathfrak{H}=\sum_{n\in Z}\oplus\mathfrak{K}_n$, where each $\mathfrak{K}_n$ is a copy of a Hilbert space $\mathfrak{K}$. Throughout this paper, we mean by a subspace a closed subspace. Let $\mathcal{T}$ be a set of bounded linear operators on $\mathfrak{H}$. We say that $\mathfrak{M}$ is invariant under $\mathcal{T}$ if $\mathcal{T}\mathfrak{M}\subset\mathfrak{M}$ and $\mathfrak{M}$ reduces $\mathcal{T}$ if, in addition, $\mathcal{T}^*\mathfrak{M}\subset\mathfrak{M}$. Let $\mathcal{S}$ be a family of shift operators on $\mathfrak{H}$. We denote by $M(\mathcal{S})$ the von Neumann algebra generated by the set $\mathcal{S}\cup\mathcal{S}^*$. If $\mathfrak{M}$ is invariant under $\mathcal{T}$, then $\mathfrak{M}$ is also invariant under the strongly closed linear span of $\mathcal{T}$. Hence we immediately have the following proposition.

PROPOSITION 1.1. *A subspace $\mathfrak{M}$ reduces $\mathcal{S}$ if and only if $\mathfrak{M}=P\mathfrak{H}$ for some projection P in the commutant $M(\mathcal{S})'$.*

Let us consider the von Neumann algebra $M(\mathcal{S})$. We denote by $L^2(T)$ the Hilbert space of square integrable scalar valued functions on the unit circle T in the complex plane with respect to the normalized Lebesgue measure and $L^\infty(T)$ the set of all essentially bounded functions on T. When $\mathcal{S}=\{S\}$, $M(\mathcal{S})$ is spatially isomorphic to the von Neumann algebra $M_{L^\infty(T)}\otimes\boldsymbol{C}(\mathfrak{K})$ on $L^2(T)\otimes\mathfrak{K}$, where $M_{L^\infty(T)}$ is the von Neumann algebra of all the multiplication operators M_f on $L^2(T)$ by a function f in $L^\infty(T)$ and $\boldsymbol{C}(\mathfrak{K})$ is the scalar multiples of the identity on $\mathfrak{K}$. We denote by $\mathfrak{S}$ the set of all shift operators on $\mathfrak{H}$, then $M(\mathfrak{S})$ becomes the full operator algebra $\boldsymbol{B}(\mathfrak{H})$. Hence, for an arbitrary family $\mathcal{S}$, $M(\mathcal{S})$ contains $M_{L^\infty(T)}\otimes\boldsymbol{C}(\mathfrak{K})$ and is contained in $B(\mathfrak{H})$. For a family $\mathcal{S}$ of shift operators on $\mathfrak{H}$, we put $W(\mathcal{S})=\{W: W=US^*,\ U\in\mathcal{S}\}$. Every operator W in $W(\mathcal{S})$ is a unitary operator on $\mathfrak{H}$ such that $W\mathfrak{K}_n=\mathfrak{K}_n$ for all n in $\boldsymbol{Z}$, hence W is of the form $\sum_{n\in Z}\oplus u_n$, where each u_n is a unitary operator on $\mathfrak{K}$. We denote by $U(\mathfrak{K})$ the group of all unitary operators on $\mathfrak{K}$. In the following, we shall show some examples of $W(\mathcal{S})$ and von Neumann algebras $M(\mathcal{S})$.

EXAMPLE 1.2. Let $W(\mathcal{S})=\{W=\sum_{n\in Z}\oplus u_n: u_n=u_0$ for all n in $\boldsymbol{Z}$, $u_0\in A\}$, where A is a subset of $U(\mathfrak{K})$. Then we can consider $\mathcal{S}$ as the tensor product $s\otimes A$ on $L^2(T)\otimes\mathfrak{K}$, where s means the usual shift operator on $L^2(T)$. Hence we have $M(\mathcal{S})=M_{L^\infty(T)}\otimes M$, where M is the von Neumann algebra generated by A.

EXAMPLE 1.3. Let α be a *-automorphism of $B(\mathfrak{K})$ and A a subset of $U(\mathfrak{K})$.

Let $W(\mathcal{S})=\{W=\sum_{n\in\boldsymbol{Z}}\oplus u_n : u_n=\alpha^{-n}(u_0)$ for all n in $\boldsymbol{Z}$, $u_0\in A\}$. Then $M(\mathcal{S})$ is the crossed product $\mathcal{R}(M, \alpha)$ of M by $\boldsymbol{Z}$ with respect to α, where M is the von Neumann algebra on $\mathfrak{K}$ generated by the set $\{\alpha^n(u_0): u_0\in A, n\in\boldsymbol{Z}\}$ (cf. [**10**; p. 364]).

EXAMPLE 1.4. Let $W(\mathcal{S})=\{W=\sum_{n\in\boldsymbol{Z}}\oplus u_n : u_n=1$ or -1 for each n in $\boldsymbol{Z}\}$. Then we have $M(\mathcal{S})=\boldsymbol{B}(L^2(T))\otimes\boldsymbol{C}(\mathfrak{K})$.

Let $\mathfrak{M}$ be an invariant subspace of $\mathcal{S}$ which does not reduce $\mathcal{S}$. Then we have two possibilities, that is, either $[\mathcal{S}\mathfrak{M}]\subsetneqq\mathfrak{M}$ or $[\mathcal{S}\mathfrak{M}]=\mathfrak{M}$, where $[\mathcal{S}\mathfrak{M}]$ is the closure of linear span of $\mathcal{S}\mathfrak{M}$. If $\mathfrak{M}$ reduces $\mathcal{S}$, we have $[\mathcal{S}\mathfrak{M}]=\mathfrak{M}$ because $\mathfrak{M}=SS^*\mathfrak{M}\subset S\mathfrak{M}\subset\mathcal{S}\mathfrak{M}\subset\mathfrak{M}$. Moreover, in the case where $\mathcal{S}=\{S\}$, only reducing subspaces have the property $[\mathcal{S}\mathfrak{M}]=\mathfrak{M}$. But, in general, there are non-reducing subspaces $\mathfrak{M}$ such that $[\mathcal{S}\mathfrak{M}]=\mathfrak{M}$ even if $\dim\mathfrak{K}=1$ (cf. Example 1.5). When $\mathfrak{M}$ is a non-reducing invariant subspace such that $[\mathcal{S}\mathfrak{M}]=\mathfrak{M}$, the structure of $\mathfrak{M}$ seems to be very complicated. It is our purpose to analyze the structure of invariant subspaces $\mathfrak{M}$ of $\mathcal{S}$ such that $[\mathcal{S}\mathfrak{M}]$ is a proper subspace of $\mathfrak{M}$. Such a subspace $\mathfrak{M}$ is said to be *simply* invariant. It is then asked whether $[\mathcal{S}^2\mathfrak{M}]\subsetneqq[\mathcal{S}\mathfrak{M}]$ or not if $\mathfrak{M}$ is a simply invariant subspace. Unfortunately, we have examples of simply invariant subspaces $\mathfrak{M}$ such that $[\mathcal{S}\mathfrak{M}]\subsetneqq\mathfrak{M}$ but $[\mathcal{S}^2\mathfrak{M}]=[\mathcal{S}\mathfrak{M}]$.

EXAMPLE 1.5. Let $\mathfrak{H}=L^2(T)=\sum_{n\in\boldsymbol{Z}}\oplus[e_n]$, where $e_n(z)=z^n$ on T. Let $W(\mathcal{S})=\{W=\sum_{n\in\boldsymbol{Z}}\oplus u_n : u_n=1$ for all $n\leqq-1$, $u_n=1$ or -1 for $n\geqq0\}$. For $f=\sum_{n=2}^{\infty}(1/2^n)e_{-n}$ in $\mathfrak{H}$ we put $\mathfrak{M}=[f]\oplus[e_{-1}]\oplus H^2(T)$. Then we have $\mathfrak{M}\ominus[\mathcal{S}\mathfrak{M}]=[g]$, where $g=f-(1/6)e_{-1}$ and $[\mathcal{S}\mathfrak{M}]=[f']\oplus H^2(T)=[\mathcal{S}^2\mathfrak{M}]$ where $f'=\sum_{n=1}^{\infty}(1/2^n)e_{-n}$.

In the next example, we give an example of a simply invariant subspace such that $[\mathcal{S}\mathfrak{M}]\subset\mathfrak{M}$, $[\mathcal{S}^2\mathfrak{M}]\subsetneqq[\mathcal{S}\mathfrak{M}]$ but $\mathcal{S}(\mathfrak{M}\ominus[\mathcal{S}\mathfrak{M}])\not\subset[\mathcal{S}\mathfrak{M}]\ominus[\mathcal{S}^2\mathfrak{M}]$.

EXAMPLE 1.6. Let $\mathfrak{H}=L^2(T)\otimes L^2(T)$ and $\mathcal{S}=s\otimes A$, where $A=\{s^n\}_{n=0}$. We put $\mathfrak{M}=\{f\in\mathfrak{H}: \hat{f}(n, m)=0$ for all $(n, m)\notin L_1\cup L_2\}$, where $\hat{f}$ is the Fourier transform of f and $L_1=\{(n, m)\in\boldsymbol{Z}^2: m\geqq0\}$, $L_2=\{(n, m)\in\boldsymbol{Z}^2: n\geqq0, m=-1\}$. Then we have that $\mathfrak{M}\ominus[\mathcal{S}\mathfrak{M}]=\{f\in\mathfrak{H}: \hat{f}(n, m)=0$ for all $(n, m)\neq(0, -1)\}$, $[\mathcal{S}\mathfrak{M}]\ominus[\mathcal{S}^2\mathfrak{M}]=\{f\in\mathfrak{H}: \hat{f}(n, m)=0$ for all $(n, m)\neq(1, -1)\}$ but $\mathcal{S}(\mathfrak{M}\ominus[\mathcal{S}\mathfrak{M}])=\{f\in\mathfrak{H}: \hat{f}(n, m)=0$ for all $(n, m)\notin L_3\}$, where $L_3=\{(n, m)\in\boldsymbol{Z}: n=1, m\geqq-1\}$.

Let $\mathfrak{M}$ be an invariant subspace of $\mathcal{S}$ such that $\mathfrak{M}_\infty=\bigcap_{n=1}^{\infty}[\mathcal{S}^n\mathfrak{M}]=\{0\}$. Then $[\mathcal{S}^{n+1}\mathfrak{M}]$ is a proper subspace of $[\mathcal{S}^n\mathfrak{M}]$ for all $n\geqq0$. Indeed, if we have $[\mathcal{S}^{n_0+1}\mathfrak{M}]=[\mathcal{S}^{n_0}\mathfrak{M}]$ for some integer $n_0>0$, then $[\mathcal{S}^{n_0+k}\mathfrak{M}]=[\mathcal{S}^{n_0}\mathfrak{M}]$ for all integer $k\geqq1$, that is, $\bigcap_{n=1}^{\infty}[\mathcal{S}^n\mathfrak{M}]=[\mathcal{S}^{n_0}\mathfrak{M}]\neq\{0\}$. We put $\mathfrak{M}_n=[\mathcal{S}^n\mathfrak{M}]\ominus[\mathcal{S}^{n+1}\mathfrak{M}]$, then we have $\mathfrak{M}=\mathfrak{M}_0\oplus\mathfrak{M}_1\oplus\mathfrak{M}_2\oplus\cdots$. An invariant subspace $\mathfrak{M}$ of $\mathcal{S}$ such that $\mathfrak{M}_\infty=\{0\}$ is called a *pure* simply invariant subspace.

PROPOSITION 1.7. *Suppose that $W(\mathcal{S})$ is a group such that $S^*W(\mathcal{S})S\subset W(\mathcal{S})$. Then, for each pure simply invariant subspace $\mathfrak{M}$ of $\mathcal{S}$, we have $\mathfrak{M}_n=[\mathcal{S}^n\mathfrak{M}_0]$ for all $n\geqq 0$. Moreover, in the case where $S^*W(\mathcal{S})S=W(\mathcal{S})$, we have $\mathfrak{M}_n=S^n[W(\mathcal{S})\mathfrak{M}_0]$ for all $n\geqq 1$.*

PROOF. Obviously $\mathcal{S}\mathfrak{M}_0\subset\mathcal{S}\mathfrak{M}$. For each $x\in\mathfrak{M}_0$, $y\in\mathfrak{M}$ and each W_1, W_2, W_3 $\in W(\mathcal{S})$, we have

$$\langle W_1S(x),\ W_3SW_2S(y)\rangle=\langle x,\ S^*W_1^*W_3SW_2S(y)\rangle=0\,.$$

Hence $\mathcal{S}\mathfrak{M}_0$ is contained in $\mathfrak{M}_1=[\mathcal{S}\mathfrak{M}]\ominus[\mathcal{S}^2\mathfrak{M}]$. Similarly we have $\mathcal{S}\mathfrak{M}_n\subset\mathfrak{M}_{n+1}$. Since $\mathfrak{M}$ has a decomposition

$$\mathfrak{M}=\mathfrak{M}_0\oplus\mathfrak{M}_1\oplus\mathfrak{M}_2\oplus\cdots,$$

$[\mathcal{S}\mathfrak{M}]$ also has a decomposition

$$[\mathcal{S}\mathfrak{M}]=[\mathcal{S}\mathfrak{M}_0]\oplus[\mathcal{S}\mathfrak{M}_1]\oplus[\mathcal{S}\mathfrak{M}_2]\oplus\cdots.$$

By the definition of $\mathfrak{M}_0$, we have

$$\mathfrak{M}_0=\mathfrak{M}\ominus[\mathcal{S}\mathfrak{M}]=\mathfrak{M}_0\oplus(\mathfrak{M}_1\ominus[\mathcal{S}\mathfrak{M}])\oplus(\mathfrak{M}_2\ominus[\mathcal{S}\mathfrak{M}_1])\oplus\cdots.$$

Thus $\mathfrak{M}_n=[\mathcal{S}\mathfrak{M}_{n-1}]$ for all $n\geqq 0$. If $S^*W(\mathcal{S})S=W(\mathcal{S})$, that is, $W(\mathcal{S})S=SW(\mathcal{S})$, then we have $\mathcal{S}^n=S^nW(\mathcal{S})$. Hence the second assertion holds.

Next, we give a decomposition theorem concerning simply invariant subspaces.

THEOREM 1.8. *Suppose that $W(\mathcal{S})$ is a group such that $S^*W(\mathcal{S})S\subset W(\mathcal{S})$. Then, for each simply invariant subspace $\mathfrak{M}$ of $\mathcal{S}$, $\mathfrak{M}$ has a decomposition*

$$\mathfrak{M}=\mathfrak{M}_p\oplus\mathfrak{M}_r$$

such that $\mathfrak{M}_p$ is a non-zero pure simply invariant subspace and $\mathfrak{M}_r$ is a reducing subspace.

PROOF. Put $\mathfrak{M}_r=\bigcap_{n=1}^{\infty}[\mathcal{S}^n\mathfrak{M}]$ and $\mathfrak{M}_p=\mathfrak{M}\ominus\mathfrak{M}_r$. Since $S^*W(\mathcal{S})^*W(\mathcal{S})S\subset W(\mathcal{S})$, we have $\mathcal{S}^*[\mathcal{S}^n\mathfrak{M}]\subset[\mathcal{S}^{n-1}\mathfrak{M}]$ for $n\geqq 2$. Thus, $\mathfrak{M}_r$ reduces $\mathcal{S}$ and $\mathfrak{M}_p$ is invariant under $\mathcal{S}$. Since $\mathfrak{M}_p\ominus[\mathcal{S}\mathfrak{M}_p]=\mathfrak{M}\ominus[\mathcal{S}\mathfrak{M}]\neq\{0\}$, $\mathfrak{M}_p$ is a simply invariant subspace. Moreover $(\mathfrak{M}_p)_\infty=\{0\}$ because $(\mathfrak{M}_p)_\infty=\bigcap_{n=0}^{\infty}[\mathcal{S}^n(\mathfrak{M}_p)]$ is contained in $\mathfrak{M}_p$ and $\mathfrak{M}_r$.

Under the hypothesis of Theorem 1.8, we can easily find that for each non-reducing invariant subspace $\mathfrak{M}$ of $\mathcal{S}$, we have $[\mathcal{S}^{n+1}\mathfrak{M}]\subsetneqq[\mathcal{S}^n\mathfrak{M}]$ for all $n\geqq 0$. Moreover, we find that the decomposition of $\mathfrak{M}$ in Theorem 1.8 does not hold if we drop any of the conditions of $W(\mathcal{S})$. In fact, we have examples of simply invariant subspaces such that $[\mathcal{S}\mathfrak{M}]=[\mathcal{S}^2\mathfrak{M}]$ in Example 1.5 ($W(\mathcal{S})$ is a group) and Example 1.6 ($S^*W(\mathcal{S})S=W(\mathcal{S})$). In this case, $\mathfrak{M}_r=[\mathcal{S}\mathfrak{M}]$ is not a reducing subspace of $\mathcal{S}$.

For an invariant subspace $\mathfrak{M}$ of $\mathcal{S}$, we denote by $\mathfrak{N}_{\mathfrak{M}}$ the smallest reducing subspace containing $\mathfrak{M}$. We put $[W(\mathcal{S})]=\bigcup_{n\in \boldsymbol{Z}} S^{*n}G(\mathcal{S})S^n$ where $G(\mathcal{S})$ is the group generated by $W(\mathcal{S})$, and $[\mathcal{S}]=[W(\mathcal{S})]S$. Then we have $\mathfrak{N}_{\mathfrak{M}}=[\bigcup_{n\in \boldsymbol{Z}}[\mathcal{S}]^n\mathfrak{M}]$. Thus, if $W(\mathcal{S})$ is a group such that $S^*W(\mathcal{S})S=W(\mathcal{S})$ then we have $\mathfrak{N}_{\mathfrak{M}}=\mathfrak{M}_{-\infty}=[\bigcup_{n\in \boldsymbol{Z}}\mathcal{S}^n\mathfrak{M}]$.

THEOREM 1.9. *Suppose that $W(\mathcal{S})$ is a group such that $S^*W(\mathcal{S})S\subset W(\mathcal{S})$. Then, for each simply invariant subspace $\mathfrak{M}$ of $\mathcal{S}$, $\mathfrak{H}$ is decomposed into three reducing subspaces;*

(i) $$\mathfrak{H}=\mathfrak{N}_{\mathfrak{M}_p}\oplus\mathfrak{M}_r\oplus\mathfrak{M}_c$$

such that $\mathfrak{N}_{\mathfrak{M}}=\mathfrak{N}_{\mathfrak{M}_p}\oplus\mathfrak{M}_r$. Moreover, in the case where $S^*W(\mathcal{S})S=W(\mathcal{S})$, we have

(ii) $$\mathfrak{N}_{\mathfrak{M}_p}=(\mathfrak{M}_p)_{-\infty}=\sum_{n\in \boldsymbol{Z}}\oplus S^n[W(\mathcal{S})\mathfrak{M}_0]\,.$$

PROOF. For a simply invariant subspace $\mathfrak{M}$ of $\mathcal{S}$, we have the decomposition $\mathfrak{M}=\mathfrak{M}_p\oplus\mathfrak{M}_r$ by Theorem 1.8. Since $\mathfrak{M}_r$ reduces $\mathcal{S}$, $\mathfrak{N}_{\mathfrak{M}_p}$ is orthogonal to $\mathfrak{M}_r$ and $\mathfrak{N}_{\mathfrak{M}}=\mathfrak{N}_{\mathfrak{M}_p}\oplus\mathfrak{M}_r$. We put $\mathfrak{M}_c=\mathfrak{H}\ominus\mathfrak{N}_{\mathfrak{M}}$. To show the decomposition (ii), we assume that $S^*W(\mathcal{S})S=W(\mathcal{S})$ and $\mathfrak{M}$ is pure. By Proposition 1.7, we have $\mathfrak{M}=\mathfrak{M}_0\oplus\mathfrak{M}_1\oplus\mathfrak{M}_2\oplus\cdots$ and $\mathfrak{M}_n=S^n[W(\mathcal{S})\mathfrak{M}_0]$ for all $n\geqq 1$. Moreover we have that $[\mathcal{S}^n\mathfrak{M}_0]=S^n[W(\mathcal{S})\mathfrak{M}_0]$ for all $n\leqq -1$ and the subspaces $\{S^n[W(\mathcal{S})\mathfrak{M}_0]\}_{n\in Z}$ are mutually orthogonal. In fact, for $x, y\in\mathfrak{M}_0$, $W_1, W_2\in W(\mathcal{S})$ and $n, m\in \boldsymbol{Z}$ $(n<m)$, it follows that

$$\langle S^nW_1(x),\, S^mW_2(y)\rangle=\langle x,\, W_1S^{m-n}W_2(y)\rangle=\langle x,\, W_1(S^{m-n}W_2S^{n-m})S^{m-n}(y)\rangle=0\,.$$

Hence we have, for all $n\neq 0$,

$$\begin{aligned}\mathcal{S}^n\mathfrak{M}&=[\mathcal{S}^n\mathfrak{M}_0]\oplus[\mathcal{S}^{n+1}W(\mathcal{S})\mathfrak{M}_0]\oplus[\mathcal{S}^{n+2}W(\mathcal{S})\mathfrak{M}_0]\oplus\cdots\\&=S^n[W(\mathcal{S})\mathfrak{M}_0]\oplus S^{n+1}[W(\mathcal{S})\mathfrak{M}_0]\oplus S^{n+2}[W(\mathcal{S})\mathfrak{M}_0]\oplus\cdots.\end{aligned}$$

Consequently we have $\mathfrak{N}_{\mathfrak{M}}=[\bigcup_{n\in \boldsymbol{Z}}\mathcal{S}^n\mathfrak{M}]=\sum_{n\in \boldsymbol{Z}}\oplus S^n[W(\mathcal{S})\mathfrak{M}_0]$.

COROLLARY 1.10. *Let u be a unitary operator on $\mathfrak{K}$ with the spectrum $\sigma(u)=\{e^{i\theta_1}, e^{i\theta_2}\}$. Let $A=\{1, u\}$ and $\mathcal{S}=s\otimes A$ on $L^2(T)\otimes\mathfrak{K}$. Then, for each simply invariant subspace $\mathfrak{M}$ of $\mathcal{S}$, $\mathfrak{H}$ has the decomposition in Theorem* 1.9.

PROOF. We may assume that $e^{i\theta_1}\neq e^{i\theta_2}$. We show that the linear span $\mathrm{Lin}\,A$ of A contains a unitary operator v on $\mathfrak{K}$ such that $v^2=1$. Since A contains the identity 1 and $u=e^{i\theta_1}e_1+e^{i\theta_2}e_2$ where e_1 and e_2 are the spectral projections of u, $\mathrm{Lin}\,A$ contains e_1 and e_2. We put $v=e_1-e_2$ and $\mathcal{S}'=\{s\otimes 1, s\otimes v\}$. Then $W(\mathcal{S}')=\{I, 1\otimes v\}$ is a group such that $S^*W(\mathcal{S}')S=W(\mathcal{S}')$ and $\mathrm{Lin}\,W(\mathcal{S})=\mathrm{Lin}\,W(\mathcal{S}')$. Since $\mathrm{Lin}\,\mathcal{S}=\mathrm{Lin}\,\mathcal{S}'$, the structure of invariant subspaces of $\mathcal{S}$ is the same as that of $\mathcal{S}'$. Thus the assertion holds by Theorem 1.9.

From Corollary 1.10, we immediately have the following.

COROLLARY 1.11. *Let u be a unitary operator on $\mathfrak{K}$. Let $A=\{1, u\}$ and $\mathcal{S}=s\otimes A$. If $\dim\mathfrak{K}\leqq 2$, then for each simply invariant subspace $\mathfrak{M}$ of $\mathcal{S}$, $\mathfrak{H}$ has the decomposition in Theorem* 1.9.

In the above theorems, we have shown the structure of reducing subspaces and simply invariant subspaces of $\mathcal{S}$. However, in general, there are many non-reducing subspaces $\mathfrak{M}$ such that $[\mathcal{S}\mathfrak{M}]=\mathfrak{M}$. For example, the subspace $[\mathcal{S}\mathfrak{M}]=[f']\oplus H^2(T)$ in Example 1.5 is such an invariant subspace. In the next proposition, we give a condition of $\mathcal{S}$ under which every non-reducing invariant subspace has the property $[\mathcal{S}\mathfrak{M}]\subsetneqq\mathfrak{M}$.

PROPOSITION 1.12. *Suppose that $W(\mathcal{S})$ satisfies the following conditions.*

(1) *$W(\mathcal{S})^k$ is a group for some integer $k\geqq 1$.*

(2) *$S^*W(\mathcal{S})S=W(\mathcal{S})$.*

Then a subspace $\mathfrak{M}$ is simply invariant under $\mathcal{S}$ if and only if $\mathfrak{M}$ is a non-reducing invariant subspace of $\mathcal{S}$.

PROOF. Let $\mathfrak{M}$ be an invariant subspace such that $[\mathcal{S}\mathfrak{M}]=\mathfrak{M}$. Since $W(\mathcal{S})^k$ is a group, we have $W(\mathcal{S})^{k+n}=W(\mathcal{S})^k$ for all $n\geqq 0$ and $W(\mathcal{S})^k$ contains $W(\mathcal{S})^*$. By condition (2), we have $\mathcal{S}^n=W(\mathcal{S})^nS^n$ for all $n\geqq 1$, so that $\mathcal{S}^{k+1}\mathfrak{M}=W(\mathcal{S})^{k+1}S^{k+1}\mathfrak{M}=W(\mathcal{S})^kS^{k+1}\mathfrak{M}=SW(\mathcal{S})^kS^k\mathfrak{M}=S\mathcal{S}^k\mathfrak{M}$. Since $[\mathcal{S}^n\mathfrak{M}]=\mathfrak{M}$ for all $n\geqq 1$, we have $\mathfrak{M}=[\mathcal{S}^{k+1}\mathfrak{M}]=S[\mathcal{S}^k\mathfrak{M}]=S\mathfrak{M}$. Thus $S^*\mathfrak{M}=\mathfrak{M}$. Moreover we have $\mathcal{S}^*\mathfrak{M}\subset\mathfrak{M}$. In fact, $\mathcal{S}^*\mathfrak{M}=S^*W(\mathcal{S})^*\mathfrak{M}\subset S^*W(\mathcal{S})^k\mathfrak{M}=S^{*k+1}W(\mathcal{S})^kS^k\mathfrak{M}=S^{*k+1}\mathcal{S}^k\mathfrak{M}\subset\mathfrak{M}$.

COROLLARY 1.13. *Let u be a unitary operator on $\mathfrak{K}$ such that $u^k=1$ for some integer $k>0$. Let $\mathcal{S}=\{s\otimes 1, s\otimes u\}$. Then $\mathfrak{M}$ is simply invariant under $\mathcal{S}$ if and only if $\mathfrak{M}$ is a non-reducing invariant subspace of $\mathcal{S}$.*

We now denote by $\mathrm{Alg}(\mathcal{S})$ the algebra generated by $\mathcal{S}$. Suppose that $W(\mathcal{S})$ is a group and $S^*W(S)S=W(S)$. Then $\mathrm{Alg}(\mathcal{S})=\{T\in\boldsymbol{B}(\mathfrak{H}): T=D_1S+\cdots+D_nS^n, D_i\in D(\mathcal{S}), 1\leqq i\leqq n, n\geqq 1\}$ and $M(\mathcal{S})$ is the closure of the algebra $\{T\in\boldsymbol{B}(\mathfrak{H}): T=D_mS^{-m}+\cdots+D_0+\cdots+D_nS^n, D_i\in D(\mathcal{S}), -m\leqq i\leqq n, m, n\geqq 0\}$ with respect to the σ-weak topology on $\boldsymbol{B}(\mathfrak{H})$. Hence $\mathrm{Alg}(\mathcal{S})+D(\mathcal{S})$ is a subdiagonal algebra with respect to the natural conditional expectation ε of $M(\mathcal{S})$ onto $D(\mathcal{S})$ (cf. [**1**, Definition 2.1.1]). We here remark that, in the case of general subdiagonal algebras, the decomposition (i) in Theorem 1.9 has been shown by Loebl and Muhly [**5**, Theorem V.2].

§ 2. The structure of invariant subspace.

Let $\mathcal{S}$ be a family of shift operators on $\mathfrak{H}$. We shall study relations between the structure of invariant subspaces of $\mathcal{S}$ and the properties of the von Neumann algebra $M(\mathcal{S})$ generated by $\mathcal{S}$. Throughout this paper, we assume that $W(\mathcal{S})$ is a group and $S^*W(\mathcal{S})S=W(\mathcal{S})$. Then the von Neumann algebra $M(\mathcal{S})$ is generated by the two groups $W(\mathcal{S})$ and $\{S^n\}_{n\in\boldsymbol{Z}}$. We now consider the von

Neumann algebra $D(\mathcal{S})$ generated by $W(\mathcal{S})$. For each n in $\boldsymbol{Z}$, the projection P_n of $\mathfrak{H}$ onto $\mathfrak{K}_n$ belongs to the commutant $D(\mathcal{S})'$. The induced von Neumann algebras $D(\mathcal{S})_{P_n}$ with respect to P_n are all isomorphic each other because of the hypothesis $S^*W(\mathcal{S})S=W(\mathcal{S})$. In fact, the natural isomorphism Φ_n from $D(\mathcal{S})_{P_0}$ onto $D(\mathcal{S})_{P_n}$ is given as follows;

$$\Phi_n(TP_0)=S^nTS^{*n}P_n \qquad (T\in D(\mathcal{S}))\,.$$

We point out that if $D(\mathcal{S})$ is a factor, then $D(\mathcal{S})_{P_n}$ are all isomorphic to $D(\mathcal{S})$ (cf. [**9**; 3.13, Proposition]). Among the induced von Neumann algebras $\{D(\mathcal{S})_{P_n}\}_{n\in\boldsymbol{Z}}$, we especially denote by $D(\mathcal{S})_0$ the von Neumann algebra $D(\mathcal{S})_{P_0}$ on $\mathfrak{K}_0$ and we sometimes regard $D(\mathcal{S})_0$ as a subalgebra of the full operator algebra $B(\mathfrak{K})$ on $\mathfrak{K}$. Furthermore, we denote by $D(\mathcal{S})_0'$ the reduced von Neumann algebra $D(\mathcal{S})'_{P_0}$ on $\mathfrak{K}_0$. For an element x in $B(\mathfrak{K})$, we denote by $i(x)$ the operator $\sum_{n\in\boldsymbol{Z}}\oplus x_n$ on $\mathfrak{H}=\sum_{n\in\boldsymbol{Z}}\oplus\mathfrak{K}_n$, where $x_n=x$ for all n in $\boldsymbol{Z}$.

At first, we show some typical simply invariant subspaces. We put

$$\mathcal{H}^2=\sum_{n=0}^{\infty}\oplus\mathfrak{K}_n\,.$$

Then $\mathcal{H}^2$ is a pure simply invariant subspace of an arbitrary family $\mathcal{S}$. Furthermore, for each subspace $\mathfrak{M}_0$ of $\mathfrak{K}_0$, we put

$$\mathfrak{M}=\sum_{n=0}^{\infty}\oplus[\mathcal{S}^n\mathfrak{M}_0]\,,$$

then $\mathfrak{M}$ is a pure simply invariant subspace of $\mathcal{S}$ and $\mathcal{S}^n\mathfrak{M}_0$ is contained in $\mathfrak{K}_n$ for all $n\geqq 0$.

DEFINITION 2.1. A simply invariant subspace $\mathfrak{M}$ is said to be of type-H^2 if $\mathfrak{M}_0=\mathfrak{M}\ominus[\mathcal{S}\mathfrak{M}]$ is contained in $\mathfrak{K}_0$.

PROPOSITION 2.2. *Let $\mathfrak{M}$ be a type-H^2 simply invariant subspace of $\mathcal{S}$, then $\mathfrak{M}_{-\infty}$ is expressed as*

$$\mathfrak{M}_{-\infty}=\sum_{n\in\boldsymbol{Z}}\oplus e\mathfrak{K}_n=i(e)\mathfrak{H}$$

for some projection e in the commutant $D(\mathcal{S})_0'$. Moreover $i(e)$ is a projection in the commutant $M(\mathcal{S})'$.

PROOF. By Theorem 1.9, the reducing subspace $\mathfrak{M}_{-\infty}$ has a decomposition $\mathfrak{M}_{-\infty}=\sum_{n\in\boldsymbol{Z}}\oplus S^n[W(\mathcal{S})\mathfrak{M}_0]$. Since P_0 commutes with $W(\mathcal{S})$, $[W(\mathcal{S})\mathfrak{M}_0]$ is a subspace of $\mathfrak{K}_0$. The projection e of $\mathfrak{K}_0$ onto the subspace $[W(\mathcal{S})\mathfrak{M}_0]$ is an element of the commutant $(D(\mathcal{S})_0)'=D(\mathcal{S})_0'$. Furthermore $i(e)$ belongs to $M(\mathcal{S})'$. In fact, for $W=\sum_{n\in\boldsymbol{Z}}\oplus u_n\in W(\mathcal{S})$, we have

$$\begin{aligned} i(e)W &= \sum_{n\in Z}\oplus eu_n = \sum_{n\in Z} S^n(S^{*n}i(e)S^n)(S^{*n}WS^n)S^{*n}P_n \\ &= \sum_{n\in Z} S^n i(e)P_0(S^{*n}WS^n)P_0S^{*n}P_n = \sum_{n\in Z} S^nP_0(S^{*n}WS^n)P_0 i(e)S^{*n}P_n \\ &= \sum_{n\in Z}\oplus u_n e = Wi(e)\,. \end{aligned}$$

Obviously, $i(e)$ commutes with S and S^*.

Here we recall Beurling's theorem [**4**; Lecture II, Theorem 3] for invariant subspaces of the usual shift S on $L^2(T)$. Namely, every simply invariant subspace $\mathfrak{M}$ of S is expressed as $\mathfrak{M}=M_uH^2(T)$, where M_u is the multiplication operator by a unitary function u in $L^\infty(T)$. Beurling's theorem has been generalized to the case of arbitrary multiplicity by Halmos [**5**]. The shift S on $\mathfrak{H}=\sum_{n\in Z}\oplus\mathfrak{K}_n$ can be regarded as a multiplication operator on the Hilbert space $L^2(T, \mathfrak{K})$ of all $\mathfrak{K}$-valued L^2-functions on T. In fact, $(SG)(z)=zG(z)$ for $G=G(z)$ in $L^2(T, \mathfrak{K})$. By Halmos' theorem, every pure simply invariant subspace $\mathfrak{M}$ of S is expressed as $\mathfrak{M}=M_F\mathcal{H}^2$, where M_F is the multiplication operator on $L^2(T,\mathfrak{K})$ by a $\boldsymbol{B}(\mathfrak{K})$-valued measurable function $F=F(z)$ on T whose values are isometries of $\mathfrak{K}'\subset\mathfrak{K}$ onto $\mathfrak{K}$.

We now consider the von Neumann algebra $M(S)$ on $L^2(T, \mathfrak{K})$ generated by S and its commutant. For a von Neumann algebra M on $\mathfrak{K}$, let $L^\infty(T, M)$ denote the M-valued essentially bounded measurable function on T. We denote by $M_{L^\infty(T,M)}$ the von Neumann algebra of all the multiplication operator M_F on $L^2(T, \mathfrak{K})$ by F in $L^\infty(T, M)$. Then we have $M(S)=M_{L^\infty(T,C(\mathfrak{K}))}$ and $M(S)'=M_{L^\infty(T,B(\mathfrak{K}))}$ (cf. [**10**; Theorem 7.10]). In the case of multiplicity one, we have $M(S)=M_{L^\infty(T)}$ and $(M_{L^\infty(T)})'=M_{L^\infty(T)}$. In connection with Beurling's and Halmos' theorems we give the following definition, which is fundamental in this paper.

DEFINITION 2.3. A family $\mathcal{S}$ is said to have Property (B) if every pure simply invariant subspace $\mathfrak{M}$ of $\mathcal{S}$ is expressed as $\mathfrak{M}=U\mathfrak{N}$, where $\mathfrak{N}$ is a type-H^2 invariant subspace of $\mathcal{S}$ and U is a partial isometry in the commutant $M(\mathcal{S})'$ whose initial space is $\mathfrak{N}_{-\infty}$ and whose final space is $\mathfrak{M}_{-\infty}$.

We here note the equivalence of projections in a von Neumann algebra M. Two projections e and f in M are said to be equivalent and this relation is denoted by $e\sim f$, if there exists a partial isometry u in M such that $e=u^*u$ and $f=uu^*$. We say that e is dominated by f, and we denote by $e\prec f$ this relation, if e is equivalent to a subprojection of f (cf. [**9**; Chapter 4]). For a subspace $\mathfrak{M}$ of $\mathfrak{H}$, we denote by $P_{\mathfrak{M}}$ the projection of $\mathfrak{H}$ onto $\mathfrak{M}$.

Let $\mathfrak{M}$ be a subspace of the form $\mathfrak{M}=U\mathfrak{N}$ in the above definition. Then $\mathfrak{N}$ has a decomposition

$$\mathfrak{N}=\mathfrak{N}_0\oplus\mathfrak{N}_1\oplus\mathfrak{N}_2\oplus\cdots$$

such that $\mathfrak{N}_0=\mathfrak{N}\ominus[\mathcal{S}\mathfrak{N}]\subset\mathfrak{K}_0$ and $\mathfrak{N}_n=S^n[W(\mathcal{S})\mathfrak{N}_0]\subset\mathfrak{K}_n$. Since $U^*U=P_{\mathfrak{N}_{-\infty}}$, we have

$$\mathfrak{M}=U\mathfrak{N}=U\mathfrak{N}_0\oplus U\mathfrak{N}_1\oplus U\mathfrak{N}_2\oplus\cdots$$

and it follows that $\mathcal{S}U\mathfrak{N}_n=U\mathcal{S}\mathfrak{N}_n=U\mathfrak{N}_{n+1}$ for all $n\geqq 0$. Hence we have $\mathfrak{M}_0=U\mathfrak{N}\ominus[\mathcal{S}U\mathfrak{N}]=U\mathfrak{N}_0$, so that $\mathfrak{M}_n=U\mathfrak{N}_n$ for all $n\geqq 0$. Since $\mathfrak{N}$ is of type-H^2, we have $P_{\mathfrak{N}_{-\infty}}=i(e)$ for some projection e in the commutant $D(\mathcal{S})'_0$. Hence we have $P_{\mathfrak{M}_{-\infty}}\sim i(e)$ in $M(\mathcal{S})'$ by the partial isometry U. Moreover, it follows that $P_{[W(\mathcal{S})\mathfrak{M}_0]}\sim P_{[W(\mathcal{S})\mathfrak{N}_0]}=i(e)P_0\leqq P_0$ in $D(\mathcal{S})'$. In fact, $V=UP_0$ is a partial isometry in $D(\mathcal{S})'$ such that $V^*V=P_{[W(\mathcal{S})\mathfrak{N}_0]}\leqq P_0$ and $VV^*=P_{[W(\mathcal{S})\mathfrak{M}_0]}$, because $U[W(\mathcal{S})\mathfrak{N}_0]=[W(\mathcal{S})U\mathfrak{N}_0]=[W(\mathcal{S})\mathfrak{M}_0]$. Furthermore, we find that $P_{\mathfrak{M}_n}\sim i(e)P_n\leqq P_n$ for all $n\geqq 1$. We here remark that the subspace $[W(\mathcal{S})\mathfrak{M}_0]$ and $\mathfrak{K}_0$ are wandering subspaces, that is, $\{S^n[W(\mathcal{S})\mathfrak{M}_0]\}_{n\in \boldsymbol{Z}}$ and $\{S^n\mathfrak{K}_0\}_{n\in \boldsymbol{Z}}$ are sequences of mutually orthogonal subspaces.

DEFINITION 2.4. A projection P in $\boldsymbol{B}(\mathfrak{H})$ is called a wandering projection for $\mathcal{S}$, if P commutes with $W(\mathcal{S})$ and $PS^nP=0$ for all $n\neq 0$.

DEFINITION 2.5. A family $\mathcal{S}$ is said to have Property (W), if every wandering projection for $\mathcal{S}$ is dominated by P_0.

In the above, we proved that if $\mathcal{S}$ has Property (B) then $\mathcal{S}$ has Property (W). In the following, we show that Property (W) is also a sufficient condition for $\mathcal{S}$ to have Property (B).

PROPOSITION 2.6. *A family $\mathcal{S}$ has Property* (B) *if and only if $\mathcal{S}$ has Property* (W).

PROOF. Suppose that $\mathcal{S}$ has Property (W). Let $\mathfrak{M}$ be a pure simply invariant subspace of $\mathcal{S}$. Then the wandering projection $P_{[W(\mathcal{S})\mathfrak{M}_0]}$ is dominated by P_0. Namely, there exists a partial isometry V in $D(\mathcal{S})'$ such that $V^*V\leqq P_0$ and $VV^*=P_{[W(\mathcal{S})\mathfrak{M}_0]}$. We put $\tilde{V}=\sum_{n\in \boldsymbol{Z}}S^nVS^{*n}$. Since $\mathcal{S}W(\mathcal{S})S^*=W(\mathcal{S})$, $\tilde{V}$ commutes with $W(\mathcal{S})$, and obviously $\tilde{V}S=S\tilde{V}$. Hence $\tilde{V}$ is a partial isometry in $M(\mathcal{S})'$ such that $V^*S^n[W(\mathcal{S})\mathfrak{M}_0]\subset\mathfrak{K}_n$ for all n in $\boldsymbol{Z}$. If we put $\mathfrak{N}=V^*\mathfrak{M}$, then $\mathfrak{N}$ is a desired invariant subspace of type-H^2.

There are many kinds of families $\mathcal{S}$ such that $W(\mathcal{S})$ is a group and $S^*W(\mathcal{S})S=W(\mathcal{S})$. In general, $\mathcal{S}$ does not necessarily have Property (B). For example, the family $\mathfrak{S}$ of all shift operators on $\mathfrak{H}$ does not have Property (B). In fact, $\mathfrak{M}=\sum_{n=1}^{\infty}\oplus\mathfrak{K}_n$ is a pure simply invariant subspace of $\mathfrak{S}$ but the commutant $M(\mathfrak{S})'=\boldsymbol{B}(\mathfrak{H})'=\boldsymbol{C}(\mathfrak{H})$ contains no partial isometry U such that $U\mathcal{H}^2=\mathfrak{M}$. Furthermore, McAsey, Muhly and Saitô [6], [7] has given a necessary and sufficient condition for $\mathcal{S}$ to have Property (B) in the case where $M(\mathcal{S})$ is the crossed product determined by a finite von Neumann algebra M in the standard form and a trace invariant *-automorphism. Though the von Neumann algebra $M(\mathcal{S})$ in their paper are special ones in $\boldsymbol{B}(\mathfrak{H})$, we can apply their technique to the general case. In the following, we give a necessary condition for $\mathcal{S}$ to have Property (B).

PROPOSITION 2.7. *Suppose that $\mathcal{S}$ has Property* (B). *Then $M(\mathcal{S})$ is the crossed product of $D(\mathcal{S})_0$ by $\boldsymbol{Z}$ with respect to a spatial *-automorphism $\alpha=\mathrm{Ad}\,u$ of $D(\mathcal{S})_0$ for some unitary operator u on $\mathfrak{K}$. Moreover, $\alpha=\mathrm{Ad}\,u$ leaves the set of all central finite projections in $D(\mathcal{S})'_0$ element-wise fixed.*

PROOF. Since P_1 (resp. P_{-1}) is a wandering projection, P_1 (resp. P_{-1}) dominated by P_0 in $D(\mathcal{S})'$ from Proposition 2.6. Let V be a partial isometry in $D(\mathcal{S})'$ such that $V^*V=P_{-1}$ and $VV^*\leqq P_0$. Then $V_S=SVS^*$ is a partial isometry in $D(\mathcal{S})'$ such that $V_S^*V_S=P_0$ and $V_SV_S^*\leqq P_1$ because V is in $D(\mathcal{S})'$ and $S^*D(\mathcal{S})S=D(\mathcal{S})$. Hence $P_1\succ P_0$ and $P_0\prec P_1$, so that $P_0\sim P_1$ by virtue of the Bernstein type theorem [**9**; Theorem 4.7]. Let U be a partial isometry in $D(\mathcal{S})'$ such that $U^*U=P_0$ and $UU^*=P_1$. We put

$$u=\text{the restriction of } U^*S \text{ to } \mathfrak{K}_0 .$$

Then u is a unitary operator on $\mathfrak{K}_0$ and, regarding u as an operator on $\mathfrak{K}$, we have $U=Si(u^*)P_0$. Let $X=\sum_{n\in\boldsymbol{Z}}\oplus x_n$ be an element of $D(\mathcal{S})$. Since $UX=XU$, we have

$$Si(u^*)P_0(\sum_{n\in\boldsymbol{Z}}\oplus x_n)P_0i(u)S^*=(\sum_{n\in\boldsymbol{Z}}\oplus x_n)P_1 .$$

Namely, we have $u^*x_0u=x_1$. Since $U_n=S^nUS^{*n}$ is also a partial isometry in $D(\mathcal{S})'$ and $U_n=Si(u^*)P_n$, it follows that $u^*x_nu=x_{n+1}$ for all n in $\boldsymbol{Z}$. Therefore, we have $x_n=u^{*n}x_0u^n$ for all n in $\boldsymbol{Z}$. Since $S^*W(\mathcal{S})S=W(\mathcal{S})$, ux_0u^* and u^*x_0u are also elements of $D(\mathcal{S})_0$, that is, $\mathrm{Ad}u(D(\mathcal{S})_0)=u(D(\mathcal{S})_0)u^*=D(\mathcal{S})_0$. Hence $\alpha=\mathrm{Ad}u$ is a *-automorphism of $D(\mathcal{S})_0$ and $X=\sum_{n\in\boldsymbol{Z}}\oplus\alpha^{-n}(x_0)$. Since $M(\mathcal{S})$ is generated by $D(\mathcal{S})$ and the group $\{S^n\}_{n\in\boldsymbol{Z}}$, $M(\mathcal{S})$ is the crossed product of $D(\mathcal{S})_0$ by $\boldsymbol{Z}$ with respect to $\alpha=\mathrm{Ad}u$.

We now remark that α is also a *-automorphism of the commutant $D(\mathcal{S})'_0$ and suppose that there exists a central finite projection e in $D(\mathcal{S})'_0$ such that $\alpha(e)\neq e$. We put $f=e-e\alpha(e)$ if $e\alpha(e)\neq e$, otherwise $f=\alpha(e)-e\alpha(e)$. Then f is a non-zero central projection in $D(\mathcal{S})'_0$ such that $\alpha(f)f=0$. We put

$$\mathfrak{M}=f\mathfrak{K}_0\oplus(\sum_{n=1}^{\infty}\oplus(f+\alpha^{-1}(f))\mathfrak{K}_n) .$$

Then $\mathfrak{M}$ is a pure simply invariant subspace of $\mathcal{S}$ such that $\mathfrak{M}_0=\mathfrak{M}\ominus[\mathcal{S}\mathfrak{M}]=f\mathfrak{K}_0\oplus\alpha^{-1}(f)\mathfrak{K}_1$. By Property (W), there exists a projection R in $D(\mathcal{S})'$ such that

$$P_{\mathfrak{M}_0}=i(f)P_0+i(\alpha^{-1}(f))P_1\sim R\leqq P_0. \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (*)$$

Hence, the central support of $P_{\mathfrak{M}_0}$ is the same as that of R in $D(\mathcal{S})'$. We put $\pi(f)=\sum_{n\in\boldsymbol{Z}}\oplus f_n$, where $f_n=\alpha^{-n}(f)$ for all n in $\boldsymbol{Z}$. Then $\pi(f)$ is a central projection in $D(\mathcal{S})'_0$, which majorizes the projection $i(f)P_0+i(\alpha^{-1}(f))P_1$. Hence $P_{\mathfrak{M}_0}\leqq\pi(f)$, so that $R\leqq\pi(f)$. Thus $R=PP_0\leqq\pi(f)P_0=i(f)P_0$. By relation $(*)$, R is of the form $R=R_1+R_2$, $R_1R_2=0$, where $R_1\sim i(f)P_0$ and $R_2\sim i(\alpha^{-1}(f))P_1$. Since

$U=Si(u^*)P_0$ is a partial isometry in $D(\mathcal{S})'$ with initial projection $i(f)P_0$ and final projection $i(\alpha^{-1}(f))P_1$, we have $i(f)P_0 \sim i(\alpha^{-1}(f))P_1$ in $D(\mathcal{S})'$. We put $g=RP_0$, $g_1=R_1P_0$ and $g_2=R_2P_0$. Then we can consider g, g_1 and g_2 as projections in $D(\mathcal{S})'_0$ and it follows that

$$f \geqq g=g_1+g_2, \quad g_1g_2=0 \quad \text{and} \quad g_1 \sim g_2 \sim f$$

in $D(\mathcal{S})'_0$. This contradicts that f is a finite projection in $D(\mathcal{S})'_0$. Therefore, $\alpha(e)=e$ for each central finite projection e in $D(\mathcal{S})'_0$ and this completes the proof.

Next, we shall show, step by step, that the condition in Proposition 2.7 is also sufficient for $\mathcal{S}$ to have Property (B). We denote by $\mathcal{R}(D(\mathcal{S})_0, \alpha)$ the crossed product of $D(\mathcal{S})_0$ by $\boldsymbol{Z}$ with respect to a spatial *-automorphism $\alpha=\mathrm{Ad}u$ (cf. [**9**; V. 7]).

PROPOSITION 2.8. *Suppose that $M(\mathcal{S})=\mathcal{R}(D(\mathcal{S})_0, \alpha)$ and $M(\mathcal{S})'$ is finite. If α leaves the center of $D(\mathcal{S})_0$ element-wise fixed, then $\mathcal{S}$ has Property* (B).

PROOF. Let P be a wandering projection for $\mathcal{S}$. By the comparability theorem [**9**; Theorem 4.6], there exists a central projection Z in $D(\mathcal{S})'$, such that

$$ZP_0 \prec ZP \quad \text{and} \quad (I-Z)P_0 \succ (I-Z)P$$

in $D(\mathcal{S})'$. We shall show that $ZP_0 \sim ZP$. Since Z is also a central projection in $D(\mathcal{S})$, Z is of the form $Z=\sum_{n\in \boldsymbol{Z}} \oplus z_n$ where $z_n=\alpha^{-n}(z_0)$ for all n in $\boldsymbol{Z}$ and z_0 is a central projection in $D(\mathcal{S})_0$. Since $\alpha(z_0)=z_0$, we have $Z=i(z_0)$. Hence Z is also in the commutant $M(\mathcal{S})'$. We put

$$\tilde{P}=\sum_{n\in \boldsymbol{Z}} S^nPS^{*n}, \quad \widetilde{ZP}=\sum_{n\in \boldsymbol{Z}} S^nZPS^{*n} \quad \text{and} \quad \widetilde{ZP_0}=\sum_{n\in \boldsymbol{Z}} S^nZP_0S^{*n}.$$

Since P, ZP and ZP_0 are wandering projections for $\mathcal{S}$, $\tilde{P}$, $\widetilde{ZP}$ and $\widetilde{ZP_0}$ converge to projections in $M(\mathcal{S})'$ and it follows that $\widetilde{ZP}=Z\tilde{P}$ and $\widetilde{ZP_0}=Z$. Since $ZP_0 \prec ZP$ in $D(\mathcal{S})'$, there exists a partial isometry V in $D(\mathcal{S})'$ such that $V^*V=ZP_0$ and $VV^* \sim Q \leqq ZP$. We put $\tilde{V}=\sum_{n\in \boldsymbol{Z}} S^nVS^{*n}$. Then $\tilde{V}$ converges to a partial isometry in $M(\mathcal{S})'$ such that $\tilde{V}^*\tilde{V}=\widetilde{ZP_0}=Z$ and $\tilde{V}\tilde{V}^*=\tilde{Q}=\sum_{n\in \boldsymbol{Z}} S^nQS^{*n} \leqq \widetilde{ZP} \leqq Z$. Namely, we have $Z \sim \tilde{Q} \leqq Z$ in $M(\mathcal{S})'$. Since $M(\mathcal{S})'$ is finite, we have $Z=\widetilde{ZP}=\tilde{Q}$. Thus $ZP=Q$, that is, $ZP_0 \sim ZP$ in $D(\mathcal{S})'$. Consequently, we have $P \prec P_0$ in $D(\mathcal{S})'$. By Proposition 2.6, $\mathcal{S}$ has Property (B).

We note that the commutant $\mathcal{R}(D(\mathcal{S})_0, \alpha)'$ in Proposition 2.8 is isomorphic to the crossed product $\mathcal{R}(D(\mathcal{S})'_0, \alpha)$ because α is a spatial *-automorphism [**10**; P. 373], so that $D(\mathcal{S})'_0$ must be finite under the condition that $M(\mathcal{S})'$ is finite [**8**; Theorem 7.11.8]. In the case where $D(\mathcal{S})'_0$ is properly infinite, we get the following.

PROPOSITION 2.9. *Suppose that $M(\mathcal{S})=\mathcal{R}(D(\mathcal{S})_0, \alpha)$. If the commutant $D(\mathcal{S})'_0$*

is properly infinite, then $\mathcal{S}$ has Property (B).

PROOF. Let u be the unitary operator on $\mathfrak{K}$ by which α is implemented. We define a unitary operator W on $\mathfrak{H}=\sum_{n\in \boldsymbol{Z}}\oplus\mathfrak{K}_n$ as follows; $W=\sum_{n\in \boldsymbol{Z}}\oplus u_n$, where $u_n=u^{*n}$ for all n in $\boldsymbol{Z}$. Then $\Phi(\cdot)=W\cdot W^*$ is a spatial *-automorphism of $\boldsymbol{B}(\mathfrak{H})$ and $\Phi(D(\mathcal{S}))=\{W=\sum_{n\in \boldsymbol{Z}}\oplus v_n : v_n=v_0$ for all n in $\boldsymbol{Z}$, $v_0\in D(\mathcal{S})_0\}$. Thus $\Phi(D(\mathcal{S}))$ is spatially isomorphic to the tensor product $\boldsymbol{C}(L^2(T))\otimes D(\mathcal{S})_0$ on $L^2(T)\otimes\mathfrak{K}$. Hence the commutant $D(\mathcal{S})'$ is isomorphic to the tensor product $\boldsymbol{B}(L^2(T))\otimes D(\mathcal{S})'_0$ and P_0 (resp. I) corresponds to $q_0\otimes 1$ (resp. $1\otimes 1$) in $\boldsymbol{B}(L^2(T))\otimes D(\mathcal{S})'_0$ where q_0 is the projection of $L^2(T)$ onto the one dimensional subspace of constant valued functions on T. Obviously, $q_0\otimes 1$ is properly infinite in $\boldsymbol{B}(L^2(T))\otimes D(\mathcal{S})'_0$ and the central support of $q_0\otimes 1$ is the identity on $L^2(T)\otimes\mathfrak{K}$. Hence we have $q_0\otimes 1 \sim 1\otimes 1$ [**9**; Proposition 4.13]. Consequently we have $P_0\sim I$ in $D(\mathcal{S})'$, so that for each wandering projection P is dominated by P_0. Hence $\mathcal{S}$ has Property (W).

In order to apply the preceding propositions, we prepare a lemma. Let e be a projection in the commutant $D(\mathcal{S})'_0$ and $\mathfrak{H}_e=\sum_{n\in \boldsymbol{Z}}\oplus e\mathfrak{K}_n$. Since $W(\mathcal{S})$ commutes with $i(e)$, $Ui(e)$ is a shift operator on $\mathfrak{H}_e$ for each shift operator U in $\mathcal{S}$. We denote by $\mathcal{S}_e$ the family of shift operators $\{Ui(e) : U\in\mathcal{S}\}$ on $\mathfrak{H}_e$. Then we have $M(\mathcal{S}_e)=M(\mathcal{S})i(e)$, $D(\mathcal{S}_e)=D(\mathcal{S})i(e)$ and $D(\mathcal{S}_e)_0=D(\mathcal{S})_0e$. If e is a central projection in $D(\mathcal{S})'_0$, then we have $M(\mathcal{S})'=M(\mathcal{S}_e)'\oplus M(\mathcal{S}_{1-e})'$ and, for each wandering projection P for $\mathcal{S}$, $Pi(e)$ (resp. $Pi(1-e)$) is a wandering projection for $\mathcal{S}_e$ (resp. $\mathcal{S}_{1-e}$). Hence we have the following lemma by Proposition 2.6.

LEMMA 2.10. *Let e be a central projection in $D(\mathcal{S})_0$. If $\mathcal{S}_e$ and $\mathcal{S}_{1-e}$ have Property* (B), *then $\mathcal{S}$ has Property* (B).

PROPOSITION 2.11. *Suppose that $M(\mathcal{S})=\mathcal{R}(D(\mathcal{S})_0, \alpha)$. If α leaves the finite central projections in $D(\mathcal{S})'_0$ element-wise fixed, then $\mathcal{S}$ has Property (B).*

PROOF. For the von Neumann algebra $D(\mathcal{S})'_0$, there exists a central projection e in $D(\mathcal{S})'_0$ such that $D(\mathcal{S})'_0e$ is finite and $D(\mathcal{S})'_0(1-e)$ is properly infinite [**10**; V. Theorem 1.19]. Since e is a finite central projection in $D(\mathcal{S})'_0$, we have $\alpha(e)=e$. Hence e commutes with the unitary operator u on $\mathfrak{K}$ by which α is implemented. We put $\alpha_e=\mathrm{Ad}(ue)$ and $\alpha_{1-e}=\mathrm{Ad}(u(1-e))$. Then α_e and α_{1-e} are *-automorphisms of $D(\mathcal{S})_0e$, $D(\mathcal{S})_0(1-e)$ and their commutants respectively. Moreover, α_e leaves the center of $D(\mathcal{S})'_0e$ element-wise fixed. Since $M(\mathcal{S}_{1-e})=\mathcal{R}(D(\mathcal{S}_{1-e})_0, \alpha_{1-e})$ and $D(\mathcal{S}_{1-e})'=D(\mathcal{S})'_0(1-e)$, $\mathcal{S}_{1-e}$ has Property (B) by Proposition 2.9. On the other hand, since the commutant $D(\mathcal{S}_e)'_0=D(\mathcal{S})'_0e$ is finite, there exists a unique faithful normal center valued trace $\boldsymbol{T}$ on $D(\mathcal{S}_e)'_0$. By the uniqueness of $\boldsymbol{T}$, we have $\boldsymbol{T}(\alpha_e(x))=\boldsymbol{T}(x)$ for each x in $D(\alpha_e)'_0$. Hence there exists a α_e-invariant faithful normal trace on $D(\mathcal{S}_e)'_0$. By Theorem 7.11.8 in [8], the crossed product $\mathcal{R}(D(\mathcal{S}_e)'_0, \alpha_e)$ is finite and this crossed product is isomorphic to the commutant $M(\mathcal{S}_e)'=\mathcal{R}(D(\mathcal{S}_e)_0, \alpha_e)'$. Thus, by Proposition 2.8, $\mathcal{S}_e$ has Property

(B) and we have the conclusion by Lemma 2.10.

The preceding propositions 2.7, 2.8, 2.9 and 2.10 implies our main theorem.

THEOREM 2.12. *A family $\mathcal{S}$ has Property* (B) *if and only if $\mathcal{S}$ satisfies the following conditions:*

(1) *$M(\mathcal{S})=\mathcal{R}(D(\mathcal{S})_0, \alpha)$, where α is a spatial *-automorphism of $D(\mathcal{S})_0$.*

(2) *α leaves the finite central projections in $D(\mathcal{S})_0'$ element-wise fixed.*

Let α be trivial in the crossed product $M(\mathcal{S})=\mathcal{R}(D(\mathcal{S})_0, \alpha)$. Then, of course, $\mathcal{S}$ satisfies the conditions in the above theorem and $D(\mathcal{S})=\{X=\sum_{n\in\boldsymbol{Z}}\oplus x_n : x_n=x_0$ for all n in $\boldsymbol{Z}\}$. Thus $D(\mathcal{S})$ is isomorphic to the tensor product $1\otimes D(\mathcal{S})_0$ on $L^2(T)\otimes\mathfrak{K}$, so that $M(\mathcal{S})$ is isomorphic to $M_{L^\infty(T)}\otimes D(\mathcal{S})_0$. In this case, $\mathcal{S}$ is regarded as the tensor product $s\otimes A$ on $L^2(T)\otimes\mathfrak{K}$. Hence we have the following.

COROLLARY 2.13. *Let $\mathfrak{H}=L^2(T)\otimes\mathfrak{K}$ and $\mathcal{S}=s\otimes A$ where A is a group of unitary operators on $\mathfrak{K}$. Then $\mathcal{S}$ has Property* (B).

We have shown, in Theorem 1.9, that for each simply invariant subspace $\mathfrak{M}$ of $A(\mathcal{S})$, $\mathfrak{H}$ has a decomposition $\mathfrak{H}=(\mathfrak{M}_p)_{-\infty}\oplus\mathfrak{M}_r\oplus\mathfrak{M}_c$. We here show that if $M(\mathcal{S})'$ is finite then this decomposition corresponds to a simple decomposition of $\mathfrak{H}$ under the conditions in Theorem 2.12.

THEOREM 2.14. *Suppose that $\mathcal{S}$ satisfies conditions* (1) *and* (2) *in Theorem* 2.12 *and $W(\mathcal{S})'$ is finite. Then, for each simply invariant subspace $\mathfrak{M}$ of $A(\mathcal{S})$, $\mathfrak{H}$ has a couple of decompositions;*

$$(1)\qquad \mathfrak{H}=(\mathfrak{M}_p)_{-\infty}\oplus\mathfrak{M}_r\oplus\mathfrak{M}_c$$

$$(2)\qquad \mathfrak{H}=\mathfrak{N}_{-\infty}\oplus\mathfrak{N}_c$$

such that

(i) *$\mathfrak{N}$ is an invariant subspace of type-H^2, that is, $\mathfrak{N}=\sum_{n=0}^{\infty}\oplus e\mathfrak{K}_n$ for some projection e in $D(\mathcal{S})_0'$.*

(ii) *$U\mathfrak{N}_{-\infty}=(\mathfrak{M}_p)_{-\infty}$, $U\mathfrak{N}=\mathfrak{M}_p$ and $U\mathfrak{N}_c=\mathfrak{M}_r\oplus\mathfrak{M}_c$ for some unitary operator U in $M(\mathcal{S})'$.*

PROOF. By Theorem 1.9, $\mathfrak{H}$ has a decomposition (1) such that $\mathfrak{M}=\mathfrak{M}_r\oplus\mathfrak{M}_p$ and $\mathfrak{M}_p$ is a non-zero pure simply invariant subspace. By Theorem 2.12, there exist an invariant subspace $\mathfrak{N}$ of type-H^2 and a partial isometry V in $M(\mathcal{S})'$ such that $V^*V=P_{\mathfrak{N}_{-\infty}}$, $VV^*=P_{(\mathfrak{M}_p)_{-\infty}}$ and $V\mathfrak{N}_n=(\mathfrak{M}_p)_n$ for all $n\geqq 0$. Since $M(\mathcal{S})'$ is finite, we have $P_{\mathfrak{H}\ominus(\mathfrak{M}_p)_{-\infty}}\sim P_{\mathfrak{H}\ominus\mathfrak{N}_{-\infty}}$.

If we drop the condition that $W(\mathcal{S})'$ is finite, then Theorem 2.14 does not necessarily hold. We shall show this fact in the case of properly infinite.

EXAMPLE 2.15. Let $\mathfrak{K}$ be an infinite dimensional Hilbert space with base $\{\xi_n\}_{n=0}^{\infty}$ and $\mathfrak{H}=L^2(T, \mathfrak{K})$. Let $\mathcal{S}=\{S\}$, where S is the usual shift on $\mathfrak{H}$, that is, $(SG)(z)=zG(z)$ a.e. z in T for each vector G in $\mathfrak{H}$. Then $M(\mathcal{S})=M_{L^\infty(T, C(\mathfrak{K}))}$ and $M(\mathcal{S})'=M_{L^\infty(T, \boldsymbol{B}(\mathfrak{K}))}$. Let u be the unilateral shift on $\mathfrak{K}$ with respect to the base $\{\xi_n\}_{n=0}^{\infty}$. We define a function F in $L^\infty(T, \boldsymbol{B}(\mathfrak{K}))$ as follows; $F(z)=1$ if $0\leqq\arg z$

$<\pi$, $=u$ if $\pi\leqq\arg z<2\pi$. Then $\mathfrak{M}=M_F H^2(T, \mathfrak{K})$ is a pure simply invariant subspace such that $P_{\mathfrak{M}_{-\infty}}(z)=1$ if $0\leqq\arg z<\pi$, $=1-e_0$ if $\pi\leqq\arg z<2\pi$, where e_0 is the projection of $\mathfrak{K}$ onto the one dimensional subspace generated by ξ_0. For this invariant subspace $\mathfrak{M}$, $\mathfrak{H}$ has a decomposition $\mathfrak{H}=\mathfrak{M}_{-\infty}\oplus\mathfrak{M}_c$, then $P_{\mathfrak{M}_c}(z)=0$ if $0\leqq\arg z<\pi$, $=e_0$ if $\pi\leqq\arg z<2\pi$. For an invariant subspace $\mathfrak{N}$ of type-H^2, we consider a decomposition of $\mathfrak{H}$, $\mathfrak{H}=\mathfrak{N}_{-\infty}\oplus\mathfrak{N}_c$. Then $P_{\mathfrak{N}_{-\infty}}=i(e)$ for some projection e in $\boldsymbol{B}(\mathfrak{K})$ by Proposition 2.2, so that $P_{\mathfrak{N}_c}=i(1-e)$. Hence $P_{\mathfrak{M}_c}$ cannot be equivalent to $P_{\mathfrak{N}_c}$ for any invariant subspace $\mathfrak{N}$ of type-H^2.

Next we note that decomposition (2) in Theorem 2.14 is unique up to equivalence in $M(\mathcal{S})'$ and $D(\mathcal{S})'$. Suppose that $\mathfrak{H}=\mathfrak{N}'_{-\infty}\oplus\mathfrak{N}'_c$ is a decomposition of $\mathfrak{H}$ satisfying conditions (i) and (ii). Then we immediately find that $P_{\mathfrak{N}_{-\infty}}\sim P_{\mathfrak{N}'_{-\infty}}$ in $M(\mathcal{S})'$ and $P_{\mathfrak{N}_n}\sim P_{\mathfrak{N}'_n}$ in $D(\mathcal{S})'$ for all n in $\boldsymbol{Z}$.

REMARK. Let $\mathcal{S}$ be a family of shift operators on $\mathfrak{H}$. When the set $W(\mathcal{S})$ does not satisfy conditions (1) $W(\mathcal{S})$ is a group (2) $S^*W(\mathcal{S})S=W(\mathcal{S})$, we cannot apply our results directly to the study of structure of invariant subspaces of $\mathcal{S}$. But, if $W(\mathcal{S}, S')=\{W: W=U(S')^*, U\in\mathcal{S}\}$ satisfies the conditions (1) and (2) for some shift operator S' on $\mathfrak{H}$, we can apply our results indirectly to the study of invariant subspaces of $\mathcal{S}$. We note that $W(\mathcal{S})$ depends on S, but the structure of invariant subspaces of $\mathcal{S}$ does not depend on S.

References

[1] W.B. Arveson, Analyticity in operator algebras, Amer. J. Math., **89** (1967), 578-642.

[2] A. Beurling, On two problems concerning linear transformations in Hilbert space, Acta Math., **81** (1949), 239-255.

[3] P.R. Halmos, Shifts on Hilbert spaces, J. Reine Angew. Math., **208** (1961), 102-112.

[4] H. Helson, Lectures on Invariant Subspaces, Academic Press, London-New York, 1964.

[5] R.I. Loeble and P.S. Muhly, Analyticity and flows in von Neumann algebras, J. Funct. Anal., **29** (1978), 214-252.

[6] M. McAsey, P.S. Muhly and K.S. Saitô, Non-self-adjoint crossed products (Invariant subspaces and maximality), Trans. Amer. Math. Soc., **248** (1978), 381-409.

[7] P.S. Muhly and K.S. Saitô, Non-self-adjoint crossed products II, J. Math. Soc. Japan, **33** (1981), 485-495.

[8] G.K. Pedersen, C*-algebras and their automorphism groups, Academic Press, London-New York-San Francisco, 1979.

[9] S. Stratila and L. Zsido, Lectures on von Neumann Algebras, Abacus Press, Tunbridge Wells-Kent, England, 1979.

[10] M. Takesaki, Theory of Operator Algebras I, Springer-Verlag, New York-Heiderberg-Berlin, 1979.

[11] S. Kawamura, Invariant subspaces for shift operators of multiplicity one, Tōhoku Math. J., **34** (1982), 15-21.

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