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Free boundary problems for a class of nonlinear parabolic equations: An approach by the theory of subdifferential operators

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Introduction.

The present paper is devoted to the study of a free boundary problem for a nonlinear parabolic equation in one-space dimension. Free boundary problems arise naturally in a number of physical phenomena with change of state (such as melting of ice and recrystallization of metals) and have been studied by many authors (e.g., [**1**, **2**, **4-10**, **14-19**] and their references).

In this paper we are concerned with the following one phase Stefan problem: For a number $l_0 \geqq 0$, functions u_0 on $[0, l_0]$, f on $[0, T]\times[0, \infty)$ and g on $[0, T]$ we find a boundary curve $x=l(t)$ ($\geqq 0$ on $[0, T]$) and a function $u=u(t, x)$ on $[0, T]\times[0, \infty)$ satisfying

(E) $$u_t-(|\beta(u)_x|^{p-2}\beta(u)_x)_x=f \qquad \text{for } l(t)>0,\ 0<x<l(t)$$

subject to

(C1) $$l(0)=l_0 \text{ and if } l_0>0, \quad \text{then } u(0, x)=u_0(x) \quad \text{for } 0<x<l_0,$$

(C2) $$\begin{cases} |\beta(u)_x(t, 0+)|^{p-2}\beta(u)_x(t, 0+)=g(t) & \text{for } 0<t<T, \\ \beta(u)(t, l(t))=0 & \text{for } 0<t<T \end{cases}$$

and

(C3) $$\frac{dl(t)}{dt}=-|\beta(u)_x(t, l(t)-)|^{p-2}\beta(u)_x(t, l(t)-) \qquad \text{for } 0<t<T,$$

where $2\leqq p<\infty$, $\beta: \boldsymbol{R}\to\boldsymbol{R}$ is a given function and $\beta(u)_x(t, x+)$ (resp. $\beta(u)_x(t, x-)$) stands for the right (resp. left) hand partial derivative of $\beta(u)(t, x)$ at x with respect to x.

This kind of problems for a certain class of nonlinear parabolic equations was treated earlier by Douglas [**6**] and Kyner [**16**] in which they showed the existence and uniqueness of solution by using a strong maximum principle for parabolic equations with variable coefficients, but their method is not applicable to our case. Our approach to problem {(E), (C1)-(C3)}, which is different from that of Douglas and Kyner in some points of view, is based upon recent results on the existence, uniqueness and stability of solutions to nonlinear evolution equations involving subdifferential operators of time-dependent convex functions on Hilbert spaces (cf. [12]).

Notations. For a (real) Banach space V we denote by $|\cdot|_V$ the norm in V, by V^* its dual and by $(\cdot, \cdot)_V$ the duality pairing between V^* and V; especially, if V is a Hilbert space and is identified with its dual space, then we mean by $(\cdot, \cdot)_V$ the inner product in V.

By an operator A from a Banach space V into another Banach space W we mean that to each v in V, A assigns a subset Av of W, namely A is a multi-valued mapping from V into W; in particular, if Av consists of at most one element of W for every v in V, then A is called singlevalued. For an operator $A: V\to W$ the set $D(A)=\{v\in V;\ Av\neq\emptyset\}$ is called the domain.

Let ϕ be a lower semi-continuous convex function on a Hilbert space H with values in $(-\infty, \infty]$ such that $\phi\not\equiv\infty$ on H. Then the set $D(\phi)=\{z\in H;\ \phi(z)<\infty\}$ is called the effective domain and the subdifferential $\partial\phi$ is an operator from H into itself defined as follows: $z^*\in\partial\phi(z)$ if and only if $z\in D(\phi)$, $z^*\in H$ and

$$(z^*, z'-z)_H\leqq\phi(z')-\phi(z), \qquad \forall z'\in H.$$

For fundamental properties of $\partial\phi$ we refer for example to a book of Brézis [3].

1. Formulation as a quasi-variational problem.

Let $2\leqq p<\infty$ and $0<T<\infty$ be numbers which are fixed, and set for simplicity

$$H=L^2(0, \infty), \qquad X=W^{1,p}(0, \infty).$$

Let $\beta: \boldsymbol{R}\to\boldsymbol{R}$ be a function with $\beta(0)=0$ and assume that β is strictly increasing bi-Lipschitz continuous on $\boldsymbol{R}$, i.e.,

$$c_\beta|r-r_1|^2\leqq(\beta(r)-\beta(r_1))(r-r_1)\leqq|r-r_1|^2/c_\beta$$

for any r, r_1 in $\boldsymbol{R}$ with a positive constant c_β.

Given a non-negative continuous function $l: [0, T]\to\boldsymbol{R}$ and a continuous function $g: [0, T]\to\boldsymbol{R}$, we define for each t in $[0, T]$

$$K_l(t)=\{z\in X;\ z(x)=0,\ \forall x\geqq l(t)\}$$

and

$$\phi^t_{l,g}(z)=\begin{cases} \dfrac{1}{p}\displaystyle\int_0^\infty|z_x|^p dx+g(t)z(0) & \text{if } z\in K_l(t), \\ \infty & \text{otherwise}. \end{cases} \tag{1.1}$$

Clearly $\phi^t_{l,g}$ is a lower semi-continuous convex function on H with $D(\phi^t_{l,g})=$

$K_l(t)$. We now consider the nonlinear evolution equation

(1.2) $$u'(t)+\partial\phi_{l,g}^t(Bu(t))\ni f(t) \qquad \text{for } 0<t<T\,,$$

where the unknown u is an H-valued function on $[0, T]$, $u'(t)=(d/dt)u(t)$ and B is the singlevalued operator from $H=D(B)$ into itself defined by

$$[Bz](x)=\beta(z(x)) \qquad \text{for } z\in H \text{ and } x\in[0, \infty)\,.$$

DEFINITION 1.1. Let l, g be as above, u_0 be in H and f in $L^2(0, T; H)$. Then we mean by $VP(l, g, u_0, f)$ the Cauchy problem for (1.2) to find a function u in $C([0, T]; H)$ such that

(A1) $u\in W^{1,2}(0, T; H)$ and $u(0)=u_0$;
(A2) the function $t\to\phi_{l,g}^t(Bu(t))$ is bounded on $[0, T]$;
(A3) $u'(t)+\partial\phi_{l,g}^t(Bu(t))\ni f(t)$ for a.e. t in $[0, T]$.

Such a function u is called a (strong) solution to $VP(l, g, u_0, f)$.

REMARK 1.1. A solution u to $VP(l, g, u_0, f)$ is able to be characterized by the following system:

(1.3) $$\begin{cases} u\in W^{1,2}(0, T; H) \qquad \text{with } u(0)=u_0\,, \\ \beta(u)\in L^\infty(0, T; X)\,, \\ \beta(u)(t, \cdot)\in K_l(t) \text{ (hence } \beta(u)(t, l(t))=0) \qquad \text{for all } t\in[0, T]\,, \end{cases}$$

(1.4) $$u_t(t, \cdot)-(|\beta(u)_x(t, \cdot)|^{p-2}\beta(u)_x(t, \cdot))_x=f(t, \cdot)$$
$$\text{in the distributional sense on } (0, l(t)) \qquad \text{for a.e. } t\in I_0\,,$$

(1.5) $$|\beta(u)_x(t, 0+)|^{p-2}\beta(u)_x(t, 0+)=g(t) \qquad \text{for a.e. } t\in I_0\,,$$

where $I_0=\{t\in[0, T];\ l(t)>0\}$. In fact, suppose that u is a solution to $VP(l, g, u_0, f)$. Then (1.3) follows immediately from (A1) and (A2). As is easily seen, (A3) can be written in the following equivalent form:

(1.7) $$\begin{cases} (u'(t)-f(t), z)_H+\int_0^\infty|\beta(u)_x(t, x)|^{p-2}\beta(u)_x(t, x)z_x(x)dx+g(t)z(0)=0\,, \\ \forall z\in K_l(t)\,, \qquad \text{for a.e. } t\in[0, T]\,. \end{cases}$$

We see from (1.7) that (1.4) holds and hence $(|\beta(u)_x(t, \cdot)|^{p-2}\beta(u)_x(t, \cdot))_x=u_t(t, \cdot)-f(t, \cdot)\in L^2(0, l(t))$ for a.e. $t\in I_0$. This implies that $|\beta(u)_x(t, x)|^{p-2}\beta(u)_x(t, x)$ is an absolutely continuous function of x on $(0, l(t))$ and $\beta(u)_x(t, 0+)$ exists for a.e. $t\in I_0$ as well as $\beta(u)_x(t, l(t)-)$, so that by integration by parts we obtain (1.5) from (1.7). Similarly we can show the converse.

Now we are going to give a quasi-variational formulation associated with our free boundary problem $\{$(E), (C1)-(C3)$\}$.

DEFINITION 1.2. Let $l_0\geqq 0$ be a number, u_0 be in H, g in $C([0, T])$ and f

in $L^2(0, T; H)$. Then we mean by $QVP(l_0, g, u_0, f)$ to find a couple $\{l, u\}$ such that

(B1) $l \in W^{1,2}(0, T)$ and $l \geqq 0$ on $[0, T]$;

(B2) u is a solution to $VP(l, g, u_0, f)$;

(B3) $l(t)=l_0-\int_0^t g(r)dr+\int_0^{l_0} u_0(x)dx+\int_0^t\int_0^{l(r)} f(r, x)dxdr-\int_0^\infty u(t, x)dx$

for all t in $[0, T]$.

Our results on $QVP(l_0, g, u_0, f)$ are stated as follows:

THEOREM 1.1. *Let $l_0 \geqq 0$, $u_0 \in H$ be non-negative, $g \in C([0, T])$ be non-positive and $f \in L^2(0, T; H)$ be non-negative. Then we have:*

(a) *If $\{l, u\}$ is a solution to $QVP(l_0, g, u_0, f)$, then u is non-negative and l is non-decreasing in t.*

(b) *Further suppose that $u_0 \in X$, $u_0(x)=0$ for all $x \geqq l_0$ and $g \in W^{1,2}(0, T)$. Then $QVP(l_0, g, u_0, f)$ has at least one solution.*

REMARK 1.2. Let l_0, g, u_0 and f be as in Definition 1.2 and let $\{l, u\}$ be a solution to $QVP(l_0, g, u_0, f)$. Then, as was seen in Remark 1.1, l and u satisfy (1.3), (1.4) and (1.5). Moreover, the following (1.5)′ and (1.6) hold:

(1.5)′ $$|\beta(u)_x(t, 0+)|^{p-2}\beta(u)_x(t, 0+)=g(t) \qquad \text{for a.e. } t \in [0, T],$$

(1.6) $$\frac{dl(t)}{dt}=-|\beta(u)_x(t, l(t)-)|^{p-2}\beta(u)_x(t, l(t)-) \qquad \text{for a.e. } t \in [0, T].$$

Indeed, from (B3) with (1.5) we derive that for a.e. $t \in I_0$ $(=\{t \in [0, T]; l(t)>0\})$

$$\begin{aligned}\frac{dl(t)}{dt}&=-g(t)+\int_0^{l(t)} f(t, x)dx-\int_0^\infty u_t(t, x)dx\\&=-g(t)-\int_0^{l(t)}(|\beta(u)_x(t, x)|^{p-2}\beta(u)_x(t, x))_x dx\\&=-|\beta(u)_x(t, l(t)-)|^{p-2}\beta(u)_x(t, l(t)-).\end{aligned}$$

Also, if $t \in (0, T]-I_0$, then $u(t, x)=0$ for all $x \geqq 0$ and (B3) implies $g(t)=0$. Therefore

$$|\beta(u)_x(t, 0+)|^{p-2}\beta(u)_x(t, 0+)=0=g(t) \qquad \text{for all } t \in (0, T]-I_0$$

and

$$\frac{dl(t)}{dt}=0=-|\beta(u)_x(t, l(t)-)|^{p-2}\beta(u)_x(t, l(t)-) \qquad \text{for a.e. } t \in [0, T]-I_0.$$

Thus (1.5)′ and (1.6) are shown, and we see that $QVP(l_0, g, u_0, f)$ is a quasi-variational problem associated with {(E), (C1)-(C3)}.

In order to demonstrate the above existence theorem we introduce a mapping P from a certain compact convex subset S of $C([0, T])$ into itself defined as follows:

$$(1.8)\qquad [Pl](t)=l_0-\int_0^t g(r)dr+\int_0^{l_0} u_0(x)dx+\int_0^t\int_0^{l(r)} f(r, x)dxdr$$
$$-\int_0^\infty u^l(t, x)dx \qquad \text{for each } l\in S \text{ and } t\in[0, T],$$

where u^l is a solution to $VP(l, g, u_0, f)$. We shall show that there is an element l of S satisfying $Pl=l$ (i.e., a fixed point l of P in S) and that the couple $\{l, u^l\}$ is a solution to $QVP(l_0, g, u_0, f)$.

The problem of uniqueness for a solution to $QVP(l_0, g, u_0, f)$ remains open, but in the special case that $p=2$ and $f\equiv 0$ we shall show

THEOREM 1.2. *If $p=2$, then $QVP(l_0, g, u_0, f)$ has at most one solution for $l_0\geqq 0$, $g\in C([0, T])$ non-positive, $u_0\in X$ non-negative with $u_0=0$ on $[l_0, \infty)$ and $f\equiv 0$.*

2. **Problem** $VP(l, g, u_0, f)$.

We begin with the following comparison theorem.

THEOREM 2.1. *Let l be a non-negative function in $C([0, T])$, g, $\bar{g}$ be in $C([0, T])$ with $g\leqq\bar{g}$ on $[0, T]$, u_0, $\bar{u}_0$ in H and f, $\bar{f}$ in $L^2(0, T; H)$. Let u and $\bar{u}$ be solutions to $VP(l, g, u_0, f)$ and $VP(l, \bar{g}, \bar{u}_0, \bar{f})$, respectively. Then we have:*

$$|(\bar{u}(t)-u(t))^+|_{L^1(0,L)}\leqq|(\bar{u}(s)-u(s))^+|_{L^1(0,L)}+\int_s^t|(\bar{f}(r)-f(r))^+|_{L^1(0,L)}dr$$

for any $0\leqq s\leqq t\leqq T$ and any positive number $L\geqq|l|_{C([0,T])}$, where $(\cdot)^+$ stands for the positive part of $(\cdot)$.

We omit the proof of this theorem, since it can be proved by a way similar to that of Bénilan [**1**] and Damlamian [**5**]. We obtain the following corollaries immediately from Theorem 2.1.

COROLLARY 1. *Let l, g, u_0 and f be as in Theorem 2.1. Then $VP(l, g, u_0, f)$ has at most one solution.*

COROLLARY 2. *Let l, g, u_0 and f be as in Theorem 2.1 and further suppose that g is non-positive and u_0, f are non-negative. Then a solution to $VP(l, g, u_0, f)$ is non-negative.*

As to the family $\{\phi^t_{l,g};\ 0\leqq t\leqq T\}$ of convex functions given by (1.1) we see the following lemma:

LEMMA 2.1. *Let $l\in C([0, T])$ be non-negative and non-decreasing in t and let $g\in C([0, T])$. Then there is a positive constant $C_{l,g}$ such that*

$$(2.1)\qquad \phi^t_{l,g}(z)-\phi^s_{l,g}(z)\leqq C_{l,g}|g(t)-g(s)|(|\phi^s_{l,g}(z)|+1)$$
$$\text{for any } 0\leqq s\leqq t\leqq T \text{ and } z\in K_l(s);$$

in fact we can take

$$C_{l,g}=(|g|_{C([0,T])}+1)^{p'}l(T)+1, \quad p'=\frac{p}{p-1}. \tag{2.2}$$

PROOF. First we have for any $z\in K_l(s)$ and any $\delta>0$

$$|z(0)|\leqq\int_0^{l(s)}|z_x|dx\leqq\int_0^{l(s)}\left\{\frac{\delta|z_x|^p}{p}+\frac{\delta^{1-p'}}{p'}\right\}dx$$

$$\leqq\delta\phi_{l,g}^s(z)-\delta g(s)z(0)+\frac{\delta^{1-p'}l(s)}{p'},$$

so

$$(1-\delta|g(s)|)|z(0)|\leqq\delta\phi_{l,g}^s(z)+\frac{\delta^{1-p'}l(T)}{p'}.$$

Since $\phi_{l,g}^t(z)-\phi_{l,g}^s(z)=(g(t)-g(s))z(0)$ for $t\geqq s$, we obtain (2.1) with (2.2) by taking $\delta=(|g|_{C([0,T])}+1)^{-1}$. Q. E. D.

This lemma allows us to apply a result of Kenmochi [**12**; Theorem 1.1] to $VP(l, g, u_0, f)$ and we get the following existence theorem.

THEOREM 2.2. *Let l be as in Lemma 2.1, g be in $W^{1,2}(0, T)$, u_0 in $K_l(0)$ and f in $L^2(0, T; H)$. Then $VP(l, g, u_0, f)$ has at least one solution.*

Now, given numbers l_0 and L such that $0\leqq l_0<L$, we consider a family

$$\mathcal{L}=\left\{l\in C([0, T]);\ \begin{array}{l} l(0)=l_0,\ l(T)\leqq L \text{ and} \\ l \text{ is non-decreasing in } t \end{array}\right\}. \tag{2.3}$$

The following stability result for solutions to $VP(l, g, u_0, f)$ with $l\in\mathcal{L}$ plays an important role in solving $QVP(l_0, g, u_0, f)$.

THEOREM 2.3. *Let $l_0\geqq 0$, $g\in W^{1,2}(0, T)$, $u_0\in X$ with $u_0=0$ on $[l_0, \infty)$ and $f\in L^2(0, T; H)$. Then there exists a constant $K>0$ such that*

$$|u^l(t)|_H\leqq K, \quad \forall t\in[0, T],$$

$$|\phi_{l,g}^t(Bu^l(t))|\leqq K, \quad \forall t\in[0, T],$$

$$\left|\frac{du^l}{dt}\right|_{L^2(0,T;H)}\leqq K$$

for every $l\in\mathcal{L}$, where u^l is a unique solution to $VP(l, g, u_0, f)$.

This stability theorem is a direct consequence of a priori estimates for approximate solutions to $VP(l, g, u_0, f)$ given in Kenmochi [**12**; section 2].

3. Operator P and proof of Theorem 1.1.

Throughout this section, assume that $l_0\geqq 0$, $u_0\in X$ is non-negative with $u_0=0$ on $[l_0, \infty)$, $g\in W^{1,2}(0, T)$ is non-positive and $f\in L^2(0, T; H)$ is non-negative and let $\mathcal{L}$ be the family as given by (2.3) with L satisfying

(3.1) $$L>l_0+\int_0^{l_0}u_0(x)dx-\int_0^T g(r)dr+\sqrt{LT}|f|_{L^2(0,T;H)}.$$

We now consider the operator P on $\mathcal{L}$ which is defined by (1.8). Concerning this operator P we have

LEMMA 3.1. $P(\mathcal{L})\subset\mathcal{L}\cap W^{1,2}(0, T)$.

PROOF. Let $l\in\mathcal{L}$. Then $Pl\in W^{1,2}(0, T)$, since a unique solution u^l to $VP(l, g, u_0, f)$ belongs to $W^{1,2}(0, T; H)$. By noting the facts in Remark 1.1, we have

$$\frac{d}{dt}[Pl](t)=-g(t)+\int_0^{l(t)}f(t, x)dx-\int_0^{l(t)}u_t^l(t, x)dx$$

$$=-g(t)-\int_0^{l(t)}(|\beta(u^l)_x(t, x)|^{p-2}\beta(u^l)_x(t, x))_x dx$$

$$=-|\beta(u^l)_x(t, l(t)-)|^{p-2}\beta(u^l)_x(t, l(t)-)$$

for a.e. $t\in I_0$ $(=\{t\in[0, T];\ l(t)>0\})$. Also u^l is non-negative by Corollary 2 to Theorem 2.1 as well as $\beta(u^l)$. Hence

$$\beta(u^l)_x(t, l(t)-)\leqq 0 \qquad \text{for a.e. } t\in I_0,$$

from which it follows that $(d/dt)[Pl](t)\geqq 0$ for a.e. $t\in I_0$. For a.e. $t\in[0, T]-I_0$ we have

$$\frac{d}{dt}[Pl](t)=-g(t)\geqq 0,$$

because $u^l(t, x)=0$ for all $x\geqq 0$ if $t\in[0, T]-I_0$. Therefore Pl is non-decreasing. Besides $[Pl](T)\leqq L$ by (3.1). Thus $Pl\in\mathcal{L}$. Q.E.D.

LEMMA 3.2. *P is continuous on $\mathcal{L}$ with respect to the topology of $C([0, T])$.*

PROOF. Suppose that $l_n\in\mathcal{L}$ and $l_n\to l$ in $C([0, T])$, and denote by u_n and u the solutions to $VP(l_n, g, u_0, f)$ and $VP(l, g, u_0, f)$, respectively. Then, on account of Theorem 2.3, there is a constant K such that

(3.2) $$\begin{cases} |u_n(t)|_H\leqq K, & \forall n,\ \forall t\in[0, T], \\ |\phi^t_{l_n, g}(Bu_n(t))|\leqq K, & \forall n,\ \forall t\in[0, T], \\ |u_n'|_{L^2(0,T;H)}\leqq K, & \forall n. \end{cases}$$

We note here that for each n the following holds:

(3.3) $$\int_0^T(u_n'(t)-f(t), Bu_n(t)-w(t))_H dt\leqq\Phi(w)-\Phi(Bu_n),$$
$$\forall w\in L^p(0, T; X) \text{ with } w(t)\in K_{l_n}(t) \text{ for a.e. } t\in[0, T],$$

where

$$\Phi(w)=\frac{1}{p}\int_0^T\int_0^\infty |w_x(t, x)|^p dxdt+\int_0^T g(t)w(t, 0)dt .$$

By (3.2), $\{u_n\}$ is relatively compact in $C([0, T]; H)$. We want to show that $u_n \to u$ in $C([0, T]; H)$. For this purpose, let $\{u_{n_k}\}$ be any subsequence of $\{u_n\}$ such that $u_{n_k}\to \bar{u}$ (hence $Bu_{n_k}\to B\bar{u}$) in $C([0, T]; H)$. Then we have

$$Bu_{n_k}(t) \longrightarrow B\bar{u}(t) \quad \text{weakly in } X \text{ for each } t\in[0, T],$$

$$Bu_{n_k} \longrightarrow B\bar{u} \quad \text{weakly in } L^2(0, T; X),$$

$$u'_{n_k} \longrightarrow \bar{u}' \quad \text{weakly in } L^2(0, T; H)$$

and by the way

$$\phi^t_{l,g}(B\bar{u}(t))\leqq K, \qquad \forall t\in[0, T],$$

$$\bar{u}\in W^{1,2}(0, T; H), \qquad \bar{u}(0)=u_0,$$

$$\liminf_{k\to\infty} \Phi(Bu_{n_k})\geqq \Phi(B\bar{u}). \tag{3.4}$$

Now denote by Z the set

$$\{v\in L^p(0, T; X);\ v(t)\in K_l(t) \quad \text{for a.e. } t\in[0, T]\}.$$

Let v be any function in Z and ε be any positive number. Putting $v_\varepsilon(t, x)=v(t, x+\varepsilon)$, we see that $v_\varepsilon(t)\in K_{l_n}(t)$ for a.e. t in $[0, T]$ and for all n sufficiently large. Hence, taking $n=n_k$ with $w=v_\varepsilon$ and letting $k\to\infty$ in (3.3), we obtain by (3.4)

$$\int_0^T(\bar{u}'(t)-f(t), B\bar{u}(t)-v_\varepsilon(t))_H dt\leqq \Phi(v_\varepsilon)-\Phi(B\bar{u}).$$

Furthermore, since $v_\varepsilon\to v$ in $L^p(0, T; X)$ and $\Phi(v_\varepsilon)\to\Phi(v)$ as $\varepsilon\downarrow 0$,

$$\int_0^T(\bar{u}'(t)-f(t), B\bar{u}(t)-v(t))_H dt\leqq \Phi(v)-\Phi(B\bar{u}).$$

This inequality holds for every v in Z, which is equivalent to

$$f(t)-\bar{u}'(t)\in\partial\phi^t_{l,g}(B\bar{u}(t)) \qquad \text{for a.e. } t\in[0, T]$$

(cf. Kenmochi [**11**; Proposition 1.1]). Thus $\bar{u}$ is a solution to $VP(l, g, u_0, f)$. By the uniqueness of solution we have $\bar{u}=u$. Therefore it must be true that $u_n\to u$ in $C([0, T]; H)$, so that

$$Pl_n \longrightarrow Pl \qquad \text{in } C([0, T]). \qquad \text{Q.E.D.}$$

PROOF OF THEOREM 1.1. The assertion (a) follows easily from (1.6) of Remark 1.2 and Corollary 2 to Theorem 2.1. To show (b), consider the following subset S of $\mathcal{L}$:

$$S=\left\{ l\in\mathcal{L};\quad \begin{array}{r} |l(t)-l(s)|\leqq|t-s||g|_{C([0,T])} \\ +\sqrt{|t-s|L}|f|_{L^2(0,T;H)}+\sqrt{|t-s|}K \\ \text{for all } s,\ t\in[0,T] \end{array} \right\},$$

where K is the same constant as in Theorem 2.3. Obviously S is a convex compact subset of $\mathcal{L}$ in $C([0, T])$ and $P(\mathcal{L})\subset S$. Taking Lemmas 3.1 and 3.2 into account, we see that P is continuous on S with respect to the topology of $C([0, T])$ and $P(S)\subset S$. Hence, by a well-known fixed point theorem there is $l\in S$ such that $Pl=l$ and it is easy to see that the couple $\{l, u^l\}$, u^l being a unique solution to $VP(l, g, u_0, f)$, is a solution to $QVP(l_0, g, u_0, f)$. Q.E.D.

4. A uniqueness theorem in a special case.

Throughout this section we assume that $p=2$, $l_0\geqq 0$, $g\in C([0, T])$ is non-positive and $u_0\in X$ $(=W^{1,2}(0, \infty))$ is non-negative with $u_0=0$ on $[l_0, \infty)$.

Let $\{l, u\}$ be an arbitrary solution to $QVP(l_0, g, u_0, 0)$. Then we know the following facts (cf. Remark 1.2 and Theorem 1.1):

(1) l is non-decreasing with $l(0)=l_0$ and u is non-negative;

(2) $u_t(t, \cdot)-\beta(u)_{xx}(t, \cdot)=0$ a.e. on $[0, l(t)]$ for a.e. $t\in[0, T]$;

(3) $u(0, x)=u_0(x)$ for $0\leqq x\leqq l_0$, $u(t, x)=0$ for $x\geqq l(t)$
and $\beta(u)_x(t, 0+)=g(t)$ for a.e. $t\in[0, T]$;

(4) $\dfrac{dl(t)}{dt}=-\beta(u)_x(t, l(t)-)$ for a.e. $t\in[0, T]$.

We define

$$v(t, x)=\int_0^t \beta(u)(r, x)dr \qquad \text{for } x\geqq 0,\ 0\leqq t\leqq T$$

and note that

$$v_t(t, x)=\beta(u)(t, x)\geqq 0, \qquad v_x(t, x)=\int_0^t \beta(u)_x(r, x)dr.$$

Now, let η be any function in X. Then we have by (1)-(4)

$$\int_0^\infty v_x(t, \cdot)\eta_x dx$$

$$=\int_0^t\int_0^{l(r)} \beta(u)_x(r, \cdot)\eta_x dx dr$$

$$=\int_0^t\left[-\int_0^{l(r)} \beta(u)_{xx}(r, \cdot)\eta dx-\beta(u)_x(r, 0+)\eta(0)+\beta(u)_x(r, l(r)-)\eta(l(r))\right]dr$$

$$=-\int_0^t\int_0^{l(r)} u_t(r,\ \cdot)\eta dxdr-\left(\int_0^t gdr\right)\eta(0)-\int_0^t \frac{dl(r)}{dr}\eta(l(r))dr$$

$$=-\int_0^\infty u(t,\ \cdot)\eta dx+\int_0^\infty u_0\eta dx-\left(\int_0^t gdr\right)\eta(0)-\int_{l_0}^{l(t)}\eta dx\,,$$

from which we get the following lemma.

LEMMA 4.1. *Let* $\{l, u\}$ *be a solution to* $QVP(l_0, g, u_0, 0)$ *and* v *be as above. Also let* ρ *be the inverse of* β. *Then*

$$I(t\,;v,\ \eta)\equiv\int_0^\infty \rho(v_t)(t,\ x)(v_t(t,\ x)-\eta(x))dx+\int_0^\infty v_x(t,\ x)(v_{xt}(t,\ x)-\eta_x(x))dx$$

$$-\int_0^\infty u_0(x)(v_t(t,\ x)-\eta(x))dx+\left(\int_0^t g(r)dr\right)(v_t(t,\ 0)-\eta(0))$$

$$+\int_{l_0}^\infty (v_t(t,\ x)-\eta(x))dx\leqq 0$$

for all $t\in[0, T]$ *and all* $\eta\in Y=\{\eta\in X;\ \eta$ *is non-negative and* $\eta(x)=0$ *for all sufficiently large* $x\}$.

PROOF. We set for simplicity

$$J(t\,;\eta)=\int_0^\infty \rho(v_t)(t,\ \cdot)\eta dx+\int_0^\infty v_x(t,\ \cdot)\eta_x dx-\int_0^\infty u_0\eta dx$$

$$+\left(\int_0^t gdr\right)\eta(0)+\int_{l_0}^\infty \eta dx \qquad \text{for } \eta\in Y\,.$$

As was seen above, $J(t\,;\eta)\geqq 0$ for all $\eta\in Y$ and $J(t\,;v_t(t,\ \cdot))=0$. Therefore $I(t\,;v,\ \eta)=J(t\,;v_t(t,\ \cdot))-J(t\,;\eta)\leqq 0$. Q.E.D.

PROOF OF THEOREM 1.2. Let $\{l, u\}$ and $\{\bar{l}, \bar{u}\}$ be two solutions to $QVP(l_0, g, u_0, 0)$. Then from Lemma 4.1 with the same notation as above it follows that

$$0\geqq I(t\,;v,\ \bar{v}_t(t,\ \cdot))+I(t\,;\bar{v},\ v_t(t,\ \cdot))$$

$$=\int_0^\infty \{\rho(v_t)(t,\ x)-\rho(\bar{v}_t)(t,\ x)\}(v_t(t,\ x)-\bar{v}_t(t,\ x))dx$$

$$+\int_0^\infty (v_x(t,\ x)-\bar{v}_x(t,\ x))(v_{xt}(t,\ x)-\bar{v}_{xt}(t,\ x))dx$$

$$\geqq\frac{1}{2}\frac{d}{dt}|v_x(t,\ \cdot)-\bar{v}_x(t,\ \cdot)|_H^2$$

for a.e. $t\in[0, T]$. This gives

$$|v_x(t,\ \cdot)-\bar{v}_x(t,\ \cdot)|_H\leqq|v_x(0,\ \cdot)-\bar{v}_x(0,\ \cdot)|_H=0$$

for all $t\in[0, T]$, so $v_t=\bar{v}_t$, i.e., $u=\bar{u}$ as well as $l=\bar{l}$. Q.E.D.

REMARK 4.1. The technic adopted above is found in Duvaut [7].

5. Some remarks.

A. Let $l_0>0$, u_0, g and f be as in (b) of Theorem 1.1 and $\{l, u\}$ be a solution to $QVP(l_0, g, u_0, f)$. Then, by definition, l belongs to $W^{1,2}(0, T)$ and u is a unique solution to $VP(l, g, u_0, f)$. It is not difficult to verify that the family $\{\phi^t_{l,g}\,;\, 0\leqq t\leqq T\}$ satisfies the following:

$$(*)\quad \begin{cases} \text{for each } s, t\in[0, T] \text{ and } z\in K_l(s) \text{ there is } z_1\in K_l(t) \text{ such that} \\ \quad |z_1-z|_H\leqq C\{|g(t)-g(s)|+|l(t)-l(s)|\}(|\phi^s_{l,g}(z)|^{1/2}+1), \\ \quad |\phi^t_{l,g}(z_1)-\phi^s_{l,g}(z)|\leqq C\{|g(t)-g(s)|+|l(t)-l(s)|\}(|\phi^s_{l,g}(z)|+1), \\ \text{where } C \text{ is a positive constant independent of } s, t \text{ and } z. \end{cases}$$

In fact, if we take for z given in $K_l(s)$

$$z_1(x)=z\left(\frac{l(s)}{l(t)}x\right), \qquad 0\leqq x<\infty,$$

then we obtain inequalities of the above forms with a positive constant C. Under $(*)$ we can show (cf. Kenmochi [13]) that the function $t\to\phi^t_{l,g}(Bu(t))$ is absolutely continuous on $[0, T]$. This implies that $t\to|\beta(u)(t, \cdot)|_X$ is continuous on $[0, T]$, so that $\beta(u)\in C([0, T]; X)$.

B. Let $l_0>0$, g, u_0, f be as in (b) of Theorem 1.1 and let h be a function in $L^2(0, T)$. Then, by $QVP(l_0, g, u_0, f, h)$ we mean the problem to find a couple $\{l, u\}$ satisfying (B1), (B2) of Definition 1.2 and (B3)′ below instead of (B3):

$$\text{(B3)}'\qquad l(t)=l_0-\int_0^t g(r)dr+\int_0^t h(r)dr+\int_0^{l_0} u_0(x)dx$$
$$+\int_0^t\int_0^{l(r)} f(r, x)dxdr-\int_0^\infty u(t, x)dx, \qquad \forall t\in[0, T].$$

This integral equation (B3)′ is corresponding to the following type of Stefan condition

$$\frac{dl(t)}{dt}=-|\beta(u)_x(t, l(t)-)|^{p-2}\beta(u)_x(t, l(t)-)+h(t), \qquad \text{for } 0<t<T.$$

In this case we should notice that the unknown boundary curve $x=l(t)$ is not necessarily non-decreasing in t. However the same approach is possible to $QVP(l_0, g, u_0, f, h)$.

C. Finally we consider the problem to find a couple $\{l, u\}$ satisfying

$$u_t - \beta(u)_{xx} = f \qquad \text{for } l(t)>0,\ 0<x<l(t) \tag{5.1}$$

subject to

$$l(0)=l_0 \text{ and if } l_0>0, \text{ then } u(0, x)=u_0(x) \text{ for } 0<x<l_0, \tag{5.2}$$

$$\begin{cases} \beta(u)(t, 0)=g_0(t) & \text{for } 0<t<T, \\ \beta(u)(t, l(t))=0 & \text{for } 0<t<T \end{cases} \tag{5.3}$$

and

$$\frac{dl(t)}{dt} = -\beta(u)_x(t, l(t)-) \qquad \text{for } 0<t<T, \tag{5.4}$$

where $l_0 \geqq 0$ is given as well as $u_0 \geqq 0$ in $W^{1,2}(0, \infty)$ with $u_0=0$ on $[l_0, \infty)$, $f \geqq 0$ in $L^2(0, T; H)$ and $g_0 \geqq 0$ in $W^{1,2}(0, T)$. By means of the family $\{\tilde{\phi}^t_{l, g_0};\ 0 \leqq t \leqq T\}$ of convex functions given by

$$\tilde{\phi}^t_{l, g_0}(z) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^\infty |z_x|^2 dx & \text{if } z \in \tilde{K}_{l, g_0}(t), \\ \infty & \text{otherwise} \end{cases}$$

with $\tilde{K}_{l, g_0}(t) = \{z \in W^{1,2}(0, \infty);\ z(0)=g_0(t),\ z=0 \text{ on } [l(t), \infty)\}$, we can similarly give a quasi-variational formulation associated with system $\{(5.1)\text{-}(5.4)\}$, in which (5.4) is transformed into the integral equation

$$l(t)^2 = l_0^2 + 2\int_0^t g_0(r)dr + 2\int_0^{l_0} x u_0(x)dx - 2\int_0^\infty x u(t, x)dx + 2\int_0^t \int_0^{l(r)} x f(r, x)dxdr, \qquad \forall t \in [0, T].$$

Also in this case we can show the existence and uniqueness (in case $f \equiv 0$) of a solution to this quasi-variational problem by modifying the arguments developed in sections 2, 3 and 4.

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