

## Rational homotopy type and self maps

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### § 1. Introduction.

D. Sullivan showed in [1] and [2] that the rational homotopy type of a simply connected simplicial complex can be algebraically described by its minimal model which is constructed from the  $Q$ -polynomial forms on it. If the minimal model of a simplicial complex  $K$  is isomorphic to the one obtained from its cohomology ring, the rational homotopy type of  $K$  is called a formal consequence of the cohomology ring. Complexes, having such a rational homotopy type, enjoy interesting homotopy properties ([2]). The purpose of this paper is to characterize such complexes by the existence of a certain kind of self maps. Since a finite  $CW$ -complex has the same homotopy type as a polyhedron we work in the category of simply connected finite  $CW$ -complexes. Our main result is

**THEOREM.** *Let  $K$  be a simply connected finite  $CW$ -complex. Then the following three conditions on  $K$  are equivalent.*

- (1) *The rational homotopy type of  $K$  is a formal consequence of the cohomology ring.*
- (2) *For any integer  $r$ , there exists a multiple  $s$  of  $r$  and a map  $f: K \rightarrow K$  such that  $f^* = s^* \text{Id}: H^*(K; Z) \rightarrow H^*(K; Z)$ .*
- (3) *There exists a rational number  $t$  ( $t \neq 0, \pm 1$ ) and a map  $F: K_{(0)} \rightarrow K_{(0)}$  such that  $F^* = t^* \text{Id}: H^*(K_{(0)}; Z) \rightarrow H^*(K_{(0)}; Z)$ , where  $K_{(0)}$  denotes the localization of  $K$  at zero and the homomorphism  $s^* \text{Id}$  denotes the homomorphism  $s^i \text{Id}$  for each degree  $i$ .*

From this Theorem, we can deduce

**COROLLARY 1.** *If the rational homotopy type of  $K$  is a formal consequence of the cohomology ring then  $K$  is 0-universal.*

**COROLLARY 2.** *Let  $K$  be a simply connected finite  $CW$ -complex. Then there exists a simply connected finite  $CW$ -complex  $\tilde{K}$  satisfying the following conditions:*

- (1)  $H^*(K; Q) = H^*(\tilde{K}; Q)$  as a ring,
- (2)  $\tilde{K}$  is 0-universal.

This paper is organized as follows:

In § 2 we give a brief account of minimal models and recall two Sullivan's

theorems which shall be quoted in the later. In §3 we prepare some lemmas and in §4 the proofs of Theorem and Corollaries are given.

After writing this paper I received a letter from Professor D. Sullivan which contained many helpful suggestions and I knew that the equivalence (1) $\Leftrightarrow$ (3) was proved in [6]. I would like to take this opportunity to thank him and also Professor S. Sasao for his helpful advice.

## §2. Minimal models.

Throughout this paper a D. G. A. means a differential graded commutative algebra  $A^*$  over  $Q$  with  $H^0(A^*)=Q$ ,  $H^1(A^*)=0$ . Let  $A^*$  be a D. G. A. Then there exists a D. G. A.  $m(A^*)$ , unique up to isomorphism, satisfying the following conditions (Theorem 1.1(a) of [1])

- (i)  $m(A^*)$  is free (i.e. relations in the algebra are only associativity, and graded commutativity).
- (ii)  $d(m(A^*))$  is decomposable, where  $d$  is the differential operator.
- (iii) There is a D. G. A. map  $\rho : m(A^*) \rightarrow A^*$  which induces an isomorphism on the cohomology ring.

The D. G. A.  $m(A^*)$  is called the minimal model of  $A^*$ .

For a simply connected finite complex  $K$ , the D. G. A.  $A^*(K)$  is constructed from  $Q$ -polynomial forms on  $K$  and its minimal model is called the minimal model of  $K$  which we denote by  $m^*(K)$ . Let  $m^*(K)(n)$  denotes the subalgebra of  $m^*(K)$  generated by elements of degree  $\leq n$ . Then  $m^*(K)$  is constructed from  $m^*(K)(n)$  inductively as follows ([2]). Suppose that the homomorphism

$$(\rho|m^*(K)(n-1))^* : H^i(m^*(K)(n-1)) \rightarrow H^i(A^*(K)^*)$$

is isomorphism for  $i \leq n-1$  and monomorphism for  $i=n$ . Then we can choose generators  $\alpha_i^n$  so that  $\rho(\alpha_i^n)$  forms basis for the cokernel of

$$(\rho|m^*(K)(n-1)) : H^n(m^*(K)(n-1)) \rightarrow H^n(A^*(K))$$

and  $\{\beta_m^n\}$  so that  $d\beta_m^n$  forms a basis for the kernel of

$$(\rho|m^*(K)(n-1))^* : H^{n+1}(m^*(K)(n-1)) \rightarrow H^{n+1}(A^*(K)).$$

Let  $m^*(K)(n) = m^*(K)(n-1)\{\alpha_i^n, \beta_m^n\}$ . Then  $(\rho|m^*(K)(n))^*$  is isomorphism for  $i \leq n$  and monomorphism for  $i=n+1$ . Here  $m^*(K)(n-1)\{\alpha_i^n, \beta_m^n\}$  denotes the polynomial algebra if  $n$  is even and exterior algebra if  $n$  is odd which are generated by  $\alpha_i^n$  and  $\beta_m^n$  over  $m^*(K)(n-1)$ .

The rational homotopy type of  $K$  is called a formal consequence of the cohomology ring if  $m^*(K)$  is isomorphic to the minimal model of  $H^*(K; Q)$  with  $d=0$ . We abbreviate “ $K$  is a formal consequence of the cohomology ring” to “ $K$  is a f. c.”.

Let  $V_i$  and  $C_i$  be subspaces spanned by  $\{\alpha_i^i; \beta_m^i\}$  and  $\{\alpha_i^i\}$  respectively. Then Sullivan showed the following theorems.

**THEOREM S<sub>1</sub>** (Theorem (4.1) of [1]). *K is a f. c. if and only if there is in each  $V_i$  a complementary subspace  $N_i$  to  $C_i$  ( $V_i=C_i\oplus N_i$ ) such that any closed form in  $I(\oplus N_i)$  is exact, where  $I(\oplus N_i)$  denotes the ideal generated by  $N_i$ .*

**THEOREM S<sub>2</sub>** (Theorem (1.2) of [1]). *Let  $A^*, B^*$  be D. G. A. and  $m^*( )$  be a minimal model of a D. G. A. and let  $g: m^*( )\rightarrow A^*$ ,  $\varphi: B^*\rightarrow A^*$  be D. G. A. maps and suppose that  $\varphi$  induces isomorphism on cohomology. Then there exists a D. G. A. map  $\tilde{g}: m^*( )\rightarrow B^*$ , unique up to homotopy, such that  $\varphi\circ\tilde{g}=g$ .*

Especially, we have

**COROLLARY S<sub>2</sub>**. *Let  $A^*, B^*$  be D. G. A. and  $\varphi: A^*\rightarrow B^*$  be any D. G. A. map. Then there exists a D. G. A. map  $\hat{\varphi}: m^*(A^*)\rightarrow m^*(B^*)$  uniquely up to homotopy such that the following diagram commutes.*

$$\begin{array}{ccc} m(A^*) & \xrightarrow{\quad} & A^* \\ \downarrow \hat{\varphi} & \rho_A & \downarrow \varphi \\ m(B^*) & \xrightarrow{\quad} & B^* \\ & \rho_B & \end{array}$$

where  $\rho_A$  and  $\rho_B$  are D. G. A. maps which induces isomorphisms on the cohomology.

**§ 3. Some lemmas.**

In this section we use the same notation as in § 2.

**LEMMA 3.1.** *Let K be a f. c. Then any skeleton of K is also a f. c.*

**PROOF.** Since  $K$  is a f. c., there exists a family  $N_i$  of subspaces satisfying the conditions in Theorem S<sub>1</sub>. Let  $K^n$  be the  $n$ -skeleton of  $K$ . Then there is a D. G. A. map

$$T_n: m^*(K)\rightarrow m^*(K^n)$$

such that  $m^*(K)(n-1)$  is mapped isomorphically to  $m^*(K^n)(n-1)$ . Let  $\tilde{N}_i=T_n(N_i)$  for  $i\leq n-1$ , and  $\tilde{N}_n$  be an arbitrary complement to  $C_n$ , and  $\tilde{N}_i=V_i$  for  $i>n$ . Let  $d\alpha=0$  ( $\alpha\in I(\oplus\tilde{N}_i)$ ). If  $\text{deg } \alpha>n$  it is trivial that  $\alpha$  is exact. If  $\text{deg } \alpha\leq n$  then we have

$$\alpha\in I(\bigoplus_{i=2}^{n-2}\tilde{N}_i)=T_n(I(\bigoplus_{i=2}^{n-2}N_i))$$

since there is no element in degree one. Hence  $\alpha$  is exact and the proof is completed by Theorem S<sub>1</sub>. q. e. d.

**LEMMA 3.2.** *Let K be an  $n+1$  dimensional complex. Suppose that  $K^n$  is a f. c., and that  $\tilde{N}_i$  is a complementary subspace to the subspace generated by the closed forms in  $V_i$  such that any closed forms in  $I(\oplus\tilde{N}_i)$  is exact. Then  $K$  is a*

*f. c. if and only if any closed form in  $T_n^{-1}(I(\oplus \bar{N}_i))$  is exact in  $m^*(K)$ .*

PROOF. Let  $K$  be a f. c., and let  $N_i$  be a family satisfying the conditions in Theorem  $S_1$ . Then we have

$$T_n^{-1}(I(\oplus \bar{N}_i)) = I(\oplus N_i).$$

Hence any closed element in  $T_n^{-1}(I(\oplus \bar{N}_i))$  is exact. Conversely assume that any closed element in  $T_n^{-1}(I(\oplus \bar{N}_i))$  is exact. Now let  $N_i = T_n^{-1}(N_i)$  for  $i \leq n-1$  and let  $N_i$  be any complementary subspace to the subspace spanned by the closed elements in  $V_i$  for  $i \geq n$ . Let  $\beta \in I(\oplus N_i)$  be any closed element. If  $\deg \beta > n+1$  it is trivial that  $\beta$  is exact. If  $\deg \beta = i \leq n+1$ , we have

$$\beta \in I(\bigoplus_{i=2}^{n-1} N_i) = T_n^{-1}(I(\bigoplus_{i=2}^{n-1} \bar{N}_i))$$

since there is no element in degree one. Thus  $\beta$  is exact. q. e. d.

Now we fix algebra generators  $\{\alpha_i^n, \beta_m^n; n \geq 2, 1 \leq l \leq p, 1 \leq m \leq q\}$  of  $m^*(K)$  as in (2.1) such that  $d$  is injective on the subspace  $C_n$  for each  $n$ .

LEMMA 3.3. *Suppose that there exist a rational number  $t$  ( $t \neq 0, \pm 1$ ) and a D. G. A. map  $F: m^*(K) \rightarrow m^*(K)$  such that the induced homomorphism is  $F^* = t^* Id$  on cohomology. Then we have a direct sum decomposition*

$$m^i(K) = \bigoplus_{k \geq 0} W_k^i, \quad I(\oplus N_n)^i = \bigoplus_{k \geq 1} W_k^i \quad (i=0, 1, 2, \dots)$$

such that  $F(a)$  is cohomologous to  $t^{i+k}a$  for  $a \in W_k^i$ , where  $I(\oplus N_n)^i$  denotes elements of degree  $i$  in  $I(\oplus N_n)$ .

PROOF. Since  $m^*(K) = \bigcup_n m^*(K)(n)$  we use the induction on  $n$ . Suppose that we have a direct sum decomposition in each degree  $i$

$$(*) \quad m^i(K)(n) = \bigoplus_{k \geq 0} W_{k,n}^i, \quad I(\bigoplus_{m=2}^i N_m)^i = \bigoplus_{k \geq 1} W_{k,n}^i$$

such that  $F(\alpha)$  is cohomologous to  $t^{i+k}\alpha$  for  $\alpha \in W_{k,n}^i$ . Let  $V_0^{n+1}$  be a subspace of  $m^{n+1}(K)(n+1)$  spanned by  $\{\alpha_i^{n+1}\}$ . Then we have

$$m^{n+1}(K)(n+1) = V_0^{n+1} \oplus N_{n+1} \oplus \bigoplus_{k \geq 0} W_{k,n}^{n+1} = V_0^{n+1} \oplus W_0^{n+1} \oplus I(\bigoplus_{m=2}^i N_m)^{n+1}.$$

Let  $V_k^{n+1} = \{x \in N_{n+1} \mid dx \in W_{k-1,n}^{n+2} \text{ for } k \geq 1\}$  and  $\gamma \in N_{n+1}$  be an element. Since  $d\gamma \in m^{n+2}(K)(n)$  we can obtain a decomposition by (\*)

$$d\gamma = \sum_k w_k \quad (w_k \in W_{k,n}^{n+2}).$$

Then, since  $\{dw_k\}$  is a linearly independent set and  $dd\gamma = 0$ , we have  $dw_k = 0$  ( $k \geq 0$ ). And moreover  $w_k$  is exact ( $k \geq 1$ ) because  $F(w_k)$  is cohomologous to  $t^{n+k+2}w_k$  but  $F^* = t^* Id$ . Thus there exists a unique element  $\gamma_k \in V_{k+1}^{n+1}$  such that  $d\gamma_k = w_k$ . Let  $\gamma_0 = \gamma - \sum_{k \geq 1} \gamma_k$ . Then  $\gamma_0 \in V_1^{n+1}$  and  $\gamma$  can be written uniquely

as  $\gamma = \sum_{k \geq 0} \gamma_k$  ( $\gamma_k \in V_{k+1}^{n+1}$ ). Thus we have a direct sum decomposition  $N_{n+1} = \bigoplus_{k \geq 1} V_k^{n+1}$ . Using

$$(I(\bigoplus_{m=0}^{m=n+1} N_m))^{n+1} = (I(\bigoplus_{m=0}^{m=n} N_m))^{n+1} + N_{n+1} = \sum_{k \geq 1} W_{k,n}^{n+1} + \sum_{k \geq 1} V_k^{n+1},$$

if we put  $W_{k,n+1}^{n+1} = V_k^{n+1} + W_{k,n}^{n+1}$  ( $k \geq 0$ ), we get a decomposition

$$m^{n+1}(K)(n+1) = \bigoplus_{k \geq 0} W_{k,n+1}^{n+1}.$$

Finally we show that  $F(v_k) - t^{n+k+1}v_k$  is exact for any  $v_k \in V_k^{n+1}$ . As  $dF(v_k) = F(dv_k)$ ,  $F(v_k) \in W_{k,n+1}$ , put  $F(v_k) = v_0 + w$ , where  $v_0 \in V_k^{n+1}$ ,  $w \in W_{k,n}^{n+1}$ . By hypothesis,

$$d(F(v_k) - t^{n+k+1}v_k) = F(dv_k) - t^{n+k+1}dv_k = 0.$$

Thus  $F(v_k) - t^{n+k+1}v_k$  is closed. Therefore two elements,  $v_0 - t^{n+k+1}v_k$  ( $\in V_k^{n+1}$ ) and  $w$  ( $\in W_{k,n}^{n+1}$ ), are both closed and from the assumption that  $d$  is injective on  $V_k^{n+1}$  and  $F^* = t^*Id$ , we can obtain that  $v_0 = t^{i+k+1}v_k$  and that  $w$  is exact. Thus  $F(w^i) - t^{i+k}w^i$  is exact for any  $w^i \in W_{k,n+1}^i$  and the proof is completed.

q. e. d.

By Theorem I in [2], there exists a bijection between the set of homotopy classes of maps of localized spaces at 0 and the set of homotopy classes of D.G.A. maps of minimal models, and corresponding maps induce the same homomorphism on cohomology. We use the same notations for corresponding maps. Let  $K$  be a f.c. Then for any rational number  $r$ , there exists a map  $F_r : K_{(0)} \rightarrow K_{(0)}$  such that  $F_r^* = r^*Id$  uniquely up to homotopy. Especially, since the  $i$ -skeleton of  $K$  is a f.c., there exists a map  $F_r^i : K_{(0)}^i \rightarrow K_{(0)}^i$  such that  $F_r^i = r^*Id$  uniquely up to homotopy.

LEMMA 3.4. *The following diagram is homotopically commutative.*

$$\begin{array}{ccc} K_{(0)} & \xrightarrow{\quad} & K_{(0)} \\ j \uparrow & F_r & j \uparrow \\ K_{(0)}^i & \xrightarrow{F_r^i} & K_{(0)}^i \end{array}$$

where  $j$  is the inclusion.

PROOF. Applying Theorem S<sub>2</sub> to the diagram :

$$\begin{array}{ccc} m^*(K) & \xrightarrow{\quad} & m^*(K^i) \\ j \circ F_r & & F_r^i \uparrow \\ & & m^*(K^i) \end{array}$$

then we have a map  $\nu : K_{(0)}^i \rightarrow K_{(0)}$  such that  $F_r^i \circ \nu \sim j \circ F_r$  uniquely up to homotopy. By considering  $r^* \circ \nu^* = r^* \circ j^*$  on cohomology we have  $\nu^* = j^*$ . Hence,

from the uniqueness of Corollary  $S_2$ , we obtain that  $\nu$  is homotopic to  $j$ .  
 q. e. d.

§ 4. A proof of Theorem.

Now we give a proof of the theorem.  $(2) \Rightarrow (3)$  is trivial, and  $(1) \Rightarrow (3)$  is immediate from Corollary  $S_2$ . In the proof of Lemma 3.3 we showed that any closed element in  $I(\bigoplus N_m)$  is exact if (3) is assumed. Hence by Theorem  $S_1$ ,  $(3) \Rightarrow (1)$  is proved.

We prove  $(1) \Rightarrow (2)$  by using induction on the dimension of  $K$ . Let  $K$  be a f.c. of dimension  $n+1$ . Assume that for any integer  $r$ , there exists a multiple  $s$  of  $r$  and a map  $f_s^n: K^n \rightarrow K^n$  such that  $f_s^{n*} = s^* Id$ . From the equivalence  $(3) \Leftrightarrow (1)$ , there exists  $F_s: K_{(0)} \rightarrow K_{(0)}$  such that  $F_s^* = s^* Id$ , and then  $j \circ F_s$  and  $f_{s(0)}^n \circ j$  are homotopic by Lemma 3.4. Consider the following diagram:

$$\begin{array}{ccccccc} \pi_{n+1}(K_{(0)}^n) & \longrightarrow & \pi_{n+1}(K_{(0)}) & \longrightarrow & \pi_{n+1}(K_{(0)}, K_{(0)}^n) & \longrightarrow & \pi_n(K_{(0)}^n) \longrightarrow \\ \downarrow & & \downarrow H & & \downarrow H & & \downarrow H \\ 0 & \longrightarrow & H_{n+1}(K_{(0)}) & \longrightarrow & H_{n+1}(K_{(0)}, K_{(0)}^n) & \longrightarrow & H_n(K_{(0)}^n) \longrightarrow \end{array}$$

where the vertical homomorphisms are Hurewicz homomorphism.

Let  $V = \{x \in H_{n+1}(K_{(0)}, K_{(0)}^n) \mid (F_s)_*(x) = s^n x\}$ . For any  $z$  of  $H_{n+1}(K_{(0)}, K_{(0)}^n)$  we have

$$\partial_*((F_s)_*(z) - s^n z) = (F_s)_*(\partial_* z) - s^n \partial_* z = 0 \quad (\text{by Lemma 3.4}).$$

Then, since  $H_{n+1}(K_{(0)})$  is a  $Q$ -vector space, we have

$$(F_s)_*(z) - s^n z = s^n(s-1)y \quad (y \in \text{Im } \sigma).$$

Since we have that

$$(F_s)_*(z-y) - s^n(z-y) = (F_s)_*z - s^n z - s^n(s-1)y = 0$$

and since  $\text{Im } \sigma \cap V = 0$  is trivial, the following decomposition is obtained.

$$(1) \quad H_{n+1}(K_{(0)}, K_{(0)}^n) = \text{Im } \sigma \oplus V.$$

Let  $A_{(0)} = H^{-1}(V)$  and  $B_{(0)} = H^{-1}(\text{Im } \sigma)$ . Since Hurewicz homomorphism is natural for maps, Lemma 3.4 shows

$$(2) \quad (F_s)_* \mid \partial_* A_{(0)} = s^n Id, \quad (F_s)_* \mid \partial_* B_{(0)} = s^{n+1} Id.$$

Let  $l: K \rightarrow K_{(0)}$  be a localization map and let

$$\begin{aligned} A &= \{a \in \pi_{n+1}(K, K^n) \mid l_*(a) \in A_{(0)}\} \\ B &= \{b \in \pi_{n+1}(K, K^n) \mid l_*(b) \in B_{(0)}\}, \end{aligned}$$

then we have from (1) that

$$\pi_{n+1}(K, K^n) = A \oplus B.$$

Let  $\{a_k\}$  and  $\{b_j\}$  be basis of  $A$  and  $B$  respectively. Let  $\bar{K}$  be the complex which is obtained from  $K^n$  by attaching  $(n+1)$ -cells with  $\{\partial_* a_k\}$  and  $\{\partial_* b_k\}$ .

Now we assert

$\bar{K}$  is homotopically equivalent to  $K$ .

In fact, define an isomorphism  $\theta: \pi_{n+1}(K, K^n) \rightarrow \pi_{n+1}(\bar{K}, K^n)$  by  $\theta(a_k) = \bar{a}_k$ ,  $\theta(b_j) = \bar{b}_j$ , where  $\bar{a}_k, \bar{b}_j$  denote characteristic maps for cells of  $\bar{K}$  attached by  $\partial_* a_k, \partial_* b_j$  respectively. Then the following diagram is commutative:

$$\begin{array}{ccc} \pi_{n+1}(K, K^n) & \xrightarrow{\theta} & \pi_{n+1}(\bar{K}, K^n) \\ \partial_* \searrow & & \swarrow \partial_* \\ & \pi_n(K^n) & \end{array}$$

Let  $K = K^n \cup \{e_k^{n+1}\}$  and let  $g_k$  be the attaching map for the cell  $e_k^{n+1}$  and  $f'_k: (D^{n+1}, S^n) \rightarrow (\bar{K}, K^n)$  the characteristic map, such that  $f'_k|_{S^n} = g_k|_{S^n}$ . If we define a map  $h: K \rightarrow \bar{K}$  by

$$h|_{K^n} = Id \quad \text{and} \quad hg_k(x) = f'_k(x) \quad x \in D^{n+1},$$

clearly  $h$  is a homotopy equivalence.

Thus we may regard  $K$  as  $\bar{K}$ . Let  $\partial_* a_k = \varphi_k$  and  $\partial_* b_j = \psi_j (\in \pi_n(K^n))$ , then we have

$$f_s^n \circ \varphi_k = s^n \varphi_k + \text{torsion}, \quad f_s^n \circ \psi_j = s^n \psi_j + \text{torsion}.$$

By assumption for complexes of dimension  $n$ , there exist an integer  $t$  and a map  $f_t^n: K^n \rightarrow K^n$  such that  $f_{t(s)}^n = F_t^n$ , where  $t$  is a multiple of  $s$  and the order of torsion of  $\pi_n(K^n)$ . An infinite telescope

$$K^n \xrightarrow{f_t^n} K^n \xrightarrow{f_t^n} K^n \xrightarrow{\dots}$$

gives a localization of  $K^n$  at primes which is prime to  $t$ . Therefore there exists an integer  $m$  such that

$$(f_t^n)^m \circ \varphi_k = t^{nm} \varphi_k, \quad (f_t^n)^m \circ \psi_j = t^{m(n+1)} \psi_j.$$

Thus we have an extension of  $(f_t^n)^m, f_{tm}: K \rightarrow K$  such that

$$(f_{tm})^* = (t^m)^* Id: H_*(K; Z) \rightarrow H_*(K; Z).$$

Since  $(1) \Rightarrow (2)$  is trivial in the case of  $\dim K = 1$  the proof of  $(1) \Rightarrow (2)$  is completed.

Next we prove corollaries. Corollary 1 is obtained from the fact that the condition (b) of Theorem 2-1 in [4] is satisfied by (2) of Theorem. From [2] and Lemma 2-6 in [5], the minimal model of  $H^*(K; Q)$  with  $d=0$  is realized by a simply connected finite CW-complex  $\tilde{K}$ . Then  $\tilde{K}$  is 0-universal by Corollary 1. This shows Corollary 2.

The following corollary is easily obtained from (3) of Theorem.

COROLLARY 3. (1)  $\forall K_i$  is a f. c.  $\Leftrightarrow$  Each  $K_i$  is a f. c..

(2)  $\prod K_i$  is a f. c.  $\Leftrightarrow$  Each  $K_i$  is a f. c..

(3)  $K_i$  is a f. c. ( $i=1, 2$ )  $\Rightarrow K_1 \wedge K_2$  is a f. c..

(4) Let  $K_i$  be a Poincaré complex of dimension  $\geq 3$  and a f. c. ( $i=1, 2$ ). Then connected sum  $K_1 \# K_2$  is a f. c.. (See [7])

#### Some remarks.

(1) The converse of Corollary 1 is false. For example, a complex  $L = S^3 \vee S^3 \cup_{\varphi} e^8$ ,  $\varphi = [l_3, [l_3, l_3]]$  is 0-universal from Theorem 3-2 in [3], but is not f. c., because  $L$  has no self map  $f$  inducing  $f^* = r^* Id$  on the cohomology.

(2) In the theorem, (3)  $\Leftrightarrow$  (1) holds for infinite complexes. But (1)  $\Rightarrow$  (2) does not hold in such generality. For example,  $QP^\infty$  is a f. c., but there is no map which multiples elements of a 4-dimensional cohomology an even number of times.

(3) Let  $K$  be a simply connected finite complex, then the following conditions are equivalent.

(i)  $K$  is a f. c.

(ii) For any ring homomorphism  $\rho : H^*(K; Q) \rightarrow H^*(K; Q)$ , there exists an integer  $r$  and a map  $f : K \rightarrow K$  such that  $f^* = r^* \rho$ .

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