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On the values at rational integers of the p-adic Dirichlet L functions

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Introduction.

Let χ be a primitive Dirichlet character with conductor f_χ, and $L(s, \chi)$ be the Dirichlet L series associated with χ. For any prime number p, we denote by $L_p(s, \chi)$ the p-adic L function introduced by Kubota and Leopoldt [**1**]. We fix an embedding ι of the algebraic closure $\bar{\boldsymbol{Q}}(\subset \boldsymbol{C})$ into $\bar{\boldsymbol{Q}}_p$ once for all. By this ι, we identify any formal power series $\sum_{n=0}^{\infty} a_n X^n \in \bar{\boldsymbol{Q}}[[X]] \subset \boldsymbol{C}[[X]]$ (resp. any number $a \in \bar{\boldsymbol{Q}}$) with $\sum_{n=0}^{\infty} \iota(a_n) X^n \in \bar{\boldsymbol{Q}}_p[[X]]$ (resp. $\iota(a) \in \bar{\boldsymbol{Q}}_p$). We assume that all the Dirichlet characters we consider are primitive.

In this paper, firstly, we present a formula for the values of $L_p(s, \chi)$ at positive integers. To simplify the description of the main result, we assume that p is an odd prime number (for the case of $p=2$, see Theorem A of this paper) and that f_χ is neither 1 nor p. For a fixed prime number p, let ω be the Dirichlet character defined by $\omega(x) \equiv x \bmod p$ for $x \in \boldsymbol{Z}$. We set $\chi_j = \chi \cdot \omega^{-j}$ and

$$B(X, j) = \frac{\tau(\chi_j)}{f_{\chi_j}} \sum_{m=1}^{f_{\chi_j}} \bar{\chi}_j(m) \log\left(1 + \frac{X}{1-\exp(2\pi\sqrt{-1}\, m/f_{\chi_j})}\right) \in \bar{\boldsymbol{Q}}_p[[X]],$$

$$\left(\text{viz. } \frac{\tau(\chi_j)}{f_{\chi_j}} \sum_{m=1}^{f_{\chi_j}} \bar{\chi}_j(m) \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \left(\frac{X}{1-\exp(2\pi\sqrt{-1}\, m/f_{\chi_j})}\right)^k\right)$$

for $j = 0, 1, 2, \cdots, p-2$,

where f_{χ_j} and $\tau(\chi_j)$ denote the conductor of χ_j and the Gaussian sum of $\chi_j\left(= \sum_{t=1}^{f_{\chi_j}} \chi_j(t) \exp(2\pi\sqrt{-1}\, t/f_{\chi_j})\right)$ respectively. Further we denote by S the formal integral operator acting on a certain subspace Q_K of formal power series with coefficients in a finite extesion field K of $\boldsymbol{Q}_p$, given by

$$S:\ \sum_{n=0}^{\infty} a_n X^n \longmapsto \sum_{n=0}^{\infty} \frac{a_n}{n+1} X^{n+1}.$$

We also denote by $(1+X)^{-1}$ the operator on the space Q_K, given by

$$\sum_{n=0}^{\infty} a_n X^n \longmapsto \Big(\sum_{n=0}^{\infty}(-X)^n\Big)\Big(\sum_{n=0}^{\infty} a_n X^n\Big).$$

Then we put

$$B_{1+j-(p-1)n}(X, j)=(S\circ(1+X)^{-1})^{(p-1)n-j}(B(X, j))$$

for each rational integer n with $(p-1)n-j\geqq 0$. The following result may be regarded as an answer to the question raised by Iwasawa ([2] p. 61) "It is an interesting open problem to find similar expressions for the values $L_p(n, \chi)$ $n\geqq 2$."

THEOREM. *Notations being as above, we obtain;*

$$L_p(1-j+(p-1)n, \chi)=\frac{1}{p}\sum_{\xi} B_{1+j-(p-1)n}(\xi-1, j)$$

where the summation with respect to ξ is over all p-th roots of unity. (Each power series $B_{1+j-(p-1)n}(X, j)$ converges at $X=\xi-1$ under the p-adic topology.)

We note that the value of $L_p(1-j-(p-1)n, \chi)$ for any $j+(p-1)n\geqq 0$ with $n\in \boldsymbol{Z}$ is also given by

(I), $$L_p(1-j-(p-1)n, \chi)=-B_{1+j+(p-1)n}(0, j)+\frac{1}{p}\sum_{\xi} B_{1+j+(p-1)n}(\xi-1, j),$$

where we put

$$B_{1+j+(p-1)n}(X, j)=\Big((1+X)\frac{d}{dX}\Big)^{j+(p-1)n} B(X, j).$$

The value of (I) is equal to $(B_{(p-1)n+j,\chi_j}/((p-1)n+j))(p^{(p-1)n+j-1}\chi_j(p)-1)$, (see e.g. [2]). We also give a formula for $L_p(n, \chi)$ when f_χ is equal to 1 or p, (viz. Theorem B). Lemma 4 and Proposition 1 play important roles in the proof of the above theorems. It should be mentioned that a "real analogue" of Theorems A and B is available (see Proposition 2 in § 1).

Secondly we study the p-adic interpolated function $L_p(s)$ of Dirichlet series

(II), $$L(s)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_i=1}^{\infty}\frac{A(m_1, m_2, \cdots, m_i)}{(a_0+a_1 m_1+a_2 m_2+\cdots+a_i m_i)^s}$$

$$(a_0, a_1, \cdots, a_i\in \boldsymbol{Z},\ a_1, a_2, \cdots, a_i>0)$$

where Λ is a periodic function of $Z^{(i)}$ with respect to each variable m_j and takes its values in the field $\bar{Q}$. We apply our results in § 1 on the p-adic Γ transform to the series of the above (II) type. Then we obtain a formula (Theorem C) for $L_p(1)$. It is also possible to obtain an expression for $L_p(n)$ where n is a rational integer with $(p-1)|(n-1)$ as in § 1. We omit arguments, (which were given in [6]), on the meromorphy (or holomorphy) of the above $L_p(s)$ and on p-adic analytic interpolations of some arithmetical zeta functions as our investigations on the values at rational integers are carried out without them.

The author wishes to express his gratitude to Professor Kenkichi Iwasawa for his book [2] to which he owes very much and to his teacher Professor Takuro Shintani for carefully reading the manuscript and making valuable suggestions for its improvement. This paper is an extraction from a part of the author's master's thesis presented in January 1976 at the University of Tokyo.

REMARK 0. K. Shiratani and J. Diamond investigated the above problem of Iwasawa independently of us. After submitting this paper, we received their preprints Shiratani [14] and Diamond [8]. Diamond's expressions for $L_p(n, \chi)$ are different from both Shiratani's and ours. Shiratani's approach is ingeneous and not the same as ours. He treats only Dirichlet L functions. But his expressions for $L_p(n, \chi)$ are similar to ours. Our method is applicable to any p-adic function expressed by Leopoldt's p-adic Γ-transform. Furthermore we show analogy between $L(n, \chi)$ and $L_p(n, \chi)$.

Y. Morita constructed p-adic analytic analogues for Γ function and series of the above (II) type in the case $i=1$ in Morita [4]. We generalized his method to the case of any integer i in Hatada [6]. K. Shiratani also investigated this problem in the case $i=1$ in Shiratani [15]. These are also discussed under a certain condition in P. Cassou-Noguès [7], (see Remark 4 below).

Notations.

$\boldsymbol{Z}$: rational integers. $\boldsymbol{N}$: positive rational integers. $\boldsymbol{Q}$: rational numbers. $\boldsymbol{R}$: real numbers. $\boldsymbol{C}$: complex numbers. Let p be a rational prime. $\boldsymbol{Q}_p$: p-adic numbers. $\boldsymbol{Z}_p$: the ring of integers in $\boldsymbol{Q}_p$. $\boldsymbol{Z}_p^{\times}$: the group of p-adic units in $\boldsymbol{Z}_p$. Ω_p: algebraic closure $\bar{\boldsymbol{Q}}_p$ of $\boldsymbol{Q}_p$. $|\cdot|$: the p-adic absolute value on Ω_p normalized so that $|p|=p^{-1}$. $\boldsymbol{V}$: the group $\{x\in\boldsymbol{Q}_p | x^{p-1}=1\}$ for $p\geqq 3$, $\{\pm 1\in\boldsymbol{Q}_2\}$ for $p=2$. $q=p$ when $p\geqq 3$ and $q=4$ when $p=2$. $e=p-1$ when $p\geqq 3$ and $e=2$ when $p=2$. Then $\boldsymbol{Z}_p^{\times}=\boldsymbol{V}\times(1+q\boldsymbol{Z}_p)$; and $a=\omega(a)\langle a\rangle$ where $\omega(a)$ (resp. $\langle a\rangle$) denotes the projection of a on $\boldsymbol{V}$ (resp. on $1+q\boldsymbol{Z}_p$). We set $\omega(a)=0$ for a with $(a, p)\neq 1$.

§ 1. On the Dirichlet L series.

Let K be a finite extension of $\boldsymbol{Q}_p$ in Ω_p, and C_K be the set of all continuous functions $\boldsymbol{Z}_p \rightarrow K$. Then C_K has a structure of commutative Banach algebra with the norm $\|f\|=\max_{s\in \boldsymbol{Z}_p}|f(s)|$. Let Q_K denote the set of all formal power series $A(X)=\sum_{n=0}^{+\infty} a_n X^n$, in $K[[X]]$ such that $|a_n n!| \rightarrow 0$ as $n \rightarrow +\infty$. For each $A \in Q_K$, put $\|A\|_{Q_K}=\sup_{n\geq 0}|n!\,a_n|$. Then Q_K becomes a Banach algebra over K with the norm $\|\cdot\|_{Q_K}$. The ring of polynomials $K[X]$ is contained in Q_K as an everywhere dense subalgebra of Q_K. Note that $\log(1+X)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}X^n \in Q_K$. We denote by ϕ the continuous map $\begin{array}{ccc}\boldsymbol{Z}_p\times \boldsymbol{Z}_p & \rightarrow & \boldsymbol{Z}_p \\ (x, s) & \mapsto & \phi(x, s)\end{array}$ defined as follows.

$$\begin{aligned} \phi(x, s)&=0 && \text{if } x\in p\boldsymbol{Z}_p \\ &=\langle x\rangle^s && \text{if } x\in \boldsymbol{Z}_p^{\times},\ p\geqq 3 \\ &=x^s && \text{if } x\in \boldsymbol{Z}_2^{\times},\ p=2. \end{aligned}$$

For each integer $n\geqq 0$, let $\gamma_n(s)=\sum_{i=0}^{n}(-1)^{n-1}\binom{n}{i}\phi(i, s)$, $s\in \boldsymbol{Z}_p$. Then, as is well known there exists a unique bounded linear map $\Gamma: Q_K \rightarrow C_K$ such that $\Gamma(X^n)=\gamma_n$ for all non negative integer n. It satisfies $\|\Gamma(A)\|_{C_K}\leqq \|A\|_{Q_K}$ for all $A\in Q_K$, and $\Gamma((1+X)^n)(s)=\phi(n, s)$ for all $n\geqq 0$. For $A\in Q_K$, put $A(e^t-1)=\sum_{n=0}^{+\infty}\delta_n(A)\frac{t^n}{n!}$. Then for each non negative integer n, $\delta_n: Q_K\rightarrow K$, $A\mapsto \delta_n(A)$ defines a K linear map with $|\delta_n(A)|\leqq \|A\|_{Q_K}$. We denote by $\frac{d}{dX}$ the K linear transform $Q_K\rightarrow Q_K$; defined by $\sum_{n=0}^{+\infty}a_n X^n \mapsto \sum_{n=1}^{+\infty} n a_n X^{n-1}$. Note that $\left\|\frac{dA}{dX}\right\|_{Q_K}\leqq \|A\|_{Q_K}$ holds for all A in Q_K. For details see [2]. The case $i=1$ of the following Lemma 1 is already given in Iwasawa [2] (Lemma 3, § 5).

LEMMA 1. *For $A\in Q_K$, $s\in 2\boldsymbol{Z}_p$, $i\in \boldsymbol{N}$, we have*

$$(1), \qquad \Gamma\circ\left((1+X)\frac{d}{dX}\right)^{i-1}(A)(s)=\lim_{k\rightarrow+\infty}\delta_{n_k+i-1}(A),$$

where n_k is any sequence of non negative integers such that $e|n_k$, and such that $n_k\rightarrow+\infty$, $|s-n_k|\rightarrow 0$, as $k\rightarrow+\infty$. (Since the integers $n\geqq 0$ with $e|n$ are everywhere dense in $2\boldsymbol{Z}_p$, such a sequence always exists for any given s in $2\boldsymbol{Z}_p$.)

PROOF. For a non negative integer m, put $A_m(X)=(1+X)^m$. Then $\lim_{k\to+\infty}\delta_{n_k+i-1}(A_m)=\lim_{k\to+\infty}m^{n_k+i-1}=m^{i-1}\lim_{k\to+\infty}m^{n_k}=m^{i-1}\lim_{k\to+\infty}\delta_{n_k}(A_m)=m^{i-1}\phi(m, s)$, by Lemma 3, §5 in Iwasawa [2]. On the other hand,

$$\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^{i-1}(A_m(X))(s)=\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^{i-2}(m(1+X)^m)(s)$$

$$=\Gamma(m^{i-1}A_m(X))(s)=m^{i-1}\Gamma(A_m(X))(s)=m^{i-1}\phi(m, s).$$

Hence $\lim_{k\to+\infty}\delta_{n_k+i-1}(A_m)=\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^{i-1}(A_m)(s)$ for all $m\geqq 0$. Thus (1) is proved for all polynomials in $K[X]$. Now given any $A(X)$ in Q_K and any $\varepsilon>0$, there exists $B(X)$ in $K[X]$ such that $\|A-B\|_{Q_K}<\varepsilon$. As Q_K is a Banach algebra, it is easy to see that $\Big\|\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^{i-1}(A)\Big\|_{C_K}\leqq\|A\|_{Q_K}$ for any $A\in Q_K$. Then $\Big\|\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^{i-1}(A)-\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^{i-1}(B)\Big\|_{C_K}$

$$=\Big\|\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^{i-1}(A-B)\Big\|_{C_K}\leqq\|A-B\|_{Q_K}<\varepsilon.$$

$$|\delta_{n_k+i-1}(A)-\delta_{n_k+i-1}(B)|_p=|\delta_{n_k+i-1}(A-B)|_p\leqq\|A-B\|_{Q_K}<\varepsilon.$$

Since the formula holds for B, it also holds for A.

Note that $(1+X)^{-1}=\sum_{l=0}^{+\infty}(-1)^l X^l\in\boldsymbol{Q}[[X]]\cap Q_K$, for any finite extension K of $\boldsymbol{Q}_p$. We denote by S the map:

$$K[[X]]\to K[[X]],\ \sum_{n=0}^{+\infty}a_nX^n\mapsto\sum_{n=0}^{+\infty}\frac{a_n}{n+1}X^{n+1}.$$

LEMMA 2.

(i) $$(S\circ(1+X)^{-1})^l(1)=\frac{1}{l!}(\log(1+X))^l$$

for all $l\in\boldsymbol{Z}_+=$ *the set of non negative integers.*

(ii) $$(S\circ(1+X)^{-1})^l(A_m(X))=\frac{1}{m^l}A_m(X)-\Big(\sum_{k=0}^{l-1}\frac{1}{k!\,m^{l-k}}(\log(1+X))^k\Big),$$

for all $(l, m)\in\boldsymbol{N}^2$ *where* $A_m(X)=(1+X)^m$. $\boldsymbol{N}=\{1, 2, 3, \cdots\}=$ *the set of natural numbers.*

PROOF. We prove by induction on l.

(i) When $l=0$, the formula is trivial. We assume $(S\circ(1+X)^{-1})^{l}(1)=\frac{1}{l!}(\log(1+X))^{l}$ for some $l\in \boldsymbol{Z}_{+}$. Let $S\circ(1+X)^{-1}$ operate on both sides of this expression. Then we have

$$(S\circ(1+X)^{-1})^{l+1}(1)=(S\circ(1+X)^{-1})\left(\frac{1}{l!}(\log(1+X))^{l}\right)$$

$$=\int_{0}^{X}\frac{1}{1+X}\frac{1}{l!}(\log(1+X))^{l}dX=\int_{0}^{\log(1+X)}e^{-t}\frac{1}{l!}t^{l}e^{t}dt$$

$$=\frac{1}{(l+1)!}(\log(1+X))^{l+1}.$$

(ii) When $l=1$, $(S\circ(1+X)^{-1})(A_{m}(X))=\int_{0}^{X}(1+X)^{m-1}dX=\frac{(1+X)^{m}}{m}-\frac{1}{m}$.
Assume that for some $l\in \boldsymbol{N}$,

$$(S\circ(1+X)^{-1})^{l}(A_{m}(X))=\frac{1}{m^{l}}A_{m}(X)-\left(\sum_{k=0}^{l-1}\frac{1}{k!\,m^{l-k}}(\log(1+X))^{k}\right).$$

Then, $(S\circ(1+X)^{-1})^{l+1}((1+X)^{m})$

$$=\frac{1}{m^{l}}(S\circ(1+X)^{-1})(A_{m}(X))-\left(\sum_{k=0}^{l-1}\frac{1}{m^{l-k}k!}S\circ(1+X)^{-1}(\log(1+X))^{k}\right)$$

$$=\frac{1}{m^{l}}\left(\frac{(1+X)^{m}}{m}-\frac{1}{m}\right)-\left(\sum_{k=0}^{l-1}\frac{1}{m^{l-k}}\int_{0}^{X}\frac{1}{1+X}\frac{(\log(1+X))^{k}}{k!}dX\right)$$

$$=m^{-l-1}(1+X)^{m}-\sum_{k=0}^{l}m^{-l-1+k}\frac{(\log(1+X))^{k}}{k!}.$$

The following lemma is a generalization of Lemma 1.

LEMMA 3. *For $A\in Q_{K}$, $s\in 2\boldsymbol{Z}_{p}$, $i\in \boldsymbol{N}$, we have*

$$[\Gamma\circ(S\circ(1+X)^{-1})^{i}(A)](s)=\lim_{k\to+\infty}\delta_{n_{k}-i}(A)$$

where the limit is taken over any sequence of integers as described in Lemma 1.

PROOF. For $A_{0}(X)=(1+X)^{0}=1$, $\lim_{k\to+\infty}\delta_{n_{k}-i}(A_{0}(X))=\lim_{k\to+\infty}0^{n_{k}-i}=0$. On the other hand,

$$\Gamma\circ(S\circ(1+X)^{-1})^i(A_0(X))(s)=\Gamma\Big(\frac{1}{i!}(\log(1+X))^i\Big)(s) \qquad \text{(Lemma 2)}$$

$$=\frac{1}{i!}\Gamma((\log(1+X))^i)(s)=\frac{1}{i!}\lim_{k\to+\infty}\delta_{n_k}((\log(1+X))^i) \qquad \text{(Lemma 1)}$$

$$=\frac{1}{i!}\lim_{n\to+\infty}\delta_n((\log(1+X))^i)=0.$$

For $A_m(X)=(1+X)^m$ $(m\geqq 1)$, $\lim_{k\to+\infty}\delta_{n_k-i}(A_m(X))=\lim_{k\to+\infty}\delta_{n_k-i}((1+X)^m)=\lim_{k\to+\infty}m^{n_k-i}$ $=m^{-i}\lim_{k\to+\infty}m^{n_k}=m^{-i}\phi(m, s)$. On the other hand,

$$\Gamma\circ(S\circ(1+X)^{-1})^i(A_m(X))(s)$$

$$=\Gamma\Big(\frac{1}{m^i}A_m(X)-\Big(\sum_{k=0}^{l-1}\frac{1}{k!\,m^{l-k}}(\log(1+X))^k\Big)\Big)(s)$$

$$=m^{-i}\Gamma(A_m(X))(s)-\Big(\sum_{k=0}^{l-1}\frac{1}{k!\,m^{l-k}}\Gamma((\log(1+X))^k)(s)\Big)$$

$$=m^{-i}\phi(m, s)-\Big(\sum_{k=0}^{l-1}\frac{1}{k!\,m^{l-k}}\lim_{t\to+\infty}\delta_{n_t}((\log(1+X))^k)\Big)$$

$$=m^{-i}\phi(m, s).$$

Hence $\lim_{k\to+\infty}\delta_{n_k-i}(A_m(X))=\Gamma\circ(S\circ(1+X)^{-1})^i(A_m(X))(s)$ for all $m\geqq 0$. The formula of the lemma is proved for all polynomials in $K[X]$. It is easy to see that the operator $S: Q_K\to Q_K$ is norm preserving and that $\|(1+X)^{-1}\|_{Q_K}$ $=\Big\|\sum_{l=0}^{+\infty}(-1)^l X^l\Big\|_{Q_K}=1$. Thus

$$\|\Gamma\circ(S\circ(1+X)^{-1})^i(A)\|_{C_K}\leqq\|(S\circ(1+X)^{-1})^i(A)\|_{Q_K}\leqq\|A\|_{Q_K}$$

for all $A\in Q_K$. The proof of the remaining part is similar to that of Lemma 1, namely, for $A\in Q_K$ and $B\in K[X]$, $\|\Gamma\circ(S\circ(1+X)^{-1})^i(A-B)\|_{C_K}\leqq\|A-B\|_{Q_K}$ and $|\delta_{n-i}(A-B)|\leqq\|A-B\|_{Q_K}$.

With the notation $\Big((1+X)\dfrac{d}{dX}\Big)^{-i}=(S\circ(1+X)^{-1})^{+i}$ for $i\in\boldsymbol{N}$=natural numbers, we conclude from Lemma 1 and Lemma 3 that

$$\lim_{k\to+\infty}\delta_{n_k+i}(A)=\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^i(A)(s) \quad \text{for} \quad A\in Q_K,\ s\in 2\boldsymbol{Z}_p,\ i\in\boldsymbol{Z},$$

where the limit is taken over any sequence of integers as described in Lemma 1.

REMARK 1. $\left((1+X)\frac{d}{dX}\right)^i A\in Q_K$ for $A\in Q_K$ and $i\in \boldsymbol{Z}$.

LEMMA 4. *For* $A\in Q_K$, $s\in 2\boldsymbol{Z}_p$, $m\in \boldsymbol{Z}$, *we have*

$$\Gamma\circ\left((1+X)\frac{d}{dX}\right)^{em}(A)(s)=\Gamma(A)(s+em).$$

PROOF. We have shown that

$$\lim_{k\to+\infty}\delta_{n_k+em}(A)=\Gamma\circ\left((1+X)\frac{d}{dX}\right)^{em}(A)(s),\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2)$$

where the limit is taken over any sequence of integers as described in Lemma 1. On the other hand,

$$\begin{cases} n\to s \text{ in } \boldsymbol{Q}_p & \underset{\text{equivalent}}{\Longleftrightarrow} \quad n+em\to s+em \text{ in } \boldsymbol{Q}_p. \\ e|n & \underset{\text{equivalent}}{\Longleftrightarrow} \quad e|(n+em). \\ n\to+\infty \text{ in } \boldsymbol{R} & \underset{\text{equivalent}}{\Longleftrightarrow} \quad n+em\to+\infty \text{ in } \boldsymbol{R}. \end{cases}$$

Note that if $s\in 2\boldsymbol{Z}_p$, $s+em\in 2\boldsymbol{Z}_p$.

Hence, $$\lim_{k\to+\infty}\delta_{n_k+em}(A)=\Gamma(A)(s+em).\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (3)$$

Lemma 4 follows from (2) and (3).

Let $L(s,\chi)$ be the Dirichlet series with a primitive character χ whose conductor is f_χ. We denote by χ_l (for any $l\in\boldsymbol{Z}$) the primitive character $\chi\cdot\omega^{-l}$. We fix an embedding of the field $\bar{\boldsymbol{Q}}:\bar{\boldsymbol{Q}}\to\Omega_p$.

PROPOSITION 1. *For any* $j\in\{0,1,\cdots,e-1\}$ *and any* $s\in(\delta_{1,j}+2\boldsymbol{Z}_p)\setminus\{0\}$, $\lim_{k\to+\infty}L(1-n_k,\chi)$ *in* Ω_p *exists, where the limit is taken over any sequence of integers* n_k, $k\geqq 0$, *such that* $n_k\geqq 0$, $n_k\equiv j \bmod e$ *and such that* $n_k\to+\infty$, $|s-n_k|\to 0$, *as* $k\to+\infty$. $\delta_{1,j}$ *is Kronecker's delta. We write* $\lim_{k\to+\infty}L(1-n_k,\chi)=L_p(1-s,\chi,j)$.

PROOF. Put $L_p(s,\chi\cdot\omega^j)$ for $L_p(s,\chi,j)$. For any natural number n, $L_p(1-n,\chi\cdot\omega^j)=(1-\chi\cdot\omega^j\cdot\omega^{-n}(p)p^{n-1})L(1-n,\chi\cdot\omega^j\cdot\omega^{-n})=(1-\chi_{n-j}(p)p^{n-1})L(1-n,\chi_{n-j})=-(1-\chi_{n-j}(p)p^{n-1})\frac{B_{n,\chi_{n-j}}}{n}$. Using $\lim_{n\to+\infty}p^n=0$, we get for any $s\in\boldsymbol{Z}_p\setminus\{0\}$, $\lim_{k\to+\infty}L(1-u_k,\chi_{u_k-j})=L_p(1-s,\chi_{-j})$ where the limit is taken over any sequence of integers u_k, $k\geqq 0$ such that $u_k\geqq 0$, $u_k\to+\infty$, $|s-u_k|\to 0$ as $k\to+\infty$. In particular $L_p(1-s,\chi_{-j})=\lim_{k\to+\infty}L(1-n_k,\chi)$ for any $s\in(\delta_{1,j}+2\boldsymbol{Z}_p)\setminus\{0\}$.

It follows from Proposition 1 that $L_p(s, \chi, j)$ is continued to the open set $\{s\in\Omega_p \mid |s|_p<p^{(p-2)/(p-1)}\}$ if $p\geqq 3$, $\{s\in\Omega_2 \mid |s|_2<2\}$ if $p=2$ as a meromorphic function with no poles (when χ_{-j} is non trivial character) or with only one simple pole at $s=1$ (when $\chi=\omega^j$) (Theorem 2, p. 29 in [2]), and that $L_p(s,\chi_j,j)=L_p(s,\chi)$ for any j. Now let p be any rational prime. We use the following notations.

$$\zeta_{f_{\chi_j}}=\exp(2\pi\sqrt{-1}/f_{\chi_j}),\quad \tau(\chi_j)=\sum_{m=1}^{f_{\chi_j}}\chi_j(m)\zeta_{f_{\chi_j}}^m,$$

$$B(X, j)=(\tau(\chi_j)/f_{\chi_j})\sum_{m=1}^{f_{\chi_j}}\bar{\chi}_j(m)\log\left(1+\frac{X}{1-\zeta_{f_{\chi_j}}^m}\right)$$

$$\text{for}\quad j\in\{0, 1, \cdots, e-1\},$$

$$B_{1+j-en}(X, j)=(S\circ(1+X)^{-1})^{en-j}B(X, j)\qquad \text{for}\quad n\in\{n\in\boldsymbol{Z}\mid en-j\geqq 0\},$$

$$B_{1+j+en}(X, j)=\left((1+X)\frac{d}{dX}\right)^{en+j}B(X, j)\qquad \text{for}\quad n\in\{n\in\boldsymbol{Z}\mid en+j\geqq 0\}.$$

THEOREM A. *Assume f_χ is neither 1 nor p. Then*

(i) $$L_p(1-j+en, \chi)=p^{-1}\sum_{\xi}B_{1+j-en}(\xi-1, j)\qquad \textit{for}\quad n\in\{n\in\boldsymbol{Z}\mid en-j\geqq 0\}.$$

(ii) $$L_p(1-j-en, \chi)=-B_{1+j+en}(0, j)+\frac{1}{p}\sum_{\xi}B_{1+j+en}(\xi-1, j)$$

$$\textit{for}\quad n\in\{n\in\boldsymbol{Z}\mid j+en\geqq 0\}.$$

Here ξ ranges over all p-th roots of unity.

PROOF. We fix a finite extension K of $\boldsymbol{Q}_p$ in Ω_p which contains all $\zeta_{f_{\chi_j}}$ and $\chi_j(a)$, for $a\in\boldsymbol{Z}$ and $j\in\{0, 1, \cdots, e-1\}$. We have assumed $f_\chi\neq 1$, p so that $f_{\chi_j}\neq 1$, p for all j. Now we fix any j. The following fact is well known.

$$|1-\zeta_f|_p=\begin{cases}1 & \text{if}\quad f\times p^{-\operatorname{ord}_p f}\gneqq 1,\\ p^{-1/(p-1)p^{r-1}} & \text{if}\quad f=p^r,\ r\geqq 1.\end{cases}$$

Therefore, the formal power series $\log\left(1+\frac{X}{1-\zeta_{f_{\chi_j}}{}^k}\right)=\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1}{n}\left(\frac{X}{1-\zeta_{f_{\chi_j}}^k}\right)^n$ is an element of Q_K if $(k, f_{\chi_j})=1$. From this fact we know that $B(X, j)$ is also an element of Q_K.

Now

$$B'(X, j)=\frac{d}{dX}B(X, j)=\frac{\tau(\chi_j)}{f_{\chi_j}}\sum_{\substack{k=1\\(k,f_{\chi_j})=1}}^{f_{\chi_j}}\overline{\chi_j}(k)\frac{1}{X+1-\zeta_{f_{\chi_j}}^k}.$$

Thus,

$$te^tB'(e^t-1, j)=te^t\frac{\sum_{v=1}^{f_{\chi_j}}\chi_j(v)e^{(v-1)t}}{e^{ft}-1}=\sum_{n=0}^{\infty}B_{n,\chi_j}\frac{t^n}{n!}$$

where B_{n,χ_j} is a generalized Bernoulli number introduced by Leopoldt. Set $D_1=(\log(1+X))(1+X)\frac{d}{dX}$. The above calculation shows that $\delta_n(D_1B(X, j))=B_{n,\chi_j}=-nL(1-n, \chi_j)$ for all $n\geqq 1$. It is known that $\delta_n(D_1B(X, j))=n\delta_n(B(X, j))$. Hence, (*) $n\delta_n(B(X, j))=-nL(1-n, \chi_j)$ for all $n\geqq 1$. We take the limit of (*) over a sequence of integers as described in Proposition 1.

$$\lim_n n\delta_n(B(X, j))=s\lim_n \delta_{(n-j)+j}(B(X, j))$$

$$=s\Gamma\circ\left((1+X)\frac{d}{dX}\right)^j(B(X, j))(s-j), \qquad \text{(Lemma 1)}.$$

On the other hand,

$$\lim_n -nL(1-n, \chi_j)=-sL_p(1-s, \chi_j, j)=-sL_p(1-s, \chi).$$

If $s\neq 0$, this implies

$$\Gamma\circ\left((1+X)\frac{d}{dX}\right)^j(B(X, j))(s-j)=-L_p(1-s, \chi) \quad \text{for } 0\neq s\in\delta_{1,j}+2\boldsymbol{Z}_p.$$

However, the same holds also at $s=0$ if $0\in\delta_{1,j}+2\boldsymbol{Z}_p$ because both sides are continuous functions of $s\in\delta_{1,j}+2\boldsymbol{Z}_p$. Next we apply Lemma 4 for $A=\left((1+X)\frac{d}{dX}\right)^jB(X, j)$. Then, $\Gamma\circ\left((1+X)\frac{d}{dX}\right)^{em+j}(B(X, j))(s_1)$

$=\Gamma\circ\left((1+X)\frac{d}{dX}\right)^j(B(X, j))(s_1+em)$ for all $m\in\boldsymbol{Z}$, $s_1\in 2\boldsymbol{Z}_p$. Put $s_1=s-j$ for $s\in\delta_{1,j}+2\boldsymbol{Z}_p$. Then we get the following:

$\Gamma\circ\left((1+X)\frac{d}{dX}\right)^{em+j}(B(X, j))(s-j)=\Gamma\circ\left((1+X)\frac{d}{dX}\right)^j(B(X, j))(s-j+em)$
$=-L_p(1-s-em, \chi)$ for all $s\in\delta_{1,j}+2\boldsymbol{Z}_p$, all $m\in\boldsymbol{Z}$.

In particular, $\Gamma(B_{1+j+em}(X, j))(0)=-L_p(1-j-em, \chi)$. It follows from Lemma 5 of §5 in Iwasawa [2] that $-L_p(1-j-em, \chi)=B_{1+j+em}(0, j)-\frac{1}{p}\sum_{\xi}B_{1+j+em}(\xi-1, j)$ for all $m\in\boldsymbol{Z}$. ……(4). Since $B_{1+j+em}(0, j)=0$ if $j+em\leqq 0$, we obtain Theorem A. We note that the convergence of the series in the right hand side of (4) in Ω_p is guaranteed by lemma 5 of §5 in Iwasawa [2].

REMARK 2. The previously known expression for $L_p(1-n, \chi)$ is given by $L_p(1-n, \chi)=-(1-\chi_n(p)p^{n-1})\frac{B_{n,\chi_n}}{n}$ for all $n\in\boldsymbol{N}$ and all $f_\chi\in\boldsymbol{N}$. (See Theorem 2 in §3 in [2]).

Now we are going to derive a formula for $L_p(m, \chi)$ which is valid for any Dirichlet character χ. Let N be any natural number such that $(N, pf_\chi)=1$. We put

$$C(X, j)=\frac{\tau(\chi_j)}{f_{\chi_j}}\sum_{a=1}^{f_{\chi_j}}\sum_{\lambda\neq 1}\bar{\chi}_j(a)\log\left(1+\frac{X}{1-\lambda\zeta_{f_{\chi_j}}^{a}}\right)$$

for $j\in\{0, 1, \cdots, e-1\}$, where λ ranges over all N-th roots of unity in Ω_p except 1. We also set $C_{1+j-en}(X, j)=(S\circ(1+X)^{-1})^{en-j}C(X, j)=\left((1+X)\frac{d}{dX}\right)^{j-en}C(X, j)$.

THEOREM B. *Notations being as above,*

(i) $$(1-\chi_j(N)N^{j-en})L_p(1-j+en, \chi)=-\frac{1}{p}\sum_{\xi}C_{-en+j+1}(\xi-1, j)$$

for $n\in\{n\in\boldsymbol{Z}|en-j\geqq 0\}$ *and* $j\in\{0, 1, \cdots, e-1\}$.

(ii) $$(1-\chi_j(N)N^{j+en})L_p(1-j-en, \chi)$$

$$=-(1-\chi_j(N)N^{j+en})(1-\chi_j(p)p^{j+en+1})\frac{B_{j+en,\chi_j}}{j+en}$$

$$=C_{en+j+1}(0, j)-\frac{1}{p}\sum_{\xi}C_{en+j+1}(\xi-1, j)$$

for $n\in\{n\in\boldsymbol{Z}|j+en>0\}$ *and* $j\in\{0, 1, \cdots, e-1\}$.

Here ξ ranges over all p-th roots of unity. Especially, when f_χ is 1 or p, take such N as $N \bmod q$ is a generator of the multiplicative group $(\boldsymbol{Z}/q\boldsymbol{Z})^\times$ where either $q=p$ or $q=4$ (for $p=2$).

PROOF. We fix a finite extension K of $\boldsymbol{Q}_p$ in Ω_p which contains all λ, $\zeta_{f_{\chi_j}}$ and $\chi(a)$, for $a\in\boldsymbol{Z}$ and $j\in\{0, 1, \cdots, e-1\}$. For a fixed j, we have (see p. 56 of Iwasawa [2]) $n\delta_n(C(X, j))=\delta_n(D_1C(X, j))=(\chi_j(N)N^n-1)B_{n,\chi_j}$ for all non negative integers n. Hence (**) $n\delta_n(C(X, j))=-n(\chi_j(N)N^n-1)L(1-n, \chi_j)$ for $n\in\boldsymbol{N}$. Applying the same argument in the proof of Theorem A, we have for $s\in\delta_{1,j}+2\boldsymbol{Z}_p$,

$$\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^j(C(X, j))(s-j)=(1-\chi_j(N)N^j\langle N\rangle^{s-j})L_p(1-s, \chi_j, j) \quad\text{if } p\geqq 3,$$

$$\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^j(C(X, j))(s-j)=(1-\chi_j(N)N^s)L_2(1-s, \chi_j, j) \quad\text{if } p=2.$$

We put $A=\Big((1+X)\frac{d}{dX}\Big)^jC(X, j)$ in Lemma 4. Then we get

$$\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^{em+j}(C(X, j))(s_1)=\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^j(C(X, j))(s_1+em)$$

for $s_1\in 2\boldsymbol{Z}_p$, $m\in\boldsymbol{Z}$. Hence for $s\in\delta_{1,j}+2\boldsymbol{Z}_p$ and $m\in\boldsymbol{Z}$,

$$\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^{em+j}C(X, j)(s-j)=(1-\chi_j(N)N^{s+em})L_2(1-s-em, \chi_j, j) \quad\text{if } p=2,$$

$$\Gamma\circ\Big((1+X)\frac{d}{dX}\Big)^{em+j}C(X, j)(s-j)=(1-\chi_j(N)N^j\langle N\rangle^{s+em-j})L_p(1-s-em, \chi_j, j) \quad\text{if } p\geqq 3.$$

In particular, $\Gamma(C_{em+j+1}(X, j))(0)=(1-\chi_j(N)N^{em+j})L_p(1-j-em, \chi)$ $=C_{em+j+1}(0, j)-\frac{1}{p}\sum_{\xi}C_{em+j+1}(\xi-1, j)$, for all $m\in\boldsymbol{Z}$. Since $C_{em+j+1}(0, j)=0$ if $em+j\leqq 0$, and $L_p(1-j-em, \chi)=-(1-\chi_j(p)p^{j+em-1})\frac{B_{em+j,\chi_j}}{em+j}$ if $em+j>0$, Theorem B follows.

Leopoldt calculated $L_p(1, \chi)$ applying his method of Γ transform, which is introduced in Iwasawa [2]. His result is that for a non principal Dirichlet character χ,

$$L_p(1, \chi)=\frac{-1}{p(1-\chi(N))}\Big(\sum_{\xi}C_1(\xi-1, 0)\Big)$$

$$=-\frac{\tau(\chi)}{f_\chi}\Big(1-\frac{\chi(p)}{p}\Big)\sum_{a=1}^{f_\chi}\bar{\chi}(a)\log_p(1-\zeta_{f_\chi}^{-a}),$$

with the p-adic log function of which domain is expanded by Iwasawa to the whole $\Omega_p^\times$ (see p. 61 [2]). In [2], it says that if we compare this formula for $L(1, \chi)$;

$$L(1, \chi) = -\frac{\tau(\chi)}{f_\chi} \sum_{a=1}^{f_\chi} \bar{\chi}(a) \log(1-\zeta_{f_\chi}^{-a})$$

we find a remarkable similarity between these two. The assertions of Theorems A and B for the values of $L_p(n, \chi)$, $n \geqq 2$, are not so strong as the above Leopoldt's formula for $L_p(1, \chi)$ because ξ remains in the final expressions for $L_p(n, \chi)$, $n \geqq 2$. In the case of Leopoldt, the p-adic log function $\log_p$ is a group homomorphism: $\Omega_p^\times \to \Omega_p$, the expression $\sum_\xi \log_p\left(1+\frac{\xi-1}{1-\zeta_{f_\chi}^m}\right)$ is equal to $\log_p\left(\prod_\xi \left(1+\frac{\xi-1}{1-\zeta_{f_\chi}^m}\right)\right)$ and we can expel ξ from this expression. But $B_{1+j-en}(X, j)$ or $C_{1+j-en}(X, j)$ for $en-j \geqq 1$ does not consist of homomorphisms.

Now we fix a character χ whose conductor f_χ is neither 1 nor p. For simplicity, let ζ_j denote $\zeta_{f_{\chi_j}}$ and let f_j denote f_{χ_j}. We define the double sequence $\{b_k^{(l)}\}$, for $l \geqq 2$ and $k \geqq 1$, inductively as follows. (i) $b_1^{(2)} = -1$, $b_2^{(2)} = -1$. (ii) $b_k^{(l)} = 0$ if $k > l$. (iii) $b_1^{(l+1)} = -b_1^{(l)}$, $b_2^{(l+1)} = -b_1^{(l)} - 2b_2^{(l)}$, $b_3^{(l+1)} = -2b_2^{(l)} - 3b_3^{(l)}$, $\cdots$, $b_l^{(l+1)} = -(l-1)b_{l-1}^{(l)} - lb_l^{(l)}$, $b_{l+1}^{(l+1)} = (-1)lb_l^{(l)}$.

COROLLARY OF THEOREM A. *Notations being as above,*

$$L_p(1-j-em, \chi) = -(1-\chi_j(p)\, p^{j+em-1}) \frac{B_{em+j,\chi_j}}{em+j}$$

$$= \sum_{\substack{k=1 \\ (k, f_j)=1}}^{f_j} \sum_{r=1}^{em+j} \frac{\tau(\chi_j)}{f_j} b_r^{(em+j)} \bar{\chi}_j(k) \zeta_j^{rk} \left\{ \frac{-1}{(1-\zeta_j^k)^r} + \frac{1}{p} \sum_\xi \frac{1}{(\xi-\zeta_j^k)^r} \right\}, \quad \cdots\cdots (5)$$

for any $m \geqq 1$ and $j \in \{0, 1, \cdots, e-1\}$.

PROOF. The above sequence satisfies the following.

$$\left((1+X)\frac{d}{dX}\right)^l \log\left(1+\frac{X}{1-a}\right) = b_1^{(l)} \frac{a}{X+1-a} + b_2^{(l)} \frac{a^2}{(X+1-a)^2} + \cdots + \frac{b_l^{(l)} a^l}{(X+1-a)^l},$$

for $l \geqq 2$ and any $a \in K$ such that $a \neq 0$. If we replace χ by $\chi\omega^j$ in (5), we get the following expression.

$$-(1-\chi(p)\, p^{N-1}) \frac{B_{N,\chi}}{N} = \sum_{k=1}^{f_\chi} \sum_{r=1}^{N} \frac{\tau(\chi)}{f_\chi} b_r^{(N)} \bar{\chi}(k) \zeta_{f_\chi}^{rk} \left\{ \frac{-1}{(1-\zeta_{f_\chi}^k)^r} + \frac{1}{p} \sum_\xi \frac{1}{(\xi-\zeta_{f_\chi}^k)^r} \right\}$$

for all $N \geqq 2$. Here χ is any Dirichlet character such that $f_\chi \neq 1, p$.

The following proposition is a "real-analogue" of Theorems A and B.

PROPOSITION 2. *Let* $L(s, \chi)$ *be a Dirichlet* L *series associated with a non principal character* χ, *which takes values in* $\boldsymbol{C}$ *for all* $s\in\boldsymbol{C}$. *Then* $L(k, \chi)=G_k(0)$ *for all* $k\geqq 1$, *where* $G_l(X)=\int_{-1}^{X}(1+t)^{-1}G_{l-1}(t)\,dt$, *for* $l\geqq 2$, *and*

$$G_1(X)=-\frac{\tau(\chi)}{f_\chi}\sum_{m=1}^{f_\chi}\bar{\chi}(m)\log(-\zeta_{f_\chi}^{-m}X+1-\zeta_{f_\chi}^{-m}).$$

Here $-1\leqq X\leqq 0$ *and* log *is defined on* $\{s\in\boldsymbol{C}|\operatorname{Re}s>0\}$ *such that* $\log 1=0$.

PROOF. For $Y\in\boldsymbol{C}$ such that $|Y|_C<1$, put $F(X)=\sum_{n=1}^{+\infty}\chi(n)Y^n$, where $X=Y-1$. Then $\int_0^Y\frac{1}{Y}F(X)\,dY=\sum_{n=1}^{+\infty}\frac{\chi(n)}{n}Y^n$. Since χ is nonprincipal, it follows from Abel's Theorem

$$\lim_{Y\to 1-0}\int_0^Y\frac{1}{Y}F(X)\,dY=\sum_{n=1}^{+\infty}\frac{\chi(n)}{n}=L(1, \chi).$$

On the other hand,

$$\frac{1}{Y}F(X)=\frac{\sum_{m=1}^{f_\chi}\chi(m)Y^{m-1}}{Y^{f_\chi}-1}=-\frac{\tau(\chi)}{f_\chi}\sum_{k=1}^{f_\chi}\frac{\bar{\chi}(k)}{Y-\zeta_{f_\chi}^k}.$$

Hence,

$$\begin{aligned}\int_0^Y\frac{1}{Y}F(X)\,dX&=-\frac{\tau(\chi)}{f_\chi}\sum_{k=1}^{f_\chi}\bar{\chi}(k)\int_0^Y\frac{dY}{Y-\zeta_{f_\chi}^k}\\&=-\frac{\tau(\chi)}{f_\chi}\sum_{k=1}^{f_\chi}\bar{\chi}(k)\log(-\zeta_{f_\chi}^{-k}X+1-\zeta_{f_\chi}^{-k}).\end{aligned}$$

Therefore,

$$\sum_{n=1}^{+\infty}\frac{\chi(n)Y^n}{n}=G_1(X)\qquad\text{for}\quad -1\leqq X<0.$$

Hence,

$$\sum_{n=1}^{+\infty}\frac{\chi(n)}{n^k}Y^n=G_k(X)\qquad\text{for}\quad -1\leqq X<0\quad\text{and}\quad k\geqq 1.$$

Abel's Theorem implies that,

$$G_k(0)=\lim_{x\to -0}G_k(X)=\sum_{n=1}^{+\infty}\frac{\chi(n)}{n^k}=L(k, \chi),$$

$$\text{for any}\quad k\in\boldsymbol{N}=\{1, 2, 3, \cdots\}.$$

We observe that the operator S and the map "$f\mapsto\frac{1}{p}\sum_{\xi}{}' f(\xi-1)$" in the p-adic case correspond respectively to the operator $\int_{-1}^{X}(\ )dX$ and the map "$f\mapsto f(0)$" in the ordinary case, and that the multiplication by $(1+X)^{-1}$ is common to both the cases.

§ **2.**

In this section we study Dirichlet series given by the following:

$$\text{(II)}\qquad L(s)=\sum_{m_1=0}^{+\infty}\sum_{m_2=0}^{+\infty}\cdots\sum_{m_i=0}^{+\infty}\frac{\Lambda(m_1, m_2, \cdots, m_i)}{(a_0+a_1m_1+\cdots+a_im_i)^s}$$

$$(a_0, a_1, \cdots, a_i\in \boldsymbol{Z},\ a_0, \cdots, a_i>0)$$

where Λ is a function of $\boldsymbol{Z}^{(i)}$ with suitable periods which takes values in the field of algebraic numbers. Our main interest is the application of Leopoldt's Γ transform to the series of the above type. The function $L(s)$ can be continued meromorphically to the whole complex plane with at most simple poles at $s=1, 2, \cdots, i$. And the value $L(1-n)$ for any $n\in N$ is in the field generated by $\Lambda(\boldsymbol{Z}\times\cdots\times\boldsymbol{Z})$ over $\boldsymbol{Q}$. Let p be any fixed prime number, and j be any integer such that $0\leqq j\leqq e-1$. Then there uniquely exists a p-adic meromorphic function $L_p(s, j)$ defined on the open set $\{s\in\Omega_p|\,|s|_p<p^{(p-2)/(p-1)}\}$ if $p\geqq 3$, $\{s\in\Omega_2|\,|s|_2<2\}$ if $p=2$, at most with simple poles at $s=1, 2, \cdots, i$, such that $L_p(1-s, j)=\lim_{k\to+\infty}L(1-n_k)$, for any $s\in(\delta_{1,j}+2\boldsymbol{Z}_p)\backslash\{0, -1, -2, \cdots, 1-i\}$, where the limit is taken over any sequence of integers as described in Proposition 1. The proof of the above which is based on Morita [**4**] and [**10**] is given in my master's thesis [**6**]. But the above statement is not necessary for the argument below if we only replace "holomorphic" by "continuous" in the sentence (C_1) of Theorem C.

We put $L_p(s)=L_p(s, 0)$. We denote by $s(m, l)$ for any $(m, l)\in\boldsymbol{N}\times\boldsymbol{N}$, the Stirling number defined by

$$m\,!\binom{X}{m}=X(X-1)\cdots(X-m+1)=\sum_{l=1}^{m}s(m, l)X^l.$$

The next lemma is easily proved by induction on m.

LEMMA 5. *For $A(X)\in K[[X]]$, $m\in\boldsymbol{N}$,*

$$e^{mt}\left[\left(\frac{d}{dX}\right)^m A(X)\right](e^t-1)=\left(\sum_{l=1}^{m}s(m, l)\left(\frac{d}{dt}\right)^l\right)A(e^t-1).$$

For every non negative integer m, let's denote by D_m the map:

$$Q_K \longrightarrow Q_K,\ A \longmapsto (1+X)^m (\log(1+X))^m \left(\frac{d}{dX}\right)^m A.$$

LEMMA 6. *For $A \in Q_K$, $m \in \boldsymbol{N}$, $s \in 2\boldsymbol{Z}_p$, we have*

$$\Gamma \circ \left((1+X)\frac{d}{dX}\right)^{m-1} \circ D_m(A)(s) = (s+m-1)(s+m-2) \times \cdots$$

$$\times (s+1) s \left(\sum_{l=1}^{m} s(m, l) \Gamma \circ \left((1+X)\frac{d}{dX}\right)^{l-1}(A)\right)(s).$$

PROOF.

$$[D_m(A)](e^t-1) = \left[(1+X)^m (\log(1+X))^m \left(\frac{d}{dX}\right)^m A(X)\right](e^t-1)$$

$$= e^{mt} t^m e^{-mt} \left(\sum_{l=1}^{m} s(m, l)\left(\frac{d}{dt}\right)^l\right) A(e^t-1)$$

$$= t^m \left(\sum_{l=1}^{m} s(m, l)\left(\frac{d}{dt}\right)^l\right)\left(\sum_{n=0}^{+\infty} \delta_n(A)\frac{t^n}{n!}\right)$$

$$= \sum_{n=0}^{+\infty}\left(\sum_{l=1}^{m} s(m, l)\delta_{n+l}(A)\right)\frac{t^{n+m}}{n!}.$$

Therefore, for any $n \geqq 0$,

$$\delta_{n+m}(D_m(A)) = (n+m)(n+m-1)\cdots(n+1)\sum_{l=1}^{m} s(m, l)\delta_{n+l}(A).$$

Then we take the limit of this equation over any sequence of integers as described in Lemma 1 substituting n_k for $n+1$, and apply Lemma 1. Lemma 6 follows.

We assume that $\Lambda(m_1, m_2, \cdots, m_i)$ has periods $f_1, f_2, \cdots, f_i$ with respect to each variable $m_1, m_2, \cdots, m_i$.

PROPOSITION 3.

$$L(1-n) = (-1)^i \frac{E_{n+i-1}}{n(n+1)\cdots(n+i-1)} \qquad \textit{for} \quad n \in \boldsymbol{N}.$$

$$\operatorname*{Res}_{s=k}(L(s)\,ds) = (-1)^{i+k} E_{i-k}\frac{1}{(i-k)!(k-1)!} \qquad \textit{for } k \textit{ with } \ 1 \leqq k \leqq i.$$

Here we put

$$\sum_{m_1=0}^{f_1-1}\sum_{m_2=0}^{f_2-1}\cdots\sum_{m_i=0}^{f_i-1}\Lambda(m_1, m_2, \cdots, m_i)e^{a_0 t}\prod_{u=1}^{i}\frac{te^{a_u m_u t}}{(e^{a_u f_u t}-1)} = \sum_{n=0}^{+\infty} E_n \frac{t^n}{n!}.$$

We omit the proof of this proposition.

In the remainings of this paper we consider the Dirichlet series given by the following:

$$L(s)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_i=1}^{\infty}\sum_{n_1=1}^{\infty}\cdots\sum_{n_j=1}^{\infty}\frac{\chi_1(m_1)\chi_2(m_2)\cdots\chi_i(m_i)\zeta_{g_1}^{n_1}\zeta_{g_2}^{n_2}\cdots\zeta_{g_j}^{n_j}}{(a+b_1m_1+\cdots+b_im_i+c_1n_1+\cdots+c_jn_j)^s}$$

where $\chi_1, \chi_2, \cdots, \chi_i$ are any multiplicative characters with conductors f_1, $f_2, \cdots, f_i$ respectively, each ζ_{g_l} is primitive g_l-th root of unity. We put $k=i+j$. For the above $L(s)$, the equation given in Proposition 3 is

$$e^{at}\left(\prod_{l=1}^{i} t\frac{\sum_{m_l=1}^{f_l}\chi_l(m_l)e^{b_l m_l t}}{e^{b_l f_l t}-1}\right)\left(\prod_{u=1}^{j}\frac{t\zeta_{g_u}e^{c_u t}}{\zeta_{g_u}e^{c_u t}-1}\right)=\sum_{m=0}^{\infty}E_m\frac{t^m}{m!}.$$

Let $K=\boldsymbol{Q}_p(\chi_1, \cdots, \chi_i, \zeta_{b_1f_1}, \cdots, \zeta_{b_if_i}, \zeta_{c_1g_1}, \cdots, \zeta_{c_jg_j})$. If each f_l and each g_u is neither 1 nor p, it is not hard to see that the following formal power series in $K[[X]]$ is an element of Q_K.

$$R(X)=Y^{a+b_1+\cdots+b_i-i}\left(\prod_{l=1}^{i}\frac{\sum_{m_l=1}^{f_l}\chi_l(m_l)Y^{b_l(m_l-1)}}{Y^{b_l m_l}-1}\right)\left(\prod_{u=1}^{j}\frac{\zeta_{g_u}Y^{c_u-1}}{\zeta_{g_u}Y^{c_u}-1}\right),$$

where we put $Y=1+X$.

We set $G(X)=S^{i+j}(R(X))$ where S is the formal integral operator such that

$$\sum_{n=0}^{\infty}a_nX^n\longmapsto\sum_{n=0}^{+\infty}\frac{a_n}{n+1}X^{n+1},$$

and $\quad G_m(X)=\left((1+X)\dfrac{d}{dX}\right)^{m-1}G(X) \quad$ for $\quad m\in\boldsymbol{Z}$.

THEOREM C. *If each f_l and each g_u is neither* 1 *nor p,*

(C$_1$) $L_p(s)$ *is holomorphic everywhere on* $2\boldsymbol{Z}_p$.

(C$_2$) $L_p(1)=\dfrac{1}{p}(-1)^{k+1}\sum_{m=1}^{k}\sum_{\xi}s(k, m)G_m(\xi-1)$,

where ξ ranges over all p-th roots of unity.

PROOF. It follows from Proposition 3 that

$$\delta_{n+k-1}(D_k(G))=(-1)^k n(n+1)\cdots(n+k-1)L(1-n) \qquad \text{for all} \quad n\in\boldsymbol{N}.$$

Then we take the limit of this equation over any sequence of integers as described in Lemma 1 substituting n_k for n. Apply Lemma 1 and Lemma 6. We get

$$s(s+1)\times\cdots\times(s+k-1)\sum_{l=1}^{k}s(k,l)\Gamma(G_l)(s)$$
$$=(-1)^k s(s+1)\times\cdots\times(s+k-1)L_p(1-s) \qquad \text{for} \quad s\in 2\boldsymbol{Z}_p.$$

By continuity we obtain

$$L_p(1-s)=(-1)^k\sum_{l=1}^{k}s(k,l)\Gamma(G_l)(s) \qquad \text{for} \quad s\in 2\boldsymbol{Z}_p.$$

In particular $L_p(1)=(-1)^k\sum_{l=1}^{k}s(k,l)\Gamma(G_l)(0)$. We note that $G_l(0)=0$ for $l\in\{1, 2, \cdots, k\}$.

We give a few examples of Theorem C. They show some similarity between $L(1)$ and $L_p(1)$.

EXAMPLE 1. On the Dirichlet series $L(s)=\sum_{n=1}^{\infty}\frac{\zeta^n}{n^s}$, where $\zeta=\exp\left(\frac{2\pi\sqrt{-1}a}{f}\right)$ and $(a,f)=1$.

Case	$L_p(1)$	$L(1)$
$f\neq 1, p$	$\frac{1}{p}\log_p(\zeta^p-1)-\log_p(\zeta-1)$	$-\log(1-\zeta)$
$f=p$	$L_p(s)$ has a simple pole at $s=1$ with residue $-\frac{1}{p}$.	
$f=1$	$L_p(s)=\zeta_p(s)$ has a simple pole at $s=1$ with residue $\left(1-\frac{1}{p}\right)$.	$\zeta(s)$ has a simple pole at $s=1$ with residue 1.

EXAMPLE 2. On the Dirichlet series $L(s)=\sum_{n=1}^{+\infty}\frac{\chi(n)}{(a+bn)^s}$, where $a, b\in\boldsymbol{N}$, χ is a Dirichlet character with conductor f which is neither 1 nor p. We put $\eta=\exp\left(\frac{2\pi\sqrt{-1}}{bf}\right)$.

$$L_p(1)=-\frac{\tau(\chi)}{bf}\sum_{\substack{k=1\\(k,f)=1}}^{f}\sum_{x=1}^{b}\bar{\chi}(k)\Big\{\sum_{i=0}^{a-1}\frac{\eta^{(k+fx)(a-1-i)}\delta(p,i+1)}{i+1}$$

$$+\eta^{a(k+fx)}\log_p(1-\eta^{k+fx})\Big\}$$

$$+\frac{\tau(\chi)}{pbf}\sum_{k=1}^{f}\sum_{x=1}^{b}\bar{\chi}(k)\,\eta^{(k+fx)a}\log_p(1-\eta^{p(k+fx)}),$$

where we put $\delta(p, i+1)=0$ if $p|(i+1)$, 1 if $p\not|(i+1)$.

$$L(1)=-\frac{\tau(\chi)}{bf}\sum_{k=1}^{f}\bar{\chi}(k)\sum_{x=1}^{b}\Big\{\sum_{i=0}^{a-1}\frac{\eta^{(k+fx)(a-1-i)}}{i+1}+\eta^{a(k+fx)}\log(1-\eta^{-k-fx})\Big\}.$$

EXAMPLE 3. On the Dirichlet series

$$L(s)=\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{\chi_1(m)\chi_2(n)}{(m+n)^s},$$

where χ_1, χ_2, are Dirichlet characters with conductors f_1, f_2 respectively which are neither 1 nor p.

(3.1) In case that $f_1\neq f_2$.

$$L_p(1)=\frac{\tau(\chi_1)\tau(\chi_2)}{f_1f_2}\sum_{\{*\}}\frac{\bar{\chi}_1(m_1)\,\bar{\chi}_2(m_2)}{(\zeta_{f_1}^{m_1}-\zeta_{f_2}^{m_2})}$$

$$\times\Big\{\zeta_{f_1}^{m_1}\log_p(1-\zeta_{f_1}^{m_1})-\frac{1}{p}\zeta_{f_1}^{m_1}\log_p(1-\zeta_{f_1}^{pm_1})$$

$$-\zeta_{f_2}^{m_2}\log_p(1-\zeta_{f_2}^{m_2})+\frac{1}{p}\zeta_{f_2}^{m_2}\log_p(1-\zeta_{f_2}^{pm_2})\Big\},$$

where $\{*\}=\{(m_1, m_2)\,|\,1\leqq m_1\leqq f_1,\ 1\leqq m_2\leqq f_2,\ \zeta_{f_1}^{m_1}\neq\zeta_{f_2}^{m_2}\}$.

$$L(1)=\frac{\tau(\chi_1)\tau(\chi_2)}{f_1f_2}\sum_{\{*\}}\frac{\bar{\chi}_1(m_1)\,\bar{\chi}_2(m_2)}{(\zeta_{f_1}^{m_1}-\zeta_{f_2}^{m_2})}\Big\{\zeta_{f_1}^{m_1}\log(1-\zeta_{f_1}^{-m_1})-\zeta_{f_2}^{m_2}\log(1-\zeta_{f_2}^{-m_2})\Big\}.$$

(3.2) In case $f_1=f_2=f$

$$L_p(1)=\frac{\tau(\chi_1)\tau(\chi_2)}{f^2}\Big[\sum_{m=1}^{f}\bar{\chi}_1(m)\,\bar{\chi}_2(m)\Big(\frac{1}{1-\zeta_f^{pm}}-\frac{1}{1-\zeta_f^{m}}$$

$$+\log_p(1-\zeta_f^{m})-\frac{1}{p}\log_p(1-\zeta_f^{pm})\Big)$$

$$+\sum_{\substack{1\leq m_1<m_2\leq f\\ 1\leq m_2<m_1\leq f}}\frac{\bar{\chi}_1(m_1)\,\bar{\chi}_2(m_2)}{(\zeta_f^{m_1}-\zeta_f^{m_2})}\Big(\zeta_f^{m_1}\log_p(1-\zeta_f^{m_1})$$

$$-\frac{1}{p}\zeta_f^{m_1}\log_p(1-\zeta_f^{pm_1})-\zeta_f^{m_2}\log_p(1-\zeta_f^{m_2})+\frac{1}{p}\zeta_f^{m_2}\log_p(1-\zeta_f^{pm_2})\Big)\Big].$$

$$L(1)=\frac{\tau(\chi_1)\tau(\chi_2)}{f^2}\Big[\sum_{m=1}^{f}\bar{\chi}_1(m)\,\bar{\chi}_2(m)\Big(\log(1-\zeta_f^{-m})-\frac{1}{1-\zeta_f^m}\Big)$$

$$+\sum_{\substack{1\leq m_1<m_2\leq f\\ 1\leq m_2<m_1\leq f}}\frac{\bar{\chi}_1(m_1)\,\bar{\chi}_2(m_2)}{(\zeta_f^{m_1}-\zeta_f^{m_2})}(\zeta_f^{m_1}\log(1-\zeta_f^{-m_1})-\zeta_f^{m_2}\log(1-\zeta_f^{-m_2}))\Big].$$

REMARK 3. We can get the expressions of $L_p(1+en)$ for $n\in\boldsymbol{Z}$ using Lemma 4.

REMARK 4. P. Cassou-Noguès constructed the p-adic meromorphic function $L_p(s, (1-i) \bmod e)$ for the series of (II) type given in §2 of this paper, in the case that $\Lambda(\cdot)\equiv 1$ and every $a_j (1\leqq j\leqq i)$ is an algebraic integer smaller than 1, independently of us. (See Cassou-Noguès [**7**]). In [**7**] there are other examples of p-adic functions. In our [**6**], we showed that $L_p(s, j)$ $(0\leqq j\leqq e-1)$ is meromorphic for any series of (II) type where $a_0, a_1, a_2, \cdots, a_i$ are arbitrary positive algebraic (over $\boldsymbol{Q}$) p-adic integers. This is a generalization of the p-adic analogue for the Hurwits L function in Y. Morita [**4**]. Y. Morita investigated a p-adic analogue for the Hurwits Lerch L function in Morita [**5**] and [**10**].

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