J. Math. Soc. Japan
Vol. 30, No. 4, 1978

Scattering theory for Schrödinger operators with long-range potentials, II, spectral and scattering theory

By Hitoshi KITADA

(Received May 4, 1976)
(Revised March 31, 1978)

This paper is a direct continuation of Part I [4] and deals with the spectral and scattering theory for Schrödinger operators with real long-range potentials. Throughout the paper, the same notations as in Part I will be used, and Theorems etc. given in Part I will be quoted as Theorem I. 5. 1 for theorems, as (I. 3. 9) for formulas, as [I. 1] for references, etc.

The present paper is divided into five sections. In §1 the summary of our main results concerning the scattering theory, that is, the completeness and invariance principle for modified wave operators, will be presented. Our assumption on the long-range potentials, which will be assumed throughout the paper, is slightly stronger than Hörmander's [I. 8] which was assumed to prove the existence of modified wave operators. §2 is assigned to developing the spectral theory for Schrödinger operators, which forms our another main result and will play an important role in establishing the results summarized in §1. In §2 the results of Y. Saitō [I. 27], [I. 28] will be used. §§3~4 are then devoted to proving the results presented in §1 applying the abstract framework given in Part I and using the result of §2. In §5, some related problems will be considered.

We remark here that except for developing spectral theory, we only need assume the same assumption as that of Hörmander [I. 8] (cf. footnotes 6), 10) and [5]).

Here the author wishes to express his sincere appreciation to Professor S. T. Kuroda for unceasing encouragement in the course of the preparation of this paper.

§1. Assumption and main results.

In this section, we summarize our main results concerning the scattering theory for Schrödinger operators

$$(1.1)\qquad \begin{cases} H_1=-\Delta=-\sum_{j=1}^{N}\partial^2/\partial x_j^2,\text{[1]} \\ H_2=H_1+U \end{cases}$$

defined in a Hilbert space $\mathfrak{H}=L^2(R^N)$, $N\geqq 1$, under the following assumption on the potential U.

ASSUMPTION 1.1. U is a maximal multiplication operator in $\mathfrak{H}$ defined by a real-valued function $U(x)=V(x)+V_S(x)$, where functions V and V_S satisfy the following conditions (L) and (S), respectively:

(L) V is a real-valued C^4 function on R^N and satisfies

$$(1.2)\qquad |\partial_x^\alpha V(x)|\leqq C(1+|x|)^{-m(|\alpha|)}$$

for any multi-index α with $|\alpha|\leqq 4$, where $C>0$; $m(k)=k+\varepsilon_0$, $0\leqq k\leqq 3$, $0<\varepsilon_0<1$, $m(1)+m(4)>5$; and $\partial_x=(\partial/\partial x_1, \cdots, \partial/\partial x_N)$.

(S) V_S is a real-valued Borel measurable function on R^N and satisfies

$$(1.3)\qquad |V_S(x)|\leqq C(1+|x|)^{-1-\varepsilon_0},$$

where constants C and ε_0 are the same as in (L).

Under this assumption, H_2 is a self-adjoint operator in $\mathfrak{H}$ with $\mathcal{D}(H_2)=\mathcal{D}(H_1)=H^2(R^N)$, where $H^2(R^N)$ denotes the Sobolev space of order two.

By Lemmas 3.2 and 3.3 of [I.8] we may replace V and V_S by another V and V_S which satisfy $U=V+V_S$ and the following assumption.

ASSUMPTION 1.1′. V_S satisfies (S) of Assumption 1.1, and V is a real-valued C^∞ function on R^N and satisfies

$$(1.2)'\qquad |\partial_x^\alpha V(x)|\leqq C_\alpha(1+|x|)^{-m(|\alpha|)}$$

for all α. Here $C_\alpha>0$ and

$$\begin{cases} m(k)=k+\varepsilon_0 & \text{for } 0\leqq k\leqq 3, \\ m(k)=\rho k+d & \text{for } k\geqq 4, \end{cases}$$

where $1/2<\rho<1$, $0<\varepsilon_0<\min(\rho-1/2, 1-\rho)$, $m(3)\leqq 3\rho+d$, $m(4)<4$, and $m(1)+m(4)>5$.

In order to formulate our main results, we record a theorem essentially due to Hörmander [I.8].

THEOREM 1.2. *Let Assumption* 1.1′ *be satisfied. Then there exists a real-valued function* $X(\xi, t)\in C^\infty((R^N-\{0\})\times R^1)$ *which satisfies* $X(\xi, 0)=0$ *for all*

1) H_1 is a self-adjoint realization of $-\Delta$ restricted to $C_0^\infty(R^N)$ in $\mathfrak{H}=L^2(R^N)$.

$\xi\in R^N-\{0\}$ and the following two conditions for any compact subset Ω of $R^N-\{0\}$:

i) *There is a positive constant T such that*

$$\partial_t X(\xi, t)=V(2t\xi+\partial_\xi X(\xi, t)) \tag{1.4}$$

for any $\xi\in\Omega$ and $|t|>T$, where $\partial_t=\partial/\partial t$, and $\partial_\xi=(\partial/\partial\xi_1, \cdots, \partial/\partial\xi_N)$.

ii) *For any multi-index α, there is a positive constant C such that*

$$\begin{cases} |\partial_\xi^\alpha W(\xi, t)|\leqq C(1+|t|)^{1+\mu(|\alpha|-1)}, \\ |\partial_\xi^\alpha X(\xi, t)|\leqq C(1+|t|)^{1+\mu(|\alpha|)-\varepsilon_0} \end{cases} \tag{1.5}$$

for any $\xi\in\Omega$ and $t\in R^1$, where $W(\xi, t)=t|\xi|^2+X(\xi, t)$ and $\mu(k)=\max(0, k+1-m(k+1))$ for $k\geqq-1$.

The proof of this theorem under our Assumption 1.1′ is easily reconstructed from that of Lemma 3.7 and Theorem 3.8 of Hörmander [I.8] so we shall omit the proof. From this theorem, we can deduce the following fundamental estimates, which will play a crucial role in the subsequent sections.

PROPOSITION 1.3. *Let Assumption 1.1′ be satisfied. Then the function $X(\xi, t)$ defined in Theorem 1.2 satisfies the following estimates: For any compact subset Ω of $R^N-\{0\}$ and any multi-indices α, β[2], there is a positive constant C such that*

$$\begin{cases} |\partial_t^\alpha\partial_\xi^\beta X(\xi, t)|\leqq C(1+|t|)^{1-|\alpha|-\varepsilon_1}, & |\alpha|+|\beta|\leqq 3, \\ |\partial_t^\alpha\partial_\xi^\beta X(\xi, t)|\leqq C(1+|t|)^{h'-\rho|\alpha|+(1-\rho)|\beta|}, & |\alpha|+|\beta|\geqq 3 \end{cases} \tag{1.6}$$

for any $\xi\in\Omega$ and $t\in R^1$. Here ε_1 and h' is defined by

$$\begin{cases} \varepsilon_1=m(4)+m(1)-5>0, \\ h'=2-\rho-d-\varepsilon_0<3\rho-2. \end{cases}$$

Moreover, when $\alpha\neq 0$, (1.6) holds with ε_1 and h' replaced by ε_0 and $h''=3\rho-2-\varepsilon_0$.

The deduction of this proposition from Theorem 1.2 is not difficult by induction so the proof is omitted.

DEFINITION 1.4. For any $t\in R^1$, define

$$X(t)=\mathcal{F}^{-1}[X(\xi, t)\cdot]\mathcal{F}. \tag{1.7}$$

2) Here α and β denote 1- and N-dimensional multi-indices, respectively. In the following, multi-indices will be used without any remark on their dimensions, as no confusion will arise.

Here $\mathcal{F}$ denotes the ordinary Fourier transformation in $\mathfrak{H}=L^2(R^N)$:

$$(\mathcal{F}u)(\xi)=\hat{u}(\xi)\equiv(2\pi)^{-N/2}\underset{M\to\infty}{\text{l.i.m.}}\int_{|x|\leq M}e^{-i\langle x,\xi\rangle}u(x)\,dx,\quad u\in\mathfrak{H},$$

and $X(\xi, t)\cdot$ denotes the maximal multiplication operator in $L^2(R_\xi^N)$ defined by the function $X(\xi, t)$.

Obviously $X(t)$ defines a self-adjoint operator in $\mathfrak{H}$ which commutes with H_1 for any $t\in R^1$. Now we can formulate our main result as the following theorem.

THEOREM 1.5 (Completeness of modified wave operators). *Suppose that Assumption 1.1 is satisfied. Then the following (strong) limit*

$$W_D^{\pm}=\underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH_2}e^{-itH_1-iX(t)} \tag{1.8}$$

exists. This $W_D^{\pm}$ defines a partially isometric operator in $\mathfrak{H}$ with initial set $\mathfrak{H}_{1,ac}=\mathfrak{H}$ and final set $\mathfrak{H}_{2,ac}$, and satisfies the intertwining property: For any Borel set Δ of R^1,

$$W_D^{\pm}E_1(\Delta)=E_2(\Delta)W_D^{\pm}. \tag{1.9}$$

(That is, the modified wave operator $W_D^{\pm}$ is complete.)

Next let us formulate the invariance principle.

THEOREM 1.6 (Invariance principle). *Suppose that Assumption 1.1 is satisfied. Let Γ be a bounded Borel set in R^1 such that $\bar{\Gamma}\subset(0,\infty)$, and let $I\subset(0,\infty)$ be a bounded open interval containing $\bar{\Gamma}$. Let $\varphi\in C^\infty(I)$ be real-valued and let $\eta\in C_0^\infty(R^1)$ satisfy $\eta(\lambda)=1$ on Γ and $\operatorname{supp}\eta\subset I$. Then we can define $Q_\varphi(t)\in B(\mathfrak{H})$, $t\in R^1$, as in (I.2.17). Suppose further that $\varphi'>0$ and $\varphi''\neq 0$ on I. Then the following strong limit*

$$W_\varphi^{\pm}(\Gamma)=\underset{t\to\pm\infty}{\text{s-lim}}\, e^{it\varphi(H_2)}Q_\varphi(t)E_{1,ac}(\Gamma) \tag{1.10}$$

exists and we have

$$W_\varphi^{\pm}(\Gamma)=W_D^{\pm}E_{1,ac}(\Gamma). \tag{1.11}$$

§ 2. Eigenfunction expansions.

In this section, we shall construct an eigenoperator $\mathcal{F}_j^{\pm}(\lambda)$ and state an eigenfunction expansion theorem for H_j $(j=1, 2)$ under Assumption 1.1$'$. The eigenoperator $\mathcal{F}_j^{\pm}(\lambda)$ constructed below will be used in the subsequent sections to prove Theorems 1.5 and 1.6. First of all, we make the following definition.

DEFINITION 2.1. For any $r>0$, $\omega\in R^N$, $\lambda\in R^1$, $\xi\in R^N$, and $t\in R^1$, put

$$f(r,\omega,\lambda;\xi,t)=\langle\omega,\xi\rangle+t(\lambda-|\xi|^2)-X(\xi,rt)/r, \tag{2.1}$$

where X is the function constructed in Theorem 1.2[3].

Then the following proposition holds.

PROPOSITION 2.2. *Let Assumption* 1.1′ *be satisfied. Let K be a compact interval contained in* $(0,\infty)$. *Then there exist a positive constant $R=R_K>1$, an open neighbourhood $U=U_K$ of $S^{N-1}\times K$ in $(R^N-\{0\})\times R^1$, and a unique C^∞ function $(\xi_c^\pm, t_c^\pm):(R,\infty)\times U\to R^N\times R^1$ satisfying the following properties* i)$\sim$iii)[4]:

i) *For any $r>R$ and $(\omega,\lambda)\in U$,*

$$\begin{cases}(\partial_\xi f)(r,\omega,\lambda;\xi_c^\pm(r,\omega,\lambda),t_c^\pm(r,\omega,\lambda))=0,\\ (\partial_t f)(r,\omega,\lambda;\xi_c^\pm(r,\omega,\lambda),t_c^\pm(r,\omega,\lambda))=0.\end{cases} \tag{2.2}$$

ii) *There is a positive constant C such that*

$$|(\xi_c^\pm(r,\omega,\lambda),t_c^\pm(r,\omega,\lambda))-(\pm\sqrt{\lambda}\,\omega/|\omega|,\ \pm|\omega|/2\sqrt{\lambda})|<Cr^{-\varepsilon_1} \tag{2.3}$$

for any $r>R$ and $(\omega,\lambda)\in U$.

iii) *There exist positive constants a and b such that $a<|J|<b$ and $a<|\det J|<b$ for any $r>R$ and $(\omega,\lambda)\in U$, where*

$$\begin{aligned}J&\equiv J(r,\omega,\lambda;\xi_c^\pm(r,\omega,\lambda),t_c^\pm(r,\omega,\lambda))\\ &=\begin{pmatrix}\partial_\xi\partial_\xi f & \partial_\xi\partial_t f\\ \partial_t\partial_\xi f & \partial_t\partial_t f\end{pmatrix}(r,\omega,\lambda;\xi_c^\pm(r,\omega,\lambda),t_c^\pm(r,\omega,\lambda)).\end{aligned} \tag{2.4}$$

PROOF. Take positive numbers c_1, c_2, d_1 and d_2 such that $K\subset(c_1^2,c_2^2)$, $d_1<1/2c_2<1/2c_1<d_2$ and put

$$B=\{\xi\mid c_1<|\xi|<c_2\},\quad D^\pm=\{\pm t\mid d_1<t<d_2\}.$$

---

3) At this stage, this definition may be abrupt to the reader. But the meaning and the importance of the function f will become clear as we proceed. Note especially that the function f will appear as the phase function of the *modified* resolvent $S^\pm(z)$ if we rewrite it using Fourier transform and then make a change of variable $t\to rt$ (see (3.25)).

4) This transformation $(\xi_c^\pm, t_c^\pm)$ will play a crucial role in connecting the time dependent modified wave operator with the eigenoperators of H_1 and H_2. The origin of our discovery of this transformation lies in the investigation of the asymptotic behavior of the modified resolvent $S^\pm(z)u$ for $u\in\mathcal{D}=\mathcal{F}^{-1}(C_0^\infty(R^N-\{0\}))$ (see §3 and [5]).

Take μ such that $\mu>1/\varepsilon_1$ and put

$$(2.5)\qquad \begin{cases} g(\rho,\omega,\lambda;\xi,t)=\begin{cases}(\partial_\xi f)(\rho^{-\mu},\omega,\lambda;\xi,t) & \text{for } \rho>0,\\ \omega-2t\xi & \text{for } \rho\leqq 0,\end{cases}\\ h(\rho,\omega,\lambda;\xi,t)=\begin{cases}(\partial_t f)(\rho^{-\mu},\omega,\lambda;\xi,t) & \text{for } \rho>0,\\ \lambda-|\xi|^2 & \text{for } \rho\leqq 0.\end{cases}\end{cases}$$

Then using Proposition 1.3 and the identity

$$(2.6)\qquad \begin{cases}(\partial_\xi f)(r,\omega,\lambda;\xi,t)=\omega-2t\xi-(\partial_\xi X)(\xi,rt)/r,\\ (\partial_t f)(r,\omega,\lambda;\xi,t)=\lambda-|\xi|^2-(\partial_t X)(\xi,rt),\end{cases}$$

it can be easily seen that the function (g,h) belongs to $C^1(R^1\times R^N\times R^1\times B\times D^\pm)$. Moreover this function satisfies the following relations: For any $(\omega,\lambda,\xi,t)\in R^N\times R^1\times B\times D^\pm$,

$$(2.7)\qquad \det\begin{pmatrix}\partial_\xi g & \partial_t g\\ \partial_\xi h & \partial_t h\end{pmatrix}\Bigg|_{(0,\omega,\lambda,\xi,t)}=2^{N+1}(-1)^N|\xi|^2t^{N-1}\neq 0,$$

and for any $(\omega,\lambda)\in S^{N-1}\times K$,

$$(2.8)\qquad \begin{cases} g(0,\omega,\lambda;\pm\sqrt{\lambda}\,\omega,\pm 1/2\sqrt{\lambda})=0,\\ h(0,\omega,\lambda;\pm\sqrt{\lambda}\,\omega,\pm 1/2\sqrt{\lambda})=0.\end{cases}$$

Thus we can apply the implicit function theorem to (g,h) and obtain the following result: For any $(\omega,\lambda)\in S^{N-1}\times K$, there exist bounded open neighbourhoods $A^\pm_{\omega,\lambda}\subset R^1\times(R^N-\{0\})\times R^1_+$ [5] of $(0,\omega,\lambda)$ and $C^\pm_{\omega,\lambda}\subset B\times D^\pm$ of $(\pm\sqrt{\lambda}\,\omega,\pm 1/2\sqrt{\lambda})$ such that for any $(\rho,\omega',\lambda')\in A^\pm_{\omega,\lambda}$ there is a unique point $(\tilde{\xi}^\pm_c,\tilde{t}^\pm_c)\in C^\pm_{\omega,\lambda}$ satisfying

$$(2.9)\qquad \begin{cases} g(\rho,\omega',\lambda';\tilde{\xi}^\pm_c,\tilde{t}^\pm_c)=0,\\ h(\rho,\omega',\lambda';\tilde{\xi}^\pm_c,\tilde{t}^\pm_c)=0.\end{cases}$$

The function $(\tilde{\xi}^\pm_c,\tilde{t}^\pm_c): A^\pm_{\omega,\lambda}\to C^\pm_{\omega,\lambda}$ is of course C^1. Furthermore, from (2.9) and Proposition 1.3, the following estimate holds good: There exists a positive constant C such that

$$(2.10)\qquad |(\tilde{\xi}^\pm_c(\rho,\omega',\lambda'),\tilde{t}^\pm_c(\rho,\omega',\lambda'))-(\pm\sqrt{\lambda'}\,\omega'/|\omega'|,\pm|\omega'|/2\sqrt{\lambda'})|\leqq C|\rho|^{\mu\varepsilon_1}$$

for any $(\rho,\omega',\lambda')\in A^\pm_{\omega,\lambda}$. Now, since $S^{N-1}\times K$ is compact, we can easily prove the existence of a positive number ρ_0, an open neighbourhood U of $S^{N-1}\times K$,

5) $R^1_\pm=\{t\in R^1\,|\,t\gtrless 0\}$.

and a unique C^1 function $(\tilde{\xi}_c^{\pm}, \tilde{t}_c^{\pm}) : (-\rho_0, \rho_0)\times U \to B\times D^{\pm}$ satisfying (2.9) and (2.10), with (ω', λ') replaced by (ω, λ), for any $(\rho, \omega, \lambda)\in(-\rho_0, \rho_0)\times U$, where $C>0$ is independent of (ρ, ω, λ).

Thus if we put

$$\left\{\begin{array}{l}\xi_c^{\pm}(r, \omega, \lambda)=\tilde{\xi}_c^{\pm}(r^{-1/\mu}, \omega, \lambda), \\ t_c^{\pm}(r, \omega, \lambda)=\tilde{t}_c^{\pm}(r^{-1/\mu}, \omega, \lambda)\end{array}\right. \tag{2.11}$$

for $r>R\equiv\rho_0^{-\mu}$ and $(\omega, \lambda)\in U$, then the function $(\xi_c^{\pm}, t_c^{\pm})$: $(R, \infty)\times U\to B\times D^{\pm}$ satisfies i) and ii). Moreover by Proposition 1.3 and (2.7) the matrix

$$J(r, \omega, \lambda ; \xi, t)=\begin{pmatrix} -2tI_N-(\partial_{\xi}^2 X)(\xi, rt)/r & -2\xi-(\partial_{\xi}\partial_t X)(\xi, rt) \\ {}^t[-2\xi-(\partial_{\xi}\partial_t X)(\xi, rt)] & -r(\partial_t^2 X)(\xi, rt)\end{pmatrix} \tag{2.12}$$

clearly satisfies condition iii) if $R>0$ is sufficiently large. The smoothness of the function $(\xi_c^{\pm}, t_c^{\pm})$ can be easily proved by using the implicit function theorem. Q.E.D.

Now the following definition makes sense.

DEFINITION 2.3. Let K be a compact interval contained in $(0, \infty)$. Let $R=R_K$ and $U=U_K$ be the same as in Proposition 2.2. Define $Y^{\pm}$ by

$$Y^{\pm}(r\omega ; \lambda)=\pm\sqrt{\lambda}\, r-rf(r, \omega, \lambda ; \xi_c^{\pm}(r, \omega, \lambda), t_c^{\pm}(r, \omega, \lambda)) \tag{2.13}$$

for $r>R$ and $(\omega, \lambda)\in S^{N-1}\times K$ where $(\xi_c^{\pm}, t_c^{\pm})$ is the C^{∞} function on $(R, \infty)\times U$ defined in Proposition 2.2.

For this function $Y^{\pm}$, the following theorem holds.

THEOREM 2.4. *Suppose that Assumption* 1.1′ *is satisfied. Let K be a compact interval contained in $(0, \infty)$. Then there exists a positive number $R'=R'_K>R=R_K$ such that the following two assertions hold:*

i) *There exists a positive constant C such that*

$$|\partial_x^{\alpha} Y^{\pm}(x ; \lambda)|\leqq C|x|^{1-|\alpha|-\varepsilon_1} \tag{2.14}$$

for any $\lambda\in K$, $|x|>R'$, and $|\alpha|\leqq 3$.

ii) *For any $r>R'$, $\omega\in S^{N-1}$ and $\lambda\in K$, one has*

$$\pm2\sqrt{\lambda}\,\partial_r Y^{\pm}(r\omega ; \lambda)=V(r\omega)+|\partial_x Y^{\pm}(r\omega ; \lambda)|^2. \tag{2.15}$$

PROOF. By definition we have

$$\begin{aligned} Y^{\pm}(r\omega ; \lambda)= & \pm\sqrt{\lambda}\, r-r\langle\omega, \xi_c^{\pm}(r, \omega, \lambda)\rangle \\ & -rt_c^{\pm}(r, \omega, \lambda)(\lambda-|\xi_c^{\pm}(r, \omega, \lambda)|^2) \\ & +X(\xi_c^{\pm}(r, \omega, \lambda), rt_c^{\pm}(r, \omega, \lambda)). \end{aligned} \tag{2.16}$$

Thus, when $|\alpha|=0$, the estimate (2.3) proves (2.14). Let $|\alpha|=1$. By (2.2) and a straightfoward computation, we get

$$\tag{2.17} \begin{cases} (\partial_r Y^\pm)(r\omega\,;\lambda)=\pm\sqrt{\lambda}-\langle\omega,\ \xi_c^\pm(r,\omega,\lambda)\rangle, \\ (\partial_\omega Y^\pm)(r\omega\,;\lambda)=-r\xi_c^\pm(r,\omega,\lambda). \end{cases}$$

Thus using the identity

$$\tag{2.18} (\partial_x g)(r\omega)=\omega(\partial_r\tilde{g})(r,\omega)+r^{-1}\{(\partial_\omega\tilde{g})(r,\omega)-\langle(\partial_\omega\tilde{g})(r,\omega),\omega\rangle\omega\}$$

which holds for $g(r\omega)=\tilde{g}(r,\omega)$, $r>0$, $\omega\in S^{N-1}$, we obtain

$$\tag{2.19} (\partial_x Y^\pm)(r\omega\,;\lambda)=\pm\sqrt{\lambda}\,\omega-\xi_c^\pm(r,\omega,\lambda).$$

This and the estimate (2.3) prove (2.14) for $|\alpha|=1$.

Next let $|\alpha|=2$. Then using (2.18) and (2.19), we can easily obtain

$$\tag{2.20} \begin{aligned} &(\partial_{x_i}\partial_{x_j} Y^\pm)(r\omega\,;\lambda) \\ &\quad=-\omega_i\partial_r\xi_{cj}^\pm+r^{-1}\{\pm\sqrt{\lambda}\,\delta_{ij}-\partial_{\omega_i}\xi_{cj}^\pm\mp\sqrt{\lambda}\,\omega_i\omega_j+\langle\partial_\omega\xi_{cj}^\pm,\omega\rangle\omega_i\}, \end{aligned}$$

where $\partial_{x_i}=\partial/\partial x_i$ and $\xi_{cj}^\pm$ denotes the j-th component of $\xi_c^\pm$. Therefore we have only to prove the following estimates:

$$\tag{2.21} \begin{cases} |\partial_r\xi_{cj}^\pm|<Cr^{-1-\varepsilon_1}, \\ |\pm\sqrt{\lambda}\,\delta_{ij}-\partial_{\omega_i}\xi_{cj}^\pm\mp\sqrt{\lambda}\,\omega_i\omega_j|<Cr^{-\varepsilon_1}, \\ |\langle\partial_\omega\xi_{cj}^\pm,\omega\rangle|<Cr^{-\varepsilon_1}. \end{cases}$$

By differentiating (2.2) with respect to r and ω, we obtain

$$J(r,\omega,\lambda\,;\xi_c^\pm,t_c^\pm)\begin{pmatrix}\partial_r\xi_c^\pm \\ \partial_r t_c^\pm\end{pmatrix}=-\begin{pmatrix}(\partial_r\partial_\xi f)(r,\omega,\lambda\,;\xi_c^\pm,t_c^\pm) \\ (\partial_r\partial_t f)(r,\omega,\lambda\,;\xi_c^\pm,t_c^\pm)\end{pmatrix},$$

and

$$J(r,\omega,\lambda\,;\xi_c^\pm,t_c^\pm)\begin{pmatrix}\partial_{\omega_i}\xi_c^\pm \\ \partial_{\omega_i} t_c^\pm\end{pmatrix}=-\begin{pmatrix}(\partial_{\omega_i}\partial_\xi f)(r,\omega,\lambda\,;\xi_c^\pm,t_c^\pm) \\ (\partial_{\omega_i}\partial_t f)(r,\omega,\lambda\,;\xi_c^\pm,t_c^\pm)\end{pmatrix}.$$

From these identities, Proposition 1.3, and (2.12), we can easily obtain (2.21). Similar but somewhat more complicated consideration proves i) for $|\alpha|=3$.

The relation (2.15) can be easily shown by using (2.17), (2.19), (1.4), and (2.2). Q. E. D.

Next let us construct an eigenoperator for H_j $(j=1, 2)$[6]. For this purpose, we set

$$H_3=H_1+V, \tag{2.22}$$

where V is the long-range potential satisfying Assumption 1.1′. Then we have

$$H_2=H_3+V_S. \tag{2.23}$$

DEFINITION 2.5. For $\lambda>0$, $\omega\in S^{N-1}$ and $r>R_\lambda(>0)$, put

$$\begin{cases} \theta_1^{\pm}(r, \omega, \lambda)=\mp\sqrt{\lambda}\, r, \\ \theta_3^{\pm}(r, \omega, \lambda)=\mp\sqrt{\lambda}\, r+Y^{\pm}(r\omega\,;\lambda) \\ \qquad\qquad =-rf(r, \omega, \lambda\,; \xi_c^{\pm}(r, \omega, \lambda), t_c^{\pm}(r, \omega, \lambda)), \end{cases} \tag{2.24}$$

where R_λ remains bounded when λ varies over a compact subset of $(0, \infty)$.

The following theorem can now be proved by using the results of Y. Saitō [I.**27**], [I.**28**][7].

THEOREM 2.6. *Suppose that Assumption* 1.1′ *is satisfied. Let* γ *be fixed as* $1/2<\gamma\leqq 1/2+\varepsilon_0/4$. *Then:*

i) *For any* $\lambda>0$, $g\in L_\gamma^2(R^N)$[8], *and* $j=1$ *or* 3, *there exists a sequence* $\{r_k\}$ *of positive numbers diverging to* ∞ *such that for* $k\to\infty$

$$\begin{cases} r_k^{1-2\gamma}\displaystyle\int_{S_{r_k}} |(R_j(\lambda\pm i0)g)(x)|^2\, dS\to 0, \\ r_k^{2\gamma-1}\displaystyle\int_{S_{r_k}} |(\mathcal{D}_{\pm,r}R_j(\lambda\pm i0)g)(x)|^2\, dS\to 0. \end{cases} \tag{2.25}$$

Here $S_r=\{x\in R^N|\ |x|=r\}$ *and* $\mathcal{D}_{\pm,r}=\partial/\partial r+(N-1)/2r\mp i\sqrt{\lambda}$, $r=|x|$.

ii) *For any* $\lambda>0$, $g\in L_\gamma^2(R^N)$, *and* $j=1$ *or* 3, *the following limit*

$$\lim_{k\to\infty} r_k^{(N-1)/2}e^{i\theta_j^{\pm}(r_k,\cdot,\lambda)}(R_j(\lambda\pm i0)g)(r_k\cdot) \tag{2.26}$$

exists in $\mathfrak{h}\equiv L^2(S^{N-1})$ *for any sequence* $\{r_k\}$ *satisfying* (2.25) *and does not depend on the choice of such* $\{r_k\}$.

---

6) In the remainder of this section, we shall only use the estimate (2.14) and the relation (2.15) just proved. The reason we adopted Assumption 1.1 which is stronger than Hörmander's [I.8] is only to assure the estimate (2.14) even for $|\alpha|=3$ which will be needed to construct eigenoperators. In all the other parts of this paper except for constructing eigenoperators, we only need assume the same assumption as that of Hörmander.

7) The following Theorem 2.6 was first proved by Y. Saitō [I.**27**] for the case $\gamma>3/2-\varepsilon_0$. Professor Y. Saitō suggested the author that this theorem would probably hold even for $\gamma>1/2$.

8) $L_\gamma^2(R^N)=L^2(R^N, (1+|x|)^{2\gamma}dx)$.

In the above $R_j(\lambda\pm i0)g$ *denotes the boundary value of* $R_j(\lambda\pm i\varepsilon)g$ *as* $\varepsilon\to+0$ *in* $L^2_{-\gamma}(R^N)$, *the existence of which is assured by the limiting absorption principle proved by Ikebe and Saitō* [I. **10**].

PROOF. i) is obvious by the limiting absorption principle (cf. Theorem 1.5 of [I. **10**]). Next let us outline the proof of ii). Without loss of generality we may assume $j=3$. Y. Saitō [I. **27**] proved ii) for $\gamma=\beta$, where β is taken as $\beta>3/2-\varepsilon_0$. (See Theorem 5.6 of [I. **27**] and note that its proof depends only on the estimate (2.14) and the relation (2.15)[9].) We shall use this result. Put for $g\in L^2_\gamma(R^N)$ and $r>0$,

$$w^\pm(r)=r^{(N-1)/2}e^{i\theta_3^\pm(r,\cdot,\lambda)}(R_3(\lambda\pm i0)g)(r\cdot)\in\mathfrak{h}$$

and denote the limit (2.26) by $w^\pm_\infty=w^\pm_\infty(g)\in\mathfrak{h}$ for $g\in L^2_\beta(R^N)$. Then the mapping $w^\pm_\infty: L^2_\beta(R^N)\ni g\mapsto w^\pm_\infty(g)\in\mathfrak{h}$ can be extended to a continuous mapping

$$L^2_\gamma(R^N)\ni g\longmapsto w^\pm_\infty(g)\in\mathfrak{h}$$

(cf. Proposition 3.6 of Saitō [I. **28**]). Put

$$v^\pm(r)=r^{-(N-1)/2}e^{-i\theta_3^\pm(r,\cdot,\lambda)}\varphi(\cdot)\rho(r)\in\mathfrak{h}$$

for $r>0$, where $\varphi\in\mathfrak{h}$ is an arbitrary smooth function on S^{N-1} and ρ is a real-valued smooth function on $[0,\infty)$ such that $\rho(r)=1$ near ∞ and $=0$ near 0. For this $v^\pm$ we can prove the following estimates for $r\to\infty$:

$$\left\{\begin{aligned}&v^\pm(r)=O(r^{-(N-1)/2}),\\&(\mathcal{D}_{\pm,r}v^\pm)(r)=O(r^{-(N-1)/2-\varepsilon_1}),\\&h^\pm\equiv(-\Delta+V-\lambda)v^\pm=O(r^{-(N-1)/2-1-\varepsilon_1}).\end{aligned}\right.\tag{2.27}$$

Thus we can apply the same reasoning as in the proof of Lemma 3.2 in Ikebe [I. **9**] to our case, and prove the existence of the following weak limit in $\mathfrak{h}$ and the relation

$$\underset{k\to\infty}{\text{w-lim}}\, w^\pm(r_k)=w^\pm_\infty$$

9) Precisely speaking, Saitō used the so-called "cutting-off" argument in proving the estimate (2.14) in [I. **27**] which plays a fundamental role in the proof of our Theorem 2.6 for the case $\gamma=\beta$. Namely, he first proved the estimate (2.14) of [I. **27**] for potentials with compact supports, and then extended it to general long-range potentials by taking a limit of the approximate sequence (3.56) in [I. **27**]. In our case, such an argument seems to be difficult to do. But the remedy comes from the proof of Lemma 2.2 of Isozaki [I. **11**] in which no cutting-off arguments were used and only the estimates (2.14) and (2.15) of our Theorem 2.4 were used. This fact assures our Theorem 2.6 for $\gamma=\beta$.

for any $g \in L_\gamma^2(R^N)$. Furthermore, using (2.25) and Proposition 3.6 of [I.**28**] we can prove

$$\lim_{k\to\infty} \|w^\pm(r_k)\|_{\mathfrak{h}} = \|w_\infty^\pm\|_{\mathfrak{h}}$$

in quite the same way as in the proof of Lemma 2.1 of [I.**9**]. Therefore we have proved ii).

For the sake of completeness, we outline the proof of (2.27). The first estimate is obvious by definition. As to the second estimate, by a straightforward computation we obtain

$$\mathcal{D}_{\pm,r} v^\pm = -i\langle \omega, \partial_x Y^\pm(r\omega\,;\lambda)\rangle v^\pm.$$

Thus (2.14) proves the second estimate of (2.27). Moreover a direct computation gives

$$\Delta v^\pm = (\pm 2\sqrt{\lambda}\,\partial_r Y^\pm - |\partial_x Y^\pm|^2 - \lambda) v^\pm + O(r^{-(N-1)/2-1-\varepsilon_1}),$$

where use was made of the estimate (2.14). Thus we get

$$\begin{aligned} h^\pm &= (-\Delta + V - \lambda) v^\pm \\ &= (\mp 2\sqrt{\lambda}\,\partial_r Y^\pm + V + |\partial_x Y^\pm|^2) + 0(r^{-(N-1)/2-1-\varepsilon_1}) \\ &= O(r^{-(N-1)/2-1-\varepsilon_1}), \end{aligned}$$

from the relation (2.15). This completes the proof of (2.27). Q.E.D.

Now the following definition makes sense.

DEFINITION 2.7. Let $\gamma > 1/2$. For any $\lambda > 0$, $g \in L_\gamma^2(R^N)$ and $j=1$ or 3, put

$$\mathcal{F}_j^\pm(\lambda) g \equiv \pi^{-1/2}\lambda^{1/4} \lim_{k\to\infty} r_k^{(N-1)/2} e^{i\theta_j^\pm(r_k\cdot,\lambda)} (R_j(\lambda \pm i0)g)(r_k\cdot), \tag{2.28}$$

where $\{r_k\}$ is any sequence satisfying (2.25) with the number γ replaced by some other γ' such that $1/2 < \gamma' \leqq 1/2 + \varepsilon_0/4$ when $\gamma > 1/2 + \varepsilon_0/4$. Moreover following Ikebe [**2**], we define for $\lambda > 0$ and $g \in L_\gamma^2(R^N)$,

$$\mathcal{F}_2^\pm(\lambda) g \equiv \mathcal{F}_3^\pm(\lambda)(g - V_S R_2(\lambda \pm i0) g). \tag{2.29}$$

The following proposition is due to Y. Saitō [I.**28**] (cf. Proposition 3.6 of [I.**28**]).

PROPOSITION 2.8. *Let Assumption* 1.1′ *be satisfied and let* $\gamma > 1/2$. *Let* $j=1$, 2 *or* 3. *Then the linear operator* $\mathcal{F}_j^\pm(\lambda) : L_\gamma^2(R^N) \to \mathfrak{h} = L^2(S^{N-1})$ *satisfies*

$$(\mathcal{F}_j^\pm(\lambda) g, \mathcal{F}_j^\pm(\lambda) h)_{\mathfrak{h}} = e_j(\lambda\,; g, h) \tag{2.30}$$

for any $\lambda>0$ *and* $g, h\in L_{\gamma}^{2}(R^{N})$. *Here* e_{j} *is defined as follows:*

$$e_{j}(\lambda; g, h)=\frac{1}{2\pi i}(\{R_{j}(\lambda+i0)g-R_{j}(\lambda-i0)g\}, h)_{\mathfrak{H}} \tag{2.31}$$

for $\lambda>0$ *and* $g, h\in L_{\gamma}^{2}(R^{N})$.

Now the following expansion theorem can be proved in a standard way and hence we shall omit the proof (cf. Ikebe [I. **9**], [**2**], Saitō [I. **28**]).

THEOREM 2.9 (Eigenfunction expansion). *Suppose that Assumption* 1.1 *is satisfied. Let* $\gamma>1/2$ *and let* $j=1, 2$, *or* 3. *Then:*

i) *For any* $g, h\in L_{\gamma}^{2}(R^{N})$ *and any Borel set* $B\subset(0, \infty)$, *we have*

$$(E_{j}(B)g, h)_{\mathfrak{H}}=\int_{B}(\mathcal{F}_{j}^{\pm}(\lambda)g, \mathcal{F}_{j}^{\pm}(\lambda)h)_{\mathfrak{h}}\,d\lambda.$$

ii) *Define* $\mathcal{F}_{j}^{\pm}$ *by*

$$(\mathcal{F}_{j}^{\pm}g)(\lambda)=\mathcal{F}_{j}^{\pm}(\lambda)g \qquad \text{for} \quad g\in L_{\gamma}^{2}(R^{N}).$$

Then $\mathcal{F}_{j}^{\pm}: L_{\gamma}^{2}(R^{N})\rightarrow\hat{\mathfrak{H}}\equiv L^{2}((0, \infty); \mathfrak{h})$ *can be extended to a partial isometry on* $\mathfrak{H}$ *with initial set* $\mathfrak{H}_{j,ac}$ *and final set* $\hat{\mathfrak{H}}$ *(which we shall denote also by* $\mathcal{F}_{j}^{\pm}$*), and the following relation holds: For any bounded Borel function* $\alpha(\lambda)$ *and for any* $g\in\mathfrak{H}_{j,ac}$

$$(\mathcal{F}_{j}^{\pm}\alpha(H_{j})g)(\lambda)=\alpha(\lambda)(\mathcal{F}_{j}^{\pm}g)(\lambda), \qquad a.e.\ \lambda>0.$$

iii) *For any bounded Borel set* B *satisfying* $\bar{B}\subset(0, \infty)$, *define*

$$\mathcal{F}_{j,B}^{\pm *}\hat{g}=\int_{B}\mathcal{F}_{j}^{\pm}(\lambda)^{*}\hat{g}(\lambda)\,d\lambda, \quad \hat{g}\in\hat{\mathfrak{H}}.$$

Then $\mathcal{F}_{j,B}^{\pm *}$ *belongs to* $B(\hat{\mathfrak{H}}, \mathfrak{H})$ *and we have* $\mathcal{F}_{j,B}^{\pm *}=E_{j}(B)\mathcal{F}_{j}^{\pm *}$. *For any Borel set* B *put* $B_{N}=B\cap[N^{-1}, N]$ $(N>1)$. *Then the following strong limit exists and we have*

$$\operatorname*{s-lim}_{N\rightarrow\infty}\mathcal{F}_{j,B_{N}}^{\pm *}=E_{j}(B)\mathcal{F}_{j}^{\pm *}.$$

In particular, the following inversion formula holds:

$$g=\operatorname*{s-lim}_{N\rightarrow\infty}\int_{N^{-1}}^{N}\mathcal{F}_{j}^{\pm}(\lambda)^{*}(\mathcal{F}_{j}^{\pm}g)(\lambda)\,d\lambda, \quad g\in\mathfrak{H}_{j,ac}.$$

iv) $\mathcal{F}_{j}^{\pm}(\lambda)^{*}$ *is an eigenoperator for* H_{j} *with eigenvalue* $\lambda>0$ *in the sense that for any* $\varphi\in\mathfrak{h}$ *and any smooth function* g *with compact support in* R^{N} *we have*

$$(H_{j}\mathcal{F}_{j}^{\pm}(\lambda)^{*}\varphi, g)_{\mathfrak{H}}=\lambda(\mathcal{F}_{j}^{\pm}(\lambda)^{*}\varphi, g)_{\mathfrak{H}},$$

where H_{j} *should be interpreted as the differentiation in the distribution sense.*

§ 3. Completeness of modified wave operators (proof of Theorem 1.5).

This section is devoted to giving a proof of Theorem 1.5, that is, to proving the completeness of time dependent modified wave operators intertwining H_1 and H_2. For this purpose we shall use Theorem I.2.2 under the following replacement:

Γ=a bounded Borel set of R^1 such that $\bar{\Gamma} \subset (0, \infty)$,

$\mathfrak{H}=L^2(R^N)$,

$H_1=-\Delta$,

$H_2=H_1+U$,

$X(t)=X(t)$ defined in Definition 1.4,

$\mathfrak{X}_1=\mathscr{D}\equiv\mathscr{F}^{-1}(C_0^\infty(R^N-\{0\}))$ endowed with the norm of $L^2_\delta(R^N)$, where δ is fixed as $1/2<\delta<1/2+\varepsilon_0$,

$\mathfrak{X}_2=L^2_\delta(R^N)$,

$\mathfrak{h}=L^2(S^{N-1})$,

$\mathscr{F}_j^\pm(\lambda)=\mathscr{F}_j^\pm(\lambda)$ defined in Definition 2.7 with $\gamma=\delta$.

Then $\mathscr{D}(H_1)=\mathscr{D}(H_2)$, and $\mathfrak{X}_j$ is a dense linear subspace of $\mathfrak{H}$ and obviously satisfies i)$\sim$iv) of § I.2. The condition (L. A. P.) is assured by the result of Ikebe and Saitō [I.**10**]. Conditions (X), (BC), and (XA) are also clearly satisfied by Definition 1.4 and Proposition 1.3. The operator $\mathscr{F}_j^\pm(\lambda)$ satisfies (a) of Theorem I.2.2 by Proposition 2.8 of the preceding section. Thus there remains to prove conditions $(Q^\pm)$, (b) and (c) of Theorem I.2.2. To prove these conditions, we prepare the following theorem.

THEOREM 3.1. *Let Assumption* 1.1′ *be satisfied*[10]. *Let* $a^\pm(\xi, t)\in C^\infty(R^N\times R^1)$ *satisfy the following two conditions:*

a1) *For some positive numbers* a_1, a_2 *satisfying* $0<a_1<a_2<\infty$,

$$\operatorname{supp} a^\pm \subset \{\xi \mid a_1<|\xi|<a_2\}\times R^1_\pm. \tag{3.1}$$

10) This theorem holds under the same assumption on V as that of Hörmander [I.8] with $h'=1-d$. Hence all results proved in this section except the first equality of (3.22) which contains $\mathscr{F}_2^\pm(\lambda)$ hold under Hörmander's assumption with δ being fixed as $1/2<\delta<1/2+\varepsilon_1$, $\varepsilon_1=m(1)+m(3)-4>0$ and taking $h_2=1-d-\rho-\varepsilon_0$ for $a_2^\pm$. Here ρ and d are the numbers corresponding to ρ and d of our Assumption 1.1′ in the case of Hörmander. Thus the existence and invariance principle hold under Hörmander's assumption, as condition $(f^\pm)$ of Part I can be proved under Hörmander's assumption quite similarly to the proof of Proposition 2.4 of [I.**15**] and as the results of §4 also hold under Hörmander's assumption.

a2) *For some real numbers* h_1, h_2 *with* $h_1 \leqq 0$, $h_2 \leqq 2\rho - 2$,

$$(3.2)\quad \begin{cases} |\partial_t^\alpha \partial_\xi^\beta a^\pm(\xi, t)| \leqq C(1+|t|)^{h_1 - |\alpha|}, & \text{when } |\alpha|+|\beta| \leqq 1, \\ |\partial_t^\alpha \partial_\xi^\beta a^\pm(\xi, t)| \leqq C(1+|t|)^{h_2 - \rho|\alpha| + (1-\rho)|\beta|}, & \text{when } |\alpha|+|\beta| \geqq 2, \end{cases}$$

for any $\xi \in R^N$, $t \in R^1$ *and multi-indices* α, β, *where* $C>0$ *is independent of* ξ, t, *and* ρ *is the number given in Assumption* 1.1$'$. *Put*

$$(3.3)\quad A_{\varepsilon,\lambda,a^\pm}(\xi) = \int_{-\infty}^{\infty} e^{it(\lambda - |\xi|^2) - iX(\xi,t)} a^\pm(\xi, t) \chi(\varepsilon t)\, dt$$

for $\varepsilon>0$, $\lambda \in R^1$, *and* $\xi \in R^N$, *where* $X(\xi, t)$ *is the function defined in Theorem* 1.2 *and* χ *is a rapidly decreasing function on* R^1 *such that* $\chi(0)=1$. *Then obviously* $A_{\varepsilon,\lambda,a^\pm} \in \mathcal{D}'(R^N)$ *for any* $\varepsilon>0$ *and* $\lambda \in R^1$. *Now take a compact interval* K *in* $(0, \infty)$ *such that* $[a_1^2, a_2^2] \subset K^i$, *where* K^i *denotes the interior of* K. *Then the following assertions hold:*

i) *For every* $\varepsilon>0$ *and* $u \in C_0^\infty(R^N)$, *there is a positive constant* C *such that*

$$(3.4)\quad |\langle A_{\varepsilon,\lambda,a^\pm}, ue^{i\langle x,\xi\rangle}\rangle - \langle A_{\varepsilon,\lambda',a^\pm}, ue^{i\langle x',\xi\rangle}\rangle| \leqq C(|\lambda-\lambda'|+|x-x'|)$$

for every λ, $\lambda' \in R^1$ *and* x, $x' \in R^N$.

ii) *For every* $x \in R^N$, $\lambda \in R^1$ *and* $u \in C_0^\infty(R^N)$, *the following limit*

$$(3.5)\quad \lim_{\varepsilon \to +0} \langle A_{\varepsilon,\lambda,a^\pm}, ue^{i\langle x,\xi\rangle}\rangle$$

exists and this defines a distribution $A_{0,\lambda,a^\pm} \in \mathcal{D}'(R^N)$.

iii) *For any* $u \in C_0^\infty(R^N)$ *and any integer* ν *such that* $\nu > ((N+2)(1-\rho+\varepsilon_0) + 2+\varepsilon_0)/(2\rho - 2\varepsilon_0 - 1)$, *there exist positive constants* R *and* C *such that the following four assertions hold:*

a) *For every* $\lambda \in R^1 - K$, $\varepsilon \geqq 0$ *and* $x \in R^N$,

$$(3.6)\quad |\langle A_{\varepsilon,\lambda,a^\pm}, ue^{i\langle x,\xi\rangle}\rangle| \leqq C(1+|x|)^{-(N-1)/2-2}(1+|\lambda|)^{-1}.$$

b) *For every* $\lambda \in K$, $\varepsilon \geqq 0$ *and* $|x| \leqq R$,

$$(3.7)\quad |\langle A_{\varepsilon,\lambda,a^\pm}, ue^{i\langle x,\xi\rangle}\rangle| \leqq C.$$

c) *For every* $\lambda \in K$, $\varepsilon \geqq 0$ *and* $|x| \geqq R$,

$$(3.8)\quad \begin{aligned} &|\langle A_{\varepsilon,\lambda,a^\pm}, ue^{i\langle x,\xi\rangle}\rangle \\ &\quad -(2\pi)^{(N+1)/2} e^{\mp\pi i(N-1)/4} r^{-(N-1)/2} e^{irf(r,\omega,\lambda;\,\xi_c^\pm(r,\omega,\lambda),\,t_c^\pm(r,\omega,\lambda))} \\ &\quad \times |\det J|^{-1/2} a^\pm(\xi_c^\pm(r, \omega, \lambda), rt_c^\pm(r, \omega, \lambda)) \\ &\quad \times \sum_{j=0}^{\nu-1} (\langle J^{-1}D, D\rangle^j v_{r,\omega,\lambda}^{\pm,\varepsilon})(0) r^{-j} (2i)^j / j!| \\ &\leqq Cr^{-(N-1)/2-1+\max(h_1,\,h_1+h'+2-3\rho,\,h_2+2-2\rho)}. \end{aligned}$$

Here the function f and $(\xi_c^\pm, t_c^\pm)$ are the same as in Proposition 2.2; *h' is the number defined in Proposition* 1.3; *$J=J(r, \omega, \lambda; \xi_c^\pm(r, \omega, \lambda), t_c^\pm(r, \omega, \lambda))$; $r=|x|$, $\omega=x/r$; $D=-i\partial_y$; and*

$$(3.9)\qquad v_{r,\omega,\lambda}^{\pm,\varepsilon}(y)$$
$$=u(\xi+\xi_c^\pm(r, \omega, \lambda))\chi(\varepsilon r(t+t_c^\pm(r, \omega, \lambda)))\tilde{\chi}(\xi, t)|_{(\xi,t)=\varphi_{r,\omega,\lambda}(y)}$$
$$\times|\det\partial_y\varphi_{r,\omega,\lambda}(y)|,$$

where $\varphi_{r,\omega,\lambda}$ and $\tilde{\chi}$ are the functions determined by Morse lemma (*cf. Lemma* 2.1 *and* (2.16) *of* [**5**]) *such that $\tilde{\chi}\in C_0^\infty(R^N)$, $\tilde{\chi}=1$ near* 0, *and $\varphi_{r,\omega,\lambda}(0)=0$. Moreover $\varphi_{r,\omega,\lambda}$ and $\tilde{\chi}$ can be taken the same as long as X remains the same.*

d) *For every $\lambda\in K$, $\varepsilon\geqq 0$ and $|x|\geqq R$,* (3.8) *holds with ν and the right-hand side replaced by* 1 *and $Cr^{-(N-1)/2+\min(-1/2,1-2\rho)}$, respectively.*

Proof. i) is obvious. Assertion iii), a) can be proved by integration by parts. It is easy to see that the functions $a^\pm$, X, $\varphi(\lambda; \xi, t)=t(\lambda-|\xi|^2)$, and $\psi(\omega; \xi)=-\langle\omega, \xi\rangle$ satisfy the conditions (Ca), (CX), $(C\varphi)$, and $(C\psi)$ of Theorem 1.2 of [**5**] with $\Omega=S^{N-1}\times K$, $\Gamma=\{\xi \mid a_1<|\xi|<a_2\}\times R_\pm^1$, $\rho=\rho$, $\delta=\varepsilon_1$, $h_1=h_1$, $h_2=h_2$, $h'=h'$, and $\varepsilon_0=\varepsilon_0$. So assertions ii) and b)$\sim$d) of iii) follow immediately from i)$\sim$iii) of Theorem 1.2 of [**5**]. Q.E.D.

Now let us prove $(Q^\pm)$ using Theorem 3.1. In quite the same way as in the proof of (1.16) of [I.**15**], we get

$$(3.10)\qquad (H_1-z)S^\pm(z)u=\mp\int_{-\infty}^\infty \chi_\pm(t)e^{-iX(t)}\frac{d}{dt}(e^{it(z-H_1)}u)dt$$

for $u\in\mathfrak{X}_1$ and $z\in\boldsymbol{C}^\pm$. Take a real-valued function $\eta\in C_0^\infty(R^N-\{0\})$ so that $\eta(\xi)=1$ on $\operatorname{supp}\hat{u}$ and put

$$X_\eta(t)=\mathcal{F}^{-1}[X(\xi, t)\eta(\xi)\cdot]\mathcal{F}.$$

Then (3.10) holds with $X(t)$ replaced by $X_\eta(t)$. Thus in Lemma 1.5 of [I.**15**], putting $X=\mathfrak{H}$, $I=[a, b]$, $f(t)=e^{it(z-H_1)}u$ and $A(t)=e^{-iX_\eta(t)}$, and letting $a\to-\infty$ or $b\to\infty$, we obtain

$$(3.11)\qquad (H_1-z)S^\pm(z)u=u\mp i\int_{-\infty}^\infty \chi_\pm(t)(\partial_t X)(t)e^{-iX(t)}e^{it(z-H_1)}u\,dt,$$

where $(\partial_t X)(t)=\mathcal{F}^{-1}[(\partial_t X)(\xi, t)\cdot]\mathcal{F}$. From this we get

$$(3.12)\qquad Q^\pm(z)u=US^\pm(z)u\mp i\int_{-\infty}^\infty \chi_\pm(t)(\partial_t X)(t)e^{-iX(t)}e^{it(z-H_1)}u\,dt$$

for $u\in\mathfrak{X}_1$ and $z\in\boldsymbol{C}^\pm$. Using inverse Fourier transformation, we then obtain for $x\in R^N$

$$(3.13)\qquad \mp i(2\pi)^{N/2}(Q^{\pm}(z)u)(x)$$
$$=U(x)\int_{-\infty}^{\infty}\int_{R^N}\chi_{\pm}(t)\,e^{i\langle x,\xi\rangle+it(z-|\xi|^2)-iX(\xi,t)}\,\hat{u}(\xi)\,d\xi dt$$
$$-\int_{-\infty}^{\infty}\int_{R^N}\chi_{\pm}(t)\,e^{i\langle x,\xi\rangle+it(z-|\xi|^2)-iX(\xi,t)}(\partial_t X)(\xi,\,t)\,\hat{u}(\xi)\,d\xi dt.$$

Choose $\rho^{\pm}\in C^{\infty}(R^1)$ such that

$$(3.14)\qquad \rho^{\pm}(t)=\begin{cases}1 & \text{for}\quad \pm t\geqq 1,\\ 0 & \text{for}\quad \pm t\leqq 1/2\end{cases}$$

and put

$$(3.15)\qquad \begin{cases} a_1^{\pm}(\xi,\,t)=\rho^{\pm}(t)\,\eta(\xi),\\ a_2^{\pm}(\xi,\,t)=\rho^{\pm}(t)(\partial_t X)(\xi,\,t)\,\eta(\xi),\end{cases}$$

where $\eta\in C_0^{\infty}(R^N-\{0\})$ is taken as $\eta(\xi)=1$ on $\operatorname{supp}\hat{u}$. Then $a_1^{\pm}$ and $a_2^{\pm}$ are C^{∞} and satisfy conditions a1) and a2) of Theorem 3.1 with $h_1=0$, $h_2=2\rho-2$ and $h_1=-\varepsilon_0$, $h_2=h''-\rho$, respectively (cf. Proposition 1.3). So writing $z=\lambda\pm i\varepsilon$ ($\lambda\in R^1$, $\varepsilon>0$), we obtain

$$(3.16)\qquad (Q^{\pm}(z)u)(x)=\pm i(2\pi)^{-N/2}[I^{\pm}(\varepsilon,\,\lambda,\,x)+J^{\pm}(\varepsilon,\,\lambda,\,x)]$$

for $x\in R^N$, where

$$(3.17)\qquad I^{\pm}(\varepsilon,\,\lambda,\,x)=U(x)\langle A_{\varepsilon,\lambda,a_1^{\pm}},\,\hat{u}e^{i\langle x,\xi\rangle}\rangle-\langle A_{\varepsilon,\lambda,a_2^{\pm}},\,\hat{u}e^{i\langle x,\xi\rangle}\rangle$$

with $\chi(t)=(1-\rho^{\mp}(t))\,e^{-|t|}$ and

$$(3.18)\qquad J^{\pm}(\varepsilon,\,\lambda,\,x)=\text{the remainder term.}$$

By Theorem 3.1 and using the relation $V(r\omega)=(\partial_t X)(\xi_c^{\pm}(r,\,\omega,\,\lambda),\,rt_c^{\pm}(r,\,\omega,\,\lambda))$, we can easily show that the following three estimates hold good:

i) For every $\varepsilon>0$, there is a constant $C>0$ such that

$$(3.19)\qquad |I^{\pm}(\varepsilon,\,\lambda,\,x)-I^{\pm}(\varepsilon,\,\lambda',\,x')|\leqq C(|\lambda-\lambda'|+|x-x'|)$$

for $x,\,x'\in R^N$ and $\lambda,\,\lambda'\in R^1$.

ii) The following limit exists for every $\lambda\in R^1$ and $x\in R^N$:

$$(3.20)\qquad I^{\pm}(0,\,\lambda,\,x)\equiv\lim_{\varepsilon\to+0}I^{\pm}(\varepsilon,\,\lambda,\,x).$$

iii) There exists a positive constant C such that

$$(3.21)\qquad |I^{\pm}(\varepsilon,\,\lambda,\,x)|\leqq C(1+|x|)^{-(N-1)/2-1-\varepsilon_0}(1+|\lambda|)^{-1}$$

for any $\varepsilon\geqq 0$, $\lambda\in R^1$ and $x\in R^N$.

On the other hand, by repeated integration by parts, we can show that the same estimates i)$\sim$iii) as above hold with $I^{\pm}$ replaced by $J^{\pm}$. Thus $(Q^{\pm}(\lambda\pm i\varepsilon)u)(x)$ also satisfies i)$\sim$iii). Therefore $(Q^{\pm})$ holds. Moreover we have shown that the condition c) of Remark I. 6. 3 also holds.

Next let us prove (b) and (c) of Theorem I. 2. 2. We shall prove

$$\begin{aligned}(3.22)\qquad &\mathcal{F}_2^{\pm}(\lambda)(u+\lim_{\varepsilon\to+0}Q^{\pm}(\lambda\pm i\varepsilon)u)\\ &=2^{-1/2}\lambda^{(N-2)/4}e^{\mp\pi i(N-3)/4}\hat{u}(\pm\sqrt{\lambda}\,\omega)\\ &=\mathcal{F}_1^{\pm}(\lambda)u\end{aligned}$$

in $\mathfrak{h}=L^2(S^{N-1})$ for any $u\in\mathfrak{X}_1=\mathcal{D}$ and $\lambda>0$. (Note that the existence of $Q^{\pm}(\lambda)u\equiv\lim_{\varepsilon\to+0}Q^{\pm}(\lambda\pm i\varepsilon)u$ in $\mathfrak{X}_2=L^2_{\delta}(R^N)$ for $\lambda>0$ has already been proved in the above as the condition c) of Remark I. 6. 3.) Let us prove the first equality of (3. 22).

From the definition of $\mathcal{F}_2^{\pm}(\lambda)$ (cf. Definition 2. 7), we get using (L. A. P.),

$$\begin{aligned}(3.23)\qquad &\mathcal{F}_2^{\pm}(\lambda)(u+\lim_{\varepsilon\to+0}Q^{\pm}(\lambda\pm i\varepsilon)u)\\ &=\mathcal{F}_3^{\pm}(\lambda)\lim_{\varepsilon\to+0}(I-V_S R_2(\lambda\pm i\varepsilon))G^{\pm}(\lambda\pm i\varepsilon)u\\ &=\mathcal{F}_3^{\pm}(\lambda)\lim_{\varepsilon\to+0}(G^{\pm}(\lambda\pm i\varepsilon)u-V_S S^{\pm}(\lambda\pm i\varepsilon)u)\\ &=\mathcal{F}_3^{\pm}(\lambda)\lim_{\varepsilon\to+0}(H_3-(\lambda\pm i\varepsilon))S^{\pm}(\lambda\pm i\varepsilon)u.\end{aligned}$$

From the definition of $\mathcal{F}_3^{\pm}(\lambda)$, we thus obtain

$$\begin{aligned}(3.24)\qquad &\mathcal{F}_2^{\pm}(\lambda)(u+\lim_{\varepsilon\to+0}Q^{\pm}(\lambda\pm i\varepsilon)u)\\ &=\pi^{-1/2}\lambda^{1/4}\lim_{k\to\infty}r_k^{(N-1)/2}e^{i\theta_3^{\pm}(r_k,\cdot,\lambda)}\\ &\quad\times[R_3(\lambda\pm i0)\lim_{\varepsilon\to+0}(H_3-(\lambda\pm i\varepsilon))S^{\pm}(\lambda\pm i\varepsilon)u](r_k\cdot)\\ &=\pi^{-1/2}\lambda^{1/4}\lim_{k\to\infty}r_k^{(N-1)/2}e^{i\theta_3^{\pm}(r_k,\cdot,\lambda)}[\lim_{\varepsilon\to+0}S^{\pm}(\lambda\pm i\varepsilon)u](r_k\cdot).\end{aligned}$$

Here $\lim_{\varepsilon\to+0}S^{\pm}(\lambda\pm i\varepsilon)u$ denotes the limit in $\mathfrak{X}_2^*=L^2_{-\delta}(R^N)$, and $\{r_k\}$ is any sequence satisfying (2. 25) with $j=3$. Using $a_1^{\pm}$ defined by (3. 15), we can then write

$$\begin{aligned}(3.25)\qquad &(S^{\pm}(\lambda\pm i\varepsilon)u)(x)\\ &=\pm i(2\pi)^{-N/2}[\langle A_{\varepsilon,\lambda,a_1^{\pm}},\hat{u}e^{i\langle x,\xi\rangle}\rangle+K^{\pm}(\varepsilon,\lambda,x)],\end{aligned}$$

where

$$\begin{aligned}(3.26)\qquad &K^{\pm}(\varepsilon,\lambda,x)\\ &=\int_{-\infty}^{\infty}\int_{R^N}(1-\rho^{\pm}(t))\chi_{\pm}(t)e^{i\langle x,\xi\rangle+it(\lambda-|\xi|^2)-iX(\xi,t)}\hat{u}(\xi)e^{-\varepsilon t}d\xi dt.\end{aligned}$$

Repeated integration by parts proves the existence of the limit $\lim_{\varepsilon\to+0} K^{\pm}(\varepsilon, \lambda, x)$ for any $\lambda\in R^1$ and $x\in R^N$, and the following estimate: For any integer $k\geqq 0$, there is a positive constant C such that

$$(3.27)\qquad |K^{\pm}(\varepsilon, \lambda, x)|\leqq C(1+|x|)^{-k}$$

for any $\varepsilon>0$, $x\in R^N$, and $\lambda\in R^1$. On the other hand, by ii) of Theorem 3.1, the limit

$$(3.28)\qquad \lim_{\varepsilon\to+0}\langle A_{\varepsilon,\lambda,a_1^{\pm}},\ \hat{u}e^{i\langle x,\xi\rangle}\rangle$$

also exists for any $\lambda\in R^1$ and $x\in R^N$. Furthermore iii), d) of Theorem 3.1 shows that

$$\begin{aligned}(3.29)\qquad & |\lim_{\varepsilon\to+0}\langle A_{\varepsilon,\lambda,a_1^{\pm}},\ \hat{u}e^{i\langle x,\xi\rangle}\rangle \\ & \quad -(2\pi)^{(N+1)/2}e^{\mp\pi i(N-1)/4}r^{-(N-1)/2}e^{-i\theta_3^{\pm}(r,\omega,\lambda)}\hat{u}(\xi_c^{\pm}(r,\omega,\lambda)) \\ & \quad\quad \times|\det J(r,\omega,\lambda;\xi_c^{\pm}(r,\omega,\lambda),t_c^{\pm}(r,\omega,\lambda))|^{-1/2}| \\ & \quad \leqq C_\lambda r^{-(N-1)/2+\min(-1/2,1-2\rho)}\end{aligned}$$

for $\lambda>0$ and $x=r\omega$ with $r=|x|>R_\lambda$, where positive numbers C_λ and R_λ remain bounded when λ varies over a compact subset of $(0,\infty)$. Thus we get from (3.24), (3.27), and (3.29),

$$\begin{aligned}(3.30)\qquad & \mathcal{F}_2^{\pm}(\lambda)(u+\lim_{\varepsilon\to+0}Q^{\pm}(\lambda\pm i\varepsilon)u) \\ & \quad =\sqrt{2}\,\lambda^{1/4}e^{\mp\pi i(N-3)/4}\lim_{k\to\infty}\hat{u}(\xi_c^{\pm}(r_k,\omega,\lambda)) \\ & \quad\quad \times|\det J(r_k,\omega,\lambda;\xi_c^{\pm}(r_k,\omega,\lambda),t_c^{\pm}(r_k,\omega,\lambda))|^{-1/2}\end{aligned}$$

for $\lambda>0$ and $u\in\mathfrak{X}_1$. From this, using ii) of Proposition 2.2, we can easily prove the first equality of (3.22).

The second equality of (3.22) can be proved using (3.30) if we put $X(\xi, t)\equiv 0$ in (3.24). Thus we have proved the conditions ($\mathcal{F}$) and ($\mathcal{F}G$) of Theorem I.5.1 and Remark I.5.2 with $\mathcal{F}_j(\lambda)=\mathcal{F}_j^{\pm}(\lambda)$ and $G(\lambda)=1+Q^{\pm}(\lambda)$. In particular we have proved the conditions (a), (b), and (c) of Theorem I.2.2. Therefore we have established the existence and completeness of modified wave operator

$$(3.31)\qquad W_D^{\pm}(\Gamma)=\operatorname*{s-lim}_{t\to\pm\infty}e^{itH_2}e^{-itH_1-iX(t)}E_{1,ac}(\Gamma)$$

for any bounded Borel subset Γ in R^1 such that $\bar{\Gamma}\subset(0,\infty)$. Since Γ is arbitrary, it follows from this that Theorem 1.5 holds.

§ 4. Invariance principle (proof of Theorem 1.6).

In this section, we shall prove Theorem 1.6, that is, the invariance principle. For this purpose, we shall use Theorem I.2.3 and Remark I.2.4. In the preceding section, we have already proved all the assumptions of Theorem I.2.3 except conditions (Q_φ^{as}) and $(\mathfrak{X}_1)$. But $(\mathfrak{X}_1)$ is clearly satisfied since $\mathfrak{X}_1=\mathcal{D}$. So we have only to prove (Q_φ^{as}). Without loss of generality we may assume that Γ is a compact interval contained in $(0, \infty)$.

To prove (Q_φ^{as}), let us investigate the asymptotic behavior of $Q_\varphi(t)u$ as $t\to\pm\infty$ for $u\in\mathfrak{H}_{1,ac}(\Gamma)$. For this purpose we prepare several lemmas.

LEMMA 4.1. *Let Assumption 1.1′ be satisfied. Let $\varphi\in C^2(I)$ be real-valued and satisfy*

$$\varphi'>0,\quad \varphi''\neq 0 \qquad on \quad I. \tag{4.1}$$

Then φ' has an inverse $l:\varphi'(I)\to I$. Put

$$g(\xi, t; y)=yl(y)-\varphi(l(y))-y|\xi|^2-X(\xi, ty)/t \tag{4.2}$$

for $\xi\in R^N$, $t\in R^1-\{0\}$, and $y\in\varphi'(I)$. Take an open interval $\tilde{I}$ so that

$$\Gamma\subset \operatorname{supp}\eta\subset\tilde{I}\subset\bar{\tilde{I}}\subset I\subset(0, \infty) \tag{4.3}$$

and put

$$c_1=\min\varphi'|_{\bar{\tilde{I}}},\quad c_2=\max\varphi'|_{\bar{\tilde{I}}}. \tag{4.4}$$

Then there exist a positive constant T, an open interval $\tilde{\Gamma}$ such that $\Gamma\subset\tilde{\Gamma}\subset\bar{\tilde{\Gamma}}\subset(0, \infty)$, and a unique C^1 function $y_c: B_{\tilde{\Gamma}}\times(-\infty, -T)\cup(T, \infty)\to\varphi'(I)$, where $B_{\tilde{\Gamma}}=\{\xi\mid |\xi|^2\in\tilde{\Gamma}\}$, satisfying the following properties:

i) *For any $\xi\in B_{\tilde{\Gamma}}$ and $|t|>T$,*

$$(\partial_y g)(\xi, t; y_c(\xi, t))=0. \tag{4.5}$$

ii) *There is a positive constant C such that for any $|t|>T$ and $\xi\in B_{\tilde{\Gamma}}$,*

$$|y_c(\xi, t)-\varphi'(|\xi|^2)|<C|t|^{-\varepsilon_1}. \tag{4.6}$$

iii) *There exist positive constants a and b such that for any $|t|>T$ and $\xi\in B_{\tilde{\Gamma}}$,*

$$a<|(\partial_y^2 g)(\xi, t; y_c(\xi, t))|<b. \tag{4.7}$$

PROOF. From (4.2), we have

$$(\partial_y g)(\xi, t; y)=l(y)-|\xi|^2-(\partial_t X)(\xi, ty)$$

for $\xi\in R^N$, $t\neq 0$ and $y\in\varphi'(I)$. Then if $\mu>1/\varepsilon_1$, it can be easily seen that the function

$$h(\xi, \tau; y)=\begin{cases}(\partial_y g)(\xi, \mathrm{sgn}(\tau)|\tau|^{-\mu}; y) & \text{for } \tau\neq 0,\\ l(y)-|\xi|^2 & \text{for } \tau=0\end{cases}$$

belongs to $C^1(B_{\tilde{\Gamma}}\times R^1\times\varphi'(I))$. Furthermore, we have

$$(\partial_y h)(\xi, 0; y)=l'(y)=\frac{1}{\varphi''(l(y))}\neq 0$$

for any $(\xi, y)\in B_{\tilde{\Gamma}}\times\varphi'(I)$, and

$$h(\xi, 0; \varphi'(|\xi|^2))=0$$

for $\xi\in B_{\tilde{\Gamma}}$. Thus using the implicit function theorem, we get to the following result: For any $\xi\in B_{\tilde{\Gamma}}$, there exist bounded open neighbourhoods $A_\xi\subset B_{\tilde{\Gamma}}\times R^1$ of $(\xi, 0)$ and $C_\xi\subset\varphi'(\tilde{I})$ of $\varphi'(|\xi|^2)$ such that for any $(\xi', \tau)\in A_\xi$ there is a unique point $\tilde{y}_c\in C_\xi$ satisfying $h(\xi', \tau; \tilde{y}_c)=0$. The function $\tilde{y}_c: A_\xi\to C_\xi$ is C^1 and satisfies

$$|\tilde{y}_c(\xi', \tau)-\varphi'(|\xi'|^2)|\leqq C|\tau|^{\mu\varepsilon_1}$$

for $(\xi', \tau)\in A_\xi$, where constant $C>0$ is independent of (ξ', τ). Now using the compactness of $B_\Gamma\subset B_{\tilde{\Gamma}}$, we can prove the existence of a positive number τ_0, an open interval $\tilde{\Gamma}\subset I$ containing Γ, and a unique C^1 function $\tilde{y}_c: B_{\tilde{\Gamma}}\times(-\tau_0, \tau_0)\to\varphi'(I)$ satisfying $h(\xi, \tau; \tilde{y}_c(\xi, \tau))=0$ and

$$|\tilde{y}_c(\xi, \tau)-\varphi'(|\xi|^2)|\leqq C|\tau|^{\mu\varepsilon_1}$$

for $(\xi, \tau)\in B_{\tilde{\Gamma}}\times(-\tau_0, \tau_0)$, where $C>0$ is independent of (ξ, τ). Thus the function $y_c(\xi, t)\equiv\tilde{y}_c(\xi, \mathrm{sgn}(t)|t|^{-1/\mu})$ satisfies i)~iii) with $T=|\tau_0|^{-\mu}$. Q. E. D.

The following Lemmas 4.2 and 4.3 are essentially the same as Lemmas 2 and 3, respectively, of Matveev [I.**21**] so we shall omit the proof.

LEMMA 4.2. *Let $\varphi\in C^2(I)$ be real-valued and satisfy $\varphi'>0$, $\varphi''\neq 0$ on I. Put*

$$a_\varphi(t, r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta(\lambda)e^{-it\varphi(\lambda)+ir\lambda}d\lambda, \tag{4.8}$$

where $\eta\in C_0^\infty(R^1)$ is taken as $\eta(\lambda)=1$ on Γ and supp $\eta\subset I$. *Then there exists a positive constant C such that*

$$|a_\varphi(t, r)-|t|^{-1/2}e^{i[rl(rt^{-1})-t\varphi(l(rt^{-1}))]}\phi(rt^{-1})|<C|r|^{-3/2} \tag{4.9}$$

for $|r|>1$ with $y=rt^{-1}\in[c_1, c_2]$, where

$$\phi(y)=\frac{1}{\sqrt{2\pi}}e^{-\pi i\,\mathrm{sgn}(t\varphi'')/4}|\varphi''(l(y))|^{-1/2}\eta(l(y)) \tag{4.10}$$

for $y\in[c_1, c_2]$.

LEMMA 4.3. *Let* φ *and* a_φ *be as in Lemma* 4.2. *Let* A *be an open interval such that* $\varphi'(\mathrm{supp}\,\eta)\subset A\subset\bar{A}\subset(c_1, c_2)=\varphi'(I)$. *Then there exists a positive constant* C *such that*

$$|a_\varphi(t, r)|\leqq C\min(|t|^{-2}, |r|^{-2}) \tag{4.11}$$

for any r, t *satisfying* $rt^{-1}\in R^1-A$.

Now take $\tilde{\eta}\in C_0^\infty(R^1)$ so that $0\leqq\tilde{\eta}\leqq 1$, $\mathrm{supp}\,\tilde{\eta}=[c_1, c_2]$, and $\tilde{\eta}(y)=1$ on some open interval A such that $\varphi'(\Gamma)\subset\varphi'(\mathrm{supp}\,\eta)\subset A\subset\bar{A}\subset(c_1, c_2)$. Put

$$\begin{cases} I_1(\xi, t)=\displaystyle\int_{-\infty}^{\infty}a_\varphi(t, r)e^{-ir|\xi|^2-iX(\xi,r)}(1-\tilde{\eta}(rt^{-1}))dr, \\ I_2(\xi, t)=\displaystyle\int_{-\infty}^{\infty}a_\varphi(t, r)e^{-ir|\xi|^2-iX(\xi,r)}\tilde{\eta}(rt^{-1})dr. \end{cases} \tag{4.12}$$

Then we can write

$$\mathcal{F}(Q_\varphi(t)E_{1,ac}(\Gamma)u)(\xi)=[I_1(\xi, t)+I_2(\xi, t)]\chi_\Gamma(\xi)\hat{u}(\xi), \tag{4.13}$$

where χ_Γ is the characteristic function of B_Γ. From Lemma 4.3, we obtain

$$|I_1(\xi, t)|\leqq C|t|^{-1} \tag{4.14}$$

for $\xi\in R^N$ and $t\neq 0$, where $C>0$ is independent of ξ, t.

In order to estimate I_2, put

$$J(\xi, t)=\int_{-\infty}^{\infty}a_\varphi(t, ty)e^{-ity|\xi|^2-iX(\xi,ty)}\tilde{\eta}(y)dy. \tag{4.15}$$

Then we have

$$I_2(\xi, t)=|t|J(\xi, t) \tag{4.16}$$

and from Lemma 4.2, we get

$$|J(\xi, t)-|t|^{-1/2}\int_{-\infty}^{\infty}e^{itg(\xi,t;\,y)}\phi(y)\tilde{\eta}(y)dy|<C|t|^{-3/2}, \tag{4.17}$$

where $C>0$ is independent of $|t|>1$ and $\xi\in R^N$. Now let us denote the integral in (4.17) by $\tilde{J}(\xi, t)$ and investigate its asymptotic behavior when $t\to\pm\infty$ using the stationary phase method. As usual, we first divide $\tilde{J}$ into the sum of two integrals, that is, into the integral near the critical point $y_c(\xi, t)$ and the integral on the remainder region, which will be denoted by $\tilde{J}_1(\xi, t)$ and $\hat{J}_2(\xi, t)$, respectively. As to $\tilde{J}_1$, since Lemma 4.1 holds, we can apply Morse lemma (cf. Lemma 2.1 of [5]) and make a change of variables given by that lemma in the integral $\tilde{J}_1$. Thus Lemma A.2 of [I.8] is applicable to $\tilde{J}_1$ and hence we obtain

$$(4.18) \qquad |\tilde{J}_1(\xi, t)-|t|^{-1/2} e^{itg(\xi,t;y_C(\xi,t))}|<C|t|^{-1},$$

where constant $C>0$ is independent of $\xi\in B_\Gamma$ and $|t|>1$. Here we have also used the estimates (4.6) and (1.6). On the other hand, by Lemma A.1 of [I.8], $\tilde{J}_2(\xi, t)$ is bounded by $C|t|^{-1}$ for some constant $C>0$ independent of $\xi\in B_\Gamma$ and $|t|>1$. Thus we have proved (4.18) with $\tilde{J}_1$ replaced by $\tilde{J}$. Combining this with (4.16) and (4.17), we obtain

$$(4.19) \qquad |I_2(\xi, t)-e^{itg(\xi,t;y_C(\xi,t))}|<C|t|^{-1/2},$$

where constant $C>0$ is independent of $\xi\in B_\Gamma$ and $|t|>1$. Therefore from (4.13), (4.14) and (4.19), we obtain

$$(4.20) \qquad \|Q_\varphi(t) E_{1,ac}(\Gamma) u-Q_\varphi^{as}(t) E_{1,ac}(\Gamma) u\|_{\mathfrak{H}}\leqq C|t|^{-1/2}\|u\|_{\mathfrak{H}}$$

for $|t|>1$ and $u\in\mathfrak{H}$, where

$$(4.21) \qquad Q_\varphi^{as}(t)\equiv\mathcal{F}^{-1}[e^{itg(\xi,t;y_C(\xi,t))}\chi_\Gamma(\xi)\cdot]\mathcal{F}.$$

Thus we have proved (Q_φ^{as}) and hence Theorem 1.6. Moreover we have proved that

$$(4.22) \qquad W_D^\pm E_{1,ac}(\Gamma)=W_\varphi^\pm(\Gamma)=W_\varphi^{\pm,as}(\Gamma),$$

where

$$(4.23) \qquad W_\varphi^{\pm,as}(\Gamma)=\operatorname*{s-lim}_{t\to\pm\infty} e^{it\varphi(H_2)} Q_\varphi^{as}(t) E_{1,ac}(\Gamma).$$

§ 5. Supplementary remarks.

In the above we have proved the existence and the completeness of time dependent modified wave operators of Hörmander type. But to complete the discussion from the physical point of view as well as from the mathematical point of view, we must show that $e^{-itH_2}u$ for $u\in\mathfrak{H}_{2,ac}$ behaves like a free state when $t\to\pm\infty$. In this section, we shall first give a solution of this problem. Next we shall consider the modified wave operator of Alsholm type (cf. Alsholm [I.2]) and prove its completeness. We shall then treat the case $1>\varepsilon_0>1/2$ in which the situation is somewhat simpler. We shall next discuss the relation between the stationary wave operator of Pinchuk [I.22] or

Isozaki [I.11] and ours. Finally we shall consider a relation between eigenoperators and our stationary wave operators.

5.1. Asymptotic behavior of $e^{-itH_2}u$ as $t\to\pm\infty$ for $u\in\mathfrak{H}_{2,ac}$. The following theorem gives a solution of this problem.

THEOREM 5.1. *Suppose that Assumption* 1.1′ *is satisfied. Put for* $f\in\mathfrak{H}$, $t\neq 0$, *and* $x\neq 0$,

$$(U^{\pm}(t)f)(x)=(it)^{-N/2}e^{i(\langle x,\eta_c^{\pm}\rangle-t|\eta_c^{\pm}|^2-X(\eta_c^{\pm},t))}\hat{f}(x/2t). \tag{5.1}$$

Here $\eta_c^{\pm}=\eta_c^{\pm}(x,t)$ *denotes the regular critical point of*

$$j(x,t;\eta)=\langle x,\eta\rangle/t-|\eta|^2-X(\eta,t)/t.$$

Then $U^{\pm}(t)$ *defines a unitary operator in* $\mathfrak{H}$ *and satisfies for any* $f\in\mathfrak{H}$

$$\lim_{t\to\pm\infty}\|e^{-itH_1-iX(t)}f-U^{\pm}(t)f\|_{\mathfrak{H}}=0. \tag{5.2}$$

Therefore it follows from Theorem 1.5 *that for any* $u\in\mathfrak{H}_{2,ac}$ *there exists a free state* $u^{\pm}\in\mathfrak{H}=\mathfrak{H}_{1,ac}$ *such that*

$$\lim_{t\to\pm\infty}\|e^{-itH_2}u-U^{\pm}(t)u^{\pm}\|_{\mathfrak{H}}=0. \tag{5.3}$$

Thus, the probability density of $e^{-itH_2}u$ *converges asymptotically to* $|t|^{-N}|\widehat{u^{\pm}}(x/2t)|^2$ *as* $t\to\pm\infty$, *and hence* $e^{-itH_2}u$ $(u\in\mathfrak{H}_{2,ac})$ *behaves like a free state as* $t\to\pm\infty$.

PROOF. We have only to prove (5.2). But this is a consequence of the proof of Theorem 3.9 of Hörmander [I.8]. Q.E.D.

5.2. The modified wave operator of Alsholm type. Alsholm [I.2] considered and proved the existence of the limit (1.8) with $X(t)$ replaced by $X_A(t)$ assuming (1.2) of Assumption 1.1 for $|\alpha|\leqq K_0=[1/\varepsilon_0]$[11] with $m(k)=k+\varepsilon_0$. Here $X_A(t)\equiv X_A^{(K_0)}(t)$ is defined by iteration as follows:

$$X_A^{(0)}(\xi,t)\equiv 0,\quad X_A^{(h)}(\xi,t)=\int_0^t V(2s\xi+(\partial_\xi X_A^{(h-1)})(\xi,s))\,ds \tag{5.4}$$

and

$$X_A^{(h)}(t)=\mathcal{F}^{-1}[X_A^{(h)}(\xi,t)\cdot]\mathcal{F} \tag{5.5}$$

for $h=1,2,\cdots$.

Using our method developed in Part I, we can prove the existence and the completeness of

$$W_A^{\pm}=\text{s-}\lim_{t\to\pm\infty}e^{itH_2}e^{-itH_1-iX_A(t)}. \tag{5.6}$$

11) Here $[a]$ for $a\in R^1$ denotes the maximum integer n such that $n\leqq a$.

But the proof is somewhat long and complicated so we shall be content with an outline of the proof. Later in this subsection, we shall also give another simpler proof using Theorem 1.5.

To give a direct proof of the existence and the completeness of $W_A^{\pm}$, we shall use Theorem I.2.2. For the sake of simplicity, we assume that $V_S=0$ and that (1.2) of Assumption 1.1 is satisfied for all α with $m(k)=k+\varepsilon_0$, where constant C appearing in Assumption 1.1 depends on α. First of all, we must construct an eigenoperator. But this can be done in quite the same way as in §2, although we must replace (2.15) by

$$|\pm 2\sqrt{\lambda}\,\partial_r Y^{\pm}(r\omega\,;\lambda)-V(r\omega)-|\partial_x Y^{\pm}(r\omega\,;\lambda)|^2|\leqq Cr^{-2}. \tag{5.7}$$

Let us make a replacement in Theorem I.2.2 similar to the one stated at the beginning of §3. We can easily show that the assumptions except $(Q^{\pm})$, (b) and (c) of Theorem I.2.2 are all satisfied. The condition $(Q^{\pm})$ can be proved by using the results of Alsholm [I.2] in a way similar to the proof of Proposition 2.3 of [I.15], though it is more complicated and longer. Conditions (b) and (c) of Theorem I.2.2 can be proved by using our Theorem 3.1, as Proposition 2.2 and Theorem 3.1 also hold with $X(\xi, t)$ replaced by $X_A(\xi, t)$. Thus we can surely prove the existence and completeness of $W_A^{\pm}$ directly. The proof outlined above is rather long and complicated. But if we use Theorem 1.5, we can give another proof which is simpler than the one given above. The crucial point is the following lemma.

LEMMA 5.2. *Let* (1.2) *of Assumption* 1.1 *be satisfied for* $|\alpha|\leqq K_0=[1/\varepsilon_0]$ *with* $m(k)=k+\varepsilon_0$[12]. *Then for any* $\xi\in R^N-\{0\}$ *the following limit exists:*

$$F^{\pm}(\xi)=\lim_{t\to\pm\infty}(X(\xi, t)-X_A(\xi, t)). \tag{5.8}$$

PROOF. We have only to prove the following estimate: For any compact subset Ω of $R^N-\{0\}$ and any ε_2 satisfying $0<\varepsilon_2<\varepsilon_0$, there exist positive constants C and T such that

$$|(\partial_\xi^\alpha X)(\xi, t)-(\partial_\xi^\alpha X_A^{(h)})(\xi, t)|\leqq C|t|^{1-(h+1)\varepsilon_2}$$

for any $\xi\in\Omega$, $|t|>T$ and $1\leqq|\alpha|\leqq K_0-h$, $h=0, 1, \cdots, K_0-1$. But this can be easily proved by induction in h using Lemma 5 of Alsholm [I.2] and an appropriate version of Theorem 1.2. Q.E.D.

Thus we have proved the following theorem.

THEOREM 5.3. *Let* (1.2) *of Assumption* 1.1 *be satisfied for* $|\alpha|\leqq\max([1/\varepsilon_0], 4)$ *with* $m(k)=k+\varepsilon_0$. *Then there exists the limit*

12) Note that an appropriate version of Theorem 1.2 holds under the assumption of the lemma (cf. Hörmander [I.8]).

(5.9) $$W_A^{\pm}=\text{s-}\lim_{t\to\pm\infty} e^{itH_2}e^{-itH_1-iX_A(t)}.$$

This $W_A^{\pm}$ *satisfies*

(5.10) $$W_A^{\pm}=W_D^{\pm}e^{-iF^{\pm}(D)},$$

where $F^{\pm}(D)=\mathcal{F}^{-1}[F^{\pm}(\xi)\cdot]\mathcal{F}$, $F^{\pm}(\xi)$ *being defined by* (5.8). *Hence* $W_A^{\pm}$ *is complete.*

When $1>\varepsilon_0>1/2$, this theorem proves the completeness of $W_D^{\pm}(\Gamma)$ in §2.2 of Part I.

5.3. The case $1>\varepsilon_0>1/2$. In this case situation becomes somewhat simpler than the case $1/2\geqq\varepsilon_0>0$. That is, the function $\theta_3^{\pm}(r,\omega,\lambda)$ in Definition 2.5 can be replaced by

(5.11) $$\theta_3'^{\pm}(r,\omega,\lambda)=\mp\sqrt{\lambda}\,r\pm\frac{1}{2\sqrt{\lambda}}\int_0^r V(s\omega)\,ds.$$

The crucial point is the following lemma.

LEMMA 5.4. *Let* (1.2) *of Assumption* 1.1 *be satisfied for* $|\alpha|\leqq 3$ *with* $1>\varepsilon_0>1/2$ *and* $m(k)=k+\varepsilon_0$. *Then the limit*

(5.12) $$\lim_{r\to\infty}[\theta_3^{\pm}(r,\omega,\lambda)-\theta_3'^{\pm}(r,\omega,\lambda)]$$

exists for any $\lambda>0$ *and* $\omega\in S^{N-1}$.

PROOF. From the identity

$$\theta_3'^{\pm}(r,\omega,\lambda)=\mp\sqrt{\lambda}\,r+X_A^{(1)}(\pm\sqrt{\lambda}\,\omega,\ \pm r/2\sqrt{\lambda})$$

and Lemma 5.2, we have only to prove the existence of the limit

(5.13) $$\lim_{r\to\infty}[\theta_3^{\pm}(r,\omega,\lambda)-\theta_3''^{\pm}(r,\omega,\lambda)],$$

where

$$\theta_3''^{\pm}(r,\omega,\lambda)=\mp\sqrt{\lambda}\,r+X(\pm\sqrt{\lambda}\,\omega,\ \pm r/2\sqrt{\lambda}).$$

But using (2.2) we obtain

$$\begin{aligned}&\theta_3^{\pm}(r,\omega,\lambda)-\theta_3''^{\pm}(r,\omega,\lambda)\\&\quad=-rt_c^{\pm}|\xi_c^{\pm}-(\pm\sqrt{\lambda}\,\omega)|^2-\langle(\partial_{\xi}X)(\xi_c^{\pm},rt_c^{\pm}),\ \xi_c^{\pm}-(\pm\sqrt{\lambda}\,\omega)\rangle\\&\qquad+(X(\xi_c^{\pm},rt_c^{\pm})-X(\pm\sqrt{\lambda}\,\omega,\ \pm r/2\sqrt{\lambda})).\end{aligned}$$

Thus an appropriate version of Proposition 1.3 and (2.3) yields

$$\theta_3^{\pm}(r,\omega,\lambda)-\theta_3''^{\pm}(r,\omega,\lambda)=O(r^{1-2\varepsilon_2})\quad(r\to\infty)$$

for any ε_2 satisfying $\varepsilon_0>\varepsilon_2>1/2$. Therefore the limit (5.13) exists and is equal to zero. Q.E.D.

This lemma shows that all theorems and propositions in §2 after Definition 2.5 hold good with $\theta_3^{\pm}$ replaced by $\theta_3'^{\pm}$. This result gives another proof of Ikebe's result [I. **9**]. Furthermore if we replace $X(t)$ by $X_A^{(1)}(t)$ in $S^{\pm}(z)$ then we can prove Theorem 3 of [I. **16**] in a way similar to the proof of (3.22) and Lemma 5.4.

As to the invariance principle we can also give another proof of the result of §I.2.2. In fact we can prove the following lemma.

LEMMA 5.5. *Let* (1.2) *of Assumption* 1.1 *be satisfied for* $|\alpha|\leqq 3$ *with* $1>\varepsilon_0>1/2$ *and* $m(k)=k+\varepsilon_0$. *Let* Γ *be a compact interval contained in* $(0, \infty)$. *Then the limit*

$$\lim_{t\to\pm\infty}[t\varphi(|\xi|^2)+X_A^{(1)}(\xi, t\varphi'(|\xi|^2))-tg(\xi, t; y_c(\xi, t))] \tag{5.14}$$

exists for any $\xi\in B_\Gamma=\{\xi \mid |\xi|^2\in\Gamma\}$, *where functions* φ, g, *and* y_c *are the same as in Lemma* 4.1.

PROOF. The formula in the parenthesis [ ] of (5.14) can be rewritten using (4.5)[13] as follows:

$$t[(\partial_t X)(\xi, ty_c)y_c+\varphi(|\xi|^2)-\varphi(l(y_c))]$$
$$+[X(\xi, t\varphi'(|\xi|^2))-X(\xi, ty_c)]+[X(\xi, ty_c)-X_A^{(1)}(\xi, ty_c)].$$

The last two terms converge as $t\to\pm\infty$ by Proposition 1.3, (4.6), and Lemma 5.2. The convergence of the first term can be proved using (4.5), by differentiating and integrating the formula in [ ] with respect to t. Q.E.D.

From this lemma and (4.22), we obtain (I.2.23).

5.4. The stationary wave operator of Pinchuk and Isozaki. In [I. **22**] Pinchuk constructed a time-independent wave operator and proved its completeness for the case $\varepsilon_0>1/2$. Isozaki [I. **11**] extended Pinchuk's results to the case $\varepsilon_0>0$. Although their methods are somewhat different from each other especially in their abstract theory, their fundamental estimates are both based on those appeared in Ikebe [I. **9**] or Saitō [I. **27**], [I. **28**].

In this subsection, we shall first give another proof of the completeness of their stationary wave operators and then prove that they are equal to our stationary wave operator and hence to time dependent ones[14].

---

13) Note that Lemma 4.1 holds under the assumption of the lemma or even under the assumption of Hörmander.

14) Recently Ikebe and Isozaki [**3**] also obtained a proof that Isozaki's stationary wave operator coincides with the time dependent one.

In the following we denote the stationary wave operators of Pinchuk and Isozaki by $W_P^{\pm}$ and $W_I^{\pm}$, respectively. For the sake of simplicity, we shall assume $V_S(x)=0$ so $H_2=H_3$[15]. Let us first summarize some of the results of Pinchuk and Isozaki which we need in the sequel.

THEOREM 5.6 (due to Pinchuk [I.**22**]). *Let* (1.2) *of Assumption* 1.1 *be satisfied for* $|\alpha|\leqq 4$ *with* $1>\varepsilon_0>1/2$ *and* $m(k)=k+\varepsilon_0$[16]. *Let* Γ *be a bounded Borel set in* R^1 *such that* $\bar{\Gamma}\subset(0, \infty)$. *Put for* $\lambda\in\Gamma$

$$U^{\pm}(\lambda)=\exp\left(\mp\frac{1}{2\sqrt{\lambda}}\int_0^r V(s\omega)\,ds\right)\cdot,\quad r=|x|,\ \omega=x/r. \tag{5.15}$$

Then there exists the limit

$$G_P^{\pm}(\lambda)\equiv\lim_{\varepsilon\to+0}(H_2-(\lambda\pm i\varepsilon))U^{\pm}(\lambda)R_1(\lambda\pm i\varepsilon) \tag{5.16}$$

in $B(\mathfrak{X}_1', \mathfrak{X}_2')$ *for all* $\lambda\in\Gamma$. *Here*

$$\mathfrak{X}_1'=L^2_{\delta'+1/2}(R^N),\quad \mathfrak{X}_2'=L^2_{\delta'}(R^N),\quad 1/2<\delta'<\varepsilon_0. \tag{5.17}$$

Moreover $G_P^{\pm}(\lambda)u$ *is strongly measurable as an* $\mathfrak{X}_2'$*-valued function of* $\lambda\in\Gamma$ *for every* $u\in\mathfrak{X}_1'$.

THEOREM 5.7 (due to Isozaki [I.**11**]). *Let* (1.2) *of Assumption* 1.1 *be satisfied for* $|\alpha|\leqq[2/\varepsilon_0]+2$ *with* $1/2\geqq\varepsilon_0>0$ *and* $m(k)=k+\varepsilon_0$. *Let* Γ *be a bounded Borel set in* R^1 *such that* $\bar{\Gamma}\subset(0, \infty)$. *Define* $Z^{(j)}(x, \kappa)$ *for* $j=1, 2, \cdots$, $\kappa\in R^1-\{0\}$ *and* $x\in R^N-\{0\}$ *as in page* 602 *of* [I.**11**] *and put*

$$\begin{cases} Z(x, \kappa)=Z^{([2/\varepsilon_0])}(x, \kappa), \\ U^{\pm}(\lambda, \varepsilon)=\exp(-iZ(x, \mathrm{Re}\sqrt{\lambda\pm i\varepsilon}))\cdot,\quad \lambda\in\Gamma,\quad \varepsilon>0. \end{cases} \tag{5.18}$$

Then there exists the limit

$$G_I^{\pm}(\lambda)\equiv\lim_{\varepsilon\to+0}(H_2-(\lambda\pm i\varepsilon))U^{\pm}(\lambda)R_1(\lambda\pm i\varepsilon) \tag{5.19}$$

in $B(\mathfrak{X}_1'', \mathfrak{X}_2'')$ *for* $\lambda\in\Gamma$. *Here*

$$\mathfrak{X}_1''=L^2_{2-\delta''}(R^N),\quad \mathfrak{X}_2''=L^2_{\delta''}(R^N),\quad 1/2<\delta''<1/2+\varepsilon_0/4. \tag{5.20}$$

Moreover $G_I^{\pm}(\lambda)u$ *is strongly continuous as an* $\mathfrak{X}_2''$*-valued function of* $\lambda\in\Gamma$ *for every* $u\in\mathfrak{X}_1''$.

---

15) This restriction is not essential at all. We can prove the completeness and equality even when $V_S(x)\not\equiv 0$.

16) Pinchuk's assumption is somewhat weaker than the one adopted here. But for the sake of simplicity we assume this.

To prove the completeness and equality, we shall use Theorem I. 5. 4. We first consider $W_P^{\pm}$. As was shown in Ikebe [I.**9**] or in the preceding subsection, the following limit exists in $\mathfrak{h}=L^2(S^{N-1})$ for $\lambda>0$ and $g\in\mathfrak{X}_2'=L^2_{\delta'}(R^N)$[17]:

(5.21) $$\mathcal{F}_2'^{\pm}(\lambda)g=\pi^{-1/2}\lambda^{1/4}\lim_{k\to\infty} r_k^{(N-1)/2}e^{i\theta_2'^{\pm}(r_k,\cdot,\lambda)}(R_2(\lambda\pm i0)g)(r_k\cdot),$$

where $\{r_k\}$ is an appropriate sequence of positive numbers diverging to ∞ as $k\to\infty$, and

(5.22) $$\theta_2'^{\pm}(r,\omega,\lambda)=\mp\sqrt{\lambda}\,r\pm\frac{1}{2\sqrt{\lambda}}\int_0^r V(s\omega)\,ds.$$

This $\mathcal{F}_2'^{\pm}(\lambda)$ satisfies (a) of ($\mathcal{F}$) in Theorem I. 5. 1 with $\mathcal{F}_2(\lambda)=\mathcal{F}_2'^{\pm}(\lambda)$ and $\mathfrak{X}_j=\mathfrak{X}_j'$ (cf. Lemma 2. 1 of [I.**9**] or §5. 3). Now the following relation can be easily seen by definition: For any $\lambda\in\Gamma$ and $u\in\mathfrak{X}_1'$

(5.23) $$\mathcal{F}_2'^{\pm}(\lambda)G_P^{\pm}(\lambda)u=\mathcal{F}_1'^{\pm}(\lambda)u,$$

where $\mathcal{F}_1'^{\pm}(\lambda)$ is the one defined in Definition 2. 7 with $\gamma=\delta'$. Thus we have proved all the assumptions of Theorem I. 5. 1 for H_1 and H_2. Therefore $W_P^{\pm}$ is complete, because Pinchuk's $W_P^{\pm}$ was constructed from $G_P^{\pm}$ in essentially the same way as ours stated in Part I (compare Theorems I. 4. 1 and I. 4. 4 with Theorem 3. 11 of Pinchuk [I.**22**]). Moreover from Lemma 5. 4, we have

(5.24) $$\mathcal{F}_2^{\pm}(\lambda)u=A^{\pm}(\lambda)\mathcal{F}_2'^{\pm}(\lambda)u,\quad \lambda\in\Gamma,\quad u\in\mathfrak{X}_2\cap\mathfrak{X}_2'$$

for some unitary operator $A^{\pm}(\lambda)$ in $\mathfrak{h}$. Therefore, by Theorem I. 5. 4, we obtain

(5.25) $$L^{\pm}W_\Gamma^{\pm}=W_P^{\pm}E_{1,ac}(\Gamma),$$

where $L^{\pm}$ is the unitary operator constructed from $A^{\pm}(\lambda)$ as in Theorem I. 5. 4, and $W_\Gamma^{\pm}$ is our stationary wave operator constructed from $G^{\pm}(\lambda)=1+Q^{\pm}(\lambda)$ as in Theorem I. 5. 1.

Next let us consider $W_I^{\pm}$. Put

(5.26) $$\theta_2''^{\pm}(r,\omega,\lambda)=\mp\sqrt{\lambda}\,r+Z(r\omega,\pm\sqrt{\lambda})$$

for $\lambda>0$, $r>0$ and $\omega\in S^{N-1}$. Then the following limit exists in $\mathfrak{h}=L^2(S^{N-1})$ for $\lambda>0$ and $g\in\mathfrak{X}_2''=L^2_{\delta''}(R^N)$[18]:

(5.27) $$\mathcal{F}_2''^{\pm}(\lambda)g=\pi^{-1/2}\lambda^{1/4}\lim_{k\to\infty} r_k^{(N-1)/2}e^{i\theta_2''^{\pm}(r_k,\cdot,\lambda)}(R_2(\lambda\pm i0)g)(r_k\cdot),$$

---

17) When we use Ikebe's result, this statement can be justified by the same reasoning as in the proof of Theorem 2. 6.

18) The remark similar to footnote 17) holds here.

where $\{r_k\}$ is an appropriate sequence of positive numbers diverging to ∞ (cf. Lemma 4.1 of [I.11]). This $\mathcal{F}_2''^{\pm}(\lambda)$ satisfies (a) of $(\mathcal{F})$ in Theorem I.5.1. Moreover by definition we have

$$\mathcal{F}_2''^{\pm}(\lambda)G_I^{\pm}(\lambda)u=\mathcal{F}_1''^{\pm}(\lambda)u \tag{5.28}$$

for $\lambda\in\Gamma$ and $u\in\mathfrak{X}_1''$, where $\mathcal{F}_1''^{\pm}(\lambda)$ is the one defined in Definition 2.7 with $\gamma=\delta''$. Thus by Theorem I.5.1, we can construct a complete stationary wave operator $W_{I,\Gamma}^{\pm}$. Since Isozaki's wave operator $W_I^{\pm}$ satisfies (I.4.8) by (1.4) of [I.11], we obtain

$$W_I^{\pm}E_{1,ac}(\Gamma)=W_{I,\Gamma}^{\pm}. \tag{5.29}$$

Therefore $W_I^{\pm}$ is complete. Furthermore by (5.27), Theorem 2.6, and the fact that $\mathcal{R}(\mathcal{F}_2^{\pm}(\lambda))$ is dense in $\mathfrak{h}$ for $\lambda\in\Gamma$, we have

$$\mathcal{F}_2^{\pm}(\lambda)u=A^{\pm}(\lambda)\mathcal{F}_2''^{\pm}(\lambda)u,\quad \lambda\in\Gamma,\quad u\in\mathfrak{X}_2\cap\mathfrak{X}_2'', \tag{5.30}$$

where $A^{\pm}(\lambda)$ is a unitary operator in $\mathfrak{h}$. Therefore by Theorem I.5.4, there exists a unitary operator $L^{\pm}$ such that

$$L^{\pm}W_{\Gamma}^{\pm}=W_I^{\pm}E_{1,ac}(\Gamma). \tag{5.31}$$

5.5. Relation between eigenoperators and our stationary wave operators. As was shown in Theorem 2.9, $\mathcal{F}_j^{\pm}$ is a partial isometry from $\mathfrak{H}$ to $\hat{\mathfrak{H}}$ with initial set $\mathfrak{H}_{j,ac}$ and final set $\hat{\mathfrak{H}}$. Thus if we define

$$W_{I\text{-}S}^{\pm}=\mathcal{F}_2^{\pm *}\mathcal{F}_1^{\pm}, \tag{5.32}$$

then $W_{I\text{-}S}^{\pm}$ is a partial isometry in $\mathfrak{H}$ with initial set $\mathfrak{H}_{1,ac}=\mathfrak{H}$ and final set $\mathfrak{H}_{2,ac}$. Furthermore, by ii) and iii) of Theorem 2.9, $W_{I\text{-}S}^{\pm}$ satisfies

$$W_{I\text{-}S}^{\pm}E_1(\Delta)=E_2(\Delta)W_{I\text{-}S}^{\pm} \tag{5.33}$$

for any Borel set Δ in R^1. Thus $W_{I\text{-}S}^{\pm}$ gives a unitary equivalence between $H_1=H_{1,ac}$ and $H_{2,ac}$. In this sense $W_{I\text{-}S}^{\pm}$ can be regarded as a stationary wave operator intertwining H_1 and H_2. In fact we can prove the following theorem.

THEOREM 5.8. *Let Assumption* 1.1 *be satisfied. Then we have*

$$W_{I\text{-}S}^{\pm}E_{1,ac}(\Gamma)=W_{\Gamma}^{\pm}=W_D^{\pm}E_{1,ac}(\Gamma) \tag{5.34}$$

for any bounded Borel set Γ in $(0, \infty)$ *such that $\bar{\Gamma}\subset(0,\infty)$. Hence we have*

$$W_{I\text{-}S}^{\pm}=W_D^{\pm}. \tag{5.35}$$

We omit the proof, because it is similar to the one given in [I . 16][19].

References

[1] L. Hörmander, Fourier integral operators, I, Acta Math., **127** (1971), 79-183.
[2] T. Ikebe, Spectral representation for Schrödinger operators with long-range potentials, II, perturbation by short-range potentials, Publ. RIMS, Kyoto Univ., **11** (1976), 551-558.
[3] T. Ikebe and H. Isozaki, Completeness of modified wave operators for long-range potentials, (to appear).
[4] H. Kitada, Scattering theory for Schrödinger operators with long-range potentials, I, abstract theory, J. Math. Soc. Japan, **29** (1977), 665-691.
[5] H. Kitada, Asymptotic behavior of some oscillatory integrals, (to appear in J. Math. Soc. Japan).

Hitoshi KITADA
Department of Pure and Applied Sciences
College of General Education
University of Tokyo
Komaba, Meguro-ku
Tokyo, Japan

19) This proof gives another proof of the completeness of $W_D^{\pm}$. But note that this proof depends on Theorem 2.9 and hence on the measurability of $\mathcal{F}_j^{\pm}(\lambda)g$, $g\in L_\gamma^2(R^N)$, etc. which was not made use of in §3, where the completeness was proved.