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On linearizable irreducible projective representations of finite groups

By G. KARPILOVSKY

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Let G be a finite group and K an arbitrary field. Yamazaki ([4], Theorem 1) proved that there exists a finite central group extension of G by which all "linearizable" projective representations of G are linearized (cf. Section 1). This result motivates consideration of the following problem. Given a finite group G and an arbitrary field K of characteristic 0, what is the number of equivalence classes of irreducible linearizable projective representations of G over K? The aim of this paper is to give the solution of this problem. As a corollary we obtain the group theoretical characterization of the number of equivalence classes of irreducible projective representations of G over K, where K is an algebraically closed field of characteristic 0, or the real number field.

I. Preliminaries.

All groups in this paper are assumed to be finite.

NOTATION. K is any field and $K^*=K-\{0\}$.

$GL(V)$ is the group of all nonsingular linear transformations of a finite dimensional vector space V over K.

A K-character is a character of a linear representation of a group G over K.

K^*1_V is the centre of $GL(V)$ where 1_V denotes the identity mapping of V onto itself.

$PGL(V)=GL(V)/K^*1_V$ is the group of projective transformations of the projective space $P(V)$ associated to V.

π is the natural projection of $GL(V)$ onto $PGL(V)$.

$|S|$ is the order of the set S.

G' is the derived group of G.

$\mathrm{Hom}(G, K^*)$ is the multiplicative group of all linear characters (one-dimensional linear representations) of the group G over K.

An ordered pair (G^*, ϕ) of a group G^* and a surjective homomorphism $\phi: G^*\to G$ is called a central group extension of the group G if the kernel $\mathrm{Ker}\,\phi$ of ϕ is included in the centre $Z(G^*)$ of the group G^*.

If T is a permutation group acting on the set S then S/T is the quotient set (the orbit set) obtained from S by identification of any two points of S in the same orbit of T.

A projective representation of G in V is a homomorphism $\rho: G \to PGL(V)$. A mapping $\Gamma_\rho: G \to GL(V)$ is called a section for ρ if $\pi\Gamma_\rho(g)=\rho(g)$ for any $g \in G$.

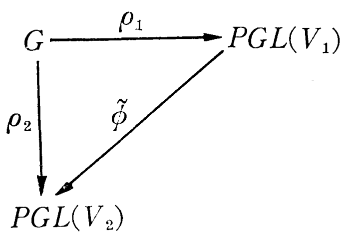
Γ_ρ determines a 2-cocycle α of G in K^* by

$$\Gamma_\rho(g_1)\Gamma_\rho(g_2)=\alpha(g_1, g_2)\Gamma_\rho(g_1 g_2), \qquad (g_1, g_2 \in G,\ \alpha(g_1, g_2) \in K^*).$$

Its cohomology class in $H^2(G, K^*)$ depends only on ρ and is denoted by C_ρ.

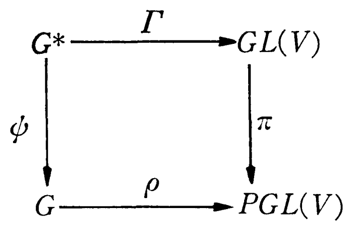
Let Γ_ρ be any section of ρ. The projective representation ρ is called irreducible if there are no non-trivial subspaces of V which are sent into themselves by all the transformations $\Gamma_\rho(g)$, $g \in G$.

Two projective representations $\rho_i: G \to PGL(V_i)$ $(i=1, 2)$ are called equivalent (written $\rho_1 \sim \rho_2$) if there exists a linear isomorphism $\phi: V_1 \to V_2$ such that the following diagram is commutative



where $\tilde{\phi}(\pi x)=\pi\phi x\phi^{-1}$ for every $x \in GL(V_1)$.

Let Γ be a linear representation of G^* in V such that $\Gamma(\mathrm{Ker}\,\psi) \subset K^* 1_V$. Then Γ induces a projective representation ρ of G in V such that the following diagram is commutative



We shall say that ρ is linearized by the group extension (G^*, ψ). It is clear that ρ is irreducible if and only if Γ is the irreducible linear representation of G^* over K.

Following Yamazaki [4] we shall call a projective representation ρ linearizable if ρ is linearized by some finite central group extension (G^*, ψ) of G. If K is algebraically closed, or the real number field, then by [4] G has a representation-group $\hat{G}$.

That is, a group $\hat{G}$ by which all projective representations of G are linearized and the order of $\hat{G}$ is equal to hm, where h is the order of G and m is the order of 2-cohomology group $H^2(G, K^*)$.

II. Some results on linear representations of finite groups.

Let G be a finite group of exponent n, K any field with characteristic not dividing $|G|$ and ε a primitive n-th root of unity over K. Let I_n be the multiplicative group consisting of those integers r, taken modulo n, for which $\varepsilon \to \varepsilon^r$ defines an automorphism of $K(\varepsilon)$ over K. It is clear that the group I_n is isomorphic to the Galois group of $K(\varepsilon)$ over K. Two elements $a, b \in G$ are called K-conjugate (written $a \underset{K}{\sim} b$) if $x^{-1}bx = a^r$ for some $x \in G$ and some $r \in I_n$. K-conjugacy is an equivalence relation and so G may be partitioned into K-conjugacy classes.

LEMMA 1. *Let $H \Delta G$, i.e. H is normal subgroup of G, and let $T_h(K_h)$ be the K-conjugacy class of the group $G(H)$ with representative $h \in H$. Then* $T_h = \bigcup_{g \in G} K_{g^{-1}hg}$.

PROOF. Let m be the exponent of H, so that $n = mk$ for some natural number k and $\delta = \varepsilon^k$ is the primitive m-th root of unity over K. Suppose $s \in T_h$. Then $s = a^{-1}h^{\mu}a$ for some $a \in G$ and some $\mu \in I_n$. If $\mu \equiv r \pmod{m}$, $0 \leqq r \leqq m-1$, then $s = (a^{-1}ha)^r$. The automorphism $\varepsilon \to \varepsilon^{\mu}$ of $K(\varepsilon)$ over K induces the automorphism $\delta \to \delta^{\mu} = \delta^r$ of $K(\delta)$ over K. Hence $r \in I_m$ and $s \in K_{a^{-1}ha}$. Conversely, let $a \in K_{g^{-1}hg}$. Then $a = h_1^{-1}g^{-1}h^{\mu}gh_1 = (gh_1)^{-1}h^{\mu}(gh_1)$ for some $h_1 \in H$ and some $\mu \in I_m$. The automorphism $\delta \to \delta^{\mu}$ of $K(\delta)$ over K can be extended to the automorphism $\varepsilon \to \varepsilon^{\lambda}$ of $K(\varepsilon)$ over K (see [3], § 52). Hence $\lambda \equiv \mu \pmod{m}$, $h^{\mu} = h^{\lambda}$ and $a = (gh_1)^{-1}h^{\lambda}(gh_1)$ for $\lambda \in I_n$. This proves the lemma.

Let χ be an irreducible K-character of G and ϕ an irreducible K-character of a subgroup H of G. ϕ induces a character ϕ^G of G and χ restricts down to a character $\chi \downarrow H$ of H. Suppose $H \triangle G$. Then G acts on the irreducible characters of H by conjugation. That is, for $g \in G$ and $x \in H$, $\phi^g(x) = \phi(gxg^{-1})$. The subgroup T fixing a given irreducible character ϕ is called the inertia group of ϕ. Clearly, $T \supseteqq H$. If $t = (G:T)$ then ϕ has precisely t distinct conjugates $\phi = \phi_1, \phi_2, \cdots, \phi_t$. Furthermore, if ϕ is an irreducible component of $\chi \downarrow H$, then ([2], Theorem 49.7)

$$\chi \downarrow H = m(\phi_1 + \phi_2 + \cdots + \phi_t) \quad \text{for some natural number } m. \tag{1}$$

If $H \Delta G$ then a character β of G/H can be regarded as a character of G with kernel containing H. Conversely, every character of G with kernel containing H arises in this manner. We shall use the same symbol to denote the character whether viewed in G or G/H. The precise situation with be clear from the context. If the field K is such that the polynomial x^n-1 splits into linear factors in K, then K contains all n-th roots of unity. We use the notation "$\sqrt[n]{1}\in K$" to denote this fact.

DEFINITION. A K-kernel of the group G is the smallest subgroup G_K, $G_K\geqq G'$, such that $\sqrt[n]{1}\in K$ where n is the exponent of the group G/G_K.

It follows from this definition that if $N\Delta G$ and $N\subseteqq G_K$ then

$$(G/N)_K=G_K/N. \tag{2}$$

LEMMA 2. *Each linear K-character of G is a character of G/G_K and the number of linear K-characters of G is $|G/G_K|$.*

PROOF. An abelian group A has $|A|$ linear K-characters if and only if $\sqrt[n]{1}\in K$ where n is the exponent of A, ([2], Theorem 9.10). Hence G/G_K has exactly $|G/G_K|$ linear characters. On the other hand, let λ be any linear K-character of G with kernel N. The mapping $g\rightarrow(gG_K)(gN)$ is a homomorphism of G into $G/G_K\times G/N$ with kernel $G_K\cap N$. Hence $G/G_K\cap N$ is isomorphic to some subgroup of $G/G_K\times G/N$. Thus $\sqrt[m]{1}\in K$ where m is the exponent of $G/G_K\cap N$, and so $G_K\cap N=G_K$ and $N\supseteqq G_K$. This proves the lemma.

LEMMA 3. *Let $H\Delta G$, K be any field and χ be an arbitrary K-character of G. Then*

$$(\chi\downarrow H)^G=\rho\chi \text{ where } \rho \text{ is the regular representation of } G/H.$$

PROOF. Let $\theta=\chi\downarrow H$. Then $\theta(g^{-1}hg)=\chi(h)$ for every $g\in G$ and $h\in H$. On the one hand,

$$\theta^G(h)=\frac{1}{|H|}\sum_{x\in G}\theta(x^{-1}hx)=(G:H)\chi(h)$$

and

$$\theta^G(x)=0 \qquad \text{for} \quad x\in G-H.$$

On the other hand,

$$(\rho\chi)(h)=\rho(h)\chi(h)=(G:H)\chi(h) \quad \text{and} \quad (\rho\chi)(x)=\rho(x)\chi(x)=0.$$

This proves the lemma.

Let K be an arbitrary field of characteristic not dividing the order of the group G and let $\hat{K}$ be the algebraic closure of K. Denote by $X=\langle\hat{\chi}_1,\cdots,\hat{\chi}_s\rangle$ the full set of irreducible $\hat{K}$-characters of G and by $Q=\langle C_1,\cdots,C_s\rangle$ the conjugacy classes of G. Under the action of the group of mappings $g\rightarrow g^\mu$, $\mu\in I_n$ the sets X and Q are partitioned into disjoint subsets $X=X_1\cup X_2\cup\cdots\cup X_q$; $Q=K_1\cup K_2\cup\cdots\cup K_q$ where the K_i are K-conjugacy classes of G, $X_i=\langle\hat{\chi}_{i1},\cdots,\hat{\chi}_{ir_i}\rangle$ $(i=1,\cdots,q)$ and q is the number of irreducible linear representations of G over K ([1], (9.1), Theorem 1.1, Theorem 5.1). Let $\Gamma_1,\Gamma_2,\cdots,\Gamma_q$ be the irreducible linear representations of G over K and χ_i the character of Γ_i

$(i=1, 2, \cdots, q)$. Then $\Gamma_i = m_i(\hat{\Gamma}_{i1} + \cdots + \hat{\Gamma}_{ir_i})$ where $\hat{\Gamma}_{ij}$ is the irreducible linear representation of G over $\hat{K}$ with character $\hat{\chi}_{ij}$ and m_i is the Schur index of any representation $\hat{\Gamma}_{ij}$ with respect to K $(i=1, 2, \cdots, q; j=1, 2, \cdots, r_i)$. Let $e_1, e_2, \cdots, e_q$ be all the minimal central idempotents of the group algebra GK. Then the following formulae hold (see [**1**], (20.1), (22.1))

$$\chi_i = m_i(\hat{\chi}_{i1} + \cdots + \hat{\chi}_{ir_i}) \qquad (i=1, 2, \cdots, q) \tag{3}$$

$$e_i = \frac{n_i}{m_i |G|} \sum_{g \in G} \chi_i(g^{-1}) g \quad \text{where } n_i = \hat{\chi}_{ij}(1) \tag{4}$$

$$(i=1, \cdots, q;\ j=1, \cdots, r_i).$$

It follows from (3) that the irreducible K-character χ of G is completely determined by any of its absolutely irreducible components. Let $t_i = \sum_{x \in K_i} x$ $(i=1, 2, \cdots, q)$ and let V be the space spanned by $t_1, \cdots, t_q$.

Then by ([**1**], Theorem 1.1 and (20.2)) the vector space V has the following two bases

$$\langle e_1, e_2, \cdots, e_q \rangle \quad \text{and} \quad \langle t_1, t_2, \cdots, t_q \rangle . \tag{5}$$

LEMMA 4 (Generalised Reciprocity Theorem) ([**1**], Theorem 2.2).

Let Γ and Γ' be irreducible linear representations of the group G and its subgroup H respectively over the field K of characteristic 0. Furthermore, suppose there corresponds to the representation Γ (Γ') a minimal two-sided ideal I (I') in the group algebra GK (HK) which is isomorphic to the full matrix ring over the skewfield D (D'). If $\Gamma \downarrow H$ contains Γ' α times, then the representation Γ'^G contains Γ $\alpha \cdot \frac{d'}{d}$ times, where $d(d')$ is the dimension of the skewfield D (D') over K.

Note that two linear representations of the group G over a field of characteristic 0 are equivalent if and only if they have the same characters ([**2**], (30.14)).

LEMMA 5. *Let $H \Delta G$ and let K be any field of characteristic* 0. *Then the number of K-characters of the group G induced from the irreducible K-characters of H is equal to the number of K-conjugacy classes of G which are in H.*

PROOF. Let α and β be irreducible K-characters of H. If $\theta = \alpha^G = \beta^G$ and χ is an irreducible component of θ then by Lemma 4 α and β are irreducible components of $\chi \downarrow H$, and from (1) it follows that α and β are G-conjugate. Conversely, if α and β are G-conjugate then a straightforward calculation shows that $\alpha^G = \beta^G$. Thus

$$\alpha^G = \beta^G \quad \text{if and only if } \beta = \alpha^g \quad \text{for some } g \in G . \tag{6}$$

Let $K_1, K_2, \cdots, K_q$ be the K-conjugacy classes of H and $t_i = \sum_{x \in K_i} x$ $(i=1, \cdots, q)$.

Consider the group $F=\left\{\phi_g \mid \phi_g=\binom{h}{g^{-1}hg}, g\in G\right\}$. Then F is the group of linear transformations of the vector space HK. Let $e_1, e_2, \cdots, e_q$ ($\chi_1, \chi_2, \cdots, \chi_q$) be the minimal central idempotents of HK (irreducible K-characters of H). Then by (4)

$$e_i=\frac{n_i}{m_i|H|}\sum_{h\in H}\chi_i(h^{-1})h\,,$$

and hence

$$e_i^{\phi_g}=\frac{n_i}{m_i|H|}\sum_{h\in H}\chi_i(h^{-1})(g^{-1}hg)=\frac{n_i}{m_i|H|}\sum_{h\in H}\chi_i^g(h^{-1})h\,.$$

Clearly $\chi_i^g=\chi_j$ implies $n_i=n_j$ and $m_i=m_j$. Therefore $\chi_i^g=\chi_j$ implies $e_i^{\phi_g}=e_j$. On the other hand, $e_i^{\phi_g}=e_j$ implies $\frac{n_i}{m_i}\chi_i^g=\frac{n_j}{m_j}\chi_j$ and hence χ_i^g and χ_j have the same absolutely irreducible components and so $\chi_i^g=\chi_j$. Thus,

$$e_i^{\phi_g}=e_j \qquad \text{if and only if } \chi_i^g=\chi_j\,. \tag{7}$$

F is the group of automorphisms of H and so each element of F permutes the K-conjugacy classes of H or the elements $t_1, t_2, \cdots, t_q$ in the group algebra HK.

Let $V=\langle t_1, t_2, \cdots, t_q\rangle=\langle e_1, e_2, \cdots, e_q\rangle$ (see (5)). Then F is the group of linear transformations of the vector space V which permutes the elements of the sets $M=\langle t_1, \cdots, t_q\rangle$ and $N=\langle e_1, \cdots, e_q\rangle$. Let $V_0=\{v\in V \mid v^{\phi_g}=v$ for every $g\in G\}$, $M/F=\{T_1, T_2, \cdots, T_k\}$, $N/F=\{S_1, \cdots, S_l\}$, and $u_i=\sum_{x\in T_i}x$, $w_j=\sum_{y\in S_j}y$ $(1\leqq i\leqq k,\ 1\leqq j\leqq l)$. If $v=\lambda_1e_1+\cdots+\lambda_ie_i+\cdots+\lambda_je_j+\cdots\in V_0$ and e_i, e_j are in the same orbit then $e_i^{\phi_g}=e_j$ for some $g\in G$. Furthermore, $v^{\phi_g}=\lambda_1e_1^{\phi_g}+\cdots+\lambda_ie_j+\cdots=\lambda_1e_1+\cdots+\lambda_je_j+\cdots$. Hence $\lambda_i=\lambda_j$ and so v is a linear combination of $\{w_1, w_2, \cdots, w_l\}$. Since $w_j\in V_0$ $(j=1, 2, \cdots, l)$ the set $\{w_1, \cdots, w_l\}$ is a basis for V_0. The same argument shows that $\{u_1, \cdots, u_k\}$ is a basis for V_0 and thus $k=l$. The number of orbits in $\{e_1, e_2, \cdots, e_q\}$ is the number of different K-characters of G induced from irreducible K-characters of H, (see (6) and (7)), while the number of orbits in $\{t_1, t_2, \cdots, t_q\}$ is the number of K-conjugacy classes of G which are in H (Lemma 1). This completes the proof of the lemma.

Let $\chi_1, \chi_2, \cdots, \chi_r$ be the irreducible K-characters of the group G where K is any field of characteristic 0.

Then $F=\left\{f_\lambda \mid f_\lambda=\binom{\chi_i}{\lambda\chi_i};\ \lambda\in T=\mathrm{Hom}\,(G, K^*)\right\}$ is the permutation group acting on the set $S=\{\chi_1, \chi_2, \cdots, \chi_r\}$.

LEMMA 6. *$|S/F|$ is equal to the number of distinct K-characters of the group G which are induced from the irreducible K-characters of G_K.*

PROOF. Let $\lambda_1, \lambda_2, \cdots, \lambda_e$ be the linear K-characters of G. Then by Lemma

2, $e=(G:G_K)$ and all linear K-characters of G are characters of G/G_K. If ϕ is an irreducible K-character of G_K and χ is an irreducible component of ϕ^G then (1) implies $\chi\downarrow G_K=m(\phi_1+\cdots+\phi_t)$ $(\phi=\phi_t)$, and thus we have

$$(\chi\downarrow G_K)^G=mt\phi^G\,.$$

On the other hand, by Lemma 3,

$$(\chi\downarrow G_K)^G=\chi\rho=\chi(\lambda_1+\cdots+\lambda_e)$$

and hence

$$mt\phi^G=\chi\lambda_1+\cdots+\chi\lambda_e\,.$$

Thus the set of irreducible components of the character ϕ^G is an element of S/F. Let $\theta_1{}^G, \theta_2{}^G, \cdots, \theta_s{}^G$ be the distinct characters of G induced from the irreducible K-characters of G_K, and let $M_i\in S/F$ be the set of irreducible components of $\theta_i{}^G$ $(i=1, 2, \cdots, s)$.

Suppose $M_i=M_j$ i.e. $\theta_i{}^G$ and $\theta_j{}^G$ have the same irreducible components. Then θ_i and θ_j are irreducible components of $\chi\downarrow G_K$ for any $\chi\in M_i=M_j$ (Lemma 4). Hence θ_i and θ_j are G-conjugate (by (1)) and $\theta_i{}^G=\theta_j{}^G$. Finally, let χ be any irreducible K-character of G. Then if ϕ is an irreducible component of $\chi\downarrow G_K$, χ is an irreducible component of ϕ^G (Lemma 4) where for some i, $1\leqq i\leqq s$, $\phi^G=\theta_i{}^G$. Hence $\chi\in M_i$, which proves the lemma.

III. The number of linearizable irreducible projective representations of G over the field K of characteristic 0.

THEOREM. *Let $(\hat{G}, \psi)$ be the finite central group extension of G by which all the linearizable projective representations of G over the field K of characteristic 0 are linearized, and let $A=\mathrm{Ker}\,\psi$. Then the number of equivalence classes of irreducible linearizable projective representations of G over K is equal to the number of K-conjugacy classes of the group $\hat{G}/A_K$ which are in $\hat{G}_K/A_K$.*

PROOF. Let S be the set of irreducible K-characters of $\hat{G}$ such that $\chi\downarrow A=\chi(1)\lambda_\chi$ $(\lambda_\chi\in\mathrm{Hom}\,(A, K^*))$ for any $\chi\in S$. It is clear that $\chi\in S$ implies $\mu\chi\in S$ for arbitrary $\mu\in\mathrm{Hom}\,(\hat{G}, K^*)$. If $\mu=\lambda\downarrow A$ where $\lambda\in\mathrm{Hom}\,(\hat{G}, K^*)$ then $\mathrm{Ker}\,\lambda\supseteqq\mathrm{Ker}\,\mu\supseteqq A_K$ by Lemma 2. Hence $A_K\subseteqq\hat{G}_K$, since from Lemma 2 it follows that $\hat{G}_K$ is the intersection of kernels of all linear K-characters of $\hat{G}$. For $\chi\in S$, $\chi\downarrow A=\chi(1)\lambda_\chi$ implies $\mathrm{Ker}\,\chi\supseteqq\mathrm{Ker}\,\lambda_\chi\supseteqq A_k$ i.e. χ is the irreducible K-character of $\hat{G}/A_K$. Conversely, let χ be any irreducible character of $\hat{G}/A_K$. Then $\chi\downarrow A$ is the character of A/A_K and by (1) $\chi\downarrow A$ is the sum of $\hat{G}$-conjugate linear characters of A. Since $A\subseteqq Z(\hat{G})$, $\chi\downarrow A=\chi(1)\lambda_\chi$ for some $\lambda_\chi\in\mathrm{Hom}\,(A, K^*)$. Hence S is the full set of irreducible K-characters of the factor group $\hat{G}/A_K$. All linear K-characters of $\hat{G}$ are characters of $\hat{G}/A_K$ and we can consider the action of F on S (see Lemma 6). It follows from Lemma 6 that $|S/F|$ is the number of distinct K-characters of the group $\hat{G}/A_K$ which are induced from irreducible K-characters of $(\hat{G}/A_K)_K$. On the other hand, $(\hat{G}/A_K)_K=\hat{G}_K/A_K$, (see (2)), and by Lemma 5 $|S/F|$ is the number of K-conjugacy classes of $\hat{G}/A_K$ which are in $\hat{G}_K/A_K$. Let $\rho_1, \rho_2, \cdots, \rho_t$ be the full set of representatives of equivalence classes of irreducible linearizable projective representations of G over K. Then for $\rho_i: G\to PGL(V_i)$ there exists a linear representation Γ_i with character $\chi_i\in S$ $(\Gamma_i: \hat{G}\to GL(V_i))$ such that $\rho_i[\psi(x)]=\pi\Gamma_i(x)$ for every $x\in\hat{G}$ $(i=1, 2, \cdots, t)$. Denote by M_i the orbit with representative χ_i under the action of F $(i=1, 2, \cdots, t)$. Suppose $M_i=M_j$ i. e. $\chi_j=\lambda\chi_i$ for some $\lambda\in\mathrm{Hom}(\hat{G}, K^*)$. Then the linear representation $\lambda\Gamma_i: \hat{G}\to GL(V_i)$ is equivalent to $\Gamma_j: \hat{G}\to GL(V_j)$. Thus there exists a linear isomorphism $\phi: V_i\to V_j$ such that $\Gamma_j(x)=\phi\lambda(x)\Gamma_i(x)\phi^{-1}$ for every $x\in\hat{G}$. Therefore

$$\begin{aligned}\rho_j[\psi(x)]&=\pi\Gamma_j(x)=\pi\phi[\lambda(x)\Gamma_i(x)]\phi^{-1}=\tilde{\phi}[\pi\lambda(x)\Gamma_i(x)]\\&=\tilde{\phi}[\pi\Gamma_i(x)]=\tilde{\phi}\rho_i[\psi(x)]\end{aligned}$$

and we have $\rho_i\sim\rho_j$.

Now let $\chi\in S$ be an irreducible K-character of the linear representation $\Gamma: \hat{G}\to GL(V)$. Then the projective representation $\rho: G\to PGL(V)$, $\rho[\psi(x)]=\pi\Gamma(x)$ $(x\in\hat{G})$ is equivalent to some ρ_i $(1\leqq i\leqq t)$, and therefore there exists a linear isomorphism $\phi: V\to V_i$ such that $\rho_i[\psi(x)]=\tilde{\phi}\rho[\psi(x)]$. Hence

$$\pi\Gamma_i(x)=\tilde{\phi}[\pi\Gamma(x)]=\pi\phi\Gamma(x)\phi^{-1}$$

or

$$\Gamma_i(x)=\alpha(x)\phi\Gamma(x)\phi^{-1}\qquad\text{for some } \alpha: G\to K^*\,.$$

It is clear that $\alpha(1)=1$. On the other hand, $\Gamma_i(xy)=\Gamma_i(x)\Gamma_i(y)$ and $\Gamma(xy)=\Gamma(x)\Gamma(y)$ imply $\alpha(xy)=\alpha(x)\alpha(y)$ and hence $\alpha\in\mathrm{Hom}(\hat{G}, K^*)$. Thus the linear representations Γ_i and $\alpha\Gamma$ of the group $\hat{G}$ are equivalent, $\chi_i=\alpha\chi$ and $\chi\in M_i$ from which follows that $t=|S/F|$. This completes the proof of the theorem.

COROLLARY 1. *Let K be an algebraically closed field of characteristic* 0. *Then the number of equivalence classes of irreducible projective representations of G over K is equal to the number of conjugacy classes of representation-group $\hat{G}$ of G which are in $\hat{G}'$.*

PROOF. This is straightforward since $\hat{G}_K=\hat{G}'$, $A_K=1$ and each K-conjugacy class of $\hat{G}$ is a conjugacy class of $\hat{G}$.

COROLLARY 2. *Let K be the real number field. Then the number of equivalence classes of irreducible projective representations of G over K is equal to the number of K-conjugacy classes of the representation-group $\hat{G}$ of G which are in $\hat{G}_K$. Here $\hat{G}_K=\hat{G}$ if $2\nmid(\hat{G}:\hat{G}')$, and $\hat{G}_K$ is a minimal normal subgroup of $\hat{G}$ such that the factor-group $\hat{G}/\hat{G}_K$ is an elementary abelian 2-group if $2\backslash(\hat{G}:\hat{G}')$.*

PROOF. Let $\hat{G}$ be a representation-group of G over K. The group $H^2(G, K^*)$ is an elementary abelian 2-group ([4], Remark 3) and from ([4], p. 32) it follows that Hom (A, K^*) is an elementary abelian 2-group. Thus A is an elementary abelian 2-group and $A_K=1$. Now apply the theorem.

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G. KARPILOVSKY
School of Mathematics
The University of New South Wales
Kensington, New South Wales
Australia 2033