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Homomorphism of the Lie algebras of vector fields

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Introduction.

The Lie algebra $\mathcal{A}(M)$ formed by vector fields on a smooth manifold M gives an important example of infinite dimensional Lie algebra and has a geometric significance for the manifold theory. A basic theorem states that the Lie algebra structure of $\mathcal{A}(M)$ completely determines the underlying smooth structure of M. Namely, for two smooth manifolds M and N and for any Lie algebra isomorphism φ of $\mathcal{A}(M)$ onto $\mathcal{A}(N)$, we can find a diffeomorphism Φ of M onto N satisfying $\varphi=\Phi_*$ ([2], [5]). Our investigation starts from the observation of this theorem. In this paper we shall consider not only isomorphisms but also homomorphisms of $\mathcal{A}(M)$ into $\mathcal{A}(N)$ and study the relation between M and N.

There is a non-trivial homomorphism of $\mathcal{A}(M)$ into $\mathcal{A}(N)$ when N has some bundle structure over a product manifold of copies of M. We shall prove that if N is compact and there is a non-trivial homomorphism, then N is necessarily related to M in such a manner; hence, in particular, we have $\dim M \leqq \dim N$, and M is also compact. We deduce these results from the fact that any homomorphism is, in a sense, of local character and can be expressed by the use of the jets of vector fields.

We shall describe an outline of this paper. In § 1 we shall determine the local form of a homomorphism. Let φ be a homomorphism of $\mathcal{A}(M)$ into $\mathcal{A}(N)$. For a generic point q of N we can find a finite number of points $p_1, \cdots, p_l$ of M and charts $\{U_\nu; (x_\nu)=(x_\nu^1, \cdots, x_\nu^n)\}$ near p_ν and $\{U; (x_*, y)\}$ near q with

$$(x_*, y)=(x_1, \cdots, x_l, y)=(x_1^1, \cdots, x_1^n, \cdots, x_l^n, y^1, \cdots, y^{d-nl}),$$

which satisfy the following property:

For any $X \in \mathcal{A}(M)$ with $X=\sum_i f_\nu^i(x_\nu)\partial_{x_\nu^i}$ on each U_ν we have

$$\varphi(X)(x_*, y)=\sum_{\nu=1}^{l}\sum_{i=1}^{n}\Big(f_\nu^i(x_\nu)\partial_{x_\nu^i}+\sum_{0<|\alpha|\leqq h}\frac{D^\alpha}{\alpha!}f_\nu^i(x_\nu)Y_{i\nu}^\alpha(y)\Big)$$

on U for some integer h and vector fields $Y_{i\nu}^\alpha(y)=\sum_j Y_{i\nu}^{\alpha j}(y)\partial_{y^j}$ where $\partial_{x_\nu^i}$ denotes the vector field $\partial/\partial x_\nu^i$.

If N is compact and φ is non-trivial, then N has a rahter restricted structure subject to M. The relation between M and N will be clarified in § 2 as follows: For any positive integer l, let M_l be a smooth manifold formed by all the sets of distinct l points of M and put $N_0=\{q\in N \mid \varphi(X)$ vanishes at q for any $X\in\mathcal{A}(M)\}$. Then N is a finite disjoint union of N_0 and some topological fibre bundles N_l over M_l. The bundle N_l is closely related to the jet bundle of the tangent bundle of $M^l=M\times\cdots\times M$. Actually, we can construct many examples of homomorphisms which yield such situations. However we have no example such that $N_0\neq\emptyset$ or N_l is not a smooth bundle. Since the behaviour of $\varphi(X)$ near q depends only on the behaviour of X near $p_1, \cdots, p_l$, we can consider the germ of φ at $(q; p_1, \cdots, p_l)$. We say φ is transitive at q if the image $\varphi(\mathcal{A}(M))$ is transitive at q. In § 3 we shall show that the classification of the transitive germs can be reduced to that of certain subalgebras of $\overset{l}{\oplus}\mathfrak{g}(n, h)$ where $\mathfrak{g}(n, h)$ is the finite dimensional Lie algebra formed by the h-jects of vector fields on $\boldsymbol{R}^n$ vanishing at 0. In § 4 we shall prove that any homomorphism of $\mathcal{A}(M)$ into $\mathcal{A}(N)$ is necessarily continuous in the C^∞-topology. As a consequence, when N is compact, it follows from [**4**; Theorem 1.3.2] that φ induces a local homomorphism between the diffeomorphism groups of M and N. This establishes an analogy to the corresponding theorem known for finite dimensional Lie algebras and Lie groups.

Some of our results were announced in [**3**].

§ 1. Local normal form of a homomorphism.

For any smooth manifold M, we denote by $\mathcal{A}(M)$ the Lie algebra formed by all the smooth vector fields on M under the usual bracket operation. Let $\varphi: \mathcal{A}(M)\to\mathcal{A}(N)$ be a Lie algebra homomorphism. In this section we shall give an explicit expression of φ in terms of local coordinate systems on M and N. For this purpose, we first establish the following theorem concerning a characterization of the subalgebra of $\mathcal{A}(M)$ with finite codimension, essentially due to I. Amemiya [**1**]. We consider $\mathcal{A}(M)$ as a $C^\infty(M)$-module under the usual multiplication. For any point p of M, we put $\mathcal{M}_p=\{f\in C^\infty(M) \mid f(p)=0\}$.

THEOREM 1. *Let $\mathcal{B}$ be a proper subalgebra of $\mathcal{A}(M)$ with* codim $\mathcal{B}=d<\infty$. *Then we can find a finite number of points $p_1, \cdots, p_l$ of M such that the relation*

$$\bigcap_{\nu=1}^{l}\mathcal{M}_{p_\nu}\mathcal{A}(M)\supset\mathcal{B}\supset\bigcap_{\nu=1}^{l}\mathcal{M}_{p_\nu}^{h+1}\mathcal{A}(M)$$

holds for $h=2((d-nl)^2+d-nl)+1$ where $n=\dim M$. Moreover we have $l\leqq d/n$.

In order to prove this theorem, we need two lemmas. For any open set

U of M, we put $\mathcal{A}_U=\{X\in\mathcal{A}(M) \mid \operatorname{supp} X\subset U\}$.

LEMMA 1. *Let* $\mathcal{B}$ *be as in Theorem* 1. *Suppose that there are* $Z\in\mathcal{A}=\mathcal{A}(M)$ *and* $g\in C^\infty(M)$ *such that* $Z(g)\equiv 1$ *on* U. *Then we can find a non-trivial polynomial* P *with* $\deg P\leqq 2(d^2+d)$ *such that* $P(g)\mathcal{A}_U\subset\mathcal{B}$.

PROOF. Put $\mathcal{B}'=\{X\in\mathcal{B} \mid [X, Y]\in\mathcal{B}$ for every $Y\in\mathcal{A}\}$. For any $X\in\mathcal{B}$, $\operatorname{ad} X: Y\to[X, Y]$ induces a linear transformation $T_X:\mathcal{A}/\mathcal{B}\to\mathcal{A}/\mathcal{B}$. Since $\mathcal{B}'$ is the kernel of the map $X\to T_X$ of $\mathcal{B}$ into the space of endomorphisms of $\mathcal{A}/\mathcal{B}$, we have $\operatorname{codim}\mathcal{B}'\leqq d^2+d$. Let $\mathcal{P}$ be the space of all polynomials and put $\mathcal{P}'=\{P\in\mathcal{P} \mid gP(g)Z, P(g)Z\in\mathcal{B}'\}$. Then we have $\operatorname{codim}\mathcal{P}'$ in $\mathcal{P}\leqq 2(d^2+d)$ since $\mathcal{P}'$ is the kernel of the map $\mathcal{P}\to\mathcal{A}/\mathcal{B}'\oplus\mathcal{A}/\mathcal{B}'$ induced by the map $P\to gP(g)Z \oplus P(g)Z$. Hence we can find a non-trivial polynomial $P\in\mathcal{P}'$ with $\deg P\leqq 2(d^2+d)$. For any $X\in\mathcal{A}_U$ we have

$$\mathcal{B}\ni[P(g)Z, gX]=g[P(g)Z, X]+P(g)Z(g)X,$$

$$\mathcal{B}\ni[gP(g)Z, X]=g[P(g)Z, X]-X(g)P(g)Z$$

and hence

$$(*)\qquad \mathcal{B}\ni P(g)X+X(g)P(g)Z.$$

Substituting $X(g)Z\in\mathcal{A}_U$ for X in (*), we obtain $\mathcal{B}\ni 2X(g)P(g)Z$, which combined with (*), gives $\mathcal{B}\ni P(g)X$. This completes the proof.

LEMMA 2. *Let* U_ν $(\nu=1, 2, \cdots)$ *be open sets of* M *such that* $\bar{U}_\nu$*'s are disjoint and locally finite and let* $(x_\nu)=(x_\nu^1, \cdots, x_\nu^n)$ *be a coordinate system on* U_ν. *Then there are a finite number of integers* $\nu_1, \cdots, \nu_l$ $(l\leqq 2(d^2+d))$ *such that* $\mathcal{B}\supset\mathcal{A}_{U'}$ *for* $U'=\bigcup_\nu U_\nu-\bigcup_{i=1}^{l} U_{\nu_i}$.

PROOF. Hereafter we shall denote by $\partial_{x_\nu^i}$ the vector field $\partial/\partial x_\nu^i$. Choose $Z\in\mathcal{A}(M)$ and $g\in C^\infty(M)$ such that $Z=\partial_{x_\nu^1}$ and $g=x_\nu^1+$constant on every U_ν. We may assume that $\nu<g<\nu+1$ on U_ν. Since $Z(g)=1$ on $U=\cup U_\nu$, by Lemma 1 we have $P(g)\mathcal{A}_U\subset\mathcal{B}$ for some polynomial P with $\deg P\leqq 2(d^2+d)$. We can take integers $\nu_1, \cdots, \nu_l$ for which we have $P(g)\neq 0$ on $U'=U-\bigcup_{i=1}^{l} U_{\nu_i}$. Then for any $Y\in\mathcal{A}_{U'}$ there is $X\in\mathcal{A}_{U'}\subset\mathcal{A}_U$ such that $Y=P(g)X$ and hence $Y\in\mathcal{B}$, which completes the proof.

PROOF OF THEOREM 1. We say a point p of M is *singular* if for any neighborhood U of p we have $\mathcal{B}\not\supset\mathcal{A}_U$. Then by Lemma 2 the number of singular points is at most $2(d^2+d)$. Let $\{p_1, \cdots, p_l\}$ be the set of singular points. We show that $\mathcal{B}\supset\bigcap_{\nu=1}^{l}\mathcal{M}_{p_\nu}^m\mathcal{A}_U$ where $m=2(d^2+d)$ and U is some neighborhood of the set $\{p_1, \cdots, p_l\}$. For each ν let (x_ν) be a coordinate system on some neighborhood U_ν of p_ν with $p_\nu=(0)$. Choose $Z\in\mathcal{A}(M)$ and $g\in C^\infty(M)$ such that $Z=\partial_{x_\nu^1}$ and $g=x_\nu^1$ on each U_ν. Then by Lemma 1 there is a polynomial $P(t)=t^p(1+at+\cdots)$ $(p\leqq m)$ for which we have $P(g)\mathcal{A}_{U'}\subset\mathcal{B}$ for $U'=\cup U_\nu$.

Since $1+ag+\cdots\neq 0$ on some neighborhood U'' of $\{p_1, \cdots\}$, we have $g^p\mathcal{A}_{U''}\subset\mathcal{B}$ and hence $g^m\mathcal{A}_{U'}\subset\mathcal{B}$. For any $g\in C^\infty(M)$ which is a homogeneous polynomial of (x_ν^i) of degree 1 on each U_ν, we have the same relation $g^m\mathcal{A}_V\subset\mathcal{B}$ for some V. Since any homogeneous polynomial of degree m is a linear combination of some m'th powers of homogeneous polynomials of degree 1, we have $\cap\mathcal{M}_{p_\nu}^m\mathcal{A}_U\subset\mathcal{B}$ for some U as desired. Next, we prove that $\mathcal{B}\supset\cap\mathcal{M}_{p_\nu}^m\mathcal{A}(M)$. For any $p\in M-\{p_1, \cdots\}$, by definition, we have $\mathcal{B}\supset\mathcal{A}_{U_p}$ for some neighborhood U_p of p. Then $\{U\}\cup\{U_p\}_p$ covers M. According to the dimension theory, M admits a finite open covering $\{U, U_1, \cdots, U_{n+1}\}$ such that for each $i\leqq n+1$, $U_i=\bigcup_j U_{ij}$ where $U_{i1}, U_{i2}, \cdots$ satisfy the conditions in Lemma 2 and each U_{ij} is contained in some U_p. By Lemma 2 there are a finite number of integers $j_1, j_2, \cdots$ such that $\mathcal{B}\supset\mathcal{A}_{U'_i}$ for $U'_i=U_i-\bigcup_k U_{ij_k}$. Since $\mathcal{B}\supset\mathcal{A}_{U_{ij_k}}$, using the partition of unity subordinate to the finite covering $\{U, U'_1, U_{1j_1}, U_{1j_2}, \cdots, U'_2, \cdots\}$ of M, we have $\mathcal{B}\supset\cap\mathcal{M}_{p_\nu}^m\mathcal{A}(M)$ as desired. Next, we show that $\cap\mathcal{M}_{p_\nu}\mathcal{A}(M)\supset\mathcal{B}$. Assume the contrary. Then there is $Z\in\mathcal{B}$ which does not vanish at some p_ν. We can take a coordinate system $(x^1, \cdots, x^n)$ on some neighborhood U of p_ν such that $Z=\partial_{x^1}$ on U. Choose $g\in C^\infty(M)$ satisfying $g=x^1$ on U. Then by Lemma 1 we have $\mathcal{B}\supset P(g)\mathcal{A}_U$ for some polynomial P. For any $Y\in\mathcal{A}_U$, we have $\mathcal{B}\ni[P(g)Y, Z]=-P'(g)Z(g)Y+P(g)[Y, Z]$ and hence $\mathcal{B}\ni -P'(g)Z(g)Y=-P'(g)Y$, which implies $\mathcal{B}\supset P'(g)\mathcal{A}_U$. Applying the same argument successively, we have $\mathcal{B}\supset\mathcal{A}_U$, which contradicts the fact that p_ν is singular. Therefore we have $\cap\mathcal{M}_{p_\nu}\mathcal{A}(M)\supset\mathcal{B}$. Since $\operatorname{codim}\cap\mathcal{M}_{p_\nu}\mathcal{A}(M)$ is nl, we have $l\leqq d/n$. We must show $\mathcal{B}\supset\cap\mathcal{M}_{p_\nu}^{h+1}\mathcal{A}(M)$ instead of $\mathcal{B}\supset\cap\mathcal{M}_{p_\nu}^m\mathcal{A}(M)$. Choose (x_ν), Z and g as in the proof of $\mathcal{B}\supset\bigcap_{\nu=1}^l\mathcal{M}_{p_\nu}^m\mathcal{A}_U$. Then the argument similar to the proof of Lemma 1 shows that there is a polynomial P with $P(0)=0$ and $\deg P\leqq 2(e^2+e)+1=h$ where $e=d-nl=\operatorname{codim}\mathcal{B}$ in $\cap\mathcal{M}_{p_\nu}\mathcal{A}(M)$ such that we have $P(g)\cap\mathcal{M}_{p_\nu}\mathcal{A}_U\subset\mathcal{B}$ for $U=\cup U_\nu$. Then the same argument as above shows that $\mathcal{B}\supset\cap\mathcal{M}_{p_\nu}^{h+1}\mathcal{A}(M)$, which completes the proof of Theorem 1.

Now, let $\varphi:\mathcal{A}(M)\to\mathcal{A}(N)$ be a *non-trivial* Lie algebra homomorphism. Throughout this paper we assume that M and N are *connected* and have no boundary and $\dim M=n$ and $\dim N=d$ are positive. Put $\mathcal{A}_q(N)=\mathcal{M}_q\mathcal{A}(N)$ and $N^+=\{q\in N \mid \varphi^{-1}\mathcal{A}_q(N)\neq\mathcal{A}(M)\}$. Then $q\in N$ belongs to N^+ if and only if there is $X\in\mathcal{A}(M)$ such that $\varphi(X)(q)$, the value of $\varphi(X)$ at q, $\neq 0$. Hence N^+ is non-empty open subset of N. For any $q\in N^+$ we have $\operatorname{codim}\varphi^{-1}\mathcal{A}_q(N)\leqq\operatorname{codim}\mathcal{A}_q(N)=d<\infty$, hence by Theorem 1, there are points $p_1, \cdots, p_l$ of M such that

$$(1)\qquad \bigcap_{\nu=1}^l\mathcal{M}_{p_\nu}\mathcal{A}(M)\supset\varphi^{-1}\mathcal{A}_q(N)\supset\bigcap_{\nu=1}^l\mathcal{M}_{p_\nu}^{h+1}\mathcal{A}(M)$$

holds for $h=2((d-nl)^2+d-nl)+1$.

LEMMA 3. *For any $X\in\mathcal{A}(M)$, $\varphi(X)(q)$ is determined by the h-jets of X at $p_1, \cdots, p_l$.*

PROOF. For each ν let $(x_\nu)=(x_\nu^1, \cdots, x_\nu^n)$ be a coordinate system on some neighborhood U_ν of p_ν and (a_ν) the coordinates of p_ν. For any multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$, choose $Y_{i\nu}^\alpha\in\mathcal{A}(M)$ such that $Y_{i\nu}^\alpha=x_\nu^\alpha\partial_{x_\nu^i}=(x_\nu^1)^{\alpha_1}\cdots(x_\nu^n)^{\alpha_n}\partial_{x_\nu^i}$ on some neighborhood of p_ν and $\operatorname{supp} Y_{i\nu}^\alpha\subset U_\nu$. We assume that U_ν's are disjoint. If $X=\sum_i f_\nu^i(x_\nu)\partial_{x_\nu^i}$ on each U_ν, we have

$$X-\sum_{\nu,i}\sum_{|\alpha|\leqq h}\frac{D^\alpha}{\alpha!}f_\nu^i(a_\nu)\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}(-a_\nu)^{\alpha-\beta}Y_{i\nu}^\beta\in\cap\mathcal{M}_{p_\nu}^{h+1}\mathcal{A}(M)\subset\varphi^{-1}\mathcal{A}_q(N)\,.$$

Here we denote by D^α the differential operator $\partial^{|\alpha|}/(\partial x^1)^{\alpha_1}\cdots(\partial x^n)^{\alpha_n}$ where $|\alpha|=\alpha_1+\cdots+\alpha_n$. Therefore we have

$$\varphi(X)(q)=\sum_{\nu,i}\sum_{|\alpha|\leqq h}\frac{D^\alpha}{\alpha!}f_\nu^i(a_\nu)\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}(-a_\nu)^{\alpha-\beta}\varphi(Y_{i\nu}^\beta)(q)\,, \tag{2}$$

which completes the proof.

Note that the set $\{p_1, \cdots, p_l\}$ is uniquely determined by (1). The set $\{p_1, \cdots, p_l\}$ is denoted by $\psi(q)$ for $q\in N^+$. Let k $(\leqq d/n)$ be the maximal number of p_ν's when q ranges over N^+, and for each $l\leqq k$, let N_l be the set of points q's of N^+ such that the number of the corresponding points is l.

EXAMPLE 1. Let $\varphi:\mathcal{A}(\boldsymbol{R}^1)\to\mathcal{A}(\boldsymbol{R}^3)$ be a homomorphism given by

$$\varphi(f(x)\partial_x)=f(x)\partial_x+f(y)\partial_y+(f'(x)+f'(y))\alpha(z)\partial_z$$

where $\alpha(z)$ is a smooth function. For a point $q=(a, b, c)\in\boldsymbol{R}^3$ we have

$$\varphi^{-1}\mathcal{A}_{(a,b,c)}(\boldsymbol{R}^3)=\{f(x)\partial_x\mid f(a)=f(b)=(f'(a)+f'(b))\alpha(c)=0\}$$

and hence

$$\mathcal{M}_a\cap\mathcal{M}_b\mathcal{A}(\boldsymbol{R}^1)\supset\varphi^{-1}\mathcal{A}_{(a,b,c)}(\boldsymbol{R}^3)\supset\mathcal{M}_a^2\cap\mathcal{M}_b^2\mathcal{A}(\boldsymbol{R}^1)\,.$$

Therefore we obtain

$$\psi((a, b, c))=\{a, b\}\ \text{when}\ a\neq b,\ =\{a\}\ \text{when}\ a=b\,,$$

$$N_1=\{(x, x, z)\in\boldsymbol{R}^3\}\ \text{and}\ N_2=\{(x, y, z)\in\boldsymbol{R}^3\mid x\neq y\}\,.$$

Now we shall study the set $\psi(q)$.

LEMMA 4. *For each $l\leqq k$, $N_k\cup N_{k-1}\cup\cdots\cup N_l$ is an open subset of N^+. Let q be a point of N_l with $\psi(q)=\{p_1, \cdots, p_l\}$ and U_ν a neighborhood of p_ν for each ν such that $U_\nu\cap U_\mu=\emptyset$ for $\nu\neq\mu$. Then there are a neighborhood U of q and a continuous map $\tilde{\psi}: U\cap N_l\to U_1\times\cdots\times U_l\subset M^l=M\times\cdots\times M$ such that for any $q'\in U\cap N_l$, $\tilde{\psi}(q')=(p_1', \cdots, p_l')$ implies $\psi(q')=\{p_1', \cdots, p_l'\}$.*

PROOF. Fix $l\leqq k$ and $q\in N_l$. It suffices to show that there is a neighborhood U of q such that for any $q'\in U$ there are points p'_ν of U_ν for all ν satisfying $\psi(q')\supset\{p'_1, \cdots, p'_l\}$. Assume the contrary. Then there is a sequence of points $\{q_i\}$ converging to q such that for some fixed ν', $\psi(q_i)\cap U_{\nu'}=\{p_{i1}, \cdots, p_{im}\}\cap U_{\nu'}=\emptyset$ for all i $(0\leqq m\leqq k)$. Choose $X\in\mathcal{A}(M)$ satisfying supp $X\subset U_{\nu'}$ and $X(p_{\nu'})\neq 0$. Then by the definition of ψ we have

$$X\in\bigcap_{\nu=1}^{m}\mathcal{M}_{pi\nu}^{h+1}\mathcal{A}(M)\subset\varphi^{-1}\mathcal{A}_{q_i}(N) \quad \text{and} \quad X\notin\bigcap_{\nu=1}^{l}\mathcal{M}_{p_\nu}\mathcal{A}(M)\supset\varphi^{-1}\mathcal{A}_q(N)\,.$$

Therefore we obtain $\varphi(X)(q_i)=0$ and $\varphi(X)(q)\neq 0$, which is a contradiction. This completes the proof.

Now let q be a point of Int N_l, the topological interior of N_l in N. Then Lemma 3 implies the next

LEMMA 5. *Let U and U_ν be the neighborhoods given in Lemma 4. Then for any $X\in\mathcal{A}(M)$, $\varphi(X)|U$, the restriction of $\varphi(X)$ to U, depends only on $X|\cup U_\nu$.*

By this lemma we may restrict our consideration to U and $\cup U_\nu$. In the following arguments we replace these neighborhoods by smaller ones if necessary. Choose a coordinate system $(x_\nu)=(x_\nu^i)=(x_\nu^1, \cdots, x_\nu^n)$ on U_ν for each ν. Since $\varphi(\partial_{x_\nu^i})(q)$'s are linearly independent and $[\varphi(\partial_{x_\nu^i}), \varphi(\partial_{x_\mu^j})]=\varphi[\partial_{x_\nu^i}, \partial_{x_\mu^j}]\equiv 0$, we can choose a coordinate system $(x_*, y)=(x_1, \cdots, x_l, y)=(x_1^1, \cdots, x_1^n, \cdots, x_l^n, y^1, \cdots, y^{d-nl})$ on U such that $\varphi(\partial_{x_\nu^i})=\partial_{x_\nu^i}$ for all i and ν.

LEMMA 6. *Let $\tilde{\psi}_\nu^i(x_*, y)$ be the (i, ν)-component of $\tilde{\psi}(x_*, y)\in U_1\times\cdots\times U_l$ with respect to the above coordinate systems. Then $\tilde{\psi}_\nu^i(x_*, y)=x_\nu^i+c_\nu^i(y)$ for some continuous function c_ν^i. Moreover $\tilde{\psi}$ is a smooth submersion on some open dense subset of U.*

PROOF. For any point $(a^*, b)\in U$, we have

$$X\equiv(x_\nu^i-\tilde{\psi}_\nu^i(a_*, b))^{h+1}\partial_{x_\nu^i}\in\varphi^{-1}\mathcal{A}_{(a^*,b)}(N)$$

and hence

$$0=\varphi(X)(a_*, b)=\sum_{s=0}^{h+1}\binom{h+1}{s}(-\tilde{\psi}_\nu^i(a_*, b))^{h+1-s}\varphi((x_\nu^i)^s\partial_{x_\nu^i})(a_*, b)\,.$$

We put $(z^1, \cdots, z^d)\equiv(x_*, y)$ and $Y=\sum Y^q\partial_{z^q}$ for $Y\in\mathcal{A}(U)$. Put $\tilde{\psi}_\nu^i(x_*, y)=x_\nu^i+c_\nu^i(x_*, y)$. Then $c_\nu^i(a_*, b)$ satisfies the equations

$$F_h^q(c_\nu^i, a_*, b)\equiv\sum\binom{h+1}{s}(-a_\nu^i-c_\nu^i)^{h+1-s}\varphi((x_\nu^i)^s\partial_{x_\nu^i})^q(a_*, b)=0, \quad q=1, \cdots, d\,.$$

For any j and μ we have

$$\begin{aligned}\frac{\partial}{\partial a_\mu^j}\varphi((x_\nu^i)^s\partial_{x_\nu^i})^q(a_*, b)&=[\varphi(\partial_{x_\mu^j}), \varphi((x_\nu^i)^s\partial_{x_\nu^i})]^q(a_*, b)\\&=\delta s\varphi((x_\nu^i)^{s-1}\partial_{x_\nu^i})^q(a_*, b)\end{aligned}$$

where $\delta=1$ when $j=i$ and $\mu=\nu$ and $\delta=0$ otherwise. Using this equation we have easily $\frac{\partial}{\partial a_\mu^j}F_h^q(c_\nu^i, a_*, b)\equiv 0$, and hence F_h^q is independent of (a_*). Since by Lemma 4 $c_\nu^i(a_*, b)$ is continuous and F_h^q is a polynomial with respect to c_ν^i, it follows that c_ν^i is independent of (a_*) as desired. Now we prove the second part of the lemma. Let U_0 be a non-empty open subset of U. Since $(x_\nu^i-\tilde{\phi}_\nu^i(a_*, b))^{h+1}\partial_{x_\nu^i}\in\varphi^{-1}\mathcal{A}_{(a_*,b)}(N)$ and $\partial_{x_\nu^i}\notin\varphi^{-1}\mathcal{A}_{(a_*,b)}(N)$ for any point (a_*, b) of U_0, we can find an integer $m\leqq h$ and a point (d_*, e) of U_0 such that $(x_\nu^i-\tilde{\phi}_\nu^i(a_*, b))^{m+1}\partial_{x_\nu^i}\in\varphi^{-1}\mathcal{A}_{(a_*,b)}(N)$ for any point (a_*, b) of U_0 and $(x_\nu^i-\tilde{\phi}_\nu^i(d_*, e))^m\partial_{x_\nu^i}\notin\varphi^{-1}\mathcal{A}_{(d_*,e)}(N)$. Then c_ν^i satisfies the equations $F_m^q(c_\nu^i, a_*, b)=0$ and we have

$$\frac{\partial F_m^q}{\partial c_\nu^i}(c_\nu^i(d_*, e), d_*, e)=-(m+1)\varphi((x_\nu^i-\tilde{\phi}_\nu^i(d_*, e))^m\partial_{x_\nu^i})^q(d_*, e)\neq 0$$

for some q. Therefore by the inverse function theorem, c_ν^i is smooth on some neighborhood of (d_*, e). The same arguments for other (i, ν)'s complete the proof.

Now we can prove the main theorem of this section. For each $l\leqq k$, put $N_l^+=\{q\in\operatorname{Int}N_l \mid \tilde{\phi} \text{ is smooth near } q\}$.

THEOREM 2. *$\bigcup_{l=1}^k N_l^+$ is dense in N^+. Let q be a point of N_l^+ with $\phi(q)=\{p_1, \cdots, p_l\}$ and $(x_\nu)=(x_\nu^1, \cdots, x_\nu^n)$ a coordinate system on some neighborhood U_ν of p_ν for each ν. Then there is a coordinate system $(x_*, y)=(x_1, \cdots, x_l, y)=(x_1^1, \cdots, x_1^n, \cdots, x_l^n, y^1, \cdots, y^{d-nl})$ on some neighborhood U of q satisfying the following properties:*

i) *$\tilde{\phi}(U)\subset U_1\times\cdots\times U_l$, $\tilde{\phi}(x_*, y)=(x_*)=(x_1, \cdots, x_l)$ and $\varphi(\partial_{x_\nu^i})=\partial_{x_\nu^i}$.*

ii) *For any $X\in\mathcal{A}(M)$ with $X|U_\nu=\sum_i f_\nu^i(x_\nu)\partial_{x_\nu^i}$ we have*

$$\varphi(X)|U=\sum_{\nu=1}^l\sum_{i=1}^n\Big(f_\nu^i(x_\nu)\partial_{x_\nu^i}+\sum_{0<|\alpha|\leqq h}\frac{D^\alpha}{\alpha!}f_\nu^i(x_\nu)Y_{i\nu}^\alpha(y)\Big). \tag{3}$$

Here $h=2((d-nl)^2+d-nl)+1$ and Y's are fixed vector fields such that $Y_{i\nu}^\alpha(y)=\sum_p Y_{i\nu}^{\alpha p}(y)\partial_{y^p}$ and satisfy the following relation:

$$[Y_{i\nu}^\alpha, Y_{j\mu}^\beta]=0 \quad \textit{for } \nu\neq\mu \textit{ and } [Y_{i\nu}^\alpha, Y_{j\nu}^\beta]=\beta_i Y_{j\nu}^{\alpha+\beta-i}-\alpha_j Y_{i\nu}^{\alpha+\beta-j}. \tag{4}$$

In the right hand side of the second equation we put $Y_{i\nu}^\gamma\equiv 0$ if $|\gamma|>h$. We use i instead of the multi-index α such that $\alpha_j=\delta_{ji}$ (Kronecker's δ).

REMARK 1. Note that the vector fields $x_\nu^\alpha\partial_{x_\nu^i}$'s satisfy the same relation as (4). It is easy to show that for all $Y(y)$'s satisfying the relation (4), the map $\varphi:\mathcal{A}(\cup U_\nu)\to\mathcal{A}(U)$ given by (3) is a homomorphism.

PROOF OF THEOREM 2. It follows easily from Lemma 4 and Lemma 6 that

$\cup N_l^+$ is dense in N^+. Let q be as in the theorem. Then we can choose a coordinate system (x_*, y) satisfying i). For any point (x_*, y) of U, since $\tilde{\phi}(x_*, y)=(x_*)$ we have, by (2),

$$\varphi(\sum_{\nu i} f_\nu^i(x_\nu)\partial_{x_\nu^i})(x_*, y)=\sum_{\nu i}\sum_{|\alpha|\leq h} D^\alpha f_\nu^i(x_\nu) Z_{i\nu}^\alpha(x_*, y)$$

where Z's are suitable smooth vector fields. Since $\varphi(\partial_{x_\nu^i})=\partial_{x_\nu^i}$, we have $Z_{i\nu}^0 =\partial_{x_\nu^i}$. We investigate Z's for $\alpha>0$. First, applying φ to the equation

$$[\partial_{x_\mu^j}, \sum_{\nu i} f_\nu^i(x_\nu)\partial_{x_\nu^i}]=\sum_i D^j f_\mu^i(x_\mu)\partial_{x_\mu^i},$$

we have $[\partial_{x_\mu^j}, Z_{i\nu}^\alpha]\equiv 0$ so that $Z_{i\nu}^\alpha(x_*, y)=Z_{i\nu}^\alpha(y)$. Putting $Z_{i\nu}^\alpha(y)=\sum_{\mu j} Z_{i\nu\mu}^{\alpha j}(y)\partial_{x_\mu^j} +\sum_p Z_{i\nu}^{\alpha p}(y)\partial_{y^p}$, we show that $Z_{i\nu\mu}^{\alpha j}\equiv 0$. Assume the contrary. Then there is some $Z_{i\nu\bar{\mu}}^{\bar{\alpha}\bar{j}}$ such that $Z_{i\nu\bar{\mu}}^{\bar{\alpha}\bar{j}}(y)\neq 0$ on some non-empty open set $U'\subset U$. Applying φ to the equation

$$[\sum_{\nu i} f_\nu^i(x_\nu)\partial_{x_\nu^i}, \sum_{\xi k} g_\xi^k(x_\xi)\partial_{x_\xi^k}]$$
$$=\sum_{\nu ik} f_\nu^i(x_\nu) D^i g_\nu^k(x_\nu)\partial_{x_\nu^k}-\sum_{\xi ki} g_\xi^k(x_\xi) D^k f_\xi^i(x_\xi)\partial_{x_\xi^i},$$

we have

$$(*)\qquad [\sum_{\nu i}(f_\nu^i\partial_{x_\nu^i}+\sum_{0<|\alpha|\leq h} D^\alpha f_\nu^i(\sum_{\mu j} Z_{i\nu\mu}^{\alpha j}\partial_{x_\mu^j}+\sum_p Z_{i\nu}^{\alpha p}\partial_{y^p})),\ \sum_{\xi k}(g_\xi^k\partial_{x_\xi^k}+\sum_{0<|\beta|\leq h} D^\beta g_\xi^k Z_{k\xi}^\beta)]$$
$$=\sum_{\nu ik}(f_\nu^i D^i g_\nu^k\partial_{x_\nu^k}+\sum_{0<|\alpha|\leq h}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma} D^\gamma f_\nu^i D^{i+\alpha-\gamma} g_\nu^k Z_{k\nu}^\alpha)$$
$$-\sum_{\xi ki}(g_\xi^k D^k f_\xi^i\partial_{x_\xi^i}+\sum_{0<|\alpha|\leq h}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma} D^\gamma g_\xi^k D^{k+\alpha-\gamma} f_\xi^i Z_{i\xi}^\alpha).$$

We claim that $Z_{i\bar{\mu}}^\alpha\equiv 0$ on U' for all α and i. Suppose it is not true. Then there is some $Z_{\tilde{i}\bar{\mu}}^{\tilde{\alpha}}\not\equiv 0$ such that $Z_{i\bar{\mu}}^\alpha\equiv 0$ on U' if $|\alpha|>|\tilde{\alpha}|$, or $|\alpha|=|\tilde{\alpha}|$ and $\alpha_{\bar{j}}>\tilde{\alpha}_{\bar{j}}$. Comparing the coefficients of $D^{\bar{\alpha}} f_\nu^{\bar{i}} D^{\tilde{\alpha}+\bar{j}} g_{\bar{\mu}}^{\tilde{i}}$ in both sides of (*), we have $Z_{\bar{i}\nu\bar{\mu}}^{\bar{\alpha}\bar{j}} Z_{\tilde{i}\bar{\mu}}^{\tilde{\alpha}}\equiv 0$ on U', which is a contradiction. Therefore we obtain $Z_{i\bar{\mu}}^\alpha\equiv 0$ on U' for all α and i. Next, comparing the coefficients of $D^\alpha f_\nu^{\bar{i}} D^{\bar{j}} g_{\bar{\mu}}^1\partial_{x_{\bar{\mu}}^1}$ in both sides of (*), we have $Z_{\bar{i}\nu\bar{\mu}}^{\bar{\alpha}\bar{j}}\equiv 0$ on U', which is a contradiction. Thus we have proved that $Z_{i\nu\mu}^{\alpha j}\equiv 0$ and hence that $Z_{i\nu}^\alpha(y)=\sum_p Z_{i\nu}^{\alpha p}(y)\partial_{y^p}$. Putting $Z_{i\nu}^\alpha(y)= \frac{1}{\alpha!} Y_{i\nu}^\alpha(y)$, it is easy to show that φ is a homomorphism if and only if (4) holds. This completes the proof of Theorem 2.

EXAMPLE 2. Let $G(n, h)$ be a Lie group consisting of all h-jets at 0 of diffeomorphisms of $\boldsymbol{R}^n$ fixing the origine 0 and $\mathfrak{g}(n, h)$ its Lie algebra. We assume that the diffeomorphisms act on $\boldsymbol{R}^n$ from the right, i.e., $(gh)(p)=h(g(p))$ for any diffeomorphisms g, h and any point p of $\boldsymbol{R}^n$. Then we can take $\{x^\alpha\partial_{x^i} \mid 0<|\alpha|\leqq h, i\leqq n\}$ as a basis of $\mathfrak{g}(n, h)$ with the usual bracket operation and the exponential mapping $\mathfrak{g}(n, h)\to G(n, h)$ is given by $\exp tX=$the h-jet of $\operatorname{Exp} tX$ at 0 where $\operatorname{Exp} tX$ is the 1-parameter group of local transformations generated by $X=\sum X_\alpha^i x^\alpha \partial_{x^i}$. Let $J^{h-1}TM^l$ be the $(h-1)$-jet bundle of the tangent bundle TM^l. It is a G-bundle where $G=\overset{l}{\oplus}G(n, h)$. Let P be its associated principal G-bundle. For a right G-manifold F, put $N=F\times_G P$. Then $\operatorname{Diff}(M)$, the group of all diffeomorphisms of M, acts on N naturally, namely there is a homomorphism $\Phi: \operatorname{Diff}(M)\to\operatorname{Diff}(N)$, hence we get a homomorphism $\varphi=\Phi_*: \mathcal{A}(M)\to\mathcal{A}(N)$. φ is given as follows. For $X\in\mathcal{A}(M)$ let $\operatorname{Exp} tX$ be the 1-parameter group of transformations generated by X (we assume that the manifolds are compact). Then $\varphi(X)=\Phi_*(X)$ is the infinitesimal transformation of $\Phi(\operatorname{Exp} tX)$. Put $Y_{i\nu}^\alpha=\rho_*(x_\nu^\alpha\partial_{x_\nu^i})\in\mathcal{A}(F)$ where $\rho: G\to\operatorname{Diff}(F)$ is a homomorphism induced by the action of G on F and $\rho_*: \overset{l}{\oplus}\mathfrak{g}(n, h)\to\mathcal{A}(F)$ is a homomorphism induced by ρ. Let $U_1\times\cdots\times U_l\times F=U\subset N$ be a local trivial structure of N. Then it is easy to show that φ is given by the formula (3) in Theorem 2. Hence the map $\tilde{\psi}$ is the projection map $N\to M^l$ in this case.

REMARK 2. Let $V_1\times\cdots\times V_l\times F=V\subset N$ be another local trivial structure of N and let g be a diffeomorphism of $W=(U_1\cap V_1)\times\cdots\times(U_l\cap V_l)\times F$ induced by the transition function of N, and let φ_U and φ_V be homomorphisms of $\mathcal{A}(M)$ into $\mathcal{A}(U_1\times\cdots\times U_l\times F)$ and $\mathcal{A}(V_1\times\cdots\times V_l\times F)$ respectively given by the formula (3). Then we have $g_*(\varphi_U(X)|W)=\varphi_V(X)|W$ for all $X\in\mathcal{A}(M)$. We shall use this fact in the proof of Theorem 3 and Theorem 3′.

In Example 2, the map $\tilde{\psi}$ can be defined globally $N\to M^l$, but this is not true in general as shown in the next example.

EXAMPLE 3. Let σ be a free involution of a manifold F and τ a free involution of $M\times M\times F$ given by $\tau(x, y, z)=(y, x, \sigma(z))$. Let $\varphi: \mathcal{A}(M)\to\mathcal{A}(M\times M\times F)$ be a homomorphism given by

$$\varphi(\sum f^i(x)\partial_{x^i})(x, y, z)=\sum f^i(x)\partial_{x^i}+\sum f^i(y)\partial_{y^i}\,.$$

Since $\tau_*\varphi(X)=\varphi(X)$, φ induces a homomorphism $\mathcal{A}(M)\to\mathcal{A}(M\times M\times F/\tau)$. Clearly in this case the map $\tilde{\psi}: M\times M\times F/\tau\to M\times M$ does not exist globally.

§ 2. Bundle structure of N_l.

In this section we shall show that N_l is a (topological) fibre bundle with the projection map ψ and study its bundle structure. It will be seen that N_l is closely related to $N=F\times_G P$ in Example 2 (cf. Theorem 3). Now, put $M(l)$

$=\{(p_1, \cdots, p_l)\in M^l \mid p_i\neq p_j \text{ for } i\neq j\}\subset M^l$. Since the symmetric group S_l acts freely on $M(l)$, we obtain a smooth manifold $M_l=M(l)/S_l$ which consists of all the sets of distinct l points of M. Put $M\{k\}=\bigcup_{l=1}^{k} M_l$ and give it the quotient topology induced by the natural map $M^k\to M\{k\}$.

PROPOSITION 1. i) *Let X be a vector field on M with compact support and q a point of N. Suppose that at q $\mathrm{Exp}\, t\varphi(X)$ (cf. Example 2) is defined for $0\leqq t\leqq 1$. Then for any $Y\in\mathcal{A}(M)$ we have $\varphi((\mathrm{Exp}\, X)_*Y)=(\mathrm{Exp}\,\varphi(X))_*\varphi(Y)$ at the point $\mathrm{Exp}\,\varphi(X)q$. Moreover if $q\in N^+$ we have $\mathrm{Exp}\,\varphi(X)q\in N^+$ and $\psi(\mathrm{Exp}\,\varphi(X)q)=\mathrm{Exp}\, X\psi(q)$.*

ii) *ψ is a continuous map of N_l into M_l. If M is not compact then $N^+=N$ and ψ is a continuous map of N into $M\{k\}$.*

REMARK 3. When $\dim M=\dim N$, the part ii) remains true even if M is compact. We do not know whether this fact holds in general.

PROOF OF PROPOSITION 1. First we prove i) under the following additional assumption:

(*) $q\in N_k^+$ and for each ν, $X(p_\nu)\neq 0$ or $X\equiv 0$ on some neighborhood of p_ν. Here $\psi(q)=\{p_1, \cdots, p_k\}$.

Choose U_ν, (x_ν), U and (x_*, y) as in Theorem 2. We may assume that $X|U_\nu=\partial_{x_\nu^1}$ for $\nu\leqq s$ and $X|U_\nu\equiv 0$ for $\nu>s$ for some s and hence that $\varphi(X)|U=\partial_{x_1^1}+\cdots+\partial_{x_s^1}$. For brevity we put $p_{\nu t}=\mathrm{Exp}\, tXp_\nu$ and $q_t=\mathrm{Exp}\, t\varphi(X)q$. We can extend the coordinate system (x_*, y) to some open set U' containing all the points q_t $(0\leqq t\leqq 1)$ so that $\mathrm{Exp}\, t\varphi(X)(x_*, y)=(x_1^1+t, x_1^2, \cdots, x_1^n, \cdots, x_s^1+t, \cdots, x_{s+1}^1, \cdots, x_k^n, y)$. Similarly we can extend (x_ν) $(1\leqq\nu\leqq s)$ to some open set U_ν'. Then we have $\varphi(X)|U'=\partial_{x_1^1}+\cdots+\partial_{x_s^1}$ and $X|U_\nu'=\partial_{x_\nu^1}$. Note that U''s are not necessarily disjoint. For each t $(0\leqq t\leqq 1)$, consider the following statement:

C_t: $q_t\in N_k^+$, $\psi(q_t)=\mathrm{Exp}\, tX\,\psi(q)=\{p_{1t}, \cdots, p_{kt}\}$ and the coordinate systems (x_ν) and (x_*, y) satisfy i) of Theorem 2 on some neighborhoods of $p_{\nu t}$ and q_t respectively.

Since the set $\{t \mid C_t \text{ is true}\}$ is open and contains a sufficiently small t, to prove C_1 it suffices to show that if C_t is true for $t<s$ then C_s is true. Take $Y_\nu^i\in\mathcal{A}(M)$ $(1\leqq\nu\leqq k)$ such that $Y_\nu^i=\partial_{x_\nu^i}$ on some neighborhood of $p_{\nu s}$ and $\mathrm{supp}\, Y_\nu^i\not\ni p_{\mu s}$ for $\mu\neq\nu$. Then $\varphi(Y_\nu^i)=\partial_{x_\nu^i}$ on some neighborhood V of the set $\{q_t \mid s-\varepsilon<t<s\}$ for some $\varepsilon>0$ and hence $\varphi(Y_\nu^i)(q_s)\neq 0$, which implies that $q_s\in N^+$. Further we have $\psi(q_s)\ni p_{\nu s}$. Really if $\psi(q_s)\not\ni p_{\nu s}$ we can choose Y_ν^i so that $\mathrm{supp}\, Y_\nu^i\cap\psi(q_s)=\emptyset$ and hence that, in view of Lemma 3, $\varphi(Y_\nu^i)(q_s)=0$, which is a contradiction. Since $p_{\nu s}$'s are distinct and k is, by definition, the maximal number of p_ν's, it follows that $\psi(q_s)=\{p_{1s}, \cdots, p_{ks}\}$ and hence $q_s\in N_k$. By Lemma 5 we may restrict our consideration to some neighborhoods of $p_{\nu s}$ and q_s. Since $L_{\varphi(X)}\varphi(\partial_{x_\nu^i})$

$=[\varphi(X), \varphi(\partial_{x^i_\nu})]=\varphi[X, \partial_{x^i_\nu}]=0$ (we denote by $L_{\varphi(X)}$ the Lie derivative with respect to $\varphi(X)$) and $\varphi(\partial_{x^i_\nu})=\partial_{x^i_\nu}$ on V, we obtain $\varphi(\partial_{x^i_\nu})=\partial_{x^i_\nu}$. By Lemma 6 we have $\tilde{\varphi}^i_\nu(x_*, y)=x^i_\nu+c^i_\nu(y)$ for some c^i_ν, and since $\tilde{\varphi}^i_\nu(x_*, y)=x^i_\nu$ on V, it follows that $\tilde{\varphi}^i_\nu(x_*, y)=x^i_\nu$. Therefore we have $q_s\in N^+_k$ and the coordinate systems (x_ν) and (x_*, y) satisfy i) of Theorem 2, which completes the proof of C_s. It remains to prove that $\varphi((\text{Exp}\, X)_* Y)=(\text{Exp}\,\varphi(X))_*\varphi(Y)$ at q_1, but this is clear since $\text{Exp}\, X$ and $\text{Exp}\,\varphi(X)$ are parallel translations and φ is given by (3) in Theorem 2.

Now we prove i) in general case. If $q\in\bar{N}_k=\bar{N}^+_k$ we can, by Theorem 2, choose a sequence $\{q_i\}$ in N^+_k converging to q such that each q_i satisfies the assumption in i) and the assumption (*). Then we have $\varphi((\text{Exp}\, X)_* Y)=(\text{Exp}\,\varphi(X))_*\varphi(Y)$ at q_{i1} and hence at q_1. It follows that $\varphi^{-1}\mathcal{A}_{q_1}(N)=(\text{Exp}\, X)_*(\varphi^{-1}\mathcal{A}_q(N))$. Therefore if $q\in N^+$ then we have $q_1\in N^+$ and $\psi(q_1)=\text{Exp}\, X\,\psi(q)$. Note that we have $q_t\in\bar{N}_k$ for $0\leqq t\leqq 1$ since $q_{it}\in N^+_k$ converges to q_t and hence that if $q\in N-\bar{N}_k$ then $q_t\in N-\bar{N}_k$. Applying the above argument to the manifold $N-\bar{N}_k$, we can prove i) for $q\in\bar{N}_{k-1}-\bar{N}_k$ and similarly for $q\in\bar{N}_1\cup\cdots\cup\bar{N}_k=\bar{N}^+$. For $q\in N-\bar{N}^+$, we have $\varphi((\text{Exp}\, X)_* Y)=(\text{Exp}\,\varphi(X))_*\varphi(Y)=0$ at $q_1=q$ since $\varphi(Z)\equiv 0$ on $N-\bar{N}^+$ for any $Z\in\mathcal{A}(M)$. This completes the proof of i). ii) Lemma 4 implies that ψ is continuous on N_l. We assume that M is not compact. Let K be a compact set of N. We prove that $L(K)=\{p\,|\,p\in\psi(q)$ for some $q\in K\cap N^+\}$ is relatively compact in M. Assume the contrary. Then there is a sequence $\{q_i\}$ in $K\cap N^+$ converging to some point q' of K such that the set $\{p_{i1}\}_i$ is discrete where $\psi(q_i)=\{p_{i1}, p_{i2}, \cdots\}$. We may assume that $p_{i1}\neq p_{j\nu}$ for all ν and $j<i$. By Lemma 3 we can choose $Y\in\mathcal{A}(M)$ such that $|\varphi(Y)(q_i)|\geqq i$, which is a contradiction. Here $|\ |$ denotes the norm of the vector with respect to some metric on M. Therefore $L(K)$ is relatively compact. Next, let $\{q_i\}\subset N^+$ be a sequence converging to a point q of N and K a compact neighborhood of q. We show $q\in N^+$. Assume the contrary. Put $\psi(q_i)=\{p_{i1}, p_{i2}, \cdots\}$. Since $L=L(K)$ is relatively compact, we may assume that the sequence $\{p_{i1}\}_i$ converges to some point p_1 of $\bar{L}$. Since M is not compact, we can choose $Y\in\mathcal{A}(M)$ such that $\text{supp}\, Y$ is compact and $\text{Exp}\, Y(U)\cap L=\emptyset$ for some neighborhood U of p_1. Since by assumption $q\notin N^+$, we have $\varphi(Y)(q)=0$. So we may assume that at q_i $\text{Exp}\, t\varphi(Y)$ is defined for $0\leqq t\leqq 1$ and $\text{Exp}\,\varphi(Y)q_i\in K$ for all i. Then by i) we have $\psi(\text{Exp}\,\varphi(Y)q_i)=\text{Exp}\, Y\psi(q_i)=\{\text{Exp}\, Y\, p_{i1}, \cdots\}$. By the definition of L we have $\text{Exp}\, Y\, p_{i1}\in L$, which contradicts the fact that $\text{Exp}\ Y(U)\cap L=\emptyset$. Thus we have $q\in N^+$. This implies that N^+ is closed. Since N^+ is open and N is connected, we have $N^+=N$. Next, we show that $\psi(q_i)=\{p_{i1}, \cdots\}\to\psi(q)=\{p_1, \cdots, p_m\}$ in $M\{k\}$. Assume the contrary. Then we have two cases:

1) There is a subsequence $\{q'_i\}$ of $\{q_i\}$ such that the sequence $\{p'_{i1}\}$ converges to a point p with $p\neq p_\nu$ for $\nu=1, 2, \cdots, m$.

2) There is a neighborhood U of p_1 such that $U \not\ni p_{i\nu}$ for all i and ν.

In case 1, we can choose $Y \in \mathcal{A}(M)$ such that supp Y is compact and does not contain p_ν for $\nu=1, \cdots, m$ and that $\operatorname{Exp} Y(U) \cap L = \emptyset$ for some neighborhood U of p. Then by Lemma 3 we have $\varphi(Y)(q)=0$, which yields a contradiction by the same argument as above. In case 2, we can choose $Y \in \mathcal{A}(M)$ such that supp $Y \subset U$ and $Y(p_1) \neq 0$. Then we have $\varphi(Y)(q_i)=0$ for all i and $\varphi(Y)(q) \neq 0$, which is a contradiction. Therefore we have that $\psi(q_i) \to \psi(q)$ in $M\{k\}$, which completes the proof of ii).

COROLLARY 1. *Suppose that N is compact. Then M is also compact and φ is injective. Moreover each non-empty N_l is a (topological) fibre bundle over M_l with the projection map ψ.*

PROOF. First we show that M is compact. Assume the contrary. Then by Proposition 1 ψ is continuous. Let q be a point of N_k $(\neq \emptyset)$ with $\psi(q)=\{p_1, \cdots, p_k\}$. For any point $\{p'_1, \cdots, p'_l\}$ of M_l, there are X_i's$\in \mathcal{A}(M)$ such that supp X_i's are compact and $\operatorname{Exp} X_i\{p_1, \cdots, p_k\} \to \{p'_1, \cdots, p'_l\}$ in $M\{k\}$ as $i \to \infty$ (recall that M is connected). Since $\psi(N)$ is compact and $\psi(\operatorname{Exp} \varphi(X_i)q)=\operatorname{Exp} X_i\psi(q) \in \psi(N)$, we have $\{p'_1, \cdots, p'_l\} \in \psi(N)$ and hence $\psi(N)=M\{k\}$. Since $\psi(N)$ is compact, so is M, which is a contradiction. Therefore M is compact. Next, let q be a point of $N_l \neq \emptyset$ with $\psi(q)=\{p_1, \cdots, p_l\}$. For any point $\{p'_1, \cdots, p'_l\}$ of M_l, there is $X \in \mathcal{A}(M)$ such that $\psi(\operatorname{Exp} \varphi(X)q)=\operatorname{Exp} \psi(q)=\{p'_1, \cdots, p'_l\}$. Thus $\psi: N_l \to M_l$ is surjective. In particular $\psi: N_k(\neq \emptyset) \to M_k$ is surjective. The injectivity of φ follows easily from this fact and the definition of ψ. It is clear that $\operatorname{Exp} \varphi(X)$ gives a homeomorphism of $\psi^{-1}\{p_1, \cdots, p_l\}$ onto $\psi^{-1}\{p'_1, \cdots, p'_l\}$. Now we give the local trivial structure of N_l. Let U_ν be a neighborhood of p_ν and (x_ν) a coordinate system on some neighborhood of $\bar{U}_\nu$ $(\nu \leqq l)$. We assume that $\bar{U}_\nu$'s are disjoint and diffeomorphic to the unit disk $\{x \in \boldsymbol{R}^n \mid |x|^2 \leqq 1\}$ by these coordinate systems. Choose $X^i_\nu \in \mathcal{A}(M)$ with $X^i_\nu | U_\nu = \partial_{x^i_\nu}$ and $X^i_\nu | U_\mu \equiv 0$ for $\mu \neq \nu$. Let (a_ν) be the coordinates of p_ν and put $F_l=\psi^{-1}\{p_1, \cdots, p_l\}$. Then the local trivial structure of N_l is given by the map $\chi_U: U_1 \times \cdots \times U_l \times F_l \to \psi^{-1}(U_1 \times \cdots \times U_l) \subset N_l$ defined by $\chi_U((x_1) \times \cdots \times (x_l) \times y)=\operatorname{Exp} \varphi(\sum_{i\nu}(x^i_\nu - a^i_\nu)X^i_\nu)y$. Here we consider $U=U_1 \times \cdots \times U_l$ as a subset of M_l. This completes the proof of Corollary 1.

To express the transition functions of the bundle N_l, we need some definitions. We assume that M is oriented and $\dim M=n \geqq 3$. Let $\mathcal{V}=\{U\}$ be an open covering of M such that each intersection of finite U's is a disk and $(x_U)=(x^i_U)$ a coordinate system on U. Let $G^+(n, h)$ be the connected component of $G(n, h)$ (cf. Example 2) containing the identity element 1 and $\tilde{G}^+(n, h)$ its universal covering Lie group. Since $G^+(n, h)$ is homotopically equivalent to $SO(n)$, $\tilde{G}^+(n, h)$ is a double covering of $G^+(n, h)$ and homotopically equivalent to $Spin(n)$. For any U and V with $U \cap V \neq \emptyset$, let $J_{UV}: U \cap V \to G^+(n, h)$ be a map given by $J_{UV}(x_U)$=the h-jet of the coordinate transformation $x_V(x_U)$ at (x_U) and let $\tilde{J}_{UV}: U\cap V\to\tilde{G}^+(n, h)$ be one of its liftings. Note that J_{UV} is a transition function of $J^{h-1}TM$. Since $J_{UV}(x_U)J_{VW}(x_V(x_U))=J_{UW}(x_U)$ for any (x_U) $\in U\cap V\cap W$, there is an element $\varepsilon_{UVW}\in \boldsymbol{Z}_2\subset\tilde{G}^+(n, h)$ such that $\tilde{J}_{UV}(x_U)\tilde{J}_{VW}(x_V(x_U))$ $=\varepsilon_{UVW}\tilde{J}_{UW}(x_U)$ for any (x_U). Here $\boldsymbol{Z}_2$ is the inverse image of 1 by the covering map $\tilde{G}^+(n, h)\to G^+(n, h)$. Note that $\{\varepsilon_{UVW}\}$ gives the second Whitney class of M, $w_2(M)\in H^2(M; \boldsymbol{Z}_2)$, and hence that if M has a spin structure we can choose the liftings $\tilde{J}_{UV}$ so that each $\varepsilon_{UVW}=1$. Let $\mathcal{U}^l=\{U\}=\{U_1\times\cdots\times U_l \mid U_\nu\in\mathcal{U}\}$ be an open covering of M^l and $(x_U)=(x_{U_1}, \cdots, x_{U_l})$ a coordinate system on U. Finally let $\tilde{J}_{UV}: U\cap V\to\overset{l}{\bigoplus}\tilde{G}^+(n, h)$ be a map given by $\tilde{J}_{UV}(x_U)$ $=\bigoplus\tilde{J}_{U_\nu V_\nu}(x_{U_\nu})$ and put $\varepsilon_{UVW}=\bigoplus\varepsilon_{U_\nu V_\nu W_\nu}\in\overset{l}{\bigoplus}\tilde{G}^+(n, h)$. With these notations we have the following

THEOREM 3. *Assume that N is compact and M is oriented with* $\dim M$ $=n\geqq 3$. i) *Let $\tilde{N}_l$ be the lifting of the bundle N_l to $M(l)$ by the map $M(l)\to M(l)/S_l=M_l$. Then there is a topological fibre bundle $\hat{N}_l$ over M^l with $\hat{N}_l|M(l)=\tilde{N}_l$.*

ii) *Put $h=2((d-nl)^2+d-nl)+1$ where $d=$*$\dim N$. *Then $G=\overset{l}{\bigoplus}\tilde{G}^+(n, h)$ acts on the fibre F_l of the bundle $\hat{N}_l$ from the right and hence we have a homomorphism $\rho: G\to$*Homeo(F_l). *The transition functions of $\hat{N}_l$ are given by $g_{UV}(x_U)=\rho(\tilde{J}_{UV}(x_U))h_{UV}$, where h_{UV}'s are elements of the centralizer of $\rho(G)$ in* Homeo(F_l) *satisfying the relation $h_{UV}h_{VW}=\rho(\varepsilon_{UVW})h_{UW}$.*

$$\begin{array}{ccccc} N_l & \longleftarrow & \tilde{N}_l & \subset & \hat{N}_l \\ \downarrow & & \downarrow & & \downarrow \\ M_l & \longleftarrow & M(l) & \subset & M^l \end{array} \qquad \begin{array}{ccccccc} L & \longrightarrow & \tilde{L} & \subset & \hat{L} & & \\ \downarrow & & \downarrow & & \downarrow & \searrow & \\ M_l & \longleftarrow & M(l) & \subset & M^l & \longleftarrow & \tilde{M}^l \end{array}$$

REMARK 4. It would seem that the fibre F_l is a smooth submanifold (with corner) of N_l. If this is true, it is easily seen that N_l is a smooth fibre bundle and that Homeo(F_l) can be replaced by Diff(F_l). Further note that for any smooth right $\overset{l}{\bigoplus}\tilde{G}^+(n, h)$-manifold F_l and for all h_{UV}'s$\in$Diff(F_l) satisfying the conditions in Theorem 3 ii), $\{g_{UV}\}$ gives a smooth fibre bundle $\hat{N}_l$ over M^l. We can construct a local homomorphism Φ: Diff$(M)\to$Diff$(\hat{N}_l)$ by using the local trivial structure of $\hat{N}_l$ and hence get a homomorphism $\varphi=\Phi_*: \mathcal{A}(M)\to\mathcal{A}(\hat{N}_l)$. If $\rho(\overset{l}{\bigoplus}\boldsymbol{Z}_2)=\{1\}\subset$Diff$(F_l)$ (which means that $\overset{l}{\bigoplus}G^+(n, h)$ acts on F_l), Φ can be extended to a global homomorphism Diff$(M)\to$Diff$(\hat{N}_l)$. In this case, since $\rho(\varepsilon_{UVW})=1$, we may put each $h_{UV}=1$. The homomorphism φ obtained in this way is exactly the same one given in Example 2.

Since $\varphi(\mathcal{A}(M))$ is a subalgebra of $\mathcal{A}(N)$ and by Proposition 1 $\operatorname{Exp} t\varphi(X)\varphi(\mathcal{A}(M))\subset\varphi(\mathcal{A}(M))$, it follows that for any point q of N_l there is a leaf L ($\subset N_l$) containing q. Clearly the map $X \to \varphi(X)|L$ gives a homomorphism $\mathcal{A}(M) \to \mathcal{A}(L)$. For this homomorphism we have a more precise theorem, namely

THEOREM 3′. i) *L is a smooth fibre bundle over M_l. Let $\tilde{L}$ be the lifting of L to $M(l)$. Then there is a smooth fibre bundle $\hat{L}$ over M^l with $\hat{L}|M(l)=\tilde{L}$. Moreover there are a covering space $\tilde{M}^l$ of M^l and a closed subgroup H of $G=\overset{l}{\bigoplus}\tilde{G}^+(n, h)$ such that $\hat{L}$ is a fibre bundle over $\tilde{M}^l$ with connected fibre $H\backslash G$ (homogeneous space).*

ii) *The transition functions of the bundle $\hat{L}$ over $\tilde{M}^l$ are given by $g_{UV}(x_U) = R(\tilde{J}_{UV}(x_U))L(k_{UV})$, where k_{UV}'s are elements of the group $H\backslash N(H)$ ($N(H)$ is the normalizer group of H in G) satisfying the relation $k_{VW}k_{UV}=\varepsilon_{UVW}k_{UW}$ and R and L are actions on $H\backslash G$ induced by the right and the left translations of G respectively. Here U, V and W are elements of the open covering of $\tilde{M}^l$ induced by $\mathcal{U}^l$.*

PROOF OF THEOREM 3 AND THEOREM 3′. We investigate the bundle $\tilde{N}_l$. Let $(p_1, \cdots, p_l)$ be a point of $U \cap M(l)$ where $U=U_1\times \cdots \times U_l \in \mathcal{U}^l$, and let U'_ν's be disjoint neighborhoods of p_ν's respectively. Then the local trivial structure of $N_l|U'_1\times \cdots \times U'_l$ given in Corollary 1 gives a foliation of $\dim nl$ of $\tilde{N}_l|U'_1\times \cdots \times U'_l$ and this foliation depends only on the coordinate system (x_U). Therefore we have a foliation of $\tilde{N}_l|U_1\times \cdots \times U_l \cap M(l)$ and each leaf is a covering space of $U_1\times \cdots \times U_l \cap M(l)=U\cap M(l)$. Since $U\cap M(l)$ is simply connected by the assumption that $\dim M \geqq 3$, each leaf is homeomorphic to $U\cap M(l)$ and hence we get a local trivial structure of $\tilde{N}_l|U\cap M(l)$. We first prove Theorem 3′. Since the groups generated by $\operatorname{Exp} X$'s and $\operatorname{Exp}\varphi(X)$'s for $X\in\mathcal{A}(M)$ act transitively on M_l and L respectively and $\psi(\operatorname{Exp}\varphi(X)q)=\operatorname{Exp} X\psi(q)$, it follows that L is a bundle over M_l and that, in view of Lemma 6, ψ is a smooth submersion of L onto M_l. Therefore $L=L_l^+$ and the expression (3) of φ in Theorem 2 holds good everywhere. Now we study the bundle $\tilde{L}$. Let $(p_1, \cdots, p_l)$ be a point of $U\cap M(l)$, (a_U) its coordinates and F_L the fibre over $(p_1, \cdots, p_l)$. We give the local trivial structure of $\tilde{L}|U\cap M(l)=(U\cap M(l))\times F_L$ as above. Choose $X^\alpha_{i\nu}\in\mathcal{A}(M)$ such that $\operatorname{supp} X^\alpha_{i\nu}\not\ni p_\mu$ for $\mu\neq\nu$ and $X^\alpha_{i\nu}\equiv(x_{U_\nu}-a_{U_\nu})^\alpha\partial_{x^i_{U_\nu}}$ on some neighborhood of p_ν. Put $Y^\alpha_{i\nu}=\varphi(X^\alpha_{i\nu})|F_L$. Then $Y^\alpha_{i\nu}$ is a vector field on F_L by ii) of Theorem 2. Moreover for $X\in\mathcal{A}(M)$ with $X|U_\nu=\sum_i f^i_{U_\nu}(x_{U_\nu})\partial_{x^i_{U_\nu}}$ we have

$$\varphi(X)=\sum_{\nu=1}^{l}\sum_{i=1}^{n}\Big(f^i_{U_\nu}(x_{U_\nu})\partial_{x^i_{U_\nu}}+\sum_{0<|\alpha|\leqq h}\frac{D^\alpha}{\alpha!}f^i_{U_\nu}(x_{U_\nu})Y^\alpha_{i\nu}\Big) \tag{5}$$

on $\tilde{L}|U\cap M(l)=(U\cap M(l))\times F_L$. For another $V\in\mathcal{U}^l$ we get a similar expression of φ with $Y^\alpha_{i\nu}$ replaced by $Y^{\alpha\prime}_{i\nu}\in\mathcal{A}(F'_L)$ where F'_L is a fibre over some point $(p'_1, \cdots, p'_l)$ of $V\cap M(l)$. Then we have

LEMMA 7. *There is a diffeomorphism $g\colon F_L\to F'_L$ such that $g_*Y^\alpha_{i\nu}=Y^{\alpha\prime}_{i\nu}$ for all α, i and ν.*

PROOF. Let $(\tilde{x}_\nu)$ be a coordinate system on some simply connected neighborhood $\tilde{U}_\nu$ of $\{p_\nu, p'_\nu\}$ which is identical with (x_{U_ν}) and (x_{V_ν}) on some neighborhoods of p_ν and p'_ν respectively. Then we have a local trivial structure of $\tilde{L}|\tilde{U}_1\times\cdots\times\tilde{U}_l\cap M(l)$ and the expression (5) of φ. This trivial structure gives the desired diffeomorphism.

By this lemma for all $V\in\mathcal{V}^l$ we have the local trivial structure of $L|V\cap M(l)=(V\cap M(l))\times F_L$ and the expression (5) with the same fibre F_L and the vector fields $Y^\alpha_{i\nu}$'s. Now we investigate the transition function $g_{UV}(x_U)$ of $\tilde{L}$. Since $Y^\alpha_{i\nu}$'s satisfy the relation (4) in Theorem 2, the map $x^\alpha_\nu\partial_{x^i_\nu}\to Y^\alpha_{i\nu}$ gives a homomorphism $\overset{l}{\oplus}\mathfrak{g}(n, h)\to\mathcal{A}(F_L)$. Since $\operatorname{Exp} t\varphi(X^\alpha_{i\nu})$ are defined for all $t\in\boldsymbol{R}$, it follows that $\operatorname{Exp} tY^\alpha_{i\nu}$ are also defined for all $t\in\boldsymbol{R}$ and hence that there is a homomorphism $\rho: G=\overset{l}{\oplus}\tilde{G}^+(n, h)\to\operatorname{Diff}(F_L)$, namely, G acts on F_L. Then $\rho(\tilde{J}_{UV}(x_U))$ gives a diffeomorphism of $(U\cap V)\times F_L$. Let φ_U be a homomorphism $\mathcal{A}(M)\to\mathcal{A}(U\times F_L)$ given by (5) and φ_V a similar one. Then we have $\rho(\tilde{J}_{UV}(x_U))_*\varphi_U(X)=\varphi_V(X)$ on $(U\cap V)\times F_L$ by Remark 2. (Remark 2 remains valid with $\overset{l}{\oplus}G(n, h)$ replaced by $\overset{l}{\oplus}\tilde{G}^+(n, h)$.) Put $h_{UV}(x_U)=\rho(\tilde{J}_{UV}(x_U))^{-1}g_{UV}(x_U)$. Then we have $h_{UV}(x_U)_*\varphi_U(X)=\varphi_U(X)$ on $(U\cap V)\times F_L$ for all $X\in\mathcal{A}(M)$. It follows easily that $h_{UV}(x_U)$ is independent of x_U and $(h_{UV})_*Y^\alpha_{i\nu}=Y^\alpha_{i\nu}$ for all α, i and ν, which implies that h_{UV} commutes with every element of $\rho(G)$. Since $\tilde{J}_{UV}(x_U)\tilde{J}_{VW}(x_V(x_U))=\varepsilon_{UVW}\tilde{J}_{UW}(x_U)$, we have $h_{UV}h_{VW}=\rho(\varepsilon_{UVW})h_{UW}$. Note that $g_{UV}(x_U)=\rho(\tilde{J}_{UV}(x_U))h_{UV}$ is defined for all $x_U\in U\cap V$. Hence $\{g_{UV}\}$ gives a bundle $\hat{L}$ over M^l as desired. The last part of i) of Theorem 3′ follows from the facts that $\hat{L}$ is connected and that the action of G is transitive. Note that for this bundle $\hat{L}$ over $\tilde{M}^l$, the same fact as in Lemma 7 holds. The action of G on $H\backslash G$ is induced by the right translation and the centralizer of $\rho(G)$ in $\operatorname{Diff}(H\backslash G)$ is isomorphic to the group $H\backslash N(H)$ and its action on $H\backslash G$ is induced by the left translation of G. This completes the proof of Theorem 3′. Since the diffeomorphism $\operatorname{Exp}\varphi(X^\alpha_{i\nu})$ of N gives a homeomorphism of the fibre F_l of the bundle N_l, Theorem 3 follows from the above argument.

When $\overset{l}{\oplus}G^+(n, h)$ acts on F_l or M has a spin structure, the relation $h_{UV}h_{VW}=\rho(\varepsilon_{UVW})h_{UW}$ reduces to $h_{UV}h_{VW}=h_{UW}$ and hence $\{h_{UV}\}$ gives a locally constant bundle over M^l. We give an example such that some $\rho(\varepsilon_{UVW})\neq 1$.

EXAMPLE 4. Assume that there is an element $v\in\operatorname{Tor} H^2(M;\boldsymbol{Z})$ reduced to $w_2(M)$ $(\neq 0)\in H^2(M;\boldsymbol{Z}_2)$. For example, $(4k+1)-$dim real projective space satisfies this condition. Let $\rho_1:\tilde{G}^+(n, 1)\to GL(N, \boldsymbol{C})$ be a complex representation such that $\rho_1(-1)=-I_N=-$identity and let $\rho_2: GL(N, \boldsymbol{C})\to\operatorname{Diff}(S^{2N-1})$ be a homomorphism induced by the action of $GL(N, \boldsymbol{C})\subset GL(2N, \boldsymbol{R})$ on the sphere S^{2N-1} considered as the real Stiefel manifold $V_{2N,1}$. Put $\rho=\rho_2\rho_1:\tilde{G}^+(n, 1)\to\operatorname{Diff}(S^{2N-1})$. Then $\rho(-1)\neq 1$. Since $v\in\operatorname{Tor} H^2(M;\boldsymbol{Z})$, there is a locally constant complex line bundle whose first Chern class is v, where locally constant means that the transition functions k_{UV}'s are constant. The assumption assures that there are complex numbers h'_{UV}'s such that $h'_{UV}{}^2=k_{UV}$ and $h'_{UV}h'_{VW}=\varepsilon_{UVW}h'_{UW}$. Here we consider $\varepsilon_{UVW}\in \boldsymbol{Z}_2=\{1, -1\}$ as a complex number. Put $h_{UV}=\rho_2(h'_{UV}I_N)$. Then it commutes with all the elements of $\rho(\tilde{G}^+(n, 1))$ and we have $h_{UV}h_{VW}=\rho(\varepsilon_{UVW})h_{UW}$ and some $\rho(\varepsilon_{UVW})\neq 1$ as desired.

In Examples 1~4 we have $\bar{N}_k=N$, but this is not true in general. The local trivial structure given in Corollary 1 gives a foliation of $N_l|U\cap M(l)/S_l$. In general the behaviour of each leaf near N_{l-1} is not simple. Really we have

EXAMPLE 5. Let $\varphi: \mathcal{A}(\boldsymbol{R}^n)\to\mathcal{A}(\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^m)$ be a homomorphism given by

$$\varphi(\sum f^i(x)\partial_{x^i})(x, y, z)=\sum f^i(x)\partial_{x^i}+\sum f^i(y)\partial_{y^i}\,.$$

For this homomorphism, we have $\psi(x, y, z)=\{x, y\}$ and hence $\bar{N}_2=N$. The leaf of the foliation of N_2 (given by the natural coordinate system (x) of $\boldsymbol{R}^n=M$) is given by z=constant. We shall deform this homomorphism. First put $(X, Y, Z)=(x-y, y, z)$. Then we have $N_2=\{(X, Y, Z)\in\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^m\,|\,X\neq 0\}$ and

$$\begin{aligned}\varphi(\sum f^i(x)\partial_{x^i})(X, Y, Z)&=\sum(f^i(X+Y)-f^i(Y))\partial_{X^i}+\sum f^i(Y)\partial_{Y^i}\\&=\sum_{ij}\int_0^1\partial_j f^i(tX+Y)dt\,X^j\partial_{X^i}+\sum f^i(Y)\partial_{Y^i}\,.\end{aligned}$$

Let $(R, \theta)=(R, \theta^1, \cdots, \theta^{n-1})$ be a polar coordinate system of $\boldsymbol{R}^n$ such that $R^2=|X|^2$ and $X^i=RS^i(\theta)$ for some S^i. Then $\partial_{X^i}=S^i(\theta)\partial_R+\sum_m A_i^m(\theta)\frac{1}{R}\partial_{\theta^m}$ for some A_i^m. Next, let $(\bar{X}, \bar{Y}, \bar{Z})=(X, Y, \alpha(R, Z))$ be another coordinate system of N_2. Then the leaf is given by $\bar{Z}=\alpha(|\bar{X}|, Z_0)$ for some constant vector Z_0. Choose a smooth function $R(r)$ with $R'(r)>0$ for $r>2$ and $R(r)=0$ for $r\leqq 2$. Let $(\tilde{X})$ and (r, θ) be the coordinate systems of $\boldsymbol{R}^n$ such that $\tilde{X}^i=rS^i(\theta)$. Then N_2 is diffeomorphic to $\{(\tilde{X}, \bar{Y}, \bar{Z})\in\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^m\,|\,|\tilde{X}|=r>2\}$ by the map $(R(r), \theta, \bar{Y}, \bar{Z})\to(r, \theta, \bar{Y}, \bar{Z})$. In this coordinate system $(r, \theta, \bar{Y}, \bar{Z})$ we have on N_2

$$\begin{aligned}X^j\partial_{X^i}=R(r)/R'(r)S^jS^i\partial_r&+\sum_m S^jA_i^m\partial_{\theta^m}\\&+S^jS^iR(r)\sum_k\partial_R\alpha^k(R(r), Z(R(r), \bar{Z}))\partial_{\bar{Z}^k}\,,\end{aligned}$$

where $Z(R, \bar{Z})$ denotes the inverse of $\bar{Z}=\alpha(R, Z)$. Now we assume that $R(r)/R'(r)$ and $R(r)\partial_R\alpha^k(R(r), Z(R(r), \bar{Z}))$ can be extended to smooth functions $g(r)$ and $h^k(r, \bar{Z})$ respectively such that $g(r)=r$ and $h^k(r, \bar{Z})=0$ for $r\leqq 1$. For example $R(r)=\exp(-\exp 1/(r-2))$ and $\alpha^k(R, Z)=\log R+Z^k$ satisfy these conditions. Put

$$P_i^j=g(r)S^jS^i\partial_r+\sum_m S^jA_i^m\partial_{\theta^m}+S^jS^i\sum_k h^k(r, \bar{Z})\partial_{\bar{Z}^k}\,.$$

Then it is a smooth vector field with respect to the coordinate systems $(\tilde{X}, \bar{Z})$ and $(r, \theta, \bar{Z})$. Let $\varphi_1: \mathcal{A}(\boldsymbol{R}^n) \to \mathcal{A}(\boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^m)$ be a map given by

$$\varphi_1(\sum f^i(x)\partial_{x^i})(\tilde{X}, \bar{Y}, \bar{Z}) = \sum_{ij} \int_0^1 \partial_j f^i(v) dt\, P_i^j + \sum f^i(\bar{Y})\partial_{\bar{Y}^i}$$

where $v = (v^1, \cdots, v^n) = (tR(r)S^1(\theta) + \bar{Y}^1, \cdots)$. Then $\varphi_1 | N_2$ is a homomorphism. If $|X| = r \leqq 2$, then $R(r) = 0$ and hence we have

$$\varphi_1(\sum f^i(x)\partial_{x^i}) = \sum_{ij} \partial_j f^i(\bar{Y}) P_i^j + \sum f^i(\bar{Y})\partial_{\bar{Y}^i}.$$

It is easy to show that P_i^j's satisfy the relation (4) in Theorem 2. Therefore by Remark 1 φ_1 is a homomorphism. For this homomorphism φ_1, we have $\psi(\tilde{X}, \bar{Y}, \bar{Z}) = \{\bar{Y}, v\}$ and hence $N_2 = \{(\tilde{X}, \bar{Y}, \bar{Z}) \in \boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^m \mid |\tilde{X}| > 2\}$, $N_1 = \{(\tilde{X}, \bar{Y}, \bar{Z}) \mid |\tilde{X}| \leqq 2\}$, $F_2 = \boldsymbol{R}^m \cup \boldsymbol{R}^m$ and $F_1 = D^n \times \boldsymbol{R}^m$. If we take $\alpha^k(R, Z) = \log R + Z^k$, then the leaf is given by $\bar{Z}^k = \alpha^k(R, Z_0) = \log R(|\tilde{X}|) + Z_0^k$ for some constant vector Z_0.

§ 3. Classification of transitive germs of homomorphisms.

In this section we shall consider the classification of germs of homomorphisms. Let $\varphi: \mathcal{A}(M) \to \mathcal{A}(N)$ be a homomorphism and q a point of N with $\psi(q) = \{p_1, \cdots, p_l\}$. Then by (1) in § 1 we have

$$\bigcap_{\nu=1}^{l} \mathcal{M}_{p_\nu}\mathcal{A}(M) / \bigcap_{\nu=1}^{l} \mathcal{M}_{p_\nu}^{h+1}\mathcal{A}(M) \supset \varphi^{-1}\mathcal{A}_q(N) / \bigcap_{\nu=1}^{l} \mathcal{M}_{p_\nu}^{h+1}\mathcal{A}(M).$$

The left hand side of this formula is isomorphic to the algebra $\mathfrak{g} = \overset{l}{\bigoplus} \mathfrak{g}(n, h)$ and hence the right hand side, denoted by B_q, is considered as a subalgebra of $\mathfrak{g}$. However, since the above isomorphism depends on the coordinate systems, B_q is not well defined as a subalgebra of $\mathfrak{g}$. We say subalgebras B and B' of $\mathfrak{g}$ are *equivalent* if $\mathrm{Ad}(g)B = B'$ for some $g \in \overset{l}{\bigoplus} G(n, h)$ and denote by $B(n, h, l)$ the set of the equivalence classes of subalgebras of $\mathfrak{g}$. Then B_q gives an element of $B(n, h, l)$, denoted by B_q also. Now we say φ is *transitive* at q if $\{\varphi(X) \in T_qN \mid X \in \mathcal{A}(M)\} = T_qN$ where T_qN denotes the tangent space of N at q. Then we have

LEMMA 8. *If φ is transitive at q, then there is a neighborhood U of q such that $B_q = B_{q'}$ for all $q' \in U$.*

PROOF. By i) of Proposition 1 we have $\varphi((\mathrm{Exp}\, X)_* Y) = (\mathrm{Exp}\, \varphi(X))_* \varphi(Y)$ at the point $q_1 = \mathrm{Exp}\, \varphi(X) q$ and hence $\varphi^{-1}\mathcal{A}_{q_1}(N) = (\mathrm{Exp}\, X)_* \varphi^{-1}\mathcal{A}_q(N)$. Let $g \in \overset{l}{\bigoplus} G(n, h)$ be the h-jet of $\mathrm{Exp}\, X$ at $\{p_1, \cdots, p_l\}$. Then we have $\mathrm{Ad}(g)B_q = B_{q_1}$. The assumption of the lemma implies that $\{\mathrm{Exp}\, \varphi(X) q \mid X \in \mathcal{A}(M)\}$ covers some neighborhood of q.

Let φ be transitive at q. Then $q \in \mathrm{Int}\, N_l^+$ and hence by Lemma 5 there are neighborhoods U and U_ν of q and p_ν respectively such that $\varphi(X)|U$ depends only on $X|\cup U_\nu$. Therefore we may consider the *germ* of φ at $(q; p_1, \cdots, p_l)$. We say the germs φ and φ' at $(q; p_1, \cdots, p_l)$ are *equivalent* if there are diffeomorphisms $g: \cup V_\nu \to \cup V'_\nu$ and $h: V \to V'$, where V_ν, V'_ν, V and V' are some neighborhoods of p_ν and q respectively, such that $g\{p_1, \cdots, p_l\} = \{p_1, \cdots, p_l\}$, $h(q)=q$ and $h_*(\varphi(X)|V) = \varphi'(g_*(X|\cup V_\nu))|V'$ for any $X \in \mathcal{A}(M)$. We do not require that φ and φ' are the restrictions of the global homomorphisms $\mathcal{A}(M) \to \mathcal{A}(N)$. We denote by $H_t(n, l, d)$ the set of equivalence classes of transitive germs at $(q; p_1, \cdots, p_l)$ (recall that $\dim M=n$ and $\dim N=d$) and by $B(n, h, l, e)$ the set of equivalence classes of the subalgebras of $\overset{l}{\oplus}\mathfrak{g}(n, h)$ of codim e. Then we have

THEOREM 4. *The correspondence $\varphi \to B_q$ gives a bijection $H_t(n, l, d) \to B(n, h, l, e)$ where $e=d-nl$ and $h=2(e^2+e)+1$.*

PROOF. We first show that the map is injective. Since codim $(\cap \mathcal{M}_{p_\nu}\mathcal{A}(\cup U_\nu))$ in $\mathcal{A}(\cup U_\nu)$ is equal to nl and φ is transitive at q, it follows that codim $B_q = d-nl=e$. By Lemma 6 and i) of Proposition 1, ϕ is a smooth submersion on some neighborhood of q and hence by Theorem 2 we have the expression (3) of φ. We use the same notations as in Theorem 2. Let (a_*, b) be the coordinates of q and put $F = \{(x_*, y) \in U \mid (x_*) = (a_*)\}$. Since the correspondence $x_\nu^\alpha \partial_{x_\nu^i} \to Y_{i\nu}^\alpha$ gives a homomorphism $f: \mathfrak{g} = \overset{l}{\oplus}\mathfrak{g}(n, h) \to \mathcal{A}(F)$, $G = \overset{l}{\oplus}G(n, h)$ acts *locally* on F in the following sense. There are a neighborhood V of $\{1\} \times F$ in $G \times F$ and a map $g: V \to F$ such that $g(\exp X, q') = \mathrm{Exp}\, f(X)q'$ for $(\exp X, q') \in V$. Since φ is transitive, this action is transitive and hence F is *locally* diffeomorphic to the *germ* of the homogeneous space $H\backslash G$, where H is a subgroup of G whose Lie algebra is $\{\sum a_{i\nu}^\alpha x_\nu^\alpha \partial_{x_\nu^i} \in \mathfrak{g} \mid \sum a_{i\nu}^\alpha Y_{i\nu}^\alpha(b) = 0\} = B_q$. More precisely, there are an open set F' of F containing q and a neighborhood W of 1 in G such that F' is diffeomorphic to $H_W \backslash W$, where H_W is a connected component of $H \cap W$ containing 1. The right translation of G induces a homomorphism $\varphi_1: \mathfrak{g} \to \mathcal{A}(H_W \backslash W)$ and $\varphi_1(x_\nu^\alpha \partial_{x_\nu^i})$ corresponds to $Y_{i\nu}^\alpha | F'$ by the above diffeomorphism. Since φ is determined by $Y_{i\nu}^\alpha$'s, it is determined by H_W and hence by B_q. Thus the map $\varphi \to B_q$ is injective. On the other hand, for any $B \in B(n, h, l, e)$ we can construct $H_W \backslash W$ and get $\varphi_1(x_\nu^\alpha \partial_{x_\nu^i}) \in \mathcal{A}(H_W \backslash W)$ and hence a homomorphism $\varphi: \mathcal{A}(\cup U_\nu) \to \mathcal{A}(\cup U_\nu \times (H_W \backslash W))$ given by the formula (3). This completes the proof.

EXAMPLE 6. For $(n, l, d) = (1, 1, 2)$ we have $H_t(1, 1, 2) = B(1, 5, 1, 1) = \{B_1, B_2, B_3\}$. The subalgebras $B_i \subset \mathfrak{g}(1, 5)$ and the corresponding transitive homomorphisms $\varphi: \mathcal{A}(\boldsymbol{R}^1) \to \mathcal{A}(\boldsymbol{R}^2)$ are given as follows.

$$B_1=\{\sum_{j=1}^{5} a_j x^j \partial_x \mid a_1=0\}, \qquad \varphi(f(x)\partial_x)=f(x)\partial_x+f'(x)\partial_y,$$

$$B_2=\{\sum_{j=1}^{5} a_j x^j \partial_x \mid a_1+a_2=0\}, \quad \varphi(f(x)\partial_x)=f(x)\partial_x+\Big(f'(x)+\frac{1}{2!}f''(x)e^y\Big)\partial_y,$$

$$B_3=\{\sum_{j=1}^{5} a_j x^j \partial_x \mid a_1+a_3=0\}, \quad \varphi(f(x)\partial_x)=f(x)\partial_x+\Big(f'(x)+\frac{1}{3!}f'''(x)e^{2y}\Big)\partial_y.$$

In general, the cardinality of $B(n, h, l, e)$ is not finite. For example, in case $n\geqq 2$, $l=1$ and $d=2n$, for any $t\in \boldsymbol{R}$ put

$$B_t=\{\sum_{i=1}^{n} \sum_{0<|\alpha|\leqq h} a_\alpha^i x^\alpha \partial_{x^i} \mid \sum_i a_k^i + t\sum_{i\neq j} a_j^i=0 \text{ for } k=1, \cdots, n\} \subset \mathfrak{g}(n, h)$$

where $h=2(n^2+n)+1$. Then $B_t=B_s$ in $B(n, h, l, n)$ if and only if $t=s$. The corresponding transitive homomorphism $\varphi_t: \mathcal{A}(\boldsymbol{R}^n)\to\mathcal{A}(\boldsymbol{R}^{2n})$ is given by

$$\varphi_t(\sum_i f^i(x)\partial_{x^i})(x, y)=\sum_i f^i(x)\partial_{x^i}+\sum_i D^i f^i(x)\partial_{y^i}$$
$$+\sum_{i\neq j} D^j f^i(x)e^{y^j-y^i}\sum_k (t+\delta_{jk})\partial_{y^k}.$$

§4. Continuity of a homomorphism.

In [4] H. Omori proved that if M and N are compact and $\varphi: \mathcal{A}(M)\to\mathcal{A}(N)$ is a homomorphism which is continuous in the C^∞-topology, then φ induces a local homomorphism Diff $(M)\to$Diff (N). We shall show that any homomorphism φ is continuous without the assumption of compactness of M and N. Lemma 3 implies the continuity of φ in the weak topology. Theorem 2 does not imply the continuity of φ, because in general the local coordinate system (x_*, y) on a neighborhood U of q does not fit with the given one $(u)=(u^p)$ on an open set U_1 of N, that is, $D_{x_\nu}^\alpha u^p$, $D_y^\alpha u^p$, $D_u^\alpha x_\nu^i$ and $D_u^\alpha y^j$ are not necessarily bounded when q tends to a point of $(N-N_l^+)\cap U_1$. Here D_z^α denotes the differential operator with respect to z. Recall that the C^∞-topology of $\mathcal{A}(N)$ is given by the seminorms $|\ |_{U,r}$ defined as follows. Let $(u)=(u^p)$ be a coordinate system on a relatively compact open set U of N which can be extended to some neighborhood of $\bar{U}$. Then for $Y\in\mathcal{A}(N)$ with $Y=\sum g^p(u)\partial_{u^p}$ on U, we put

$$|Y|_{U,r}=\sup_{|\alpha|\leqq r, u\in U, p}|D^\alpha g^p(u)|.$$

First we assume that M is compact. Then there is a finite open covering $\{V_\mu\}$ of M satisfying the following properties:

i) Each V_μ is diffeomorphic to the unit disk $\{x\in\boldsymbol{R}^n \mid |x|^2<1\}$ by the coordinate system $(x_\mu)=(x_\mu^1, \cdots, x_\mu^n)$ on some neighborhood of $\bar{V}_\mu$.

ii) Any set $\{p_1, \cdots, p_k\}\subset M$ is contained in some V_μ, where k is the integer defined in §1.

To prove the continuity of φ it suffices to show the next

LEMMA 9. *For any seminorm* $|\ |_{U,r}$, *there is a constant* C *such that for any* $X\in\mathcal{A}(M)$ *we have*

$$|\varphi(X)|_{U,r}\leqq C\sum_{\mu}|X|_{V_\mu,ar+b},$$

where $a=[d/n]=$ *the integer part of* d/n *and* $b=2a((d-n)^2+d-n+1)-1$.

PROOF. Let $\varphi(X)|U=\sum\varphi^p(X)(u)\partial_{up}$. Now we estimate $D_u^\beta\varphi^p(X)(u)$ for $|\beta|\leqq r$. For any $q\in U\cap N_l^+$, choose an open set V_μ containing $\psi(q)=\{p_1, \cdots, p_l\}$. Applying Theorem 2 to $U_\nu=V_\mu$ and $(x_\nu)=(x_\mu)$ $(\nu=1, \cdots, l)$, we can get a coordinate system (x_*, y) on some neighborhood U_q of q such that $\tilde{\psi}(x_*, y)=(x_*)=(x_1, \cdots, x_l)\in U_1\times\cdots\times U_l=V_\mu\times\cdots\times V_\mu$ and that for any $X\in\mathcal{A}(M)$ with $X=\sum f^i(x_\mu)\partial_{x_\mu^i}$ on V_μ we have

$$\varphi(X)(x_*, y)=\sum_{\nu i}\Big(f^i(x_\nu)\partial_{x_\nu^i}+\sum_{0<|\alpha|\leqq h}\frac{D^\alpha}{\alpha!}f^i(x_\nu)Y_{i\nu}^\alpha(y)\Big)$$

on U_q. It follows that

$$D_u^\beta\varphi^p(X)(u)=\sum_{\nu i}\sum_{|\gamma|\leqq h+r}D^\gamma f^i(x_\nu(u))Z_{i\nu}^{\gamma\beta p}(u)$$

on U_q, where Z's are smooth functions on U_q. To eliminate Z's we need the following lemma which will be proved at the end of this section.

LEMMA 10. *Let* $\Phi: C^\infty(\boldsymbol{R}^n)\to C^\infty(\boldsymbol{R}^{nl})[Z_\nu^\alpha]$ (*=the polynomial ring over* $C^\infty(\boldsymbol{R}^{nl})$) *be a map given by*

$$\Phi(f(x))=\sum_{\nu=1}^{l}\sum_{|\alpha|\leqq h}D^\alpha f(x_\nu)Z_\nu^\alpha.$$

Then we have

$$\Phi(f(x))=f(x_1)\Phi(1)+\sum_{k=1}^{l(h+1)-1}\sum_{j_1,\cdots,j_k=1}^{n}\int_0^1\cdots\int_0^1\partial_{j_1}\cdots\partial_{j_k}f(x(k))dt(k)$$

$$\sum_{m=0}^{k}(-1)^m\sum_{1\leqq i_1<\cdots<i_m\leqq k}x_{i_1}^{j_1}\cdots x_{i_m}^{j_m}\Phi(x^{j_{m+1}+\cdots+j_k})\Big|_{\substack{x_{\nu+le}=x_\nu\\ \nu\leqq l, e\leqq h}},$$

where

$$x(k)=(1-t_1)x_1+(1-t_2)t_1x_2+\cdots+(1-t_k)t_{k-1}\cdots t_1x_k+t_kt_{k-1}\cdots t_1x_{k+1},$$

$$dt(k)=t_1^{k-1}t_2^{k-2}\cdots t_{k-1}dt_1\cdots dt_k.$$

Put $\Phi(f(x))=\Phi(f(x))(x_*)=\sum_{\nu=1}^{l}\sum_{|\gamma|\leqq h+r}D^\gamma f(x_\nu)Z_{i\nu}^{\gamma\beta p}(u)$. Then we have $\Phi(f(x))(x_*(u))=D_u^\beta\varphi^p(f(x_\mu)\partial_{x_\mu^i})(u)$. Here we consider $f(x_\mu)\partial_{x_\mu^i}$ as a vector field on M by extending it suitably. For $u\in U_q$, the right hand side of the above equation is independent of this extension. Applying Lemma 10 to Φ and substituting

$x_*(u)$ for x_*, we have

$$D_u^\beta\varphi^p(f(x_\mu)\partial_{x_\mu^i})(u)=f(x_1)D_u^\beta\varphi^p(\partial_{x_\mu^i})(u)$$

$$+\sum_{k=1}^{l(h+r+1)-1}\sum_{j_*}\int_0^1\cdots\int_0^1\partial_{j_1}\cdots\partial_{j_k}f(x(k))dt(k)$$

$$\sum_{m=0}^{k}(-1)^m\sum_{i_*}x_{i1}^{j_1}\cdots x_{im}^{j_m}D_u^\beta\varphi^p(x_\mu^{j_{m+1}+\cdots+j_k}\partial_{x_\mu^i})(u)|_{x_\nu+le=x_\nu}.$$

Note that $|x_\nu|<1$, that $x(k)|_{x_\nu+le=x_\nu}\in V_\mu$ and that $D_u^\beta\varphi^p(x_\mu^{j_{m+1}+\cdots+j_k}\partial_{x_\mu^i})(u)$ is smooth on $\bar{U}$ and hence bounded on $\bar{U}$. If we fix the extension of $x_\mu^{j_{m+1}+\cdots+j_k}\partial_{x_\mu^i}$, there is a constant $C_{\mu l}^i$ not depending on $q\in U\cap N_l^+$ such that we have

$$|D_u^\beta\varphi^p(f(x_\mu)\partial_{x_\mu^i})(u)|\leqq C_{\mu l}^i|f(x_\mu)\partial_{x_\mu^i}|_{V_\mu,t}$$

for $u\in U_q$, where $t=l(h+r+1)-1$. Putting $C=\sum_{i\mu l}C_{\mu l}^i$, we obtain

$$|D_u^\beta\varphi^p(X)(u)|\leqq C\sum_\mu|X|_{V_\mu,t}$$

for $u\in U\cap(\cup N_l^+)$. Since $\overline{\cup N_l^+}=\bar{N}^+$ and $\varphi(X)\equiv 0$ on $N-\bar{N}^+$, it follows that the above inequality holds for all $u\in U$. By Theorem 1 we have $t\leqq ar+b$, which completes the proof of Lemma 9.

Next, we consider the case where M is not compact. Let q be a point of $N=N^+$ with $\psi(q)=\{p_1,\cdots,p_l\}$. Then by ii) of Proposition 1 there are neighborhoods U and U_ν of q and p_ν respectively such that $\psi(q')=\{p_1',\cdots,p_m'\}\subset\cup U_\nu$ for any $q'\in U$. We may assume that there is an open set V which is diffeomorphic to the unit disk and contains $\cup U_\nu$. By the similar argument as above, we can show that $|\varphi(X)|_{U,r}\leqq C|X|_{V,ar+b}$ and hence φ is continuous.

Thus we have proved

THEOREM 5. *Any homomorphism* $\varphi:\mathcal{A}(M)\to\mathcal{A}(N)$ *is continuous in the* C^∞-*topology.*

By Corollary 1 to Proposition 1 in §2 and Theorem 1.3.2 in [4] we have

COROLLARY. *If* N *is compact then* φ *induces a local homomorphism* Diff $(M)\to$Diff (N).

PROOF OF LEMMA 10. By the definition of Φ we have

(6) $\Phi(f(x))-$the right hand side of the desired equation

$$=\sum_{\nu=1}^{l}\sum_{|\alpha|\leqq h}Z_\nu^\alpha D_y^\alpha\Big[f(y)-f(x_1)-\sum_{k=1}^{s}\sum_{j_*}\int_0^1\cdots\int_0^1\partial_{j_1}\cdots\partial_{j_k}f(x(k))dt(k)$$

$$\sum_{m=0}^{k}(-1)^m\sum_{i_*}x_{i1}^{j_1}\cdots x_{im}^{j_m}y^{j_{m+1}+\cdots+j_k}\Big]\Big|_{\substack{y=x_\nu\\ x_\nu+le=x_\nu}}$$

where $s=l(h+1)-1$. Let S_k be the symmetrization operator with respect to $j_1, \cdots, j_k$. Then we have easily

$$S_k \sum_{m=0}^{k} (-1)^m \sum_{1 \leq i_1 < \cdots < i_m \leq k} x_{i_1}^{j_1} \cdots x_{i_m}^{j_m} y^{j_{m+1}+\cdots+j_k} = S_k \prod_{\nu=1}^{k} (y^{j_\nu} - x_\nu^{j_\nu}) .$$

The interior of [ ] in (6) is equal to

$$f(y)-f(x_1)-\sum_{k=1}^{s} \sum_{j_*} \int_0^1 \cdots \int_0^1 \partial_{j_1} \cdots \partial_{j_k} f(x(k)) dt(k) S_k \prod_{\nu=1}^{k} (y^{j_\nu} - x_\nu^{j_\nu})$$

$$= \int_0^1 \frac{d}{dt_1} f(x_1 + t_1(y - x_1)) dt_1 - \sum_{k=1}^{s} \cdots$$

$$= \sum_{j_1} \int_0^1 \partial_{j_1} f((1-t_1)x_1 + t_1 y) dt_1 (y^{j_1} - x_1^{j_1})$$

$$- \sum_{j_1} \int_0^1 \partial_{j_1} f((1-t_1)x_1 + t_1 x_2) dt_1 (y^{j_1} - x_1^{j_1}) - \sum_{k=2}^{s} \cdots$$

$$= \sum_{j_1} \int_0^1 \int_0^1 \frac{d}{dt_2} \partial_{j_1} f((1-t_1)x_1 + t_1 x_2 + t_2(t_1 y - t_1 x_2)) dt_1 dt_2 (y^{j_1} - x_1^{j_1}) - \sum_{k=2}^{s} \cdots$$

$$= \sum_{j_1, j_2} \int_0^1 \int_0^1 \partial_{j_1} \partial_{j_2} f((1-t_1)x_1 + (1-t_2)t_1 x_2 + t_2 t_1 y) t_1 dt_1 dt_2 (y^{j_1} - x_1^{j_1})(y^{j_2} - x_2^{j_2})$$

$$- \sum_{k=2}^{s} \cdots$$

$$= \sum_{j_1, \cdots, j_{s+1}} \int_0^1 \cdots \int_0^1 \partial_{j_1} \cdots \partial_{j_{s+1}} f((1-t_1)x_1 + \cdots + (1-t_{s+1})t_s \cdots t_1 x_{s+1}$$

$$+ t_{s+1} t_s \cdots t_1 y) dt(s+1) S_{s+1} \prod_{\nu=1}^{s+1} (y^{j_\nu} - x_\nu^{j_\nu}) .$$

Since $s+1=l(h+1)$ and $|\alpha| \leqq h$, we have

$$D_y^\alpha [\ ] \Big|_{\substack{y=x_\nu \\ x_{\nu+le}=x_\nu}} \equiv 0 ,$$

which completes the proof of Lemma 10.

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