J. Math. Soc. Japan
Vol. 28, No. 3, 1976

Fundamental groups of the spaces of regular orbits of the finite unitary reflection groups of dimension 2

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(Received Jan. 9, 1974)
(Revised Dec. 4, 1975)

§0. In [1] E. Brieskorn has calculated the fundamental groups of the regular orbit spaces of the finite real reflection groups. It is natural to extend the calculation to the finite *unitary groups generated by reflections.*

Using Shephard-Todd's classification [5] of the irreducible finite *unitary groups generated by reflections*, the author calculates in this paper the fundamental groups of their regular orbit spaces for $n=2$.

Henceforth, we shall abbreviate the italicized words as "u. g. g. r."

§1. Let G be an irreducible finite u. g. g. r. in $U(2)$, then G belongs to one of the following classes ([5]):

(1) the imprimitive groups $G(m, p, 2)$ (no. 2 in [5]) of order $2qm$ where $m=pq$, $m>1$ (these groups are derived from the dihedral group),

(2) the four primitive groups (no. 4, $\cdots$, no. 7 in [5]) generated by S and T where $S=\lambda S_1$, $T=\mu T_1$,

$$S_1=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\quad T_1=\frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon & \varepsilon^3 \\ \varepsilon & \varepsilon^7 \end{pmatrix}\qquad (\varepsilon=\exp(2\pi i/8),\ i=\sqrt{-1})$$

and no. 4: $\lambda=-1$, $\mu=-\omega$; no. 5: $\lambda=-\omega$, $\mu=-\omega$; no. 6: $\lambda=i$, $\mu=-\omega$; no. 7: $\lambda=i\omega$, $\mu=-\omega$ ($\omega=\exp(2\pi i/3)$), (these groups are derived from the tetrahedral group),

(3) the eight primitive groups (no. 8, $\cdots$, no. 15 in [5]) generated by S and T where $S=\lambda S_1$, $T=\mu T_1$,

$$S_1=\frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix},\quad T_1=\frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon^3 & \varepsilon^7 \end{pmatrix}\qquad (\varepsilon=\exp(2\pi i/8),\quad i=\sqrt{-1})$$

and no. 8: $\lambda=\varepsilon^3$, $\mu=1$; no. 9: $\lambda=i$, $\mu=\varepsilon$; no. 10: $\lambda=\varepsilon^7\omega^2$, $\mu=-\omega$; no. 11: $\lambda=i$, $\mu=\varepsilon\omega$; no. 12: $\lambda=i$, $\mu=1$; no. 13: $\lambda=i$, $\mu=i$; no. 14: $\lambda=i$, $\mu=-\omega$; no. 15: $\lambda=i$, $\mu=i\omega$ ($\omega=\exp(2\pi i/3)$) (these groups are derived from octahedral group),

(4) the seven primitive groups (no. 16, $\cdots$, no. 22 in [5]) generated by S

and T where $S=\lambda S_1$, $T=\mu T_1$,

$$S_1=\frac{1}{\sqrt{5}}\begin{pmatrix}\eta^4-\eta & \eta^2-\eta^3\\ \eta^2-\eta^3 & \eta-\eta^4\end{pmatrix},\ T_1=\frac{1}{\sqrt{5}}\begin{pmatrix}\eta^2-\eta^4 & \eta^4-1\\ 1-\eta & \eta^3-\eta\end{pmatrix}\ (\eta=\exp(2\pi i/5),\ i=\sqrt{-1})$$

and no. 16: $\lambda=-\eta^3$, $\mu=1$: no. 17: $\lambda=i$, $\mu=i\eta^3$; no. 18: $\lambda=-\omega\eta^3$, $\mu=\omega^2$; no. 19: $\lambda=i\omega$, $\mu=i\eta^3$; no. 20: $\lambda=1$, $\mu=\omega^2$; no. 21: $\lambda=i$, $\mu=\omega^2$; no. 22: $\lambda=i$, $\mu=1$ (these groups are derived from icosahedral group). (The notation follows that of Shephard and Todd [**5**].)

Let Σ be the set consisting of all the reflections in G. For $s\in\Sigma$, H_s means the hyperplane of fixed points of s. Let $Y_G=\boldsymbol{C}^2-\bigcup_{s\in\Sigma}H_s$ and $X_G=Y_G/G$. Then we obtain the following theorems.

THEOREM 1. *Let $G=G(m, p, 2)$, $m=pq$, $m>1$. Then we obtain the following:*

(i) *if $p=m$, then $\pi_1(X_G)$ is the Artin group of type $I_2(m)$,*

(ii) *if $p\neq m$ and $p=odd$, then $\pi_1(X_G)$ is the Artin group of type B_2,*

(iii) *if $p\neq m$ and $p=even$, then $\pi_1(X_G)$ is the Artin group of type $A_1\times\tilde{A}_1$.*

THEOREM 2. *Let G be a primitive finite u.g.g.r.*

(i) *If G is no. 4, no. 8 or no. 16, then $\pi_1(X_G)$ is the Artin group of type A_2.*

(ii) *If G is no. 5, no. 10 or no. 18, then $\pi_1(X_G)$ is the Artin group of type B_2.*

(iii) *If G is no. 6, no. 9, no. 13 or no. 17, then $\pi_1(X_G)$ is the Artin group of type G_2.*

(iv) *If G is no. 14, then $\pi_1(X_G)$ is the Artin group of type $I_2(8)$.*

(v) *If G is no. 20, then $\pi_1(X_G)$ is the Artin group of type $I_2(5)$.*

(vi) *If G is no. 21, then $\pi_1(X_G)$ is the Artin group of type $I_2(10)$.*

(vii) *If G is no. 7, no. 11, no. 15 or no. 19, then $\pi_1(X_G)$ is the Artin group of type $A_1\times\tilde{A}_1$.*

(viii) *If G is no. 12, then $\pi_1(X_G)$ is $K_{3,4}$.*

(ix) *If G is no. 22, then $\pi_1(X_G)$ is $K_{3,5}$.*

REMARK 1. The case (i) in Theorem 1 is due to Brieskorn [**1**], because these groups are realizable in the real field.

REMARK 2. For the definition of the Artin groups see [**2**].

REMARK 3. The Coxeter diagram associated to the Artin group of type $A_1\times\tilde{A}_1$ is $\circ\quad\circ\overset{}{\underset{\infty}{\text{———}}}\circ$, i.e., the Artin group of this type is $\langle a, b, c\,|\,ab=ba,\ ac=ca\rangle$.

REMARK 4. $K_{p,q}=\langle a, b\,|\,a^p=b^q\rangle$. (cf. [**4**]).

REMARK 5. The Artin group of type $I_2(m)$ (m=odd) is isomorphic to $K_{2,m}$(and $A_2=I_2(3)$, $B_2=I_2(4)$ and $G_2=I_2(6)$).

REMARK 6. The Artin groups of type A_2, B_2, G_2, $I_2(m)$ ($m=5, 7, 8, 9, \cdots$). $A_1\times\tilde{A}_1$, $K_{3,4}$ and $K_{3,5}$ are not isomorphic to each other.

§ 2. Proof of the theorems.

The algebra of invariant polynomials of a u. g. g. r. G is generated by two homogeneous polynomials $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ which are algebraicaly independent (cf. Shephard and Todd [5]). Moreover, the map Φ from $\boldsymbol{C}^2$ into $\boldsymbol{C}^2$ defined by $\Phi(u_1, u_2)=(f_1(u_1, u_2), f_2(u_1, u_2))$ for $(u_1, u_2)\in\boldsymbol{C}^2$ gives a homeomorphism between $\boldsymbol{C}^2/G$ and $\boldsymbol{C}^2$ (see [3]). Then we can show by certain amount of elementary calculations that the image of $\bigcup_{s\in\Sigma} H_s$ under the mapping Φ is a complex curve D.

The following table is the result of this calculation (where D is obtained by a suitable transformation of coordinates).

Imprimitive groups

Group		f_1	f_2	D
no. 2	$p=m$	x_1x_2	$x_1^m+x_2^m$	$z_1^p-z_2^2=0$
	$p<m$	$(x_1x_2)^q$	$x_1^m+x_2^m$	$z_1(z_1^p-z_2^2)=0$

groups derived from the tetrahedral group

Group	f_1	f_2	D
no. 4	f	t	$z_1^3-z_2^2=0$
no. 5	f^3	t	$z_1^4-z_2^2=0$
no. 6	f	t^2	$z_1^6-z_2^2=0$
no. 7	f^3	t^2	$z_1(z_1^2-z_2^2)=0$

where $f=x_1^4-2\sqrt{3}\,ix_1^2x_2^2+x_2^4$ and $t=x_1x_2(x_1^4-x_2^4)$.

groups derived from the octahedral group

group	f_1	f_2	D
no. 8	h	t	$z_1^3-z_2^2=0$
no. 9	h	t^2	$z_1^6-z_2^2=0$
no. 10	h^3	t	$z_1^4-z_2^2=0$
no. 11	h^3	t^2	$z_1(z_1^2-z_2^2)=0$
no. 12	h	f	$z_1^3-z_2^4=0$
no. 13	h	f^2	$z_1(z_1^2-z_2^3)=0$
no. 14	f	t^2	$z_1^8-z_2^2=0$
no. 15	f^2	t^2	$z_1(z_1^4-z_2^2)=0$

where $f=x_1x_2(x_1^4-x_2^4)$, $h=x_1^8+14x_1^4x_2^4+x_2^8$ and $t=x_1^{12}-33x_1^8x_2^4-33x_1^4x_2^8+x_2^{12}$.

groups derived from the icosahedral group

Group	f_1	f_2	D
no. 16	h	t	$z_1^3-z_2^2=0$
no. 17	h	t^2	$z_1^6-z_2^2=0$
no. 18	h^3	t	$z_1^4-z_2^2=0$
no. 19	h^3	t^2	$z_1(z_1^2-z_2^2)=0$
no. 20	f	t	$z_1^5-z_2^2=0$
no. 21	f	t^2	$z_1^{10}-z_2^2=0$
no. 22	f	h	$z_1^3-z_2^5=0$

where $f=x_1x_2(x_1^{10}+11x_1^5x_2^5-x_2^{10})$, $h=-x_1^{20}-x_2^{20}+228(x_1^{15}x_2^5-x_1^5x_2^{15})-494x_1^{10}x_2^{10}$ and $t=x_1^{30}+x_2^{30}+522(x_1^{25}x_2^5-x_1^5x_2^{25})-10005(x_1^{20}x_2^{10}+x_1^{10}x_2^{20})$.

For example, consider group no. 15. In this case, Σ consists of 18 reflections of order 2 and 16 reflections of order 3. The hyperplanes which are associated to the reflections of order 2 are defined by the following 18 equations:

$$x_1=0,\ x_2=0,\ x_1+\alpha x_2=0 \text{ where } \alpha=1,\ -1,\ i \text{ or } -i,$$
$$x_1+\beta x_2=0 \text{ where } \beta=(1+i)/\sqrt{2},\ -(1+i)/\sqrt{2},\ i(1+i)/\sqrt{2}$$
$$\text{or } -i(1+i)/\sqrt{2},\ x_1+\gamma x_2=0 \text{ where } \gamma=\sqrt{2}+1,\ -(\sqrt{2}+1),$$
$$i(\sqrt{2}+1),\ -i(\sqrt{2}+1),\ (\sqrt{2}-1),\ -(\sqrt{2}-1),\ i(\sqrt{2}-1) \text{ or } -i(\sqrt{2}-1).$$

The hyperplanes which are associated to the reflections of order 3 are defined by the following 8 equations:

$$x_1+\delta x_2=0 \text{ where } \delta=\omega+i\omega^2,\ -(\omega+i\omega^2),\ i(\omega+i\omega^2),$$
$$-i(\omega+i\omega^2),\ \omega-i\omega^2,\ -(\omega-i\omega^2),\ i(\omega-i\omega^2) \text{ or } -i(\omega-i\omega^2).$$

On the other hand we have

$$f=x_1x_2(x_1^4-x_2^4)=x_1x_2(x_1+x_2)(x_1-x_2)(x_1+ix_2)(x_1-ix_2),$$
$$h=x_1^8+14x_1^4x_2^4+x_2^8=(x_1^4-\omega+i\omega^2)^4x_2^4)(x_1^4-(\omega-i\omega^2)^4x_2^4),$$
$$t=x_1^{12}-33x_1^8x_2^4-33x_1^4x_2^8+x_2^{12}$$
$$=(x_1^4+x_2^4)(x_1^4-(\sqrt{2}+1)^4x_2^4)(x_1^4-(\sqrt{2}-1)^4x_2^4).$$

Therefore, we obtain

$$\bigcup_{s\in\Sigma}H_s=\{(u_1,u_2)\in \boldsymbol{C}^2 \mid f(u_1,u_2)=0\}\cup\{(u_1,u_2)\in \boldsymbol{C}^2 \mid h(u_1,u_2)=0\}$$
$$\cup\{(u_1,u_2)\in \boldsymbol{C}^2 \mid t(u_1,u_2)=0\}.$$

It is easily verified that f, h and t satisfy the relation

$$108f^4-h^3+t^2=0\,.$$

Since $f_1=f^2$ and $f_2=t^2$, we have

$$\Phi(\bigcup_{s\in\Sigma}H_s)=\{(z_1,z_2)\in \boldsymbol{C}^2 \mid z_1z_2(108z_1^2-z_2)=0\}\,.$$

Setting $z_1'=\sqrt{54}\,z_1$ and $z_2'=-\sqrt{54}\,z_1^2+z_2$, we have

$$D=\Phi(\bigcup_{s\in\Sigma}H_s)=\{z_1,z_2)\in \boldsymbol{C}^2 \mid z_1(z_1^4-z_2^2)=0\}\,.$$

Next we will prove Theorem 1. We can easily calculate the fundamental group of the space $\boldsymbol{C}^2-D$ by the method of Zariski [6] Chap. VIII 1.

(i) of Theorem 1 is due to Brieskorn [**1**].

If $G=G(m,p,2)$ and $p\lneqq m$, then $D=\{(z_1,z_2)\in\boldsymbol{C}^2 \mid z_1(z_1^p-z_2^2)=0\}$. Let us define the projection $\pi:\boldsymbol{C}\times\boldsymbol{C}\to\boldsymbol{C}$ by $\pi(z_1,z_2)=z_2$. The fibers of π are complex lines L_z, $z\in\boldsymbol{C}$. By restriction of π we obtain a fibering $\pi:\boldsymbol{C}^2-D-L_0\to\boldsymbol{C}-\{0\}$ whose typical fiber is $L_1-\{0,1,\theta,\theta^2,\cdots,\theta^{p-1}\}$, where $\theta=\exp(2\pi i/p)$. In this fibering, for every differential closed path $z(t)$ in $\boldsymbol{C}-\{0\}$ with $z(0)=z(1)=1$ and $t\in[0,1]$, we can find an isotopy $f_t: L_1\to L_{z(t)}$ which induces a family of diffeomorphisms on the fibers covering the path and fixes out side of a compact set K on L_1. For example, for the path $z(t)=\exp 2\pi it$, $t\in[0,1]$, let us define f_t using polar coordinates on the fibers by $f_t(r,\varphi)=(r,\varphi+4\pi th(r)/p)$, where $h(r)$ is a C^∞-function with $h(r)=1$ for $r\leqq 1$, $h(r)=0$ for $r\geqq 2$ and $h(r)$ is strictly decreasing for $1\leqq r\leqq 2$. Then f_1 induces a diffeomorphism f of $L_1-\{0,1,\theta,\cdots,\theta^{p-1}\}$ and homomorphism f_* of the fundamental group of $L_1-\{0,1,\theta,\cdots,\theta^{p-1}\}$. Now take a base point $v\notin K$ in $L_1-\{0,1,\theta,\cdots,\theta^{p-1}\}$ and represent the generators $g_1,\cdots,g_{p+1}$ of $\pi_1(L_1-\{0,1,\theta,\cdots,\theta^{p-1}\})$ by the paths shown in the following Figure 1.

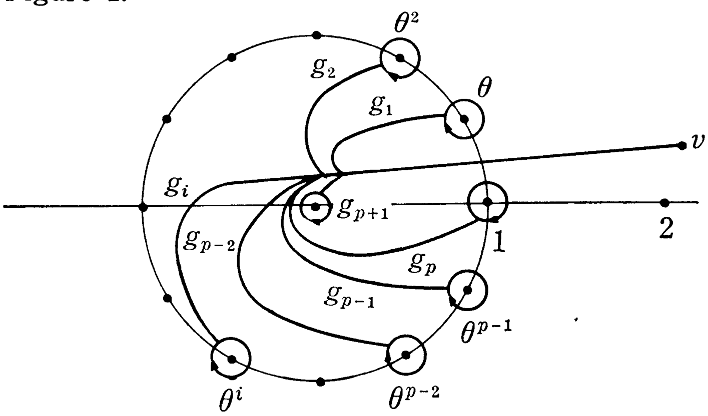

Fig. 1.

Then we have

$$f_*(g_i)=g_1^{-1}g_1^{-1}g_{i+2}g_2g_1 \qquad (i=1, \cdots, p-2),$$

$$f_*(g_{p-1})=g_1^{-1}g_2^{-1}g_{p+1}^{-1}g_1g_{p+1}g_2g_1,$$

$$f_*(g_p)=g_1^{-1}g_2^{-1}g_{p+1}^{-1}g_2g_{p+1}g_2g_1,$$

$$f_*(g_{p+1})=g_1^{-1}g_2^{-1}g_{p+1}g_2g_1.$$

If $j: L_1-\{0, 1, \theta, \cdots, \theta^{p-1}\} \to \boldsymbol{C}^2-D$ is the inclusion mapping then $\pi_1(\boldsymbol{C}^2-D, v)$ is generated by j_*g_i and generating relations are given by $j_*g_i=j_*f_*(g_i)$, $i=1, 2, \cdots, p+1$. (In the following we write g_i for j_*g_i.) Then we have easily

$$g_i=\underbrace{g_2 \cdots g_1g_2}_{i-2\text{ factors}}g_1\underbrace{g_2^{-1}g_1^{-1} \cdots g_2^{-1}}_{i-2\text{ factors}} \qquad \text{for } i=\text{odd}, \ 1<i\leqq p,$$

$$g_i=\underbrace{g_2 \cdots g_2g_1}_{i-2\text{ factors}}g_2\underbrace{g_1^{-1}g_2^{-1} \cdots g_2^{-1}}_{i-2\text{ factors}} \qquad \text{for } i=\text{even}, \ 1<i\leqq p.$$

Therefore $\pi_1(\boldsymbol{C}^2-D, v)$ is generated by g_1, g_2 and g_{p+1} and the generating relations are given by

$$\underbrace{g_2g_1g_2 \cdots}_{p-2\text{ factors}} \ \underbrace{\cdots g_2^{-1}g_1^{-1}g_2^{-1}}_{p-3\text{ factors}}=g_1^{-1}g_2^{-1}g_{p+1}^{-1}g_1g_{p+1}g_2g_1,$$

$$\underbrace{g_2g_1g_2 \cdots}_{p-1\text{ factors}} \ \underbrace{\cdots g_2^{-1}g_1^{-1}g_2^{-1}}_{p-2\text{ factors}}=g_1^{-1}g_2^{-1}g_{p+1}^{-1}g_2g_{p+1}g_2g_1,$$

$$g_{p+1}=g_1^{-1}g_2^{-1}g_{p+1}g_2g_1.$$

If p=odd, then by setting $g_{p+1}(g_2g_1)^{(p-1)/2}=a$, $g_1=b$ and $g_2=c$, we can show $\pi_1(\boldsymbol{C}^2-D, v)=\langle a, b \mid abab=baba\rangle$, i.e., the Artin group of type B_2. If p=even, then by setting $g_{p+1}(g_2g_1)^{p/2}=a$, $g_1=b$ and $g_2=c$, we can show that $\pi_1(\boldsymbol{C}^2-D, v)=\langle a, b, c \mid ab=ba, ac=ca\rangle$, i.e., the Artin group of type $A_1\times\tilde{A}_1$. Thus we have proved Theorem 1.

(i), (ii), (iii) (except no. 13), (iv), (v) and (vi) of Theorem 2 can be shown using the method of Brieskorn [**1**]. (vii) of Theorem 2 follows from an argument similar to that used for case (iii) of Theorem 1.

The remaining groups are no. 12, no. 13 and no. 22.

If G is no. 12, then $D=\{(z_1, z_2)\in\boldsymbol{C}^2 \mid z_1^3-z_2^4=0\}$ and we obtain a fibering $\pi: \boldsymbol{C}^2-D-L_0\to\boldsymbol{C}-\{0\}$ with the typical fiber $L_1-\{1, \omega, \omega^2\}$, where $\omega=\exp(2\pi i/3)$. For the path $z(t)=\exp(2\pi it)$, $t\in[0, 1]$, we define $f_t(r, \varphi)=(r, \varphi+8\pi th(r)/3)$. Let us take the generators g_1, g_2 and g_3 of $\pi_1(L_1-\{1, \omega, \omega^2\})$ represented in the following Figure 2.

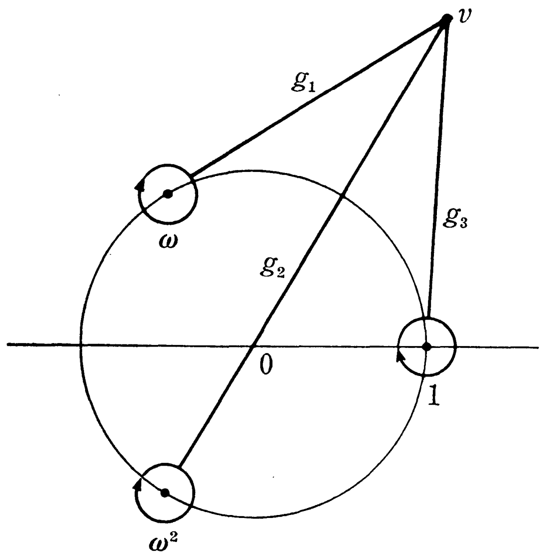

Fig. 2.

Then it follows that $\pi_1(\boldsymbol{C}^2-D, v)$ is generated by g_1, g_2 and g_3 and the generating relations are given by $g_1g_3g_2g_1=g_2g_1g_3g_2$ and $g_3g_2g_1g_3=g_1g_3g_2g_1$. By setting $g_1g_3g_2g_1=a$, $g_1g_3g_2=b$ and $g_1g_3=c$, we see that $\pi_1(\boldsymbol{C}^2-D, v)=\langle a, b \mid a^3=b^4\rangle$ $=K_{3,4}$.

If G is no. 13, then $D=\{(z_1, z_2)\in\boldsymbol{C}^2 \mid z_1(z_1^2-z_2^3)=0\}$ and we obtain a fibering $\pi: \boldsymbol{C}^2-D-L_0\to\boldsymbol{C}-\{0\}$ with the typical fiber $L_1-\{0, 1, -1\}$. For the path $z(t)$ $=\exp(2\pi it)$, $t\in[0, 1]$, we define $f_t(r, \varphi)=(r, \varphi+3\pi th(r))$ and take the generators of $\pi_1(L_1-\{0, 1, -1\})$ indicated by the following Figure 3.

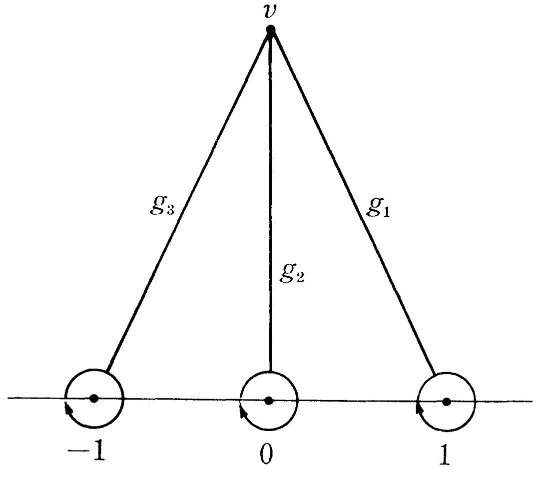

Fig. 3.

Then we obtain $\pi_1(\boldsymbol{C}^2-D)=\langle g_1, g_2, g_3 \mid g_1g_2g_3g_1=g_3g_1g_2g_3,\ g_3g_1g_2g_3g_2=g_2g_3g_1g_2g_3\rangle$. By setting $g_3g_1g_2g_3=a$, $g_3g_1g_2=ab$ and $g_3g_1=c$, we obtain $\pi_1(\boldsymbol{C}^2-D)$ $=\langle a, b \mid ababab=bababa\rangle=$Artin group of type G_2.

If G is no. 22, then $D=\{(z_1, z_2)\in \boldsymbol{C}^2 \mid z_1^3-z_2^5=0\}$, and we obtain a fibering $\pi: \boldsymbol{C}^2-D-L_0\to \boldsymbol{C}-\{0\}$ with the typical fiber $L_1-\{1, \omega, \omega^2\}$. For the path $z(t)=\exp(2\pi it)$, let us define $f_t(r, \varphi)=(r, \varphi+10\pi th(r)/3)$, and take the generators g_1, g_2 and g_3 of $\pi_1(L_1-\{1, \omega, \omega^2\})$ shown in the Figure 2. Then we can show easily $\pi_1(\boldsymbol{C}^2-D)=\langle g_1, g_2, g_3 \mid g_2g_1g_3g_2g_1=g_3g_2g_1g_3g_2,\ g_1g_3g_2g_1g_3=g_2g_1g_3g_2g_1\rangle$. By setting $g_2g_1g_3g_2g_1=a$, $g_2g_1g_3=b$ and $g_2g_1g_3g_2=c$, we obtain $\pi_1(\boldsymbol{C}^2-D)=\langle a, b \mid a^3=b^5\rangle=K_{3,5}$. Thus we have completed the proof of Theorem 2.

The author thanks Professor Lu of Ohio State University for correcting the English of this paper.

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