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A test of Picard principle for rotation free densities, II

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A nonnegative locally Hölder continuous function $P(z)$ on the punctured closed unit disk $0<|z|\leqq 1$ will be referred to as a *density* on $0<|z|\leqq 1$. The dimension of the half module of nonnegative solutions u of $\Delta u(z)=P(z)u(z)$ on $0<|z|<1$ with vanishing boundary values zero on $|z|=1$ is called the *elliptic dimension* of P at $z=0$, $\dim P$ in notation. After Bouligand we say that the *Picard principle* is valid for P if $\dim P=1$. For *rotation free* densities $P(z)$, i.e. densities $P(z)$ satisfying $P(z)=P(|z|)$ on $0<|z|\leqq 1$, it was shown in [20] that

$$\dim P=1+\alpha(P)\cdot \mathfrak{c} \tag{1}$$

where $\mathfrak{c}$ is the cardinal number of continuum and $\alpha(P)$ is the quantity in $[0, 1)$ associated with P which is referred to as the *singularity index* of P. In particular the Picard principle is valid for rotation free densities P if and only if $\alpha(P)=0$. In this context it is important to provide practical tests for $\alpha(P)=0$ and also for $\alpha(P)>0$. The purpose of this paper is, as a continuation of the paper [22] with the same title, to contribute to this latter subject.

There exists a unique bounded solution $e_P(z)$, referred to as the *P-unit*, of $\Delta u=Pu$ on $0<|z|<1$ with boundary values 1 on $|z|=1$. The first of our main results in this paper is the following complete characterization of $\alpha(P)=0$ in terms of e_P given in §2: The Picard principle is valid for a rotation free density $P(z)$ if and only if

$$\int_0^1 \frac{dr}{r\left(r\frac{d}{dr}\log e_P(r)+1\right)}=\infty\,. \tag{2}$$

As an application of this we can settle the validity of the *order comparison theorem* in the affirmative for rotation free densities (cf. [20], [21], [22]): If $P_1(z)$ and $P_2(z)$ are rotation free densities on $0<|z|\leqq 1$ such that

$$c^{-1}P_1(z)\leqq P_2(z)\leqq cP_1(z)$$

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on $0<|z|\leqq 1$ with a constant $c\geqq 1$, then $\dim P_1=\dim P_2$. Actually we shall prove a bit more. We say that P_1 and P_2 are *order equivalent* if there exists a constant $c\geqq 1$ such that

$$c^{-1}(P_1(z)+|z|^{-2})\leqq P_2(z)+|z|^{-2}\leqq c(P_1(z)+|z|^{-2})$$

on $0<|z|\leqq 1$. This is certainly an equivalence relation and therefore the class $\mathcal{D}$ of rotation free densities on $0<|z|\leqq 1$ is divided into equivalence classes $\mathcal{P}$. Then the mapping $\dim: \mathcal{D}\to\{1, \mathfrak{c}\}$ is constant on each equivalence class $\mathcal{P}$. This will be shown in § 3. We can thus speak of $\dim\mathcal{P}$ and (1) can be restated as

$$\dim\mathcal{P}=1+\alpha(\mathcal{P})\cdot\mathfrak{c} \tag{3}$$

where $\alpha(\mathcal{P})=\sup_{P\in\mathcal{P}}\alpha(P)$. Therefore our task of providing complete tests for $\alpha(P)=0$ amounts to the same of providing complete tests for $\alpha(\mathcal{P})=0$.

In our preceding paper [22] we considered the condition

$$\int_0^1\frac{dr}{r\sqrt{r^2P(r)+1}}=\infty\,. \tag{4}$$

Since (4) is valid for every $P\in\mathcal{P}$ if and only if (4) is valid for one $P\in\mathcal{P}$, the condition (4) may be viewed as a condition for $\mathcal{P}$. In [22] it was shown that the condition (4) is necessary and sufficient for $\dim\mathcal{P}=1$ for classes $\mathcal{P}$ with the condition (**I**): $\mathcal{P}$ contains a $P(z)$ such that $r^2P(r)$ is increasing as $r\to 0$. We shall discuss in §§ 4-6 whether this rather unpleasant condition (**I**) can be removed. The conclusion is that if $\dim P=1$, then (4) is valid for P without any additional requirement (§ 4), but, unfortunately, the converse is not true, which is shown by an example in § 6. However, if the class $\mathcal{P}$ satisfies the condition (**B**): $\mathcal{P}$ contains a $P(z)$ such that $(r^2P(r)+1)^{-1/2}$ is of bounded variation on $(0, 1]$, which is weaker than (**I**), then the condition (4) implies $\dim\mathcal{P}=1$ (§ 5).

§ 1. Fundamental inequality.

1.1. Consider a nonnegative locally Hölder continuous function $P(z)$ on $0<|z|\leqq 1$ which is *rotation free* in the sense that $P(z)=P(|z|)$. In this paper we only consider rotation free densities unless otherwise is explicitly stated. Thus a density $P(z)$ may be considered as a function $P(r)$ on $(0, 1]$. We briefly recall results obtained in [20]. Consider the ordinary differential equations

$$\frac{1}{r}\Big(\frac{d}{dr}\Big(r\frac{d}{dr}u(r)\Big)\Big)=\Big(P(r)+\frac{n^2}{r^2}\Big)u(r)\qquad(n=0, 1, \cdots)\,. \tag{5}$$

For each n the equation (5) has a unique bounded solution $e_n(r)$ on $(0, 1]$ with the initial condition $e_n(1)=1$. We have that

(6) $$1 \geqq e_0(r) \geqq e_1(r) \geqq e_2(r) \geqq \cdots > 0$$

on $(0, 1]$. The functions $e_n(r)/e_{n-1}(r)$ and $e_n(r)/e_0(r)$ $(n=0, 1, \cdots; e_{-1}\equiv 1)$ are decreasing as $r \to 0$ and

(7) $$1 \geqq e_n(r)/e_{n-1}(r) \geqq e_{n+1}(r)/e_n(r).$$

Therefore the limit

(8) $$\alpha_n(P) = \lim_{r\to 0} e_n(r)/e_0(r)$$

exists and is referred to as the n^{th} *singularity index* of P at $z=0$, and in particular, $\alpha(P)=\alpha_0(P)$ as *singularity index* of P at $z=0$. We have the following fundamental inequality:

(9) $$0 \leqq \alpha(P) < 1, \qquad (\alpha(P))^{(3^n-1)/2} \leqq \alpha_n(P) \leqq (\alpha(P))^n$$

for $n=0, 1, \cdots$. The Martin compactification Ω_P^* of the punctured open unit disk $\Omega: 0<|z|<1$ with respect to a density P on $0<|z|\leqq 1$ is homeomorphic to a closed annulus, i.e.

$$\Omega_P^* \approx (\alpha(P) \leqq |z| \leqq 1).$$

As a result we have the following equality for the elliptic dimension of a density P:

(10) $$\dim P = 1 + \alpha(P)\cdot \mathfrak{c}$$

where $\mathfrak{c}$ is the cardinal number of continuum. Therefore it is important to determine whether $\alpha(P)=0$ or $\alpha(P)>0$ for given P.

1.2. As an example and also for later use, we compute the singularity index of $P(r)+3^2/r^2$ when that of $P(r)$ is given. We shall show that

(11) $$\dim (P+3^2/r^2) = \dim P,$$

(cf. 5.2 in [22]). Let e_n and $\bar{e}_n$ be the solution for P and $\bar{P}=P+3^2/r^2$, respectively. Observe that $\bar{e}_0=e_3$ and $\bar{e}_4=e_5$. By (6) and (7) we have

$$\frac{e_5}{e_0} \leqq \frac{e_5}{e_3} = \frac{\bar{e}_4}{\bar{e}_0} = \frac{e_5}{e_3} = \frac{e_5}{e_4}\,\frac{e_4}{e_3} \leqq \left(\frac{e_1}{e_0}\right)^2.$$

Thus by (9) we have

$$\alpha(P)^{(3^5-1)/2} \leqq \alpha_5(P) \leqq \alpha_4(\bar{P}) \leqq \alpha(P)^2.$$

Therefore $\alpha(P)=0$ if and only if $\alpha_4(\bar{P})=0$. Again applying (9) to $\bar{P}$, we have that $\alpha(\bar{P})=0$ if and only if $\alpha_4(\bar{P})=0$. By (10) we conclude that $\dim P=\dim \bar{P}$.

1.3. We recall another expression of $\alpha(P)$ obtained in § 2 in [22]. Change the variable $r\in(0, 1]$ to $t\in[0, \infty)$ by $r=e^{-t}$. The function $Q(t)$ on $[0, \infty)$ given by

$$Q(t)=e^{-2t}P(e^{-t}) \tag{12}$$

will be referred to as the *associated function* to $P(r)$. Consider a Riccati type equation

$$-\frac{d}{dt}a(t)+a(t)^2=Q(t). \tag{13}$$

Here $Q(t)$ may be any nonnegative continuous function on $[0, \infty)$ but we mainly consider those $Q(t)$ which are associated functions to densities $P(r)$. The equation (13) has a unique nonnegative solution $a_Q(t)$ on $[0, \infty)$ and, if $Q_1\leqq Q_2$, then $a_{Q_1}\leqq a_{Q_2}$. We shall call $a_Q(t)$ the *Riccati component* of $Q(t)$. Next consider the equation

$$-\frac{d^2}{dt^2}w(t)+2a_Q(t)\frac{d}{dt}w(t)+\left(\frac{d}{dt}w(t)\right)^2=1. \tag{14}$$

The equation (14) has a unique nonnegative solution $w_Q(t)$ on $[0, \infty)$ with the initial condition $w_Q(0)=0$. The w_Q is increasing on $[0, \infty)$ and has a limit as $t\to\infty$:

$$\lim_{t\to\infty} w_Q(t)=w_Q(\infty).$$

In terms of $w_Q(\infty)$ we have the following expression of $\alpha(P)$:

$$\alpha(P)=\exp(-w_Q(\infty)) \tag{15}$$

if $Q(t)$ is the associated function to $P(r)$. The following relation will be used later:

$$\frac{d}{dt}w_Q(t)=a_{Q+1}(t)-a_Q(t). \tag{16}$$

1.4. We recall that the relation (15) can be reformulated as the b-test (Theorem 2.6 in [22]) which is more manageable for the practical applications. The b-test reads: The singularity index $\alpha(P)=0$ ($\alpha(P)>0$, resp.) if and only if there exists a nonnegative C^2 function $b(t)$ on $[0, \infty)$ such that

$$\begin{aligned} &-b''(t)+2a_Q(t)b'(t)+b'(t)^2\leqq 1 \\ &(-b''(t)+2a_Q(t)b'(t)+b'(t)^2\geqq 1, \text{ resp.}) \end{aligned} \tag{17}$$

on $[0, \infty)$ and

$$\limsup_{t\to\infty} b(t)=\infty \qquad (\liminf_{t\to\infty} b(t)<\infty,\ \text{resp.})$$

where $Q(t)$ is the associated function to $P(r)$.

1.5. Consider the nonnegative C^1 functions $a_i(t)$ $(i=1, 2)$ on $[0, \infty)$. We set $Q_i(t)=-a_i'(t)+a_i(t)^2$. If $Q_2\geqq 0$ and $Q_1\leqq Q_2$, then

$$a_1(t)\leqq a_2(t)\,. \tag{18}$$

The assertion reduces to Lemma 2.7 in [**22**] if $Q_1\geqq 0$ is postulated. Here we do not assume anything on the sign of Q_1. To prove this, we set

$$v_i(t)=\exp\left(-\int_0^t a_i(s)ds\right)$$

$(i=1, 2)$. Since $a_i(t)\geqq 0$, the function $v_i(t)$ are bounded solutions of $v''-Q_iv=0$ on $[0, \infty)$ with the initial condition $v_i(0)=1$. The maximum principle (cf. 2.2 in [**22**]) applied to $v_1-v_2+\eta t$ $(\eta>0)$ for the operator $L_{Q_2}f=f''-Q_2f$ yields $v_1-v_2+\eta t\geqq 0$ on $[0, \tau]$ for sufficiently large $\tau>0$. Thus we have $v_1\geqq v_2$ on $[0, \infty)$. As in Lemma 2.4 in [**22**] we next set

$$u(t)=\exp\left(-\int_0^t (a_2(s)-a_1(s))ds\right),$$

which is a bounded solution of

$$u''-2a_1u'-(Q_2-Q_1)u=0$$

on $[0, \infty)$ with the initial condition $u(0)=1$. Since $Q_2-Q_1\geqq 0$ and $a_1\geqq 0$, Lemma 2.2 in [**22**] is applicable to deduce that u is decreasing, i. e. $u'(t)=u(t)(a_1(t)-a_2(t))\leqq 0$ and thus $a_1(t)\leqq a_2(t)$.

1.6. As an application of the above result, we deduce the following inequality:

$$c^*a_Q+k\geqq a_{cQ+k^2}\geqq \lambda a_Q+k\sqrt{1-\lambda} \tag{19}$$

where c and k are nonnegative numbers, $c^*=\max(1, c)$, $0\leqq\lambda\leqq\min(1, c)$, and a_Q is a unique nonnegative solution of (13).

To prove this we consider a function $\alpha a_Q+\beta$, where α and β are nonnegative numbers. Set

$$\begin{aligned}\bar{Q}&=-(\alpha a_Q+\beta)'+(\alpha a_Q+\beta)^2\\&=\alpha Q+(\alpha^2-\alpha)a_Q^2+2\alpha\beta a_Q+\beta^2\,.\end{aligned}$$

We have that $\alpha Q+\beta^2\leqq\bar{Q}$ if $\alpha\geqq 1$. Therefore, on setting $\alpha=c$ and $\beta=k$, (18) implies the first inequality of (19) if $c\geqq 1$. Next assume that $0\leqq c<1$. By choosing $\alpha=1$ and $\beta=k$, we deduce $\bar{Q}\geqq Q+k^2\geqq cQ+k^2$. Thus again by (18) the first inequality of (19) is valid. Next assume that $0<\alpha<1$. We then have that

$$\bar{Q}\leqq\alpha Q+\beta^2/(1-\alpha).$$

By taking $k^2=\beta^2/(1-\alpha)$, (18) implies that $a_{\alpha Q+k^2}\geqq\alpha a_Q+k\sqrt{1-\alpha}$. This inequality remains valid even for $\alpha=0$ and we have the second inequality of (19).

§ 2. The P-unit criterion.

2.1. The unique solution e_P of $\Delta u=Pu$ on $0<|z|<1$ with continuous boundary values 1 on $|z|=1$ will be referred to as the *P-unit* (cf. [28], [21]). For a rotation free density P, the P-unit $e_P(z)$ may be considered as a function $e_P(r)$ on $(0, 1]$ which, moreover, is equal to $e_0(r)$ in § 1 (cf. [22]). We are interested in the question how the asymptotic behavior of e_P as $r\to 0$ rules the validity of the Picard principle for P, i. e. $\dim P=1$. In view of Theorem 4.1 and Theorem 4.3 in [22], we see that if $e_P(r)$ decreases 'so slowly' as $r\to 0$, then $\dim P=1$ and if $e_P(r)$ decreases 'enough rapidly' as $r\to 0$, then $\dim P=\mathfrak{c}$. We wish to describe the rate of this decreasingness exactly. We state one of the main results in this paper:

THEOREM. *The Picard Principle is valid for a rotation free density P, i. e.* $\dim P=1$, *if and only if*

$$\int_0^1\frac{dr}{r\left(r\frac{d}{dr}\log e_P(r)+1\right)}=\infty. \tag{20}$$

In other words the Picard principle is invalid for a rotation free density P, i. e. $\dim P=\mathfrak{c}$, *if and only if*

$$\int_0^1\frac{dr}{r\left(r\frac{d}{dr}\log e_P(r)+1\right)}<\infty. \tag{21}$$

2.2. Change the variable $r\in(0, 1]$ to $t\in[0, \infty)$ by $r=e^{-t}$ and consider the Riccati component $a_Q(t)$ of the associated function $Q(t)=e^{-2t}P(e^{-t})$ to a density P. The above condition (20) is rewritten in terms of $a_Q(t)$ as follows:

$$\int_0^\infty\frac{dt}{a_Q(t)+1}=\infty. \tag{22}$$

Thus Theorem 2.1 takes the following form:

The Picard principle is valid for a density P if and only if the condition (22) *is satisfied.*

If $P(r)\geqq r^{-2}$, or equivalently, if $Q(t)\geqq 1$, then (18) assures that $a_Q(t)\geqq a_1(t)=1$. In this case the condition (22) is clearly equivalent to the following condition:

$$\int_0^\infty \frac{dt}{a_Q(t)} = \infty . \tag{23}$$

2.3. Before proving the theorem in 2.1 or equivalently the italicized assertion in 2.2, we state the following remark. Let $\bar{P}(r)=P(r)+3^2/r^2$ and $\bar{Q}(t)$ be the associated function to $\bar{P}(r)$. Observe that $\bar{Q}(t)\geqq 1$, and in fact, $\bar{Q}(t)=Q(t)+3^2$. We recall the identity (11): $\dim \bar{P}=\dim P$. Therefore the validity of the Picard principle for P is equivalent to that for $\bar{P}$. On the other hand, we have

$$a_{\bar{Q}}(t) \leqq 3(a_Q(t)+1) \leqq 6a_{\bar{Q}}(t) .$$

In fact, the first inequality of (19) with $c=k=3$ and the inequality (18) yield the first of the above inequality. The last inequality of (19) with $c=1$, $k=3$, and $\lambda=1/2$ implies the last inequality of the above. Therefore (22) is equivalent to

$$\int_0^\infty \frac{dt}{a_{\bar{Q}}(t)} = \infty .$$

These observations show that the assertion in 2.2 for P is equivalent to that for $\bar{P}$. Therefore we may assume that $Q(t)\geqq 1$ in the proof of the assertion in 2.2.

2.4. Although the essence of the proof of the sufficiency of the assertion in 2.2 is found in [22], we include here its whole proof briefly for the sake of completeness. In view of 2.3 we may assume that $Q(t)\geqq 1$ and hence $a_Q(t)\geqq 1$. We shall show that (23) implies $\dim P=1$. We set

$$b(t) = \eta \int_0^t \frac{ds}{a_Q(s)} \qquad (\eta > 0) .$$

Since $a_Q'/a_Q^2 \leqq 1$, the function $b(t)$ is of class C^2 and

$$-b''+2a_Q b'+b'^2 = \eta\cdot a_Q'/a_Q^2+2\eta+\eta^2/a_Q^2 \leqq 3\eta+\eta^2<1$$

for sufficiently small $\eta>0$. On the other hand, by the assumption (23), we have $\limsup_{t\to\infty} b(t)=\infty$. A fortiori, by 1.4, we deduce that $\alpha(P)=0$, i.e. $\dim P=1$.

2.5. We next prove the necessity of the assertion in 2.2 which is the essential part of our proof. In view of 2.2 we may again assume $Q(t)\geqq 1$ to prove that $\dim P=1$ implies (23). Consider the equation (14) in 1.3, i.e.

$$-w_Q''(t)+2a_Q(t)w_Q'(t)+w_Q'(t)^2=1 . \tag{24}$$

We observe that $a_Q\geqq 1$, $a_Q'/a_Q^2\leqq 1$, and $w_Q'=a_{Q+1}-a_Q$ (cf. (16)). Moreover by setting $c=k=1$ in (19), we have that $a_Q+1\geqq a_{Q+1}$, i.e. $0\leqq w_Q'\leqq 1$. From (24) it follows that

$$2w_Q'(t) \leqq 1/a_Q(t) + w_Q''(t)/a_Q(t).$$

Integration of both sides of the above on the interval $[0, T]$ $(0<T<\infty)$ and the integration by parts yield

$$2w_Q(T) \leqq \int_0^T \frac{dt}{a_Q(t)} + \left[\frac{w_Q'(t)}{a_Q(t)}\right]_{t=0}^{t=T} + \int_0^T \frac{a_Q'(t)}{a_Q(t)^2} w_Q'(t)dt$$

$$\leqq \int_0^T \frac{dt}{a_Q(t)} + 2 + w_Q(T).$$

A fortiori

$$w_Q(T) \leqq \int_0^T \frac{dt}{a_Q(t)} + 2$$

for any $T>0$. This shows that $w_Q(\infty)=\infty$ implies (23). By the identity (10) and (15), we conclude that $\dim P=1$ implies (23). This completes the proof of Theorem 2.1.

§ 3. Order comparisons.

3.1. There are many structures S_P associated with the equation $\Delta u = Pu$ which are invariant if P is replaced by $\hat{P}$ with $c^{-1}P \leqq \hat{P} \leqq cP$ $(c \in [1, \infty)$ (cf. [**26**], [**19**], etc.). In this section we shall show that the elliptic dimension $\dim P$ also belongs to this category. The assertion follows from the monotone property of $\dim P$ and the invariance of $\dim P$ by multiplications of P by positive constants. Both of these two properties will be derived as direct consequences of the P-unit criterion. We start with the monotone property of $\dim P$:

PROPOSITION. *If P_1 and P_2 are rotation free densities with $P_1 \leqq P_2$, then* $\dim P_1 \leqq \dim P_2$.

This was already shown in [**22**, Proposition 5.1]. Here we give an alternate proof based on Theorem 2.1. In view of $\dim P_i = 1 + \alpha(P_i) \cdot \mathfrak{c}$, we only have to show that $\dim P_2 = 1$ implies $\dim P_1 = 1$. Let Q_i be the associated function to P_i and a_{Q_i} be the Riccati component of Q_i $(i=1, 2)$. Then $P_1 \leqq P_2$ implies $Q_1 \leqq Q_2$ and which in turn implies $a_{Q_1} + 1 \leqq a_{Q_2} + 1$, i. e.

$$\int_0^\infty \frac{dt}{a_{Q_2}(t)+1} \leqq \int_0^\infty \frac{dt}{a_{Q_1}(t)+1}.$$

The left hand side of the above inequality is infinite by 2.2 since $\dim P_2 = 1$. Therefore the right hand side integral is infinite and by the same reason as above we conclude that $\dim P_1 = 1$.

3.2. We proceed to the multiplication invariance of $\dim P$. This is stated in e. g. [**21**], [**22**], but has never been proven. We shall prove the following

PROPOSITION. *If P is a rotation free density and c a positive constant,* *then the following identity is valid:*

$$\dim(cP) = \dim P. \tag{25}$$

Suppose (25) is true whenever $c \geqq 1$. Then even if $0 < c < 1$, $c^{-1} > 1$ implies that $\dim(c^{-1}(cP)) = \dim(cP)$, i.e. (25) is valid also for $0 < c < 1$. Therefore we may suppose $c \geqq 1$ to prove (25). Then since $cP \geqq P$, we have $\dim(cP) \geqq \dim P$ by 3.1. By the fact that the range of dim is two element set $\{1, \mathfrak{c}\}$, we only have to show the implication $\dim(cP) = 1$ from $\dim P = 1$. Let Q be the associated function to P and a_Q the Riccati component of Q. Then the associated function to cP is cQ. An application of (19) with $k = 0$ yields $a_{cQ} \leqq c a_Q$ and thus

$$\int_0^\infty \frac{dt}{a_{cQ}(t)+1} \geqq \frac{1}{c} \int_0^\infty \frac{dt}{a_Q(t)+1}.$$

By 2.2 we deduce that $\dim P = 1$ implies $\dim(cP) = 1$.

3.3. From Propositions 3.1 and 2 the following *order comparison theorem* follows at once: If P_1 and P_2 are rotation free densities such that $c^{-1}P_1 \leqq P_2 \leqq cP_1$ with a constant $c \geqq 1$, then $\dim P_1 = \dim P_2$. For the later purpose we shall reformulate this in a slightly sharper form. We first remark that the number 3 in the identity (11) is only chosen for the technical reason, i.e. the number 3 is one of the convenient numbers for the application of fundamental inequalities (9). Actually 3 may be replaced by the smallest number of any Pythagorean triple. In view of the above order comparison theorem, we can sharpen (11) as

$$\dim c(P(r) + \phi(r^{-1})) = \dim P \tag{26}$$

where c is a positive constant and ϕ a quadratic polynomial with nonnegative coefficients. We observe that

$$P_2(r) + \phi_2(r^{-1}) \leqq c(P_1(r) + \phi_1(r^{-1}))$$

where c is a positive constant and ϕ_i are quadratic polynomials with nonnegative coefficients ($i = 1, 2$), for two rotation free densities P_1 and P_2 is equivalent to

$$P_2(r) + r^{-2} \leqq c_1(P_1(r) + r^{-2})$$

for a positive constant c_1. Based on these observations we say that two rotation free densities P_1 and P_2 are *order equivalent* if there exists a constant $c \geqq 1$ such that

$$c^{-1}(P_1(z) + |z|^{-2}) \leqq P_2(z) + |z|^{-2} \leqq c(P_1(z) + |z|^{-2}) \tag{27}$$

in a neighborhood $0 < |z| < r$ ($r \in (0, 1]$) of $z = 0$. Since $\dim P$ depends only on the behavior of P in small vicinity of $z=0$ (cf. 2.5 in [22]), we obtain the following stronger form of the order comparison

THEOREM. *If* P_1 *and* P_2 *are order equivalent rotation free densities, then* $\dim P_1=\dim P_2$.

3.4. The order equivalence (27) is clearly an equivalence relation on the class $\mathcal{D}$ of rotation free densities on $0<|z|\leqq 1$. Therefore the family $\mathcal{D}$ is divided into order equivalence classes $\mathcal{P}$. Theorem 3.3 means that the mapping $\dim : \mathcal{D}\to\{1, \mathfrak{c}\}$ is constant on each equivalence class $\mathcal{P}$. We can thus speak of $\dim \mathcal{P}$ and (1) can be restated as

$$\dim \mathcal{P}=1+\alpha(\mathcal{P})\cdot\mathfrak{c} \tag{28}$$

where $\alpha(\mathcal{P})=\sup_{P\in\mathcal{P}}\alpha(P)$. Let Q_i be the associated functions to densities P_i $(i=1, 2)$. Then P_1 and P_2 are order equivalent if and only if there exists a constant $c\geqq 1$ such that

$$c^{-1}(Q_1+1)\leqq Q_2+1\leqq c(Q_1+1)\,. \tag{29}$$

We shall also say that Q_1 and Q_2 are *order equivalent* if (29) is satisfied. For example, if there exists a positive constant k such that $|Q_1-Q_2|\leqq k$, then Q_1 and Q_2 are order equivalent.

3.5. We shall call a density P *normal* if the set of zeros of the function

$$\frac{d}{dr}\Big(r\frac{d}{dr}\log e_P(r)\Big)$$

is isolated in $(0, 1]$ where e_P is the P-unit. The associated function $Q(t)$ to $P(r)$ will also be called *normal* if P is normal. The normality of Q is then defined by the isolatedness of the set of zeros of the derivative $da_Q(t)/dt$ of the Riccati component a_Q of Q on $[0, \infty)$. If P (Q, resp.) is real analytic on $(0, 1]$ ($[0, \infty)$, resp.), then P (Q, resp.) is normal.*) That the normality is only technical and not essential restriction for the elliptic dimensions is seen by the following (cf. e.g. Milnor [14])

PROPOSITION. *Any order equivalence class of densities always contains a normal density.*

The proof will be given in this and the next two nos. Let $\mathcal{P}$ be an order equivalence class and let $Q(t)$ be the associated functions of densities P in $\mathcal{P}$. We only have to show that there exists a $\hat{Q}(t)$ associated to a density $\hat{P}$ in $\mathcal{P}$ such that $|Q(t)-\hat{Q}(t)|\leqq 1$ and $\hat{Q}$ is normal. Since Q and $Q+1$ are order equivalent, we may assume that $Q\geqq 1$ and $a_Q\geqq 1$ where a_Q is the Riccati component

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*) Of course we must exclude the case $P(r)\equiv\text{const.}\, r^{-2}$, i.e. $Q(t)\equiv\text{const.}$

of Q. First we approximate the function $a_Q(t)$ by a function $f(t)$ on $[0, \infty)$ such that the graph of f is a polygonal line and that $f>0$ and $-f'+f^2>0$ except corner points. For the purpose take a strictly increasing and divergent sequence $\{s_n\}_{n=0}^{\infty}$ in $[0, \infty)$ with $s_0=0$ such that: for a given $\eta\in(0, 1)$ and for each n,

$$|a_Q(s')^2-a_Q(s'')^2|<\eta/3 \tag{30}$$

for points s' and s'' in $[s_n, s_{n+1}]$ and

$$|a_Q'(s')-a_Q'(s'')|<\min(\eta/3, k_n) \tag{31}$$

for points s' and s'' in $[s_n, s_{n+1}]$ where $k_n=\min_{s\in[s_n,s_{n+1}]} Q(t)/2$. We can actually take such sequence $\{s_n\}$, for example, by the following way. Since $a_Q(t)$ and $a_Q'(t)$ are both uniformly continuous on each interval $[n, n+1]$, we can divide each interval $[n, n+1]$ into finite subintervals on which conditions (30) and (31) are satisfied. With the aid of the sequence $\{s_n\}$ we define a polygonal function $f(t)$ to be $f(s_n)=a_Q(s_n)$ and linear on (s_n, s_{n+1}) for all $n=0, 1, 2, \cdots$. By the mean value theorem applied to a_Q, there exists an $s_n'\in(s_n, s_{n+1})$ such that $a_Q'(s_n')=f'(t)$ for $t\in(s_n, s_{n+1})$. Therefore we deduce that

$$\begin{aligned}-f'(t)+f(t)^2 &= -a_Q'(s_n')+f(t)^2\\ &\geqq -a_Q'(\tau_n)-k_n+f(\tau_n)^2\\ &= -a_Q'(\tau_n)+a_Q(\tau_n)^2-k_n>0\end{aligned}$$

where $\tau_n=s_n$ if $f(s_n)\leqq f(s_{n+1})$ and $\tau_n=s_{n+1}$ if $f(s_n)>f(s_{n+1})$. On the other hand by (30) and (31) we obtain that

$$|Q(t)-(-f'(t)+f(t)^2)|<\eta$$

on $[0, \infty)$ except for $t=s_n$ $(n=0, 1, 2, \cdots)$.

3.6. We modify f to a function $g(t)$ if there exists an interval $[s_n, s_{n+1}]$ on which $f'(t)\equiv 0$. We take a middle point $\bar{s}_n$ of $[s_n, s_{n+1}]$ and define a continuous function $g(t)$ on $[s_n, s_{n+1}]$ such that $g(\bar{s}_n)=f(s_n)+\varepsilon_n$, $g(s_n)=g(s_{n+1})=f(s_n)$, and $g(t)$ is linear on $(s_n, \bar{s}_n)\cup(\bar{s}_n, s_{n+1})$, where $\varepsilon_n>0$ is chosen so small that $|Q(t)-(-g'(t)+g(t)^2)|<\eta$ except for corner points on $[s_n, s_{n+1}]$. If $f'(t)\neq 0$ in (s_n, s_{n+1}), then we define $g(t)=f(t)$ in $[s_n, s_{n+1}]$. We also have

$$|Q(t)-(-g'(t)+g(t)^2)|<\eta \tag{32}$$

on $[0, \infty)$ except for corner points. We denote by $\{t_n\}_0^{\infty}$ the strictly increasing divergent sequence such that $\{t_n\}_0^{\infty}=\{s_n\}_0^{\infty}\cup\{\bar{s}_n\}$.

3.7. The last step is a reguralization of $g(t)$. Set $A_n=(t_n-\delta_n', g(t-\delta_n'))$,

$B_n=(t_n, g(t_n))$, and $C_n=(t_n+\delta_n, g(t+\delta_n))$ for $\delta_n>0$ and $\delta'_n>0$ such that $\overline{A_nB_n}=\overline{B_nC_n}$ and that each neighborhood $(t_n-\delta'_n, t_n+\delta_n)$ of t_n are mutually disjoint. We replace $g(t)$ by a parabola in each interval $[t_n-\delta'_n, t_n+\delta_n]$ which is tangent to $g(t)$ at A_n and C_n, and then the resulting function $\hat{a}(t)$ is of class C^1 and equal to $g(t)$ except for a neighborhood of t_n.

Since $\hat{a}'(t)$ varies from $g'(t_n-\delta'_n)$ to $g'(t_n+\delta_n)$ monotone as t varies from $t_n-\delta'_n$ to $t_n+\delta_n$ and $g(t_n)>0$, we have by (32) that $-\hat{a}'(t)+\hat{a}(t)^2>0$ for sufficiently small δ_n and δ'_n. Moreover we see that $|\hat{Q}(t)-Q(t)|\leqq 1$ if δ_n and δ'_n are chosen small enough, where $\hat{Q}(t)=-\hat{a}'(t)+\hat{a}(t)^2$. Since $g'(t)\neq 0$ for $t\neq t_n$, the zero set of $\hat{a}'(t)$ contains at most one point in each neighborhood of t_n. Thus the zeros of $a'(t)$ form an isolated point set in $[0, \infty)$. Let $\hat{P}(r)=r^{-2}Q(-\log r)$. Then $\hat{P}\in\mathscr{P}$, the $\hat{Q}$ is the associated function to $\hat{P}$, and $\hat{a}$ is the Riccati component $a_{\hat{Q}}$ of $\hat{Q}$ with isolated zero set, and the proof is herewith complete.

§ 4. A necessary condition on Picard principle.

4.1. There have been given various conditions for the validity of the Picard principle, i. e. $\dim P=1$, even for general densities P (cf. [**1**], [**4**], [**5**], [**20**], [**21**], [**22**] etc.), but any of them is either incomplete or implicit in the sense that the condition is not stated only in terms of P even for rotation free densities. Especially we are interested in the condition

$$\int_0^1 \frac{dr}{r\sqrt{r^2P(r)+1}}=\infty \tag{33}$$

for rotation free densities P introduced in [**22**]. Clearly (33) is valid for every P in an order equivalence class $\mathscr{P}$ if and only if (33) is valid for one P in $\mathscr{P}$. Thus the condition (33) may be viewed as a condition on $\mathscr{P}$. At the first sight the following result obtained in [**22**] seemed to us quite promising to complete the study of the Picard principle: *The condition* (33) *is necessary and sufficient for* $\dim\mathscr{P}=1$ *for classes* $\mathscr{P}$ *with the condition* (**I**): $\mathscr{P}$ *contains a* P *such that* $r^2P(r)$ *is increasing as* $r\to 0$. In this assertion if the restriction (**I**) were redundant, then the study would be complete, and it is natural to ask whether (**I**) can be dispensed with. In this section we shall show that the condition (33) is necessary for $\dim P=1$ without any additional requirement. Namely, we shall prove the following which is another of our main result in this paper:

THEOREM. *If the Picard principle is valid for a rotation free density* P, *then* P *must satisfy* (33). *In other words the Picard principle is invalid for* P *if*

$$\int_0^1 \frac{dr}{r\sqrt{r^2P(r)+1}}<\infty\,. \tag{34}$$

4.2. As before, change the variable $r\in(0,1]$ to $t\in[0,\infty)$ by $r=e^{-t}$. We consider the associated function $Q(t)=e^{-2t}P(e^{-t})$ to $P(r)$. The condition (33) is equivalent to the following:

$$\int_0^\infty \frac{dt}{\sqrt{Q(t)+1}}=\infty . \tag{35}$$

Since both of the Picard principle and the condition (33) are invariant under the order equivalence, we may replace Q by $Q+2$ and then by a normal $\widehat{Q+2}$ with $|\widehat{Q+2}-(Q+2)|\leqq 1$ (cf. 3.5) to prove the theorem. Thus we may assume that $Q(t)$ is normal and that $Q(t)\geqq 1$. Then the condition (35) is equivalent to

$$\int_0^\infty \frac{dt}{\sqrt{Q(t)}}=\infty . \tag{36}$$

Consider the defining equation

$$-a_Q'(t)+a_Q(t)^2=Q(t) \tag{37}$$

of the Riccati component a_Q of Q. Since $Q\geqq 1$, $a_Q(t)\geqq 1$. Suppose that $a_Q'\leqq 0$ on an interval $[\alpha, \beta]$. Rewriting (37) in the form

$$\frac{1}{a_Q^2}=\frac{1}{Q}+\frac{1}{Q}\left(\frac{1}{a_Q}\right)'$$

and we have the following inequality similar to those in 6.3 and 6.4 in [22]:

$$\frac{1}{a_Q}\leqq\frac{1}{\sqrt{Q}}+\frac{1}{2}\frac{1}{Q}\left(\frac{1}{a_Q}\right)'\Big/\frac{1}{\sqrt{Q}}\leqq\frac{1}{\sqrt{Q}}+\frac{1}{2}\frac{1}{a_Q}\left(\frac{1}{a_Q}\right)'$$

and a fortiori

$$\frac{1}{a_Q}\leqq\frac{1}{\sqrt{Q}}+\frac{1}{4}\left(\frac{1}{a_Q^2}\right)' . \tag{38}$$

We again stress that the inequality is valid on any interval on which $a_Q'\leqq 0$.

Next assume that $a_Q'\geqq 0$ on an interval $[\beta, \gamma]$. Observe that $a_Q\geqq\sqrt{Q}\geqq 1$ on $[\beta, \gamma]$. This time we rewrite (37) in the form

$$\frac{Q}{a_Q^3}=\frac{1}{a_Q}+\frac{1}{2}\left(\frac{1}{a_Q^2}\right)' .$$

Since $Q/a_Q^3\geqq 0$, we deduce the following inequalities

$$-\frac{1}{2}\left(\frac{1}{a_Q^2}\right)'\leqq\frac{1}{a_Q}\leqq\frac{1}{\sqrt{Q}} \tag{39}$$

if $a_Q'\geqq 0$ on $[\beta, \gamma]$.

4.3. By Theorem 2.1 we only have to show that (23) implies (36). Since

$Q(t)$ is normal, the zeros of $a_Q'(t)$ form an isolated point set. If this zero set is finite, then a_Q' is of constant sign on $[\varepsilon, \infty)$ for some $\varepsilon>0$. In this case the implication of (36) from (23) was shown in [**22**], but, for the completeness sake, we here give another proof of this briefly as an application of Theorem 2.1. If $a_Q'\geqq 0$ on $[\varepsilon, \infty)$, then the above implication is an immediate consequence of the inequality $a_Q\geqq\sqrt{Q}$ on $[\varepsilon, \infty)$. If $a_Q'\leqq 0$ on $[\varepsilon, \infty)$, then, by (38) and $a_Q\geqq 1$, we have the same conclusion.

We next assume that the zeros of $a_Q'(t)$ form an infinite point set. This case is essential in our proof. Since the zeros of a_Q' form an isolated point set on $[0, \infty)$, we may assume that there exist two divergent sequences $\{t_n\}_1^\infty$ and $\{s_n\}_1^\infty$ such that $a_Q'(t_1)=0$, $t_n<s_n<t_{n+1}$ and that $a_Q'(t)\leqq 0$ on (t_n, s_n) and $a_Q'(t)\geqq 0$ on (s_n, t_{n+1}). Therefore $a_Q'(t_n)=a_Q'(s_n)=0$. By (38) and (39) we deduce

$$\begin{aligned}\int_{t_n}^{t_{n+1}}\frac{dt}{a_Q(t)}&=\int_{t_n}^{s_n}\frac{dt}{a_Q(t)}+\int_{s_n}^{t_{n+1}}\frac{dt}{a_Q(t)}\\&\leqq\int_{t_n}^{s_n}\frac{dt}{\sqrt{Q(t)}}+\frac{1}{4}\Big[\frac{1}{a_Q(s_n)^2}-\frac{1}{a_Q(t_n)^2}\Big]+\int_{s_n}^{t_{n+1}}\frac{dt}{\sqrt{Q(t)}}\\&=\int_{t_n}^{t_{n+1}}\frac{dt}{\sqrt{Q(t)}}+\frac{1}{4}\Big[\frac{1}{a_Q(s_n)^2}-\frac{1}{a_Q(t_n)^2}\Big].\end{aligned}$$

On the other hand, by (39), we have that

$$\frac{1}{2}\Big[\frac{1}{a_Q(s_n)^2}-\frac{1}{a_Q(t_{n+1})^2}\Big]\leqq\int_{s_n}^{t_{n+1}}\frac{dt}{\sqrt{Q(t)}}\leqq\int_{t_n}^{t_{n+1}}\frac{dt}{\sqrt{Q(t)}}.$$

Therefore by adding these two inequalities we obtain

$$\int_{t_n}^{t_{n+1}}\frac{dt}{a_Q(t)}\leqq\frac{3}{2}\int_{t_n}^{t_{n+1}}\frac{dt}{\sqrt{Q(t)}}+\frac{1}{4}\Big[\frac{1}{a_Q(t_{n+1})^2}-\frac{1}{a_Q(t_n)^2}\Big].$$

Summing up these inequalities for $n=1, 2, \cdots$, we deduce

$$\int_{t_1}^{\infty}\frac{dt}{a_Q(t)}\leqq\frac{3}{2}\int_{t_1}^{\infty}\frac{dt}{\sqrt{Q(t)}}+\frac{1}{4}\Big[\limsup_{t\to\infty}\frac{1}{a_Q(t)^2}-\frac{1}{a_Q(t_1)^2}\Big].$$

Since $1/a_Q(t)^2$ is bounded by 1, we conclude that (23) implies (36).

This completes the proof of Theorem 4.1.

§5. A sufficient condition on Picard principle.

5.1. In §4 we saw that the condition (33) is a necessary condition for $\dim P=1$. Unfortunately, however, it is not sufficient, which is shown in §6 by an example. Thus the problem of finding a complete condition on the Picard principle is still widely open. In this section we consider order equivalence classes $\mathscr{P}$ which satisfy the *condition* (**B**): *$\mathscr{P}$ contains a $P(z)$ such that $(r^2P(r)+1)^{-1/2}$ is of bounded variation on* $(0, 1]$ *with respect to* $\log(1/r)$. Although the condition (**B**) is weaker than (**I**), it is quite far from being necessary. We include the following only for comparison:

THEOREM. *If an order equivalence class $\mathscr{P}$ with the condition* (**B**) *satisfies the condition* (33), *then* $\dim\mathscr{P}=1$.

5.2. We consider an associated function $Q(t)$ to a $P\in\mathscr{P}$. As in §4, we may assume that $Q(t)$ is normal and $Q(t)\geqq 1$. Furthermore by the condition (**B**) for $\mathscr{P}$, we may assume that $1/\sqrt{Q(t)}$ is of bounded variation on $[0, \infty)$. We first prove an inequality similar to that in 6.3 in [22]. As in 4.2, from the identity (37) the following inequality follows:

$$\frac{1}{\sqrt{Q}}\leqq\frac{1}{a_Q}+\sqrt{\frac{1}{\sqrt{Q}}\left(\frac{1}{\sqrt{Q}}\left|\left(\frac{-1}{a_Q}\right)'\right|\right)}$$

$$\leqq\frac{1}{a_Q}+\frac{1}{2}\left(\frac{1}{\sqrt{Q}}+\frac{1}{\sqrt{Q}}\left|\left(\frac{-1}{a_Q}\right)'\right|\right).$$

A fortiori

$$\frac{1}{2}\frac{1}{\sqrt{Q}}\leqq\frac{1}{a_Q}+\frac{1}{2}\frac{1}{\sqrt{Q}}\left|\left(-\frac{1}{a_Q}\right)'\right|. \tag{40}$$

We observe that since $\sqrt{Q}\geqq 1$, if we replace $\sqrt{Q}$ by 1 in the last term of the above inequality, then we have the same inequality in 6.3 in [22] if $a_Q'\geqq 0$:

$$\frac{1}{2}\frac{1}{\sqrt{Q}}\leqq\frac{1}{a_Q}-\frac{1}{2}\left(\frac{1}{a_Q}\right)'. \tag{41}$$

We are ready to prove Theorem 5.1. By Theorem 2.1 we only have to show that (36) implies (23) under the assumption (**B**) i.e. $1/\sqrt{Q(t)}$ is of bounded variation on $[0, \infty)$. As in §4 we consider two cases: the first case is that a_Q' is of constant sign on $[\varepsilon, \infty)$ for some $\varepsilon>0$ and the second case is that the zeros of a_Q' form an infinite point set in $[0, \infty)$. The first case was taken care of in [22] but we also include its proof briefly as an application of Theorem 2.1. If $a_Q'\geqq 0$ on $[\varepsilon, \infty)$, then from (41) and Theorem 2.1, the required conclusion follows. If $a_Q'\leqq 0$ on $[\varepsilon, \infty)$, then the obvious inequality $a_Q\geqq\sqrt{Q}$ on $[\varepsilon, \infty)$ and Theorem 2.1 imply the same conclusion.

5.3. We next consider the second case, which is an essential part in our proof. Let sequences $\{t_n\}$ and $\{s_n\}$ be as in 4.3. Since $1/\sqrt{Q(t)}$ is of bounded variation, we integrate both sides of (40) using the integration by parts on the right in the sense of Stieltjes integral and obtain

$$\frac{1}{2}\int_{s_n}^{t_{n+1}}\frac{dt}{\sqrt{Q(t)}}\leqq\int_{s_n}^{t_{n+1}}\frac{dt}{a_Q(t)}-\frac{1}{2}\Big[\frac{1}{\sqrt{Q(t)}}\cdot\frac{1}{a_Q(t)}\Big]_{t=s_n}^{t=t_{n+1}}+\frac{1}{2}\int_{s_n}^{t_{n+1}}\frac{1}{a_Q(t)}d\Big(\frac{1}{\sqrt{Q(t)}}\Big).$$

Here observe that $(-1/a_Q)'\geqq 0$ on $[s_n, t_n]$. Since $\sqrt{Q(s_n)}=a_Q(s_n)$ and $\sqrt{Q(t_{n+1})}=a_Q(t_{n+1})$, combining (39) and the above inequality, we deduce that

$$\frac{1}{2}\int_{s_n}^{t_{n+1}}\frac{dt}{\sqrt{Q(t)}}\leqq 2\int_{s_n}^{t_{n+1}}\frac{dt}{a_Q(t)}+\frac{1}{2}\int_{s_n}^{t_{n+1}}\frac{1}{a_Q(t)}d\Big(\frac{1}{\sqrt{Q(t)}}\Big).$$

Since $0<1/a_Q(t)\leqq 1$ and $1/\sqrt{Q(t)}$ is of bounded variation, we have that

$$\sum_{n=1}^{\infty}\int_{s_n}^{t_{n+1}}\frac{1}{a_Q(t)}d\Big(\frac{1}{\sqrt{Q(t)}}\Big)<+\infty.$$

On the other hand, since $a_Q'(t)\leqq 0$ on $[t_n, s_n]$ i.e. $\sqrt{Q(t)}\geqq a_Q(t)$ on $[t_n, s_n]$, we have

$$\sum_{n=1}^{\infty}\int_{t_n}^{s_n}\frac{dt}{\sqrt{Q(t)}}\leqq\sum_{n=1}^{\infty}\int_{t_n}^{s_n}\frac{dt}{a_Q(t)}.$$

Therefore we conclude that (36) implies (23).

This completes the proof of Theorem 5.1.

5.4. The condition that $p(r)=(r^2P(r)+1)^{-1/2}$ is of bounded variation on $(0, 1]$ with respect to $-\log r$ is equivalent to that $p(r)$ can be represented as $p(r)=p_1(-\log r)-p_2(-\log r)$ with p_i $(i=1, 2)$ bounded monotone functions on $[0, \infty)$. Observe that $p_i(-\log r)$ $(i=1, 2)$ are also bounded monotone functions of r on $(0, 1]$. Therefore the condition (**B**) may be restated as follows: $\mathcal{P}$ *contains a* $P(z)$ *such that* $(r^2P(r)+1)^{-1/2}$ *is of bounded variation with respect to* r *on* $(0, 1]$. Such a $P(r)$ is differentiable almost everywhere and satisfies

$$\int_0^1\left|\frac{d}{dr}(r^2P(r)+1)^{-\frac{1}{2}}\right|dr<\infty. \tag{42}$$

The converse is only true under an additional condition, e.g. $P(r)$ is of class C^1. As a more computational reformulation of (**B**) we state: *The class* $\mathcal{P}$ *satisfies* (**B**) *if and only if* P *contains a* $P(r)$ *of class* C^1 *with* (42).

§ 6. A counter example.

6.1. We have shown in § 4 that the condition (33) is necessary for the validity of the Picard principle. As to the converse implication we showed in § 5 that (33) is also a sufficient condition for the validity of the Picard principle if the additional requirement that $(r^2P(r)+1)^{-1/2}$ is of bounded variation with respect to $\log(1/r)$ on $(0, 1]$ (up to the order equivalence) is imposed upon. In this section we shall show that this unpleasant requirement, however, cannot be got rid of. Therefore there still remains the question what the complete condition for the validity of Picard principle is even for the rotation free densities.

6.2. We shall construct a rotation free density P on $(0, 1]$ with (33) such that $\dim P\neq 1$ (and hence $=c$) so that $(r^2P(r)+1)^{-1/2}$ is not of bounded variation with respect to $\log(1/r)$ on $(0, 1]$ in the sense of order equivalence.*) For the purpose we take two sequences $\{t_n\}$ and $\{s_n\}$ $(n=1, 2, \cdots)$ in $[0, \infty)$ as follows:

$$t_1=0, \quad t_{n+1}=t_n+(n^2+1)/n^3 \qquad (n=1, 2, \cdots);$$

$$s_n=t_n+(n^2-n+1)/n^3 \qquad (n=1, 2, \cdots).$$

Then clearly we have $\lim_{n\to\infty} t_n=\lim_{n\to\infty} s_n=\infty$ and

$$t_n<s_n<t_{n+1} \qquad (n=1, 2, \cdots).$$

With the aid of these sequences we define a function $a(t)$ on $[0, \infty)$ by

$$a(t)=\begin{cases} (n^2-n+1)/(n^2(s_n-t)+n(t-t_n)), & (t\in[t_n, s_n]); \\ (n+1)/((n+1)(t_{n+1}-t)+n^2(t-s_n)), & (t\in[s_n, t_{n+1}]) \end{cases}$$

for $n=1, 2, \cdots$. Observe that $1/a(t)$ is a polygonal line on $[0, \infty)$. Set

$$Q(t)=-a'(t)+a(t)^2$$

for $t\in[0, \infty)-\{t_n\}\cup\{s_n\}$. Once more observe that $(1/a(t))'>0$ for $t\in(s_n, t_{n+1})$,

$$(1/a(t))'=(-n^2+n)/(n^2-n+1)$$

for $t\in(t_n, s_n)$, and $Q(t)=a(t)^2(1+(1/a(t))')$. Therefore

$$Q(t)\geqq 1 \tag{43}$$

on $[0, \infty)$ except for $\{t_n\}\cup\{s_n\}$. We compute

$$\int_{t_n}^{s_n}\frac{dt}{a(t)}=(n+1)(n^2-n+1)/2n^5$$

and similarly

$$\int_{t_n}^{s_n}\frac{dt}{a(t)}=(n^2+n+1)/2n^4(n+1).$$

On the other hand, $1/Q(t)=(n^2-n+1)/a(t)^2$ for $t\in(t_n, s_n)$ and hence we deduce

$$\int_{t_n}^{s_n}\frac{dt}{\sqrt{Q(t)}}=(n+1)(n^2-n+1)^{3/2}/2n^5.$$

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*) The authors owe much to Dr. Akio Osada in constructing this example.

Therefore we conclude that

$$\int_0^\infty \frac{dt}{a(t)} = \sum_{n=1}^\infty ((n+1)(n^2-n+1)/2n^5 + (n^2+n+1)/2n^4(n+1)) < \infty . \tag{44}$$

However we have

$$\int_0^\infty \frac{dt}{\sqrt{Q(t)}} \geqq \sum_{n=1}^\infty \int_{t_n}^{s_n} \frac{dt}{\sqrt{Q(t)}} = \sum_{n=1}^\infty (n+1)(n^2-n+1)^{3/2}/2n^5 = \infty . \tag{45}$$

6.3. Next we take two sequences $\{\tau_n\}$ $(n=2, 3, \cdots)$ and $\{\sigma_n\}$ $(n=1, 2, \cdots)$ of positive numbers with the following properties. First $\{\tau_n\}$ and $\{\sigma_n\}$ are chosen so small that $T_n=[t_n-\tau_n, t_n+\tau_n]\subset(s_{n-1}, s_n)$ $(n=2, 3, \cdots)$, $S_n=[s_n-\sigma_n, s_n+\sigma_n]\subset(t_n, t_{n+1})$ $(n=1, 2, \cdots)$, and any T_n does not meet any S_m. We replace $1/a(t)$ by the parabola on T_n $(n=2, 3, \cdots)$ and S_n $(n=1, 2, \cdots)$ tangent to $1/a(t)$ at the end points of the intervals so that the resulting function $1/\hat{a}(t)$ is of class C^1 on $[0, \infty)$. Observe that

$$(1/\hat{a}(t))' \geqq (1/a(t))'_{t=s_n-\sigma_n} = (-n^2+n)/(n^2-n+1)$$

on $[t_n-\tau_n, s_n+\sigma_n]$ and therefore if we set

$$\hat{Q}(t) = -\hat{a}'(t)+\hat{a}(t)^2 = \hat{a}(t)^2(1+(1/\hat{a}(t))') ,$$

then $a_{\hat{Q}}(t)=\hat{a}(t)$ and the counterpart to (43) is also valid, i.e.

$$\hat{Q}(t) \geqq 1 \tag{46}$$

on $[0, \infty)$ since $\hat{a}(t)\geqq a(t_n)=n$ on $[t_n-\tau_n, s_n+\sigma_n]$. We moreover choose τ_n $(n=2, 3, \cdots)$ and σ_n $(n=1, 2, \cdots)$ so small that

$$\left| \int_{T_n \cup S_n} \frac{dt}{a(t)} - \int_{T_n \cup S_n} \frac{dt}{\hat{a}(t)} \right| \leqq \frac{1}{n^2}$$

and

$$\left| \int_{T_n \cup S_n} \frac{dt}{\sqrt{Q(t)}} - \int_{T_n \cup S_n} \frac{dt}{\sqrt{\hat{Q}(t)}} \right| \leqq \frac{1}{n^2} .$$

Then the counterpart to (44) and (45) are again valid, i.e.

$$\int_0^\infty \frac{dt}{a_{\hat{Q}}(t)} < \infty \tag{47}$$

and

$$\int_0^\infty \frac{dt}{\sqrt{\hat{Q}(t)}} = \infty . \tag{48}$$

Finally we set

$$P(r) = r^{-2}\hat{Q}(-\log r)$$

on $(0, 1]$, which is the required density. In fact, the condition (33) is satisfied by this P in view of (47) and (48). However the condition (47) implies that $\dim P = \mathfrak{c}$ (cf. Theorem 2.1), and therefore $(r^2P(r)+1)^{-1/2}$ cannot be of bounded variation in the sense of order equivalence.

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