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On the support of the solution of some variational inequalities of evolution

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§ 1. Introduction.

It has been observed by Berkovitz and Pollard [1] that some problems of calculus of variations for a *non differentiable* functional have a solution with *compact support*. This result naturally leads to the following general question, proposed in Lions [1]: *When is it true that the solution of a variational inequality*(1) *is with a compact support?*

For *stationary* variational inequalities (V. I.), the first result answering positively this kind of question was given by H. Brezis [1]; his method was based on a systematic use of comparison functions, and it was clear that this method could be extended to V. I. of *parabolic type*; this is done, together with a number of estimates on the behaviour of the support of the solution, in H. Brezis and A. Friedman [1].

We present here a number of results along these lines. Our method is entirely different from the one of Brezis and Friedman [1]. We require less assumptions, but our estimate on the support is a little bit less accurate.

We use here in an essential manner the fact that, as we have shown in our papers [1] and [2], the solution of V. I. of parabolic type, for *linear* and *non linear* second order operators, or the solution of V. I. of first order hyperbolic type, can be interpreted as *the optimal cost function in a problem of optimal stopping time.*

The plan is as follows.

1—Introduction
2—Variational inequalities of evolution-linear operators
  2.1. Statement of the main result
  2.2. Proof of theorem 2.1
3—Other estimates on the support of u
4—Hyperbolic operators

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(1) We shall write V. I. for "Variational Inequality".

§ 2. Variational inequalities of evolution—Linear operators.

2.1. STATEMENT OF THE MAIN RESULT.

We consider, for $x\in \boldsymbol{R}^n$, $t<T$, the problem:

$$\left\{\begin{aligned} & u=u(x,t)\leqq 0\,, \\ & -\frac{\partial u}{\partial t}-\Delta u-f\leqq 0\,, \\ & u\Big(-\frac{\partial u}{\partial t}-\Delta u-f\Big)=0\,, \end{aligned}\right. \tag{2.1}$$

$$u(x,T)=\bar{u}(x)\,, \tag{2.2}$$

where f and $\bar{u}$ are given functions.

We assume that[1]:

$$f\in L^\infty(\boldsymbol{R}^n\times]0,T[) \tag{2.3}$$

$$\bar{u}\in L^2(\boldsymbol{R}^n)\,,\qquad \bar{u}\leqq 0\,. \tag{2.4}$$

Then it is known that there exists a unique function u which satisfies (2.1), (2.2) and which is such that, $\forall \gamma>0$:

$$ue^{-\gamma|x|}\in L^2(0,T;H^1(\boldsymbol{R}^n)) \tag{2.5}$$

$$\frac{\partial u}{\partial t}e^{-\gamma|x|}\in L^2(0,T;L^2(\boldsymbol{R}^n)) \tag{2.6}$$

(in (2.5) $H^1(\boldsymbol{R}^n)$ denotes the Sobolev space of functions $\zeta\in L^2(\boldsymbol{R}^n)$ such that $\frac{\partial \zeta}{\partial x_i}\in L^2(\boldsymbol{R}^n)$, $i=1,\cdots,n$); the results (2.5), (2.6) are not the best possible but are sufficient for what we want to obtain.

The problem (2.1), (2.2), (2.5), (2.6) is a *variational inequality* (V. I.) of *parabolic type* for the linear operator

$$u\to -\frac{\partial u}{\partial t}-\Delta u\,.$$

REMARK 2.1. We consider here the *backward* problem, with the "final" data (2.2), because it simplifies the interpretation of the solution u as the optimal cost functional of a stopping time problem (see below).

(1) These hypotheses are not the most general we could consider.

REMARK 2.2. Analogous V.I. for other operators will be considered later in the paper.

In what follows we want to study some properties of u by using the following interpretation of u; let us consider a system whose *stochastic state* y in $\boldsymbol{R}^n$ is given by the solution of the Ito's differential equation

$$\begin{cases} dy=\sqrt{2}\,dw(s) \\ y(t)=x \end{cases} \tag{2.7}$$

where w is a standard Wiener process in $\boldsymbol{R}^n$ (with respect to an increasing family of σ algebras $\mathcal{F}^t$).

We denote by $y_{x,t}(s)$ the solution of (2.7); given a *stopping time* $\tau\leqq T$ we define the "*cost function*"

$$J_{xt}(\tau)=E\Big[\int_t^\tau f(y_{xt}(s),\, s)ds+\chi_{\tau=T}\bar{u}(y_{xt}(T))\Big] \tag{2.8}$$

where E denotes the expectation and where $\chi_{\tau=T}$ equals 1 if $\tau=T$ and equals 0 if $\tau<T$ (note that we have ruled out the possibility $\tau>T$).

It has been proved in Bensoussan and Lions [**1**] that the solution *u of problem* (2.1), (2.2), (2.5), (2.6) is *given* by

$$u=\inf_\tau J_{xt}(\tau)\,. \tag{2.9}$$

Using this formula, our first goal is to prove the following

THEOREM 2.1. *We suppose that hypotheses* (2.3), (2.4) *hold true. Moreover we assume that*

$$f(x,\, t)\geqq f_0>0 \tag{2.10}$$

$$\bar{u} \text{ is bounded, say } \bar{u}\geqq -c \qquad (c>0)\,. \tag{2.11}$$

We denote by S the support of $\bar{u}$ and by $d(x, S)$ the distance from x to S. Given $\alpha>0$ and $p\geqq 2$ we consider those points x, t in $R_x^n\times R_t$ such that

$$\begin{aligned} d(x,\, S)\geqq & (2+\alpha)(T-t)^{1/2}\log(T-t)^{1/2} \\ & +\Big(\frac{c}{f_0}\Big)^{1/p}\Big(\frac{p}{p-1}\Big)^{p/2}(p(p-1))^{1/2}(T-t)^{1/2-1/p} \quad (t<T)\,; \end{aligned} \tag{2.12}$$

let us denote by $Q_{\alpha p}$ the set of points satisfying (2.12). *Then*

$$u=0 \text{ for } x,\, t\in Q_{\alpha p} \text{ and for } T-t \text{ small enough}\,. \tag{2.13}$$

REMARK 2.3. In Brezis and Friedman [**1**], a better estimate is given, namely

$$d(x,\, S)\geqq C_0(T-t)^{1/2}\log(T-t)^{1/2}\,, \qquad C_0>2+\varepsilon$$

but with assumptions on the regularity of the boundary of S (and as we have already said, using an entirely different method based upon the construction of an adequate comparison function).

OPEN QUESTION: It seems very likely that the second term in the right hand side of (2.12) can be omitted, but we have not been successful in proving it. Further more, the estimate of $T-t$ for which (2.13) holds true, does not depend on this second term (see below the proof of the theorem).

2.2. PROOF OF THEOREM 2.1.

Let us introduce the function $\chi_S(x)=\begin{cases}1 & \text{if } x\in S\\ 0 & \text{otherwise.}\end{cases}$

Since for any τ

$$J_{xt}(\tau)\geqq f_0E(\tau-t)-cE\chi_S(y_{xt}(T))\chi_{\tau=T}$$

we can, without loss of generality, restrict ourselves to the case when

$$f=f_0\,; \qquad \bar{u}=-c\chi_S\,. \tag{2.14}$$

Let us now consider the penalized payoff

$$\begin{aligned}J^{\varepsilon}_{xt}(\tau)&=f_0E(\tau-t)+f_0E\int_\tau^T\exp-\frac{s-\tau}{\varepsilon}\,ds\\&\quad-cE\Big[\chi_S(y_{xt}(T))\exp-\frac{T-\tau}{\varepsilon}\Big].\end{aligned}\tag{2.15}$$

We have

$$\begin{aligned}J^{\varepsilon}_{xt}(\tau)-J_{xt}(\tau)&=f_0E\int_\tau^T\exp-\frac{s-\tau}{\varepsilon}\,ds\\&\quad+cE\chi_S(y_{xt}(T))\Big[\chi_{\tau=T}-\exp-\frac{T-\tau}{\varepsilon}\Big]\end{aligned}$$

hence

$$J^{\varepsilon}_{xt}(\tau)-J_{xt}(\tau)\leqq f_0\varepsilon\,. \tag{2.16}$$

Let now θ be defined (in a unique way) by the equation

$$\int_\tau^T\exp-\frac{s-\tau}{\varepsilon}\,ds=(T-\tau)\exp-\frac{\theta-\tau}{\varepsilon}\,. \tag{2.17}$$

Since θ is obtained through a continuous mapping of τ, it is clear that θ is $\mathcal{F}^\tau$ *measurable.* Now[1]

$$\begin{aligned}J^{\varepsilon}(\tau)&=f_0E(\tau-t)+f_0E(T-\tau)\exp-\frac{\theta-\tau}{\varepsilon}\\&\quad-cE\chi_S(y_{xt}(T))\exp-\frac{T-\tau}{\varepsilon}\\&\geqq f_0E(\tau-t)+f_0E(T-\tau)\exp-\frac{\theta-\tau}{\varepsilon}-cE\chi_S(y_{xt}(T))\exp-\frac{\theta-\tau}{\varepsilon}\end{aligned}\tag{2.18}$$

but

(1) We drop the indices x, t for convenience.

$$E\chi_S(y_{xt}(T))\exp-\frac{\theta-\tau}{\varepsilon}=E\Big[\exp-\frac{\theta-\tau}{\varepsilon}(E\chi_S(y_{xt}(T))\,|\,\mathcal{F}^\tau)\Big]$$

$$=E\Big[\exp-\frac{\theta-\tau}{\varepsilon}P(y_{xt}(T)\in S\,|\,\mathcal{F}^\tau)\Big].$$

Using the strong Markov property of y we get

$$=E\Big[\exp-\frac{\theta-\tau}{\varepsilon}P(y_{xt}(\tau),\tau\,;S,T)\Big]$$

where

(2.19) $$P(\xi,s\,;S,T)=\int_S\frac{1}{(4\eta)^{n/2}(T-s)^{n/2}}\exp-\frac{|\xi-\lambda|^2}{4(T-s)}d\lambda.$$

We have next:

$$y(\tau)=y_{xt}(\tau)=x+\sqrt{2}\,(w(\tau)-w(t))$$

hence for $p\geqq 2$

(2.20) $$E|y(\tau)-x|^p=2^{p/2}E|w(\tau)-w(t)|^p$$

$$\leqq 2^{p/2}c_pE(\tau-t)^{p/2}$$

(see for instance Priouret [**1**]) where

$$c_p=\Big(\frac{p}{p-1}\Big)^{p^2/2}\frac{(p(p-1))^{p/2}}{2^{p/2}},$$

hence

(2.21) $$E|y(\tau)-x|^p\leqq\Big(\frac{p}{p-1}\Big)^{p^2/2}(p(p-1))^{p/2}E(\tau-t)^{p/2}$$

$$=c'_pE(\tau-t)^{p/2}.$$

For $h>0$ we have

(2.22) $$P(|y(\tau)-x|\geqq h)\leqq\frac{E|y(\tau)-x|^p}{h^p}\leqq\frac{c'_pE(\tau-t)^{p/2}}{h^p}.$$

We have next:

(2.23) $$E\exp-\frac{\theta-\tau}{\varepsilon}P(y(\tau),\tau\,;S,T)$$

$$=\int dP(\omega)\Big(\exp-\frac{\theta-\tau}{\varepsilon}P(y(\tau),\tau\,;S,T)\Big)\{|y(\tau)-x|\geqq h\}$$

$$+E\chi_{|y(\tau)-x|<h}\exp-\frac{\theta-\tau}{\varepsilon}P(y(\tau),\tau\,;S,T))$$

$$\leqq c'_p\frac{E(\tau-t)^{p/2}}{h^p}+E\Big\{\chi_{|y(\tau)-x|<h}\exp-\frac{\theta-\tau}{\varepsilon}P(y(\tau),\tau\,;S,T)\Big\}.$$

Using (2.23) in (2.18) we get

$$(2.24) \qquad J^{\varepsilon}(\tau) \geqq E(\tau-t)\Big[f_0-\frac{cc'_p}{h^p}(\tau-t)^{(p/2)-1}\Big]$$

$$+E\exp-\frac{\theta-\tau}{\varepsilon}[f_0(T-\tau)-c\chi_{|y(\tau)-x|<h}P(y(\tau),\tau\,;S,T)]\,.$$

We shall now choose h. We take

$$(2.25) \qquad h=\Big(\frac{c}{f_0}\Big)^{1/p}c'^{1/p}_p(T-t)^{1/2-1/p}$$

which clearly implies

$$(2.26) \qquad J^{\varepsilon}(\tau) \geqq E\exp-\frac{\theta-\tau}{\varepsilon}[f_0(T-\tau)-cZ]\,,$$

where

$$Z=\chi_{|y(\tau)-x|<h}P(y(\tau),\tau\,;S,T)\,.$$

From (2.19) we have

$$P(y(\tau),\tau\,;S,T)=\int_{y(\tau)+\sqrt{2}\lambda\in S}\exp-\frac{|\lambda|^2}{2(T-\tau)}\frac{d\lambda}{(2\pi)^{n/2}(T-\tau)^{n/2}}\,;$$

but when $|y(\tau)-x|<h$ we have

$$B_x(\omega)=\{\lambda\in R^n\,|\,y(\tau)+\sqrt{2}\lambda\in S\}\subset\{\sqrt{2}\,|\lambda|\geqq d(x,S)-h\}$$

hence

$$Z(\omega)\leqq\int_{|\lambda|\geqq\frac{d(x,S)-h}{\sqrt{2}}}\exp-\frac{\lambda}{2(T-\tau)}\frac{d\lambda}{(2\pi)^{n/2}(T-\tau)^{n/2}}=I(\omega)\,.$$

Setting

$$\xi=\frac{\lambda}{(2(T-\tau))^{1/2}}$$

we get

$$I(\omega)=\frac{1}{\pi^{n/2}}\int_{|\xi|\geqq\frac{d(x,S)-h}{2(T-\tau)^{1/2}}}\exp-|\xi|^2d\xi\,,$$

and by using spherical coordinates

$$\xi=C_1\int_{\frac{d-h}{2(T-\tau)^{1/2}}}^{\infty}(\exp-r^2)r^{n-1}dr$$

where $d=d(x,S)$.

For $\eta\in]0,1[$, let $c(\eta)$ be such that

$$\int_{\beta}^{\infty}(\exp-r^2)r^{n-1}dr\leqq c(\eta)\exp-\beta^2(1-\eta) \qquad \forall\beta\geqq0\,;$$

hence it follows that

$$I(\omega)\leqq c_1c(\eta)\exp-(1-\eta)\frac{(d-h)^2}{4(T-\tau)}\,.$$

Since for $T-t$ small enough the function $x^{1/2}|\mathrm{Log}\,x|^{1/2}$ increases in $[0,T-t]$, it follows from the choice of x (cf. (2.12)) that

$$d \geqq h+(2+\alpha)(T-\tau)^{1/2}|\mathrm{Log}\,(T-\tau)|^{1/2}$$

hence it follows that

$$I(\omega) \leqq c_1 c(\eta) \exp -\frac{(1-\eta)(2+\alpha)^2}{4}|\mathrm{Log}\,(T-\tau)|\,.$$

Let us now choose η such that

$$\frac{1}{4}(1-\eta)(2+\alpha)^2 = 1+\alpha_1 \qquad \left(\alpha_1 < \frac{\alpha^2+4\alpha}{4}\right).$$

It follows that

$$I(\omega) \leqq c_1 c(\eta) \exp -(1+\alpha_1)|\mathrm{Log}\,(T-\tau)| = c_1 c(\eta)(T-\tau)^{1+\alpha_1}\,.$$

Finally we get

$$\begin{aligned} f_0(T-\tau)-cZ &\geqq f_0(T-\tau)-cc_1c(\eta)(T-\tau)^{1+\alpha_1} \\ &= (T-\tau)[f_0-cc_1c(\eta)(T-\tau)^{\alpha_1}] \\ &\geqq (T-\tau)[f_0-cc_1c(\eta)(T-t)^{\alpha_1}] \\ &\geqq 0 \end{aligned}$$

if $T-t$ satisfies the condition

$$f_0-cc_1c(\eta)(T-t)^{\alpha_1} > 0\,. \tag{2.27}$$

We thus have proved that for $(x, t) \in Q_{\alpha\beta}$ and $T-t$ small enough we have

$$J_{xt}^{\varepsilon}(\tau) \geqq 0 \qquad \text{for any } \tau \text{ and } \varepsilon\,.$$

By the estimate (2.16), it follows that

$$J_{xt}(\tau) \geqq -f_0\varepsilon \qquad \text{for any } \tau \text{ and } \varepsilon$$

which means

$$J_{xt}(\tau) \geqq 0 \qquad \text{for any } \tau\,.$$

Therefore also $u(x, t) \geqq 0$, which implies $u(x, t)=0$ (since u is always $\leqq 0$).

§ 3. Other estimates on the support of u.

One can obtain a number of other estimates on the support of u when one makes *some hypotheses on the behaviour* of u *in the neighborhood of the boundary* of S (in Theorem 2.1 the only hypothesis made was that u was *bounded*).

Along these lines we are going to give several results.

THEOREM 3.1. *We assume that the hypotheses of theorem* 2.1 *hold true. We assume moreover that*

$$|\bar{u}(x)| \leqq c_3 d(x, \partial S)^{\gamma}\,, \qquad x \in S \qquad \textit{where} \quad \gamma \in\,]0, 2[\,. \tag{3.1}$$

We define $Q_{\alpha p\gamma}$ *as the set of points* $x, t, t<T$, *such that*(*)

$$d(x, S) \geqq (2+\alpha)(T-t)^{1/2}\left(1-\frac{\gamma}{2}\right)^{1/2} |\operatorname{Log}(T-t)|^{1/2} \tag{3.2}$$

$$+\left(\frac{c}{f_0}\right)^{1/p}\left(\frac{p}{p-1}\right)^{p/2}(p(p-1))^{1/2}(T-t)^{1/2-1/p}.$$

Then

$$u=0 \quad \textit{for } x, t \in Q_{\alpha p\gamma} \textit{ and } T-t \textit{ small enough.} \tag{3.3}$$

PROOF. Again we can assume $f=f_0$, hence

$$J_{xt}(\tau)=f_0E(\tau-t)+E\chi_{\tau=T}\bar{u}(y_{xt}(T))$$

and

$$J^{\varepsilon}_{xt}(\tau)=f_0E(\tau-t)+f_0E\int_{\tau}^{T}\exp-\frac{s-\tau}{\varepsilon}ds+E\left[\bar{u}(y_{xt}(T))\exp-\frac{T-\tau}{\varepsilon}\right],$$

and again (since $\bar{u}\leqq 0$) we have

$$J^{\varepsilon}_{xt}(\tau)-J_{xt}(\tau)\leqq f_0\varepsilon .$$

As in the proof of Theorem 2.1, it is enough to consider $J^{\varepsilon}_{xt}(\tau)$. By considering θ defined by (2.17), it comes (we drop the index x, t)

$$J^{\varepsilon}(\tau)\geqq f_0E(T-\tau)\exp-\frac{\theta-\tau}{\varepsilon}+E\bar{u}(y(T))\exp-\frac{\theta-\tau}{\varepsilon}+f_0E(\tau-t). \tag{3.4}$$

Now we have

$$E\bar{u}(y(T))\exp-\frac{\theta-\tau}{\varepsilon}=E\left[\exp-\frac{\theta-\tau}{\varepsilon}E(\bar{u}(y(T))\,|\,\mathcal{F}^{\tau})\right]$$

and by the strong Markov property, this equals

$$E\exp-\frac{\theta-\tau}{\varepsilon}\int\bar{u}(\xi)P(y(\tau), \tau\,;d\xi, T)$$

$$\geqq -cP(|y(\tau)-x|\geqq h)+E\chi_{|y(\tau)-x|<h}\int\bar{u}(\xi)P(y(\tau), \tau\,;d\xi, T)$$

hence we obtain

$$J^{\varepsilon}(\tau)\geqq E(\tau-t)\left[f_0-\frac{cc'_p}{h^p}(\tau-t)^{p/2-1}\right] \tag{3.5}$$

$$+E\exp-\frac{\theta-\tau}{\varepsilon}f_0(T-\tau)+\chi_{|y(\tau)-x|<h}\int\bar{u}(\xi)P(y(\tau), \tau\,;d\xi, T).$$

Defining h by (2.25) we get

$$J^{\varepsilon}(\tau)\geqq E\exp-\frac{\theta-\tau}{\varepsilon}[f_0(T-\tau)-Z] \tag{3.6}$$

where

(*) We observe that $Q_{\alpha p\gamma}\subset Q_{\alpha p}$.

$$Z=\chi_{|y(\tau)-x|<h}\int_S |\bar{u}(\xi)|P(y(\tau),\tau\,;d\xi,T)$$

$$=\chi_{|y(\tau)-x|<h}\int_{y(\tau)+\sqrt{2}\lambda\in S}|\bar{u}(y(\tau)+\sqrt{2}\lambda)|\exp-\frac{|\lambda|^2}{2(T-\tau)}\frac{d\lambda}{(2\pi)^{n/2}(T-\tau)^{n/2}}\,.$$

Using (3.1) and noticing that when $|y(\tau)-x|<h$ and $y(\tau)+\sqrt{2}\lambda\in S$ we have

$$d(y(\tau),S)\geqq d-h$$

$$d(y(\tau)\sqrt{2}\lambda,\partial S)\leqq\sqrt{2}|\lambda|-(d-h)$$

we obtain

$$Z\leqq c_3(\sqrt{2})^\gamma\int_{|\lambda|\geqq\frac{d-h}{\sqrt{2}}}\Big(|\lambda|-\frac{d-h}{\sqrt{2}}\Big)^\gamma\exp-\frac{|\lambda|^2}{2(T-\tau)}\frac{d\lambda}{(2\pi)^{n/2}(T-\tau)^{n/2}}$$

$$=\frac{c_3(\sqrt{2})^\gamma}{\pi^{n/2}}\int_{|\xi|\geqq\frac{d-h}{(2(T-\tau))^{1/2}}}\Big(|\xi|(2(T-\tau))^{1/2}-\frac{d-h}{\sqrt{2}}\Big)^\gamma\exp-|\xi|^2d\xi\,.$$

Setting $\beta=\dfrac{d-h}{(2(T-\tau))^{1/2}}$ it follows that

$$Z\leqq c_4(T-\tau)^{\gamma/2}\int_\beta^\infty(r-\beta)^\gamma r^{n-1}\exp-r^2dr$$

$$\leqq c_4(T-\tau)^{\gamma/2}\int_\beta^\infty r^{n+\gamma-1}\exp-r^2dr\,.$$

Defining $c'(\eta)$ as $c(\eta)$ (with n changed into $n+\gamma$) we get

$$Z\leqq c_4c'(\eta)(T-\tau)^{\gamma/2}\exp-\frac{(1-\eta)(2+\alpha)^2\Big(1-\dfrac{\gamma}{2}\Big)}{4}|\mathrm{Log}\,(T-\tau)|\,,$$

and with the same choice of η as in Theorem 2.1, we obtain

$$Z\leqq c_4c'(\eta)(T-\tau)^{\gamma/2}\exp-(1+\alpha_1)\Big(1+\frac{\gamma}{2}\Big)|\mathrm{Log}\,(T-\tau)|$$

$$=c_4c'(\eta)(T-\tau)^{1+\alpha_1\left(1-\frac{\gamma}{2}\right)}.$$

Therefore, returning to (3.6) it follows that

$$J^\varepsilon(\tau)\geqq E\exp-\frac{\theta-\tau}{\varepsilon}[f_0(T-\tau)-c_4c'\eta(T-\tau)^{1+\alpha_1\left(1+\frac{\gamma}{2}\right)}]$$

$$\geqq E\exp-\frac{\theta-\tau}{\varepsilon}(T-\tau)[f_0-c_4c'(\eta)(T-t)^{\alpha_1\left(1-\frac{\gamma}{2}\right)}]$$

$$\geqq 0\qquad\text{for } T-t \text{ small enough}\,,$$

and (3.3) follows.

THEOREM 3.2. *We assume the hypotheses of Theorem* 3.1 *to hold true with*

$$\gamma=2. \tag{3.7}$$

Then there exists a constant c_5 *such that*

$$u=0 \quad \text{if} \quad d(x, S) \geqq c_5(T-t)^{1/2} + \left(\frac{c}{f_0}\right)^{1/p}\left(\frac{p}{p-1}\right)^{p/2}(p(p-1))^{1/2}(T-t)^{1/2-1/p}, \qquad \forall t<T. \tag{3.8}$$

PROOF. We note that (3.6) still holds true. We now have the following estimate on Z

$$Z \leqq c_4 c'(\eta)(T-\tau)\exp-(1-\eta)\frac{(d-h)^2}{4(T-\tau)}.$$

hence

$$J^\varepsilon(\tau) \geqq E\exp-\frac{\theta-\tau}{\varepsilon}(T-\tau)\left[f_0-c_4c'(\eta)\exp-(1-\eta)\frac{(d-h)^2}{4(T-\tau)}\right]$$

$$\geqq E\exp-\frac{\theta-\tau}{\varepsilon}(T-\tau)\left[f_0-c_4c'(\eta)\exp-\frac{(1-\eta)(d-h)^2}{4(T-\tau)}\right]$$

hence the result follows, by choosing c_5 such that

$$f_0-c_4c'(\eta)\exp-\frac{(1-\eta)c_5^2}{4} \geqq 0.$$

REMARK 3.1. A result of this type (without the second term in the estimate (3.8) but with stronger hypotheses on the regularity of $\bar{u}$ and of ∂S) is given in Brezis and Friedman [1], who do not obtain results along the lines of Theorem 3.1 and 3.3 below.

REMARK 3.2. We can take $\gamma>2$; then we can assert that (with the same definition of h)

$$u=0 \quad \text{for } d(x, S) \geqq h, \text{ and } T-t \text{ small enough}. \tag{3.9}$$

This follows from the fact

$$Z \leqq c_4c'(\eta)(T-\tau)^{\gamma/2}$$

and

$$J^\varepsilon(\tau) \geqq E\exp-\frac{\theta-\tau}{\varepsilon}(T-\tau)[f_0-c_4c'(\eta)(T-t)^{\gamma/2-1}].$$

THEOREM 3.3. *We assume that the hypotheses of Theorem* 2.1 *hold true with* (2.10) *changed into*

$$f(x, t) \geqq f_0(T-t)^\delta, \qquad \delta \geqq 0. \tag{3.10}$$

We define $Q_{\alpha p\delta}$ *as the set of points* (x, t), $t>T$, *such that*

(3.11) $$d(x, S) \geqq (2+\alpha)(T-t)^{1/2}|\operatorname{Log}(T-t)|^{1/2} + \left(\frac{c(\delta+1)}{f_0}\right)^{1/p}\left(\frac{p}{p-1}\right)^{p/2}(p(p-1))^{1/2}(T-t)^{1/2-(\delta+1/p)} \quad (p \geqq 2(\delta+1)).$$

Then

(3.12) $u=0$ *for* $(x, t) \in Q_{\alpha p \delta}$ *and* $T-t$ *small enough.*

PROOF. We have by virtue of (3.10)

$$J_{xt}(\tau) \geqq E\int_t^\tau f_0(T-s)^\delta ds - cE\chi_S(y_{xt}(T))\chi_{\tau=T}$$

$$\geqq \frac{f_0}{\delta+1} E(\tau-t)^{\delta+1} - cE\chi_S(y_{xt}(T))\chi_{\tau=T}.$$

We define

$$J_{xt}^\varepsilon(\tau) = \frac{f_0}{\delta+1} E(\tau-t)^{\delta+1} + f_0 E\int_\tau^T \exp -\frac{s-\tau}{2} ds - c\, E\chi_S(y_{xt}(T)) \exp -\frac{T-\tau}{\varepsilon}.$$

Clearly

(3.13) $$J_{xt}(\tau) - J_{xt}^\varepsilon(\tau) \geqq -f_0\varepsilon.$$

We next consider $J^\varepsilon(\tau)$ (we drop the index x, t).

By arguments similar to those of Theorem 2.1, we arrive at

(3.14) $$J^\varepsilon(\tau) \geqq E\left[\frac{f_0}{\delta+1}(\tau-t)^{\delta+1} - \frac{cc_p'}{h^p}(\tau-t)^{p/2-1}\right] + E\exp -\frac{\theta-\tau}{\varepsilon}\left[f_0(T-\tau) - c\chi_{|y(\tau)-x|<h} P(y(\tau), \tau; S, T)\right].$$

Choosing h equal to the second term of the right hand side (3.11) we obtain a situation identical to (2.26). Hence the result follows.

REMARK 3.3 The preceding results hold true for more general operators than $-\Delta$. Namely we can handle instead of (2.7) more general equations of the type

(3.15) $$dy = g(y)ds + \sigma(y)dw(s)$$

where g, σ are Lipschitz bounded functions and σ being invertible with $\|\sigma^{-1}(y)\| \leqq C$.

The operator Δ is changed into

$$A = \sum a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum g_j \frac{\partial}{\partial x_j}$$

where $a = \dfrac{\sigma\sigma^*}{2}$.

The estimate (2.20) is still true with a different constant. Of course (2.19) is no more valid but $P(\xi, s; S, T)$ can be bounded by an integral of the form (2.19) with different constants, by virtue of well known estimates on fundamental solutions of parabolic equations.

Taking into account these comments, the same proofs apply.

§ 4. Hyperbolic operators.

It is interesting to consider now the *deterministic case;* the state is given by[1]

$$dy=g(y)ds,\ y(t)=x, \text{ whose solution is given by } y_{xt}(s) \tag{4.1}$$

where, to fix ideas, we shall consider two cases:

$$\text{case (i) : } \quad \|g(x)\|\leqq c \qquad \forall x\in \boldsymbol{R}^n, \tag{4.2}$$

$$\text{case (ii): } \quad g(x)=\text{linear operator}. \tag{4.3}$$

In case (i) we have

$$\|y_{xt}(s)-x\|\leqq c(s-t) \tag{4.4}$$

and in case (ii)

$$\|y_{xt}(s)-x\|\leqq c(e^{\omega(s-t)}-1). \tag{4.5}$$

If we define

$$J_{x,t}(\tau)=\begin{cases} \int_t^\tau f(y_{xt}(s), s)ds & \text{if } \tau<T, \\ \int_t^T f(y_{xt}(s), s)ds+\bar{\psi}(y_{xt}(T)) & \text{if } \tau=T \end{cases} \tag{4.6}$$

and if we set

$$u(x, t)=\inf J_{x,t}(\tau) \tag{4.7}$$

then $u(x, t)$ is the solution of the "hyperbolic" V. I.

$$\begin{cases} -\frac{\partial u}{\partial t}-g\nabla u-f\leqq 0, \quad u\leqq 0, \\ u\Big(-\frac{\partial u}{\partial t}-g\nabla u-f\Big)=0 \quad \text{for } x\in\boldsymbol{R}^n,\ t<T, \end{cases} \tag{4.8}$$

where

$$g\cdot\nabla u=\sum g_j(x)\frac{\partial u}{\partial x_j}, \tag{4.9}$$

with the "final" condition

(1) We assume g to be Lipschitz.

$$u(x, T)=\bar{\phi}(x). \tag{4.10}$$

We assume that (2.3), (2.4) hold true and that

$$\frac{\partial f}{\partial t} \in L^{\infty}(\boldsymbol{R}^n \times]0, T[). \tag{4.11}$$

Then u is the solution of (4.8), (4.10) such that

$$ue^{-\gamma|x|}, \quad \frac{\partial u}{\partial t} e^{-\gamma|x|} \in L^2(0, T; L^2(\boldsymbol{R}^n)). \tag{4.12}$$

We have now

THEOREM 4.1. *We assume that f and $\bar{u}$ satisfy* (2.10) *and* (2.11). *Then one has $u(x, t)=0$ if*[1]

$$d(x, S) > c(T-t) \quad \textit{in case (i)}, \tag{4.13}$$

$$d(x, S) > c(e^{\omega(T-1)}-1) \quad \textit{in case (ii)}. \tag{4.14}$$

PROOF. We have $J_{x,t}(T) > 0$ if $0 < \tau < T$, so that

$$u(x, t)=\inf\{0, J_{x,t}(T)\}.$$

If (x, t) is such that (4.13) (resp. (4.14)) takes place, then $y_{x,t}(T) \notin S$ so that $\bar{u}(y_{x,t}(T))=0$ and $J_{x,t}(T) > 0$, so that $u=0$.

§ 5. Variational inequalities of evolution. Non linear operators.

5.1. STATEMENT OF THE PROBLEM.

Let us consider now functions

$$f(x, t, \lambda) \in \boldsymbol{R}, \qquad g(x, t, \lambda) \in \boldsymbol{R}^n \tag{5.1}$$

where $\lambda \in \mathcal{U} \subset \boldsymbol{R}^m$.

We consider the state to be given by

$$dy=g(y, s, v(s))ds+\sqrt{2}\,dw(s) \tag{5.2}$$

where $v(s)$ is a control function subject to

$$v(s) \in \mathcal{U}, \tag{5.3}$$

and where

$$y(t)=x. \tag{5.4}$$

We denote by $y_{x,t}(s; v(s))$ the solution of (5.2), (5.4).

(1) The constants c appearing in (4.13) (resp. (4.14)) are the same than those appearing in (4.4) (resp. (4.5)).

We assume that g is Lipschitz in x and that

(5.5) $$\|g(x, t, \lambda)\| \leqq k .$$

We now introduce the cost function

(5.6) $$J_{x,t}(\tau, v) = E\Big[\int_t^\tau f(y_{xt}(s\,; v), s, v(s))ds + \chi_{\tau \geqq T}\bar{u}(y_{x,t}(\tau\,; v))\Big],$$

and we define

(5.7) $$u(x, t) = \inf_{\tau \geqq T} J_{xt}(\tau, v)$$

v subject to (3.3).

We have proved in Bensoussan and Lions [2] that $u(x, t)$ is characterized by

(5.8) $$\begin{cases} -\dfrac{\partial u}{\partial t} - \nabla u - \mathcal{B}(u) - f \leqq 0, \\ u \leqq 0, \\ u\Big(-\dfrac{\partial u}{\partial t} - \nabla u - \mathcal{B}(u) - f\Big) = 0 \qquad \text{for } x \in \boldsymbol{R}^n,\ \tau < T \end{cases}$$

where $\mathcal{B}$ *is the non linear operator given* by

(5.9) $$\mathcal{B}(u) = \inf_{\lambda \in \mathcal{U}}\Big[f(x, t, \lambda) + \sum_{j=1}^n g_j(x, t, \lambda)\frac{\partial u}{\partial x_j}\Big],$$

with the "final" condition

(5.10) $$u(x, T) = \bar{u}(x), \qquad \bar{u} \leqq 0 .$$

5.2. ESTIMATES ON THE SUPPORT OF THE SOLUTION OF *V. I.* FOR NON LINEAR OPERATORS.

We shall assume that

(5.11) $$f \in L^\infty(\boldsymbol{R}^m \times (0, T) \times \mathcal{U}), \qquad f \geqq f_0 > 0,\ \forall x, t, \lambda,$$

(5.12) $$\bar{u} \geqq -c .$$

Let λ_0 be arbitrary in $\mathcal{U}$ and set

(5.13) $$g_0(x, t) = g(x, t, \lambda_0) .$$

We consider the stochastic differential equation

(5.14) $$\begin{cases} d\tilde{y} = g_0(\tilde{y}, s)ds + \sqrt{2}\,dw(s) \qquad s > t \\ \tilde{y}(t) = x \end{cases}$$

whose solution is denoted by $\tilde{y}_{xt}(s)$. We define

(5.15) $$\tilde{P}(\xi, s\,; S, T) = P(\tilde{y}_{\xi s}(T) \in S) .$$

We have the following estimate

$$(5.16) \qquad \tilde{P}(\xi, s; S, T) \leqq \frac{M}{(T-s)^{n/2}} \int_S \exp - \frac{m|y-\xi|^2}{T-s} dy .$$

Next if $y(\tau)=y_{xt}(\tau; v)$ (where τ is a stopping time and $v \equiv v(s)$ an admissible control), we have

$$|y(\tau)-x| \leqq (\tau-t)k+\sqrt{2}|w(\tau)-w(t)|.$$

Let β_p be a constant such that

$$(5.17) \qquad (a+b)^p \leqq \beta_p(a^p+b^p) \qquad \forall a, b>0 .$$

We have

$$E|y(\tau)-x|^p \leqq \beta_p[k^p E(\tau-t)^p + c'_p E(\tau-t)^{p/2}] ,$$

and assuming that $T-t<1$, we obtain

$$(5.18) \qquad E|y(\tau)-x|^p \leqq E(\tau-t)^{p/2}(k^p+c'_p)\beta_p .$$

We recall that $c'_p=\left(\frac{p}{p-1}\right)^{p^2/2}(p(p-1))^{p/2}$.

THEOREM 5.1. *We assume that* (5.5), (5.11), (5.12) *hold true. Let S denote the support of $\bar{u}$. Let us define $Q_{\alpha p}$ as the set of points x, t such that*

$$(5.19) \qquad d(x, S) \geqq \left(\frac{c}{f_0}\right)^{1/p} [\beta_p(k^p+c'_p)]^{1/p}(T-t)^{1/2-1/p} + \left(\sqrt{\frac{2}{m}}+\alpha\right)(T-t)^{1/2}|\operatorname{Log}(T-t)|^{1/2} .$$

Then if u denotes the solution of (5.8), (5.10) *one has*

$$(5.20) \qquad u=0 \text{ in } Q_{\alpha p} \qquad \text{for } T-t \text{ small enough} .$$

REMARK 5.1. One can give analogous extensions of the results of Theorems 3.1 to 3.3 to the present situation.

PROOF. We restrict the controls to satisfy the additional requirement

$$(5.21) \qquad v(s)=v_0 \qquad \text{if } s \geqq \tau .$$

This does not modify the value of $J_{xt}(\tau; v)$ (which does not depend on the values of v for $s \geqq \tau$).

We have (dropping the index x, t)

$$J(\tau; v) \geqq f_0 E(\tau-t) - cE\chi_S(y(T; v))\chi_{\tau=T}$$

and we define

$$J^\varepsilon(\tau; v) = f_0 E(\tau-t) + f_0 E \int_\tau^T \exp - \frac{s-\tau}{\varepsilon} ds - cE\chi_S(y(T; v)) \exp - \frac{T-\tau}{\varepsilon} .$$

Again we have

(5.22) $$J(\tau\,;v)-J^{\varepsilon}(\tau\,;v)\geqq -f_0\varepsilon\,.$$

Defining θ by (2.17) we have

(5.23) $$J^{\varepsilon}(\tau\,;v)\geqq f_0E(\tau-t)+f_0E(T-\tau)\exp-\frac{\theta-\tau}{\varepsilon}$$
$$-cE\chi_S(y(T\,;v))\exp-\frac{\theta-\tau}{\varepsilon}\,.$$

By virtue of (5.21) $\tilde{y}(T\,;v)=\tilde{y}_{y(\tau),\tau}(T)$.

By the strong Markov property of $\tilde{y}$ we have

$$E\chi_S(y(T\,;v))\exp-\frac{\theta-\tau}{\varepsilon}=E\Big[\exp-\frac{\theta-\tau}{\varepsilon}\tilde{P}(y(\tau),\tau\,;S,T)\Big],$$

hence using (5.18) we obtain

(5.24) $$J^{\varepsilon}(\tau\,;v)\geqq E(\tau-t)\Big[f_0-\frac{c\beta_p(k^p+c'_p)(\tau-t)^{p/2-1}}{h^p}\Big]$$
$$+E\exp-\frac{\theta-\tau}{\varepsilon}[f_0(T-\tau)-c\chi_{|y(\tau)-x|<h}\tilde{P}(y(\tau),\tau\,;S,T)]\,.$$

Using (5.16) we see as in Theorem 2.1 that

$$\begin{aligned}Z&=\chi_{|y(\tau)-x|<h}\tilde{P}(y(\tau),\tau\,;S,T)\\&\leqq\frac{M}{(T-\tau)^{n/2}}\int_{|\lambda|\geqq(d-h)\sqrt{m}}\exp-\frac{|\lambda|^2}{2(T-\tau)}d\lambda\\&=c_6\int_{\frac{(d-h)\sqrt{m}}{\sqrt{2(T-\tau)}}}^{+\infty}(\exp-r^2)r^{n-1}dr\\&\leqq c_6c(\eta)\exp-\frac{(1-\eta)(d-h)^2m}{2(T-\tau)}\,.\end{aligned}$$

We can then easily proceed as in the proof of Theorem 2.1 and we obtain the desired result.

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