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Some results on continuous time branching processes with state-dependent immigration

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§ 1. Introduction.

J. H. Foster and others consider Galton-Watson processes modified to allow immigration of particles whenever the number of particles is zero. Let $\{Z_n; n=0, 1, \cdots\}$ be such a process. Under some additional assumptions, J. H. Foster showed in [**5**] that the limit law of this process with offspring law having mean 1 is quite different from that of the original critical Galton-Watson process $\{Z_n^*; n=0, 1, \cdots\}$. He proved that for any $k>0$

$$\lim_{n\to\infty} P(\log Z_n/\log n \leqq \beta \mid Z_0=k)=\beta \qquad \text{for} \quad 0<\beta<1,$$

i.e., the limit distribution of $\log Z_n/\log n$ is the uniform distribution on $[0, 1]$. On the other hand, it is known for the original Galton-Watson process that

$$\lim_{n\to\infty} P(Z_n^*/n \leqq Bx/2 \mid Z_n^*>0,\ Z_0^*=1)=\int_0^x e^{-t}dt \qquad \text{for} \quad x\geqq 0,$$

where $B=\operatorname{var} Z_1^*$.

It is natural to ask whether a similar argument is possible when the time parameter is continuous. The main purpose of this paper is to prove an analogous limit theorem in the case of continuous time parameter. In addition we consider general continuous time branching processes which allow random immigration whenever the population size is zero (we call such processes CTBP-RI-0), and give necessary and sufficient conditions in terms of their infinitesimal generators in order that such processes should be transient, null-recurrent or positive-recurrent. Further we give a limit theorem also in the case where the original branching process is supercritical.

In § 2, we describe some basic properties of continuous time branching processes as a preparation for the study of CTBP-RI-0.

In § 3, we give necessary and sufficient conditions for transience, null-recurrence and possitive-recurrence.

In § 4, we calculate generating functions for the use of § 5 and § 6.

In § 5 and § 6, we will obtain limit theorems for supercritical and critical cases.

§ 2. Preliminaries.

1. Continuous time branching processes.

Let $\{Z^*(t); t \geqq 0\}$ be a time continuous Markov chain on the non-negative integers. The chain is called continuous time Markov branching process if its transition probabilities

$$P_{ij}^*(t) = P\{Z^*(t+s) = j \mid Z^*(s) = i\}$$

satisfy

a) $P_{ij}^*(t) = P \sum_{j_1+\cdots+j_i=j} P_{1j_1}^*(t) \cdots P_{1j_i}^*(t)$ for all $i>0$, $j \geqq 0$

and

$P_{00}^*(t) = 0$,

b) $P_{1i}^*(s) = \delta_{1i} + p_i s + o(s)$ as $s \downarrow 0$ for $i \geqq 0$, p_i finite.

We use an abbreviation CTBP for such a chain. Note that $p_1 \leqq 0$ and $p_i \geqq 0$ for $i \neq 1$. From now on we assume $p_1 < 0$.

For CTBP we define a generating function $F(t; s)$ and an infinitesimal generating function $f(s)$ by

$$F(t; s) = \sum_{i=0}^{\infty} P_{1i}^*(t) s^i \quad \text{and} \quad f(s) = \sum_{i=0}^{\infty} p_i s^i .$$

Now, we recall some of their properties.

1) $f(1) = 0$.

2) $$\begin{cases} \dfrac{\partial}{\partial t} F(t; s) = f(F(t; s)) \\ F(0; s) = s . \end{cases}$$

3) $$\begin{cases} \dfrac{\partial}{\partial t} F(t; s) = f(s) \dfrac{\partial}{\partial s} F(t; s) \\ F(0; s) = s . \end{cases}$$

4) $\sum_{j=0}^{\infty} P_{ij}^*(t) s^j = F(t; s)^i$.

The infinitesimal generator A^* of CTBP is easily obtained from the definition of CTBP and is given by

$$A^* = (a_{ij}^*), \text{ where } a_{ij}^* = \begin{cases} i p_{j-i+1} & \text{if } j \geqq i-1 \\ 0 & \text{if } j < i-1 . \end{cases}$$

It is known that if $a_1 = f'(1)$ is finite, then solution of (2) satisfies

$$F(t; 1) = 1 \qquad \text{for all} \quad t \geqq 0 .$$

That is, $P_{ij}^*(t)$ is uniquely determined by $\{p_i\}$.

Usually CTBP is called supercritical if $0<f'(1)<\infty$, critical if $f'(1)=0$, subcritical if $f'(1)<0$. $P_{10}^*(t)$ satisfies

$$\lim_{t\to\infty} P_{10}^*(t)=\begin{cases} 1 & \text{in critical or subcritical case}, \\ q<1 & \text{in supercritical case}. \end{cases}$$

Let $a_1=f'(1)$ and $b_1=f''(1)$. In subcritical case, it is known that

$$1-P_{10}^*(t)\sim K\exp(a_1 t) \qquad \text{for some constant } K$$

if and only if

$$\int_0^1 \frac{a_1 u+f(1-u)}{uf(1-u)}du$$

is finite. If b_1 is finite, then this integral is finite. In critical case,

$$1-P_{10}^*(t)\sim 2/b_1 t$$

if b_1 is finite. In supercritical case,

$$q-P_{10}^*(t)\sim Me^{ct},$$

where M is a positive constant and $c=f'(q)$.

For the proof of the above results, we refer to [**1**].

2. Continuous time branching process which allows random immigration whenever the population size is zero.

For the generator A^* of CTBP described above, the state 0 is an absorbing barrier. We modify this state so that the resulting infinitesimal generator is

$$A=(a_{ij}),\ \text{where } a_{ij}=\begin{cases} ip_{j-i+1} & \text{if } j\geqq i-1 \text{ and } i\geqq 1, \\ q_j & \text{if } i=0, \\ 0 & \text{otherwise}. \end{cases}$$

From now on, we assume that

$$q_0<0,\quad q_i\geqq 0 \quad i>0 \quad \text{and} \quad \sum_{i=0}^{\infty} q_i=0.$$

The processes generated by A is a continuous time analogue of "branching process with state-dependent immigration" investigated by J. H. Foster and others. Generally, processes generated by A are not unique and may approach to infinity in a finite time. We denote by CTBP-RI-0 the minimal processes generated by A, which terminate at the instant T of first infinity. There are versions such that their sample functions are right continuous step functions and if $T>t$, then the number of jumps before t is finite. Let $\{Z(t); 0\leqq t<T\}$ be such a version. We will use $Z^*(t)$ for the sample function of CTBP corresponding to $Z(t)$ and $P_{ij}^*(t)$ ($t\geqq 0$, i, $j=0, 1, \cdots$) for the transition probabilities

of $Z^*(t)$.

It is easy to see that if $a_1<\infty$, then

$$P(T=\infty)=1\,.$$

In the sequel we assume that $a_1<\infty$.

§ 3. Transience, null-recurrence and positive-recurrence.

1. Transience and recurrence.

We can easily determine by $a_1=f'(1)$ whether $\{Z(t)\,;\,t<\infty\}$ is transient or recurrent.

THEOREM 1. *$\{Z(t)\,;\,0\leqq t<\infty\}$ is recurrent if and only if $a_1\leqq 0$.*

PROOF. Let $Z(0)=i>0$. By the definition of $Z(t)$, $Z(t)$ behaves in the same way as $Z^*(t)$ till it reaches the state 0. $Z(t)$ stays there according to exponential distribution with mean $-1/q_0$. Then it jumps to state $j\geqq 1$ with probability $-q_j/q_0$. After that it repeats the same motion by the strong Markov property. From this fact, it is obvious that $Z(t)$ is recurrent if and only if the extinction probability q of $Z^*(t)$ equals 1, and $Z(t)$ is transient if and only if $q<1$. q. e. d.

2. Null-recurrence and positive-recurrence.

In § 2-1, we have defined two generating functions $F(t\,;\,s)$ and $f(s)$. Now we define one more generating function

$$g(s)=\sum_{i=0}^{\infty}q_is^i\,.$$

As is seen in § 3-1, $a_1=f'(1)$ determines whether the process is recurrent or transient. Further, does a_1 determine null-recurrence and positive-recurrence of the process? The answer is "no". We must take into consideration the function $g(s)$. But Remark 1 will show that a criterion is given by a_1 under some additional conditions.

Let $T_0=\inf\{t\,;\,Z(t)\neq Z(0)\}$ and $T_1=\inf\{t\,;\,t>T_0$ and $Z(t)=Z(0)\}$. We investigate a probability $H(t)=P\{T_1<t\,|\,Z(0)=0\}$ to give a criterion whether $\{Z(t)\,;\,0\leqq t\leqq\infty\}$ is positive-recurrent. $H(t)$ is the distribution function of the first recurrence time when the starting point is 0.

LEMMA 1.

$$H(t)=\int_0^t\{g(P_{10}^*(t-u))-q_0\}\exp(q_0u)du\,. \tag{1}$$

PROOF. T_0 is a Markov time. From the strong Markov property, we have

$$H(t)=\sum_{i=1}^{\infty}\int_0^t P_{i0}^*(t-u)P(T_0\in du,\,Z(T_0)=i\,|\,Z(0)=0)\,.$$

Since $P(T_0 \in du, Z(T_0)=i \mid Z(0)=0)=q_i \exp(q_0 u)du$ for each $i>0$,

$$H(t)=\sum_{i=0}^{\infty}\int_0^t q_i P_{i0}^*(t-u)\exp(q_0 u)du .$$

Moreover, $P_{i0}^*(t-u)=P_{10}^*(t-u)^i$. Therefore we get the lemma. q. e. d.

LEMMA 2. *Let* $a_1 \leqq 0$. *Then*

$$\int_0^1 \frac{g(s)}{f(s)}ds=-\int_0^{\infty} t\frac{d}{dt}g(P_{10}^*(t))dt .$$

PROOF. Let $s=P_{10}^*(t)$. From the property 2) of § 2-1, we get

$$\frac{ds}{dt}=f(P_{10}^*(t)) \quad \text{and} \quad t=\int_0^{P_{10}^*(t)}\frac{1}{f(s)}ds .$$

Using these relations, we obtain

$$\begin{aligned}\int_0^{\infty} t\frac{d}{dt}g(P_{10}^*(t))dt &= \int_0^1 g'(s)ds\int_0^s \frac{1}{f(u)}du \\ &= \int_0^1 \frac{1}{f(u)}du\int_u^1 g'(s)ds \\ &= -\int_0^1 \frac{g(u)}{f(u)}du .\end{aligned}$$

q. e. d.

THEOREM 2. *Let* $a_1 \leqq 0$. *If*

$$\int_0^1 \frac{g(s)}{f(s)}ds>-\infty ,$$

then $\{Z(t)\,;\,0\leqq t<\infty\}$ *is positive recurrent. If this integral diverges, then the process is null-recurrent.*

PROOF. Differentiation of (1) leads to the equation

$$\frac{d}{dt}H(t)=q_0H(t)+g(P_{10}^*(t))-q_0 . \tag{2}$$

Integrating (1) partially, we get

$$q_0H(t)=-g(P_{10}^*(t))+q_0+A(t) , \tag{3}$$

where $A(t)=\exp(q_0t)\int_0^t \exp(-q_0u)\frac{d}{du}g(P_{10}^*(u))du$. We denote the density of $H(t)$ by $h(t)$. Then we have by (2) and (3),

$$h(t)=A(t) . \tag{4}$$

Thus,

$$\begin{aligned}\int_0^{\infty} tdH(t) &= \int_0^{\infty}\exp(-q_0u)\frac{d}{du}g(P_{10}^*(u))du\int_u^{\infty} t\cdot\exp(q_0t)dt \\ &= \frac{1}{q_0}\int_0^{\infty}(1/q_0-u)\frac{d}{du}g(P_{10}^*(u))du .\end{aligned}$$

This shows that $\{Z(t);\ 0\leqq t<\infty\}$ is positive-recurrent if and only if the integral

$$\int_0^\infty u\frac{d}{du}g(P_{10}^*(u))du$$

converges. By Lemma 2, we get the theorem. q. e. d.

The representations (1) and (2) will be used in the proof of Lemma of § 5, Lemma 1 and 3 of § 6.

REMARK 1. The following statements are easily verified by Theorem 2.

a) If $a_1=0$ and $b_1<\infty$, then $\{Z(t);\ 0\leqq t<\infty\}$ is null-recurrent.

b) Suppose that $a_1<0$ and

$$\int_0^1 \frac{a_1u+f(1-u)}{uf(1-u)}du$$

is finite. If there is an $\alpha>0$ such that

$$g(s)/(1-s)^\alpha=O(1) \qquad \text{as} \quad s\uparrow 1,$$

then $\{Z(t);\ 0\leqq t<\infty\}$ is positive-recurrent.

REMARK 2. The statement "If $a_1<0$ and $b_1<\infty$, then $\{Z(t);0\leqq t<\infty\}$ is positive-recurrent" is not true.

COUNTER-EXAMPLE. Let

$$g(s)=(1/q_0+\log(1-s))^{-1}$$

and

$$f(s)=a(s-1)+b(s-1)^2/2,$$

where $-1/2<q_0<0$, $a<0$ and $b>0$. In this case, we can solve the differential equation 2) of § 2-1 and further we have

$$P_{10}^*(t)=1-e^{at}/(b(e^{at}-1)/2a+1).$$

Hence

$$g(P_{10}^*(t))=(1/q_0+at-\log(b(e^{at}-1)/2a+1))^{-1}$$
$$\sim 1/at \qquad \text{as} \quad t\to\infty.$$

By Theorem 2, the process in null-recurrent.

We must establish that $g(s)$ is a generating function of some $\{q_i\}$. For this, let $1/q_0+\log(1-s)=B(s)$. Then

$$g(s)=B(s)^{-1}$$

and

$$g'(s)=B(s)^{-2}(1-s)^{-1}>0.$$

Let $B(s)^{-2}(1-s)^{-1}=a(s)$ and $(1-s)^{-1}+2B(s)^{-1}(1-s)^{-1}=b(s)$. Then

$$g''(s)=a(s)b(s).$$

By the hypothesis $0>q_0>-1/2$, we have $b(0)>0$. Therefore

$$g''(0)=a(0)b(0)>0\,.$$

Now, since $a'(s)=a(s)b(s)$ and $b'(s)=b(s)^2/2+(1-s)^{-2}/2$, higher derivatives of $g(s)$ are represented as polynomials of $a(s)$, $b(s)$ and $(1-s)^{-1}$ with positive coefficients. Therefore all coefficients (except the constant term) of the Taylor expansion of $g(s)$ at the origin are positive.

§ 4. Generating functions and moments.

Now we want to describe limit theorems for our processes. For this we begin with calculating generating functions and moments.

1. Generating functions.

Let

$$G_i(t\,;\,s)=\sum_{j=0}^{\infty}P_{ij}(t)s^j\,,$$

where $P_{ij}(t)=P\{Z(t)=j\,|\,Z(0)=i\}$. We will represent $G_i(t\,;\,s)$ by generating functions of CTBP-RI-0. The transition probability of $Z(t)$ satisfies the following renewal equations.

$$P_{0i}(t)=\sum_{j=1}^{\infty}q_j\int_0^t P_{ji}^*(t-u)\exp(q_0u)du+\int_0^t P_{0i}(t-u)dH(u)$$

for $i>0$ and

$$P_{00}(t)=\int_0^t P_{00}(t-u)dH(u)+\exp(q_0t)\,.$$

The proof is easily obtained from the strong Markov property.

From the renewal theory, the unique solution of the above equation is given by

$$P_{0i}(t)=\int_{0-}^t dU(s)\int_0^{t-s}\sum_{j=1}^{\infty}q_jP_{ji}^*(t-s-u)\exp(q_0u)du$$

for $i>0$, and

$$P_{00}(t)=\int_{0-}^t \exp(q_0(t-u))dU(u)\,,$$

where $U(t)=\sum_{n=0}^{\infty}H^{n*}$ and H^{n*} is the n-fold convolution of H. Using (1), we obtain

$$G_0(t\,;\,s)=1+\int_{0-}^t dU(u)\int_0^{t-u}g(F(t-u-w\,;\,s))\exp(q_0w)dw\,.$$

It is easy to see that

$$P_{ij}(t)=P_{ij}^*(t)+\int_0^t P_{0j}(t-s)dP_{i0}^*(s)$$

for $i>0$ and

$$P_{i0}(t)=\int_0^t P_{00}(t-s)dP_{i0}^*(s),$$

by the strong Markov property. Therefore we have

$$G_i(t;s)=F^i(t;s)+\int_0^t (G_0(t-u;s)-1)dP_{i0}^*(u). \tag{5}$$

By this result, it is easily seen that if

$$F(t;1)=1 \quad \text{for all} \quad t\geqq 0,$$

then

$$G_i(t;1)=1 \quad \text{for all} \quad t\geqq 0 \text{ and } i\geqq 0.$$

This is obvious also by the fact that $Z(t)$ behaves in the same way as $Z^*(t)$ till it reaches the state zero.

2. Moments.

Let $E_i(\cdot)$ stand for $E(\cdot\,|\,Z(0)=i)$.

THEOREM 3. a) *If $a_1=f'(1)$ and $a_2=g'(1)$ are finite, then $E_i(Z(t))$ is finite for all $t\geqq 0$.*

b) *If a_1, a_2, $b_1=f''(1)$ and $b_2=g''(1)$ are finite, then $E_i(Z(t)^2)$ is finite for all $t\geqq 0$.*

PROOF. a) If $\lim_{s\uparrow 1}\frac{\partial}{\partial s}G_i(t;s)$ is finite, then this is equal to $E_i(Z(t))$. From the expression of $G(t;s)$ we get

$$\begin{aligned}&\lim_{s\uparrow 1}\frac{\partial}{\partial s}G_0(t;s)\\&=\lim_{s\uparrow 1}\int_{0-}^t dU(u)\int_0^{t-u}\frac{d}{dF}g(F)\frac{\partial}{\partial s}F(t-u-w;s)\exp(q_0w)dw\\&=\frac{a_2}{q_0-a_1}\int_{0-}^t \exp(q_0(t-u))dU(u).\end{aligned}$$

Thus, $\lim_{s\uparrow 1}\frac{\partial}{\partial s}G_0(t;s)$ is finite and continuous. Hence it is bounded on closed intervals. From (5) we have

$$\begin{aligned}&\lim_{s\uparrow 1}\frac{\partial}{\partial s}G_i(t;s)\\&=\lim_{s\uparrow 1}\frac{\partial}{\partial s}F^i(t;s)+\lim_{s\uparrow 1}\int_0^t\frac{\partial}{\partial s}G_0(t-u;s)dP_{i0}^*(u).\end{aligned}$$

It is well known in the theory of branching process that the first term is finite. The second term is finite since $\lim_{s\uparrow 1}\frac{\partial}{\partial s}G_0(t-u;s)$ is bounded on $u\in[0,t]$. Therefore $\lim_{s\uparrow 1}\frac{\partial}{\partial s}G_i(t;s)$ is finite.

b) Let us calculate

$$\lim_{s\uparrow 1}\frac{\partial^2}{\partial s^2}G_0(t\,;\,s)=E_0(Z(t)(Z(t)-1))\,.$$

It follows from

$$\frac{\partial^2}{\partial s^2}G_0(t\,;\,s)=\int_{0-}^{t}dU(u)\int_0^{t-u}\Big\{\frac{d^2}{dF^2}g(F)\Big(\frac{\partial}{\partial s}F(t-u-w\,;\,s)\Big)^2 + \frac{d}{dF}g(F)\frac{\partial^2}{\partial s^2}F(t-u-w\,;\,s)\Big\}\exp(q_0w)dw$$

that

$$\lim_{s\uparrow 1}\frac{\partial^2}{\partial s^2}G_0(t\,;\,s) =\int_{0-}^{t}dU(u)\int_0^{t-u}\{b_2\exp(2a_1(t-u-w))+a_2B(t-u-w)\}\exp(q_0w)dw\,,$$

where $B(t)=E_1(Z^*(t)(Z^*(t)-1))$. Therefore $E_0(Z(t)(Z(t)-1))$ is finite. In the same way as the proof of a), we can conclude that $\lim_{s\uparrow 1}\frac{\partial^2}{\partial s^2}G_i(t\,;\,s)$ is finite for each $i\geqq 0$. q. e. d.

THEOREM 4. *If a_1, a_2, b_1 and b_2 are finite, then $E_i(Z(t))$ and $E_i(Z(t)^2)$ satisfy*

a) $$\begin{cases}\frac{d}{dt}E_i(Z(t))=a_1E_i(Z(t))+a_2P_{i0}(t)\\ E_i(Z(0))=i\end{cases}$$

and

b) $$\begin{cases}\frac{d}{dt}E_i(Z(t))=2a_1E_i(Z(t)^2)+C(t)\\ E_i(Z(0)^2)=i^2\,,\end{cases}$$

where $C(t)=(b_1-a_1)E_i(Z(t))+(a_2+b_2)P_{i0}(t)$.

PROOF. Using the forward equation

$$\frac{d}{dt}P_{ij}(t)=\sum_{k=0}^{\infty}P_{ik}(t)a_{kj}\qquad\text{where}\quad (a_{ij})=A$$

and the preceding theorem, we have

$$\begin{aligned}\frac{d}{dt}E_i(Z(t))&=\frac{d}{dt}\Big(\sum_{j=0}^{\infty}jP_{ij}(t)\Big)\\ &=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}jP_{ik}(t)a_{kj}\\ &=\sum_{k=1}^{\infty}\sum_{j=k-1}^{\infty}jkp_{j-k+1}P_{ik}(t)+\sum_{i=1}^{\infty}jP_{i0}(t)q_j\\ &=a_1E_i(Z(t))+a_2P_{i0}(t)\,.\end{aligned}$$

b) is obtained by similar calculation. q. e. d.

THEOREM 5. *Under the hypothesis of Theorem 4,*

$$E_i(Z(t))=\left(a_2\int_0^t P_{i0}(u)\exp(-a_1u)du+i\right)\exp(a_1t)$$

and

$$E_i(Z(t)^2)=\left(\int_0^t C(u)\exp(-2a_1u)du+i^2\right)\exp(2a_1t).$$

In critical case $(a_1=0)$,

$$E_i(Z(t))=a_2\int_0^t P_{i0}(u)du+i$$

and

$$E_i(Z(t)^2)=(a_2(b_1+1)+b_2)\int_0^t P_{i0}(u)du+i^2.$$

PROOF. Solve the differential equations obtained in Theorem 4. q. e. d.

§ 5. Limit theorems in supercritical case.

For supercritical CTBP $(a_1>0)$, it is known that $\xi^*(t)=Z^*(t)\exp(-a_1t)$ converges almost surely to some random variable ξ^*. If $\sum_{j=2}^{\infty} jp_j\log j=\infty$, then $P(\xi^*=0)=1$. If $\sum_{j=2}^{\infty} jP_j\log j<\infty$, then ξ^* is non-degenerate and the Laplace transform $\phi(\lambda)=E(\exp(-\lambda\xi^*)\,|\,Z^*(0)=1)$ satisfies a differential equation

$$\frac{d}{d\lambda}\phi(\lambda)=\frac{f(\phi(\lambda))}{a_1}, \qquad \phi(0)=1.$$

Note that

$$\phi(\lambda)=\lim_{t\to\infty}F(t\,;\,\exp(-\lambda\exp(-a_1t))). \tag{6}$$

We will prove similar limit theorems for CTBP-RI-0.

THEOREM 6. *Let* a_1, a_2 *be finite and* $a_1>0$. *Let* $Z(0)=i$. *Then* $\xi_i(t)=Z(t)\exp(-a_1t)$ *converges almost surely to some random variable* ξ_i.

PROOF. It follows from § 4, Theorem 5 that

$$E_i(\xi_i(t)\,|\,\xi_i(u)\,;\,u\leqq s)=a_2\int_0^{t-s}P_{\xi_i(s)0}(u)\exp(-a_1u)du+\xi_i(s)$$
$$\geqq \xi_i(s) \qquad \text{for all} \quad t>s.$$

Hence $\{\xi_i(t)\,;\,t\geqq 0\}$ is a submartingale. Since

$$E_i(\xi_i(t))\leqq a_2(i+1/a_1) \qquad \text{for all} \quad t\geqq 0,$$

$\xi_i(t)$ converges almost surely to some ξ_i such that $E_i(\xi_i)<\infty$ by the convergence theorem for submartingales. q. e. d.

THEOREM 7. *Under the hypothesis of Theorem* 6, *the Laplace transform* $\varphi_i(\lambda)=E_i(\exp(-\lambda\xi_i))$ *is represented as follows.*

$$\varphi_i(\lambda)=\phi^i(\lambda)+\int_0^\infty dP_{i0}^*(u)\int_{0-}^\infty dU(v)\int_0^\infty g(\phi(\lambda\exp(-a_1(u+v+w))))\exp(q_0w)dw\,.$$

PROOF. From the preceding theorem and § 4-1 it follows that

$$\varphi_0(\lambda)=\lim_{t\to\infty}G_0(t\,;\,\exp(-\lambda\exp(-a_1t)))$$

$$=\lim_{t\to\infty}\left\{1+\int_{0-}^t dU(u)\int_0^{t-u}g(F(t-u-w\,;\,D(t,u,w)))\exp(q_0w)dw\right\},$$

where $D(t,u,w)=\exp(-\lambda\exp(-a_1(t-u-w))\exp(-a_1(u+w)))$. Since U is a bounded measure, we have by the bounded convergence theorem,

$$\varphi_0(\lambda)=1+\int_{0-}^\infty dU(u)\lim_{t\to\infty}\int_0^{t-u}g(F(t-u-w\,;\,D(t,u,w)))\exp(q_0w)dw$$

$$=1+\int_{0-}^\infty dU(u)\int_0^\infty g(\phi(\lambda\exp(-a_1(u+w))))\exp(q_0w)dw\,.$$

Similarly we can prove the theorem when $i>0$ by the boundedness of $P_{i0}^*(t)$.

q. e. d.

By the absolute continuity of $H(x)$, $U(x)$ is written in the form

$$U(x)=1+\int_0^x u(y)dy\,,$$

where $u(x)=\int_0^x h(x-y)dU(y)$. We need the following lemma in order to prove the absolute continuity of the distribution of ξ_i.

LEMMA. *We assume the hypothesis of Theorem* 6. *If* $q>0$, *then*

$$u(x)\sim L\cdot\exp(-\gamma x)\qquad as\quad x\to\infty$$

for some constants $L>0$ *and* $\gamma>0$ *satisfying*

$$\gamma<-\max(c,q_0)\,.$$

PROOF. Let $\lambda<\lambda_0=-\max(c,q_0)$. Let $I(\lambda)=\int_0^\infty e^{\lambda t}dH(t)$. By (4), we have

$$I(\lambda)=\int_0^\infty\exp((\lambda+q_0)t)dt\int_0^t\exp(-q_0u)\frac{d}{du}g(P_{10}^*(u))du\,.$$

Since $\lambda+q_0<0$, we have

$$I(\lambda)=\int_0^\infty\exp(-q_0u)\frac{d}{du}g(P_{10}^*(u))du\int_u^\infty\exp((\lambda+q_0)t)dt$$

$$=-\frac{1}{\lambda+q_0}\int_0^\infty\frac{d}{du}g(P_{10}^*(u))\exp(\lambda u)du\,.$$

It follows from $q-P_{10}^*(t)\sim Me^{ct}$ that

$$\frac{d}{du}g(P_{10}^*(u))\sim M'e^{cu}$$

where M' is a positive constant. From this, it is obvious that for fixed $\varepsilon>0$, we can choose $T>0$ so that

$$\frac{d}{du}g(P_{10}^*(u)) > M'(1-\varepsilon)e^{cu}$$

for $u>T$. Hence

$$I(\lambda) > -\frac{1}{\lambda+q_0}\int_T^\infty M'(1-\varepsilon)\exp((\lambda+c)t)dt$$

$$= \frac{M'(1-\varepsilon)}{(\lambda+q_0)(\lambda+c)}\exp((\lambda+c)T).$$

This shows that $I(\lambda)$ tends to infinity as $\lambda \uparrow \lambda_0$. Thus there is some $\gamma>0$ such that

$$\int_0^\infty e^{\gamma t}dH(t)=1.$$

Let $e^{\gamma t}h(t)=h^*(t)$ and $e^{\gamma t}u(t)=u^*(t)$. Then $h^*(t)$ and $u^*(t)$ satisfy a renewal equation

$$u^*(t)=h^*(t)+\int_0^t u^*(t-y)h^*(y)dy.$$

Applying L'Hospital's rule to (4), we have for $\gamma<\lambda<\lambda_0$

$$h^*(t)=o(\exp((\gamma-\lambda)t)).$$

This shows that for large t, $h^*(t)$ is estimated from above by integrable decreasing function. Then $h^*(t)$ is directly Riemann integrable. The definition of directly Riemann integrable was given by Feller [4]. Since

$$\int_0^\infty h^*(y)dy=1,$$

by the renewal theorem [4, Chap. 11, Sect. 1], we have

$$u^*(x)\to L \quad \text{as} \quad x\to\infty,$$

where $L^{-1}=\int_0^\infty yh^*(y)dy$. It is obvious that L^{-1} is finite. Therefore

$$u(x)\sim L\cdot\exp(-\gamma x) \quad \text{as} \quad x\to\infty. \qquad \text{q. e. d.}$$

THEOREM 8. *For ξ_k, either $P(\xi_k=0)=1$ or $P(\xi_k=0)=0$. The latter holds if and only if*

$$\sum_{j=2}^\infty jp_j\log j<\infty,$$

and then the distribution of ξ_k has a continuous density on $(0, \infty)$.

PROOF. If $\sum_{j=2}^\infty jp_j\log j=\infty$, then $\phi(\lambda)=1$ for all $\lambda\geqq 0$, and hence $\varphi_k(\lambda)=1$ by Theorem 7. Therefore $P(\xi_k=0)=1$. If the series converges, $\phi(\lambda)$ is known to tend to q as $\lambda\to\infty$. Hence, we have by Theorem 7

$$\lim_{\lambda\to\infty}\varphi_k(\lambda)=q^k+\int_0^\infty dP_{k0}^*(u)\int_{0-}^\infty dU(u)\int_0^\infty g(q)\exp(q_0w)dw$$

$$=q^k-q^kg(q)U(\infty)/q_0\,.$$

Since $U(\infty)=q_0/g(q)$, $\lim_{\lambda\to\infty}\varphi_k(\lambda)=0$. This means $P(\xi_k=0)=0$.

If $\sum_{j=2}^{\infty} jp_j\log j<\infty$, then applying Theorem 2 of [**1**, p. 110] to Lemma 7-(ii) of [**1**, p. 35], we have for any $0<\delta_0<\delta$,

$$\sup_\lambda|\psi'(i\lambda)|\cdot|\lambda|^{1+\delta_0}=R<\infty\,,$$

where $\delta=-c/a_1$, $c=f'(q)$ and $i=\sqrt{-1}$. Differentiation of $\varphi_k(\lambda)$ leads to the inequality

$$\sup_\lambda|\varphi_k'(i\lambda)|\cdot|\lambda|^{1+\delta_0}$$

$$\leqq kR+\int_0^\infty dP_{k0}^*(u)\int_{0-}^\infty dU(v)\int_0^\infty a_2M\cdot\exp(a_1\delta_0(u+v+w)+q_0)dw\,.$$

If $q>0$, we can choose δ_0 so that $a_1\delta_0<\gamma$ where γ is the positive number obtained in the preceding lemma. Then the right-hand side of the above inequality is smaller than

$$kR+\frac{a_2M}{a_1\delta_0+q_0}\int_0^\infty dP_{k0}^*(u)\int_{0-}^\infty\exp(a_1\delta_0(u+v))dU(u)\,.$$

Thus, by the preceding lemma and the fact that

$$\frac{d}{du}P_{k0}^*(u)\sim M''e^{cu}\qquad\text{as}\quad u\to\infty$$

where M'' is a positive constant, we have

$$\sup_\lambda|\varphi_k'(i\lambda)|\cdot|\lambda|^{1+\delta_0}<\infty\,. \tag{7}$$

If $q=0$, then $U(t)=1$ for $t\geqq0$. Choosing δ_0 so that $a_1\delta_0+q_0<0$, we also have (7). Thus, $\varphi_k'(i\lambda)$ is integrable. Let $W_k(x)$ be the distribution function of ξ_k. Since $E_k(\xi_k)<\infty$,

$$\int_0^\infty xdW_k(x)<\infty\,.$$

Hence

$$-\int_0^\infty x\cdot\exp(-i\lambda x)dW_k(x)=\varphi_k'(i\lambda)$$

and by the integrability of $\varphi_k'(i\lambda)$, $xdW_k(x)$ has a continuous density. q. e. d.

§ 6. Limit theorem in critical case.

Let $a_1=0$. We assume $Z(0)=0$ in this section.

LEMMA 1. *If $f''(1)=b_1<\infty$, then*

$$(1-F(t;s))^{-1}-(1-s)^{-1}\sim b_1t/2$$

uniformly in $s\in[0,1)$ as $t\to\infty$.

PROOF. See [**1**, p. 113, Lemma 2]. q. e. d.

LEMMA 2. *If $b_1<\infty$ and $g'(1)=a_2<\infty$, then*

a) $1-H(t)\sim 1/Qt$

and

b) $h(t)\sim 1/Qt^2$

as $t\to\infty$, where $Q=-\dfrac{b_1q_0}{2a_2}$.

PROOF. By Lemma 1 of § 3-2, we have

$$t(1-H(t))=t\cdot\exp(q_0t)\Big(1-\int_0^t g(P_{10}^*(u))\exp(-q_0u)du\Big).$$

Since $1-P_{10}^*(t)\sim 2/b_1t$ as $t\to\infty$,

$$g(P_{10}^*(t))\sim -2a_2/b_1t \qquad \text{as} \quad t\to\infty.$$

By L'Hospital's rule, we have

$$\lim_{t\to\infty}t(1-H(t))=\lim_{t\to\infty}\frac{g(P_{10}^*(t))\exp(-q_0t)}{(1/t^2+q_0/t)\exp(-q_0t)}$$

$$=\lim_{t\to\infty}\frac{t\cdot g(P_{10}^*(t))}{q_0+1/t}=1/Q.$$

Similarly we obtain (b). q. e. d.

LEMMA 3. *Under the condition of Lemma 2, $u(t)=\int_0^t h(t-y)dU(y)$ satisfies; $u(t)\sim Q/\log t$ as $t\to\infty$.*

PROOF. Put

$$m(t)=\int_0^t(1-H(x))dx.$$

It follows from (a) of Lemma 2 that

$$m(t)\sim(\log t)/Q \qquad \text{as} \quad t\to\infty.$$

By the same reason as $h^*(t)$, $h(t)$ is directly Riemann integrable. Thus, we have by Theorem 1 of [**3**, p. 264]

$$u(t)\sim\frac{1}{m(t)}\int_0^\infty h(t)dt=Q/(\log t) \qquad \text{as} \quad t\to\infty. \qquad \text{q. e. d.}$$

Now, we can prove the following:

THEOREM 9. *If* $a_1=0$, $b_1<\infty$ *and* $a_2<\infty$, *then, for* $0<\beta<1$,

$$\lim_{t\to\infty} G_0(t\,;\ \exp(-s/t^{\beta}))=\beta\,,\qquad s\in(0,\infty)\,.$$

Equivalently,

$$\lim_{t\to\infty} P(\log Z(t)/\log t\leqq\beta\,|\,Z(0)=0)=\beta\,.$$

PROOF. Let $I_0=G_0(t\,;\ \exp(-s/t^{\beta}))-1$. Then, by the representation of $G_0(t\,;s)$ we have

$$I_0=\int_{0-}^{t}dU(u)\int_0^{t-u}g(F(t-u-v\,;\ \exp(-s/t^{\beta})))\exp(q_0v)dv\,.$$

Let $t-u-v=w$. Changing the order of integration, we have

$$I_0=\int_0^{t}g(F(w\,;\exp(-s/t^{\beta})))\exp(-q_0w)dw\int_{0-}^{t-w}\exp(q_0(t-u))dU(u)\,.$$

Let

$$I_1=\int_0^{t/2}g(F(w\,;\exp(-s/t^{\beta})))\exp(-q_0w)dw\int_{t/2}^{t-w}\exp(q_0(t-u))dU(u)$$

and

$$I_2=\int_0^{t}g(F(w\,;\exp(-s/t^{\beta})))\exp(-q_0w)dw\int_{0-}^{\omega}\exp(q_0(t-u))dU(u)\,,$$

where $\omega=\min(t-w,t/2)$. Then $I_0=I_1+I_2$. Let

$$I_3=\int_{t/2}^{t-w}\exp(q_0(t-u))dU(u)\,.$$

By Lemma 3, we have

$$I_3=Q(1+o(1))\int_{t/2}^{t-w}\frac{\exp(q_0(t-u))}{\log u}du\,.$$

From this equality, we have

$$\begin{aligned}&\frac{b_1(1+o(1))}{2a_2\log(t-w)}\{\exp(q_0w)-\exp(q_0t/2)\}\\ &\leqq I_3 \\ &\leqq\frac{b_1(1+o(1))}{2a_2\log(t/2)}\{\exp(q_0w)-\exp(q_0t/2)\}\,.\end{aligned}\tag{8}$$

For arbitrary $\varepsilon>0$, we can choose T such that if $t>T$, then

$$-(a_2-\varepsilon)(1-F(t\,;s))\geqq g(F(t\,;s))\geqq-a_2(1-F(t\,;s))\tag{9}$$

for all $s\in(0,1]$. By Lemma 1,

$$\begin{aligned}(2t^{\beta}/s+w(b_1+\varepsilon)/2)^{-1}&\leqq1-F(w\,;\exp(-s/t^{\beta}))\\ &\leqq(t^{\beta}/s+w(b_1-\varepsilon)/2)^{-1}\,.\end{aligned}\tag{10}$$

Therefore for sufficiently large t we get by (8), (9) and (10)

$$I_4 \leqq I_1 \leqq I_5$$

where

$$I_4 = -\frac{b_1(1+o(1))}{2\log(t/2)}\int_0^{t/2}\frac{1-\exp(q_0(t/2-w))}{t^\beta/s+w(b_1-\varepsilon)/2}dw$$

and

$$I_5 = -\frac{b_1(a_2-\varepsilon)(1+o(1))}{2a_2\log t}\int_0^{t/2}\frac{1-\exp(q_0(t/2-w))}{2t^\beta/s+w(b_1+\varepsilon)/2}dw$$

Dividing the integral I_2 into two parts, we have

$$-I_2 \leqq a_2\int_0^{t/2}\frac{\exp(-q_0w)}{t^\beta/s+w(b_1-\varepsilon)/2}dw\int_{0-}^{t/2}\exp(q_0(t-u))dU(u)$$

$$+a_2\int_{t/2}^{t}\frac{\exp(-q_0w)}{t^\beta/s+w(b_1-\varepsilon)/2}dw\int_{0-}^{t-w}\exp(q_0(t-u))dU(u).$$

Denote the first and second terms by I_6 and I_7 respectively. Changing the order of the integration, we have

$$I_7 = a_2\int_{0-}^{t/2}\exp(q_0(t-u))dU(u)\int_{t/2}^{t-u}\frac{\exp(-q_0w)}{t^\beta/s+w(b_1-\varepsilon)/2}dw.$$

Let $C=2/(s(b_1-\varepsilon))$ and let $x=Ct^\beta+w$. Then

$$I_4 = \frac{-b_1(1+o(1))}{(b_1-\varepsilon)\log(t/2)}\left\{I(t/2,0)\exp(q_0(Ct^\beta+t/2))+\int_{Ct^\beta}^{Ct^\beta+t/2}\frac{dx}{x}\right\},$$

$$I_6 = \frac{2a_2}{b_1-\varepsilon}\cdot I(t/2,0)\cdot I_8\cdot\exp(q_0Ct^\beta)$$

and

$$I_7 = \frac{2a_2}{b_1-\varepsilon}\int_{0-}^{t/2}I(t-u,t/2)\exp(q_0(t-u+Ct^\beta))dU(u),$$

where

$$I(u,v) = \int_{Ct^\beta+v}^{Ct^\beta+u}x^{-1}\exp(-q_0x)dx$$

and

$$I_8 = \int_{0-}^{t/2}\exp(q_0(t-u))dU(u).$$

By L'Hospital's rule, we have

$$I(t/2,0)\sim\frac{\exp(-q_0(Ct^\beta+t/2))}{q_0(Ct^\beta+t/2)} \quad \text{as} \quad t\to\infty. \tag{11}$$

Thus, we have

$$\lim_{t\to\infty} I_4 = \frac{b_1}{b_1-\varepsilon}(\beta-1). \tag{12}$$

Similarly, we have

$$\lim_{t\to\infty} I_5 = \frac{b_1(a_2-\varepsilon)}{a_2(b_1+\varepsilon)}(\beta-1). \tag{13}$$

Since ε is arbitrary, we have by (12) and (13)

(14) $$\lim_{t\to\infty} I_1=\beta-1 .$$

Now we estimate I_2. By Lemma 3 and L'Hospital's rule,

$$I_8=o(\exp(q_0t/2)) .$$

By this estimation and (11), we have

$$I_6=o((Ct^\beta+t/2)^{-1}) .$$

That is,

(15) $$\lim_{t\to\infty} I_6=0 .$$

Since

$$I(t-u,\ t/2)\leqq-\frac{2\{\exp(-q_0(t-u))-\exp(-q_0t/2)\}}{q_0(b_1-\varepsilon)(Ct^\beta+t/2)} ,$$

we have

$$I_7\leqq-\frac{2a_2}{q_0(b_1-\varepsilon)(Ct^\beta+t/2)}\int_{0-}^{t/2}\{1-\exp(q_0(t/2-u))\}dU(u)$$

$$\leqq-\frac{2a_2}{q_0(b_1-\varepsilon)(Ct^\beta+t/2)}\int_{0-}^{t/2}dU(u) .$$

By Lemma 3 and L'Hospital's rule, we have

(16) $$\lim_{t\to\infty} I_7=0 .$$

Thus, by (15) and (16) we have

(17) $$\lim_{t\to\infty} I_2=0 .$$

Therefore by (14) and (17), we can conclude that

$$G_0(t;\exp(-s/t^\beta))=\beta \qquad \text{for} \quad 0<\beta<1 . \qquad \text{q. e. d.}$$

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Bibliography

[1] K. B. Athreya and P. Ney, Branching processes, Springer, Berlin-Heidelberg-New York, 1972.
[2] K. L. Chung, Markov chains with stationary transition probabilities, 2nd ed., Springer, Berlin-Heidelberg-New York, 1967.
[3] K. B. Erickson, Strong renewal theorems with infinite mean, Trans. Amer.

Math. Soc., 151 (1970), 263-293.

[4] W. Feller, An introduction to probability theory and its application 2, Wiley, New York, 1966.

[5] J. H. Foster, A limit theorem for a branching process with state-dependent immigration, Ann. of Math. Statist., 42 (1971), 1773-1776.

ADDED IN PROOF. The following paper deals with a related problem in discrete case; A. G. Pakes, A branching process with state dependent immigration component, Adv. Appl. Prob., 3 (1971), 301-314.

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