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On the stability of incompressible viscous fluid motions past objects

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§ 1. Introduction.

Let us consider a stationary flow, in 3 dimensions, of an incompressible viscous fluid past a finite number of isolated rigid bodies Ω (of finite size) which are bounded by surfaces $\Sigma_1, \cdots, \Sigma_n$. It is assumed that the fluid extends to infinity in the domain $\mathcal{E}$, and that the velocity $\boldsymbol{w}=(w_1, w_2, w_3)$ and the pressure $\bar{p}$ of the fluid motion are governed by the Navier-Stokes equation

$$(1) \qquad \begin{cases} -\nu\Delta \boldsymbol{w}+(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}-\nabla\bar{p}=\boldsymbol{f}_0(x) \\ \operatorname{div} \boldsymbol{w}=0 \end{cases}$$

with the condition at infinity

$$(1_a) \qquad \lim_{|x|\to\infty} \boldsymbol{w}(x)=\boldsymbol{w}^\infty$$

and with the condition on the boundary $\Sigma=\Sigma_1\cup\cdots\cup\Sigma_n$ of Ω

$$(1_b) \qquad \lim_{x\to\Sigma} \boldsymbol{w}(x)=\boldsymbol{w}^*(x)$$

where the viscousity coefficient ν is a positive constant, $\boldsymbol{f}_0(x)$ is a given function on $\mathcal{E}$, $\boldsymbol{w}^\infty$ is some fixed constant vector, $\boldsymbol{w}^*(x)$ is some prescribed function on Σ, and $(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}=\sum_{j=1}^3 w_j(\partial/\partial x_j)\boldsymbol{w}$. In what follows we shall assume that $\nu=1$; the general case can always be reduced to this one by coordinate transformation.

Now, Finn [**3, 4**] introduced the notion of a physically reasonable solution of (1), (1_a), (1_b), and showed: *if Σ is sufficiently smooth, $\boldsymbol{f}_0(x)$ is sufficiently small, and $\boldsymbol{w}^*-\boldsymbol{w}^\infty$ is sufficiently "small", then there is a solution $\boldsymbol{w}(x)$ of* (1), (1_a), (1_b). *The solution is locally smooth, and there holds*

$$(2) \qquad |\boldsymbol{w}(x)-\boldsymbol{w}^\infty|\leqq\frac{C}{|x|}, \qquad (x\in\mathcal{E}),$$

$$(2_a) \qquad \nabla\boldsymbol{w}\in L^3(\mathcal{E})$$

where C is a constant.

In this paper, we are concerned with the perturbation problem for the above flow. If $\boldsymbol{w}$ and $\boldsymbol{f}_0$ are perturbed by $\boldsymbol{u}_0$ and $\boldsymbol{f}$ respectively, then the perturbed flow $\boldsymbol{v}(x, t)$ is governed by the time dependent Navier-Stokes equation

$$
(3) \qquad \begin{cases} \partial \boldsymbol{v}/\partial t - \Delta \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla \bar{\bar{p}} = \boldsymbol{f}_0(x) + \boldsymbol{f}(x, t), \\ \operatorname{div} \boldsymbol{v} = 0, \\ \lim_{|x|\to\infty} \boldsymbol{v}(x, t) = \boldsymbol{w}^{\infty}, \quad \lim_{x\to\Sigma} \boldsymbol{v}(x, t) = \boldsymbol{w}^*, \end{cases}
$$

satisfying the initial condition

$$
(3_a) \qquad \lim_{t\to 0} \boldsymbol{v}(x, t) = \boldsymbol{w}(x) + \boldsymbol{u}_0(x) \qquad (x \in \mathcal{E}).
$$

J. Heywood [**6**, **18**] showed: *if the initial disturbance* $\boldsymbol{u}_0$ *is small enough, if* $\boldsymbol{f}=0$ *and if* (2) *holds with* $C<1/2$ *for all* $x\in\mathcal{E}$, *then there is a unique solution* $\boldsymbol{v}$ *of* (3) *and* (3_a), *and the integrals*

$$
\int_{\mathcal{E}} |\nabla(\boldsymbol{v}(x, t) - \boldsymbol{w}(x))|^2 dx \qquad \text{and} \qquad \int_{\substack{x\in\mathcal{E} \\ |x|<R}} |\boldsymbol{v}(x, t) - \boldsymbol{w}(x)|^2 dx
$$

converge to zero as $t\to\infty$, *where* R *is any positive number.*

In this paper we shall show, among others, that *any solution of* (3), (3_a) *tends to* $\boldsymbol{w}(x)$ *uniformly on* $\mathcal{E}$ *like* $t^{-1/8}$ *as* $t\to\infty$, *provided that the* (2) *holds with* $C<1/2$, *and that* $\boldsymbol{f}(x, t)$ *decays sufficiently rapidly.*

In the case where $\mathcal{E}$ is bounded, the stability problems were studied by Sobolevski [**13**], Kiselev-Ladyzhenskaya [**8**], Serrin [**11**], Velte [**14**] and others. For the case of the exterior domain, besides Heywood [**6**, **18**] and Ladyzhenskaya [**9**], see also the interesting paper of J. Cannon-G. Knightly [**2**] in which they discussed the continuous dependence of $\boldsymbol{v}$ on $\boldsymbol{u}_0$, $\boldsymbol{f}$, $\boldsymbol{w}^*$. G. Knightly[1] [**19**] showed the point-wise convergence of $\boldsymbol{v}(x, t)$ to $\boldsymbol{w}(x)$ in the case that $\mathcal{E}$ is the exterior domain, and that $\boldsymbol{f}=\boldsymbol{w}=0$.

Before stating our results more precisely, we introduce some notations. Let $C_{0,\sigma}^{\infty}(\mathcal{E})$ denote the set of all C^{∞}-real vector functions φ with compact support in $\mathcal{E}$, and such that $\operatorname{div}\boldsymbol{\varphi}=0$ $(x\in\mathcal{E})$. Let X_0 be the completion of $C_{0,\sigma}^{\infty}(\mathcal{E})$ with respect to the norm $\|\boldsymbol{u}\| = \left(\int_{\mathcal{E}} |\boldsymbol{u}(x)|^2 dx\right)^{1/2}$. Then X_0 is the Hilbert space with the usual inner product $(\cdot, \cdot)$, which is the closed subspace of $L^2(\mathcal{E})$. P denotes the projection operator from $L^2(\mathcal{E})$ to X_0. The Hilbert space X_1 is the subspace of the Sobolev space $\overset{\circ}{W}{}_2^1(\mathcal{E})$, consisting of all vector functions $\boldsymbol{u}$ in $\overset{\circ}{W}{}_2^1(\mathcal{E})$ with $\operatorname{div}\boldsymbol{u}=0$; for the definitions of $\overset{\circ}{W}{}_2^1(\mathcal{E})$, and $W_2^2(\mathcal{E})$, see, e.g., Ladyzhenskaya [**9**; p. 10]. If X is a Hilbert space, then $L^p((0, \infty); X)$, $1\leqq p<\infty$, denotes the set

1) Professor Knightly kindly informed me of his recent work (whose preprint I received after the submission of the present paper) in which, by a method quite different from ours, he showed that a generalized solution constructed by Heywood [**6**; 18] converges to a stationary solution uniformly on $\mathcal{E}$, but he does not give the rate of decay: Heywood constructed a generalized solution in the case that the initial disturbance is sufficiently small.

of all measurable function $\boldsymbol{u}(t)$ $(t>0)$ with values in X such that $\int_0^\infty \|\boldsymbol{u}(t)\|_X^2 dt < \infty$ ($\|\cdot\|_X$ is the norm of X). $L^\infty((0, \infty); X)$ denotes the set of all essentially bounded (in the norm of X) measurable functions of t with values in X. $H^1((0, \infty); X)$ is the set of all functions $\boldsymbol{u}$ in $L^2((0, \infty); X)$ such that $\boldsymbol{u}(t)$ is differentiable (in the norm of X) for almost all $t>0$ and $\dot{\boldsymbol{u}}(t)$ belongs to $L^2((0, \infty); X)$; $\dot{\boldsymbol{u}}(t)=(d/dt)\boldsymbol{u}(t)$. Let $C([t_1, t_2]; X)$ denote the set of all X-valued continuous function of t $(t_1 \leqq t \leqq t_2)$. An X-valued function $\boldsymbol{u}(t)$, defined in an interval I, of t is said to be *X-continuous* (*X-continuously differentiable*) if $\boldsymbol{u}(t)$ is continuous (continuously differentiable) for t in I with respect to the norm of X, respectively. Let $C_\alpha^1((0, T); X)$ be the set of all X-valued continuous functions $\boldsymbol{u}(t)$ of t $(0 \leqq t \leqq T)$ that are X-continuously differentiable for t $(0<t<T)$, and that satisfy the estimate $\sup_{0<t<T}(t^\alpha \|\dot{\boldsymbol{u}}(t)\|_X) < \infty$.

Now, if $\boldsymbol{w}$ is a solution of (1) with $\nu=1$, and if $\boldsymbol{v}$ is a solution of (3), then a pair of the functions $\boldsymbol{u}=\boldsymbol{v}-\boldsymbol{w}$, $p=\bar{p}-\bar{\bar{p}}$ satisfies, formally,

$$
(4) \qquad \begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - \nabla p = \boldsymbol{f}(x, t) \qquad (x \in \mathcal{E},\ t>0) \\ \operatorname{div} \boldsymbol{u} = 0 \end{cases}
$$

with the conditions at infinity and at the boundary

$$
(4_a) \qquad \lim_{|x|\to\infty} \boldsymbol{u}(x, t) = 0\,, \qquad \lim_{x\to\Sigma} \boldsymbol{u}(x, t) = 0 \qquad (t>0)
$$

and with the initial condition

$$
(4_b) \qquad \lim_{t\to 0} \boldsymbol{u}(x, t) = \boldsymbol{u}_0(x) \qquad (x \in \mathcal{E})\,.
$$

Thus, the stability problem for (1), (1_a), (1_b) is now reduced to the decay problem for (4), (4_a), (4_b).

Now we state the assumptions made throughout the present paper.

ASSUMPTION 1. *The boundary surface Σ consists of a finite number of separate Lyapunov surfaces Σ_j $(j=1, \cdots, n)$ of type $\Lambda_{2,h}$ $(0<h<1)$*, (for the definition of a Lyapunov surface, see, e.g., [**9**]).

ASSUMPTION 2. *$\boldsymbol{w}(x)$ is a continuously differentiable solenoidal vector function on $\bar{\mathcal{E}}$ with $\nabla w \in L^3(\mathcal{E})$ and such that the estimate holds;*

$$
\sup_{x\in\mathcal{E}} |x|\,|\boldsymbol{w}(x) - \boldsymbol{w}^\infty| (\equiv C) < 1/2
$$

where $\boldsymbol{w}^\infty$ is a fixed constant vector.

ASSUMPTION 3. *$(P\boldsymbol{f})(\cdot, t)$, as a function of t, is continuously differentiable for $t \geqq 0$ with respect to the norm of X_0, and satisfies:*

$$
(P\boldsymbol{f})(\cdot, t) \in L^1((0, \infty); X_0)\,, \qquad \sup_{t>0} \Big(\int_t^{t+1} \|(d/ds)(P\boldsymbol{f})(\cdot, s)\|^2 ds\Big)^{1/2} < \infty
$$

and

$$\int_0^\infty s^{1/2}\|(d/ds)P\boldsymbol{f}(\cdot,s)\|ds<\infty\,. \tag{5}$$

We next define a weak solution of (4), (4_a), (4_b), where $\boldsymbol{u}_0\in X_0$. By a *weak solution* $\boldsymbol{u}$ of (4), (4_a), (4_b), we mean a measurable function on $\mathcal{E}\times(0,\infty)$ satisfying the following conditions:

(a) $\boldsymbol{u}\in L^\infty((0,\infty)\,;X_0)\cap L^2((0,\infty)\,;X_1)$;

(b) For any $\boldsymbol{\varphi}\in X_0$, $(u(\cdot,t),\boldsymbol{\varphi})$ is continuous in $t\geqq 0$ and $(\boldsymbol{u}(\cdot,0),\boldsymbol{\varphi})=(u_0,\boldsymbol{\varphi})$;

(c) The relation

$$\int_s^t\{(\boldsymbol{u},\boldsymbol{\Phi}_t)-(\nabla\boldsymbol{u},\nabla\boldsymbol{\Phi})+(\boldsymbol{u},(\boldsymbol{u}\cdot\nabla)\boldsymbol{\Phi})+(\boldsymbol{u},(\boldsymbol{w}\cdot\nabla)\boldsymbol{\Phi})+(\boldsymbol{w},(\boldsymbol{u}\cdot\nabla)\boldsymbol{\Phi})+(\boldsymbol{f},\boldsymbol{\Phi})\}dt$$
$$=(\boldsymbol{u}(\cdot,t),\boldsymbol{\Phi}(\cdot,t))-(\boldsymbol{u}(\cdot,s),\boldsymbol{\Phi}(\cdot,s))\qquad(0<s\leqq t<\infty)$$

holds for all $\boldsymbol{\Phi}\in L^\infty((0,\infty)\,;X_1)\cap H^1((0,\infty)\,;X_0)$;

(d) The energy inequality

$$\begin{aligned}\|\boldsymbol{u}(\cdot,t)\|^2+2\int_s^t\|\nabla\boldsymbol{u}\|^2d\sigma+2\int_s^t((\boldsymbol{u}\cdot\nabla)\boldsymbol{w},\boldsymbol{u})d\sigma\\ \leqq\|\boldsymbol{u}(\cdot,s)\|^2+2\int_s^t(\boldsymbol{f},\boldsymbol{u})d\sigma\end{aligned} \tag{6}$$

holds for almost all $s\geqq 0$, including $s=0$, and all $t>s$.

Our result is now given by:

THEOREM. *Let the Assumptions* 1-3 *be satisfied. Let* $\boldsymbol{u}_0\in X_0$. *Let* $\boldsymbol{u}$ *be a weak solution of* (4), (4_a), (4_b). *Then there is a* T_0 *such that*

(i) $\boldsymbol{u}(\cdot,t)\in W_2^2(\mathcal{E})\cap\mathring{W}_2^1(\mathcal{E})$, $(t>T_0)$,

(ii) $\boldsymbol{u}(\cdot,t)$, *as a function of* t, *is* X_1*-continuously differentiable for* $t>T_0$,

(iii) u *satisfies*:

$$\begin{cases}\dfrac{\partial\boldsymbol{u}}{\partial t}=P\Delta\boldsymbol{u}-P(\boldsymbol{w}\cdot\nabla)\boldsymbol{u}-P(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}-P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+P\boldsymbol{f}(\cdot,t)\\ \operatorname{div}\boldsymbol{u}=0\qquad(t>T_0)\,,\end{cases}$$

(iv) $\boldsymbol{u}$ *decays like:*

$$\|\nabla\boldsymbol{u}(\cdot,t)\|\leqq Mt^{-1/4}\,, \tag{7}$$

$$\sup_{x\in\mathcal{E}}|\boldsymbol{u}(x,t)|\leqq Mt^{-1/8}\,,\qquad(t>T_0)\,, \tag{8}$$

where M *is some positive constant.*

REMARK 1.1. Suppose that

$$\Big(\int_0^\infty\|\boldsymbol{f}(t)\|^2dt\Big)^{1/2}+(2-4C)^{-1/2}K_0^{1/2}(\|\boldsymbol{w}\|_{L^\infty}+\|\nabla\boldsymbol{w}\|_{L^3})\leqq K_1$$

where $K_0=\|\boldsymbol{u}_0\|+\int_0^\infty\|\boldsymbol{f}(t)\|dt$, $K_1=\pi^2(2-4C)^{1/4}\Big[128\sqrt{3}\,\Gamma\Big(\dfrac{2}{3}\Big)^{3/4}\Gamma\Big(\dfrac{1}{4}\Big)^2K_0\Big]^{-1}$

($\|\cdot\|_{L^p}$ denotes the usual norm of $L^p(\mathcal{E})$-space). Then the T_0 is explicitly estimated by $T_0=K_0^2[(1-2C)K_1^2]^{-1}$ ($\Gamma(\cdot)$ is the Gamma function). We note that T_0 depends only on $\|\nabla \boldsymbol{w}\|_{L^3}$, $\|\boldsymbol{w}\|_{L^\infty}$, $\|\boldsymbol{u}_0\|$ and $\int_0^\infty \|\boldsymbol{f}(t)\|dt$, but not on a (possibly not unique) particular weak solution considered. In particular, if $\boldsymbol{f}=0$ and $\boldsymbol{w}=0$, then all weak solutions with the same data $\boldsymbol{u}_0$ become regular (even analytic: see [**20**]) after some definite time T_0 (depending only on $\|\boldsymbol{u}_0\|$). In this case, using (83) below, we can improve the rate of decay; $\sup_{x\in\mathcal{E}}|\boldsymbol{u}(x, t)| \leqq Mt^{-3/4}$.

REMARK 1.2. The existence of a weak solution of (4), (4_a), (4_b) with arbitrary disturbance $\boldsymbol{u}_0$ in X_0 can be proved by the well-known Galerkin procedure; see J. Heywood [**6, 18**].

REMARK 1.3. In addition to the Assumptions 1 and 2, if we assume, instead of the Assumption 3, that $P\boldsymbol{f}$ and $(d/dt)P\boldsymbol{f}$ are in $L^1((0, \infty); X_0)$, and that $\sup_{t>0}\left(\int_t^{t+1}\|(d/ds)P\boldsymbol{f}(\cdot, s)\|^2 ds\right)^{1/2}<\infty$, then the parts (i), (ii) and (iii) in the theorem hold and we have $\|\nabla \boldsymbol{u}(\cdot, t)\|\to 0$, $\sup_{x\in\mathcal{E}}|\boldsymbol{u}(x, t)|\to 0$ for $t\to\infty$. Furthermore, if there is an α ($0\leqq\alpha<1/2$) such that

$$\int_0^t s^{1/2}\|(d/ds)P\boldsymbol{f}(\cdot, s)\|ds\leqq Mt^\alpha \qquad (t\geqq 1)$$

(M: a positive constant), then we have

$$\|\nabla u(\cdot, t)\|\leqq Mt^{\left(\alpha-\frac{1}{2}\right)/2}; \qquad \sup_{x\in\mathcal{E}}|\boldsymbol{u}(x, t)|\leqq Mt^{\left(\alpha-\frac{1}{2}\right)/4}.$$

REMARK 1.4. We can consider the stability problem not only for the stationary flow, but also for the non-stationary flow; to see this, we have only to change a function $\boldsymbol{w}(x)$ of x into a function $\boldsymbol{w}(x, t)$ of x and t ($x\in\mathcal{E}$, $t\geqq 0$) in equation (4). Instead of the Assumption 2, we then make the following assumption.

ASSUMPTION 2′. *$\boldsymbol{w}(x, t)$ is a continuously differentiable solenoidal vector function of x and t ($x\in\bar{\mathcal{E}}$, $t\geqq 0$), $(d/dt)\boldsymbol{w}(x, t)$ being continuously differentiable for x on $\bar{\mathcal{E}}$ (for each fixed $t\geqq 0$), such that the estimate holds:*

$$\sup_{\substack{x\in\mathcal{E}\\ t\geqq 0}}|x|\,|\boldsymbol{w}(x, t)-\boldsymbol{w}^\infty(t)|<1/2$$

where $\boldsymbol{w}^\infty(t)$ is a given vector function of t, and such that

$$\boldsymbol{w}(x, t),\ (d/dt)\boldsymbol{w}(x, t)\in L^\infty((0, \infty);\ L^\infty(\mathcal{E}))\,,$$

$$\nabla_x\boldsymbol{w}(x, t),\ \nabla_x(d/dt)\boldsymbol{w}(x, t)\in L^\infty((0, \infty);\ L^3(\mathcal{E}))\,.$$

Under the Assumptions 1, 2′ and 3, the theorem still holds. (The proof of the above remark is quite similar to that of the case that $\boldsymbol{w}(x, t)$ is independent of t.)

Section 2 is devoted to some calculus for the Stokes operator $Au=-P\Delta u$, $Bu=P(u\cdot\nabla)w+P(w\cdot\nabla)u$, the nonlinear term $(u\cdot\nabla)u$, and f. The regularity of a weak solution u of (4), (4_a), (4_b) is discussed in section 3. We shall show in section 4 that a weak solution u becomes smooth after some definite time T_0. In section 5 we shall give the proof of the decay rate (7), (8). We shall conclude the present paper with the appendix which is concerned with a priori estimate of the Stokes operator.

Finally, in what follows, we shall write simply f for Pf, and denote by the same M various constants, and by M_j $(j=0,1,2)$ special constants.

§ 2. Preliminary.

2.1. *Stokes operator A.* We first define the Stokes operator A. A is the Friedrichs extension of the symmetric operator $-P\Delta$ in X_0 defined for every u in $W_2^2(\mathcal{E})\cap X_1$ where X_1 is the set of all u in $\overset{\circ}{W}{}_2^1(\mathcal{E})$ with $\operatorname{div} u=0$. Equivalently the relation $Au=g(u\in D(A)\subset X_1,\ g\in X_0)$ is true if and only if $(\nabla u, \nabla\varphi)=(g, \varphi)$ for all $\varphi\in X_1$; $D(A)$ denotes the domain of A. From the definition it follows that $D(A^{1/2})=X_1$ and that

$$\|A^{1/2}u\|=\|\nabla u\|\,, \tag{9}$$

where $\|\cdot\|$ is the L^2-norm over $\mathcal{E}$. In what follows, we denote

$$\|u\|_A=\|A^{1/2}u\| \qquad (=\|\nabla u\|)\,.$$

We can see the operator A more explicitly from the following proposition to be proved in the appendix: (i), (ii) and (iii) are due to Ladyzhenskaya, Golovkin and Solonnikov.

PROPOSITION 1. *We have*

(i) $D(A)=W_2^2(\mathcal{E})\cap X_1$,

(ii) $Au=-P\Delta u$, $(u\in D(A))$,

(iii) *the estimate holds*

$$\|u\|_{W_2^2}\leqq M(\|Au\|+\|u\|), \qquad (u\in D(A))\,, \tag{10}$$

(iv) *the estimate holds*

$$\|u\|_{L^\infty}\leqq M_0\|Au\|^{1/2}\|u\|_A^{1/2}+M_0'\|u\|_A^{1/2}\|u\|^{1/2} \qquad (u\in D(A)) \tag{11}$$

where M, M_0, and M_0' are constants independent of u.

REMARK. In the case of the bounded domain, Fujita-Kato [**5**; p. 277] established the inequality

$$\|u\|_{L^\infty}\leqq M\|Au\|^{3/4}\|u\|^{1/4} \tag{12}$$

which can be relatively easily extended to the unbounded case. In the following, instead of (11) we may make use of the generalized Fujita-Kato inequality.

Now we shall give some operational calculus for the non-negative self-adjoint operator A with the spectral resolution $\{E(\lambda)\}$, in a series of lemmas.

LEMMA 1. *We have,*

(i) $\|A^{\alpha}e^{-tA}\boldsymbol{v}\| \leqq t^{-\alpha}\|\boldsymbol{v}\|$ $(0 \leqq \alpha \leqq 2,\ \boldsymbol{v} \in X_0)$,

(ii) $\|(1-e^{-tA})\boldsymbol{v}\| \leqq t^{\alpha}\|A^{\alpha}\boldsymbol{v}\|$ $(0 \leqq \alpha \leqq 1,\ \boldsymbol{v} \in D(A^{\alpha}))$,

(iii) $\|A^{\alpha}[e^{-tA}-e^{-sA}]\boldsymbol{v}\| \leqq (t-s)^{\beta}s^{-\alpha-\beta}\|\boldsymbol{v}\|$ $(0 \leqq \alpha \leqq 1,\ 0 \leqq \beta \leqq 1,\ \boldsymbol{v} \in X_0)$,

for $0<s<t$.

PROOF. Since $|\lambda^{\alpha}e^{-t\lambda}| \leqq t^{-\alpha}$ $(0 \leqq \alpha \leqq 2)$, we have

$$\|A^{\alpha}e^{-tA}\boldsymbol{v}\|^2 \leqq \int_0^{\infty} |\lambda^{\alpha}e^{-t\lambda}|^2 d(E(\lambda)\boldsymbol{v}, \boldsymbol{v}) \leqq t^{-2\alpha}\|\boldsymbol{v}\|^2,$$

showing (i). Since $1-e^{-t\lambda} \leqq (t\lambda)^{\beta}$ $(0 \leqq \beta \leqq 1)$, we can similarly show (ii). (iii) easily follows from (i) and (ii). Q.E.D.

LEMMA 2. *We have*

$$\|A^{\alpha}\boldsymbol{v}\| \leqq \|\boldsymbol{v}\|^{1-2\alpha}\|\boldsymbol{v}\|_A^{2\alpha} \tag{13}$$

for $\boldsymbol{v} \in D(A^{1/2})$ *and* $0 \leqq \alpha \leqq 1/2$.

PROOF. By the Schwarz inequality,

$$\int_0^{\infty} \lambda^{2\alpha} d(E(\lambda)\boldsymbol{v}, \boldsymbol{v}) \leqq \Big(\int_0^{\infty} d(E(\lambda)\boldsymbol{v}, \boldsymbol{v})\Big)^{1-2\alpha}\Big(\int_0^{\infty} \lambda d(E(\lambda)\boldsymbol{v}, \boldsymbol{v})\Big)^{2\alpha}$$

which implies (13). Q.E.D.

LEMMA 3. *Let* $0<\gamma<1$. *Then*

$$(\mu+A)^{-\gamma}\boldsymbol{v} = \frac{\sin \pi\gamma}{\pi}\int_0^{\infty} s^{-\gamma}(s+\gamma+A)^{-1}\boldsymbol{v}\,ds \tag{14}$$

for $\boldsymbol{v}$ *in* X_0.

PROOF. The proof follows from the spectral representation for $(\mu+A)^{-\gamma}$ and from the relation

$$(\mu+\lambda)^{-\gamma} = \frac{\sin \pi\gamma}{\pi}\int_0^{\infty} s^{-\gamma}(s+\mu+\lambda)^{-1}ds.$$

LEMMA 4. *Let* $\lambda \geqq 0$, *and* $0 \leqq \alpha \leqq 1/2$. *Let* $\boldsymbol{g}(s) \in L^2((0, \infty); X_0)$. *We set*

$$\boldsymbol{h}(t) = \int_0^t e^{-(t-s)[A+\lambda]}\boldsymbol{g}(s)ds.$$

Then $\boldsymbol{h}(t) \in D(A^{1/2})$, *and*

$$\|A^{\alpha}\boldsymbol{h}(t)\| \leqq \Big(\frac{1}{2}\int_0^t \|\boldsymbol{g}(s)\|^2 ds\Big)^{\alpha}\Big(\int_0^t \|\boldsymbol{g}(s)\| ds\Big)^{1-2\alpha}, \qquad t \geqq 0. \tag{15}$$

PROOF. For $\boldsymbol{\varphi} \in D(A^{1/2})$, we have, by the Schwarz inequality,

$$(\boldsymbol{h}(t), A^{1/2}\boldsymbol{\varphi}) = \int_0^t (\boldsymbol{g}(s), A^{1/2}e^{-(t-s)[A+\lambda]}\boldsymbol{\varphi})ds$$

$$\leqq\left(\int_0^t\|\boldsymbol{g}(s)\|^2ds\right)^{1/2}\left(\int_0^t\|A^{1/2}e^{-(t-s)[A+\lambda]}\boldsymbol{\varphi}\|^2ds\right)^{1/2}.$$

Since

$$\int_0^t\|A^{1/2}e^{-(t-s)[A+\lambda]}\boldsymbol{\varphi}\|^2ds\leqq\int_0^t\|A^{1/2}e^{-(t-s)A}\boldsymbol{\varphi}\|^2ds\leqq\frac{1}{2}\|\boldsymbol{\varphi}\|^2,$$

which can be seen by means of the spectral representation for A, we have $|(\boldsymbol{h}(t), A^{1/2}\boldsymbol{\varphi})|\leqq\left(\frac{1}{2}\int_0^t\|\boldsymbol{g}(s)\|^2ds\right)^{1/2}\|\boldsymbol{\varphi}\|$, from which it follows that $\boldsymbol{h}(t)\in D(A^{1/2})$ and that (15) holds with $\alpha=1/2$. Since, by Lemma 1, $\|e^{-(t-s)[A+\lambda]}\boldsymbol{g}(s)\|\leqq\|\boldsymbol{g}(s)\|$, it is clear that (15) holds with $\alpha=0$. By Lemma 2, we have (15). Q.E.D.

We set

$$|\boldsymbol{g}|_{\alpha,t}=\sup_{0<s<t}[s^\alpha\|\boldsymbol{g}(s)\|]$$

and

$$|\boldsymbol{g}|_t=|\boldsymbol{g}|_{0,t}=\sup_{0<s<t}\|\boldsymbol{g}(s)\|.$$

Then:

LEMMA 5. *Let $0\leqq\alpha<1$. Let $T>0$. If $\boldsymbol{g}(t)$ is an X_0-valued continuous function of t $(0\leqq t\leqq T)$, then the function $\boldsymbol{h}(t)$ defined by*

$$\boldsymbol{h}(t)=\int_0^tA^\alpha e^{-(t-s)A}\boldsymbol{g}(s)ds \tag{16}$$

satisfies the estimate

$$\|\boldsymbol{h}(t)-\boldsymbol{h}(s)\|\leqq2(1-\alpha-\beta)^{-1}t^{1-\alpha-\beta}(t-s)^\beta|\boldsymbol{g}|_t\qquad(0\leqq s\leqq t\leqq T) \tag{17}$$

where $0\leqq\beta<1-\alpha$.

PROOF. We have, for $0\leqq\beta<1-\alpha$,

$$\boldsymbol{h}(t)-\boldsymbol{h}(s)=\int_s^tA^\alpha e^{-(t-\sigma)A}\boldsymbol{g}(\sigma)d\sigma+\int_0^sA^\alpha(e^{-(t-\sigma)A}-e^{-(s-\sigma)A})\boldsymbol{g}(\sigma)d\sigma$$
$$(\equiv\boldsymbol{h}^{(1)}(t)+\boldsymbol{h}^{(2)}(t)).$$

By Lemma 1,

$$\|A^\alpha(e^{-(t-\sigma)A}-e^{-(s-\sigma)A})\boldsymbol{g}(\sigma)\|\leqq(t-s)^\beta(s-\sigma)^{-\alpha-\beta}|\boldsymbol{g}|_t.$$

Hence

$$\|\boldsymbol{h}^{(2)}(t)\|\leqq\int_0^s\|A^\alpha(e^{-(t-\sigma)A}-e^{-(s-\sigma)A})\boldsymbol{g}(\sigma)\|d\sigma$$
$$\leqq(1-\alpha-\beta)^{-1}(t-s)^\beta s^{1-\alpha-\beta}|\boldsymbol{g}|_t.$$

Similarly, $\|\boldsymbol{h}^{(1)}(t)\|\leqq(1-\alpha)^{-1}(t-s)^{1-\alpha}|\boldsymbol{g}|_t\leqq(1-\alpha-\beta)^{-1}(t-s)^\beta t^{1-\alpha-\beta}|\boldsymbol{g}|_t$. Hence we have (17). Q.E.D.

LEMMA 6. *Let $0\leqq\alpha<1$. Let β be a positive number with $0\leqq\beta<1-\alpha$. Let $T>0$. If $\boldsymbol{g}\in C_1^1((0,T);X_0)$, then the function $\boldsymbol{h}(t)$ defined in* (16) *is in $C_\alpha^1((0,T);X_0)$. Moreover, $h(t)$ satisfies the estimate*

$$|\dot{\boldsymbol{h}}|_{\alpha,t}\leqq|\boldsymbol{g}|_t+B(\beta,1-\alpha-\beta)|\dot{\boldsymbol{g}}|_{1,t} \tag{18}$$

where $B(\cdot, \cdot)$ denotes the Beta function and $C'_\alpha((0, T); X_0)$ is defined in §1.

PROOF. Let $0<t_1<t_2<T$. Let $0<\varepsilon<t_1$. Then it is easy to see that the function $\boldsymbol{h}_\varepsilon(t)$ defined by

$$\boldsymbol{h}_\varepsilon(t)=\int_0^{t-\varepsilon} A^\alpha e^{-(t-s)A}\boldsymbol{g}(s)ds \qquad (t_1<t<t_2)$$

is continuously differentiable for t $(t_1<t<t_2)$ in the norm of X_0 and converges to $\boldsymbol{h}(t)$ uniformly on $[t_1, t_2]$ as $\varepsilon \to 0$ in the norm of X_0; note $A^n e^{-\varepsilon A}$ is a bounded operator $(n=1, 2, \cdots)$. After changing the variable $s \to s'=t-s$, we differentiate in t. The result is

$$\begin{aligned}\dot{\boldsymbol{h}}_\varepsilon(t)&=A^\alpha e^{-(t-\varepsilon)A}\boldsymbol{g}(\varepsilon)-\int_0^{t-\varepsilon} A^\alpha e^{-sA}\boldsymbol{g}_s(t-s)ds\\ &=A^\alpha e^{-(t-\varepsilon)}\boldsymbol{g}(t)-\int_0^{t-\varepsilon} A^\alpha(e^{-sA}-e^{-(t-\varepsilon)A})\boldsymbol{g}_s(t-s)ds\end{aligned}$$

where $\boldsymbol{g}_s(t-s)=(\partial/\partial s)\boldsymbol{g}(t-s)$. We shall show that $\dot{\boldsymbol{h}}_\varepsilon(t)$ converges to $\boldsymbol{h}'(t)$ uniformly in t $(t_1 \leqq t \leqq t_2)$ in the norm of X_0, where

$$\boldsymbol{h}'(t)=A^\alpha e^{-tA}\boldsymbol{g}(t)-\int_0^t A^\alpha(e^{-sA}-e^{-tA})\boldsymbol{g}_s(t-s)ds\,.$$

By a simple calculation,

$$\begin{aligned}\dot{\boldsymbol{h}}_\varepsilon(t)-\boldsymbol{h}'(t)&=A^\alpha(e^{-(t-\varepsilon)A}-e^{-tA})\boldsymbol{g}(t)\\ &\quad+\int_0^{t-\varepsilon} A^\alpha(e^{-(t-\varepsilon)A}-e^{-tA})\boldsymbol{g}_s(t-s)ds\\ &\quad+\int_{t-\varepsilon}^t A^\alpha(e^{-sA}-e^{-tA})\boldsymbol{g}_s(t-s)ds\\ &(\equiv \boldsymbol{w}_1+\boldsymbol{w}_2+\boldsymbol{w}_3)\,.\end{aligned}$$

By Lemma 1,

$$\|\boldsymbol{w}_1\| \leqq \varepsilon^\beta(t-\varepsilon)^{-\alpha-\beta}\|\boldsymbol{g}(t)\|\,.$$

Also, by Lemma 1, we have

$$\begin{aligned}\|A^\alpha(e^{-(t-\varepsilon)A}-e^{-tA})\boldsymbol{g}_s(t-s)\| &\leqq \varepsilon^\beta(t-\varepsilon)^{-\alpha-\beta}(t-s)^{-1}[(t-s)\|\boldsymbol{g}_s(t-s)\|]\\ &\leqq \varepsilon^\beta(t-\varepsilon)^{-\alpha-\beta}(t-s)^{-1}|\dot{\boldsymbol{g}}|_{1,t}\end{aligned}$$

and hence

$$\|\boldsymbol{w}_2\| \leqq \int_0^{t-\varepsilon} \|A^\alpha(e^{-(t-\varepsilon)A}-e^{-tA})\boldsymbol{g}_s(t-s)\|ds \leqq (t-\varepsilon)^{-\alpha-\beta}\varepsilon^\beta \log(t/\varepsilon)|\dot{\boldsymbol{g}}|_{1,t}\,.$$

Similarly, using the estimate $\|A^\alpha(e^{-sA}-e^{-tA})\boldsymbol{g}_s(t-s)\| \leqq (t-s)^{\beta-1}s^{-\alpha-\beta}|\dot{\boldsymbol{g}}|_{1,t}$, we have

(20) $$\|\boldsymbol{w}_3\| \leqq \int_{t-\varepsilon}^t (t-s)^{\beta-1}s^{-\alpha-\beta}ds|\dot{\boldsymbol{g}}|_{1,t}\,.$$

As can be easily seen from the above estimate, $\|\boldsymbol{w}_1\|+\|\boldsymbol{w}_2\|+\|\boldsymbol{w}_3\|$ tends to zero as $\varepsilon\to 0$ uniformly on $[t_1, t_2]$. Hence $\dot{\boldsymbol{h}}_\varepsilon(t)$ converges to $\boldsymbol{h}'(t)$ uniformly on $[t_1, t_2]$ in the norm of X_0. Since $\boldsymbol{h}_\varepsilon(t)$ converges to $\boldsymbol{h}(t)$, this implies that the uniform limit $\boldsymbol{h}(t)$ of $\boldsymbol{h}_\varepsilon(t)$ is X_0-continuously differentiable in t $(t_1\leqq t\leqq t_2)$, and equals to $\boldsymbol{h}'(t)$. Since $\|A^\alpha e^{-tA}\boldsymbol{g}(t)\|\leqq t^{-\alpha}\|\boldsymbol{g}(t)\|$, and since, by (20) with $\varepsilon=t$,

$$\int_0^t \|A^\alpha(e^{-sA}-e^{-tA})\boldsymbol{g}_s(t-s)\|ds\leqq t^{-\alpha}B(\beta, 1-\alpha-\beta)|\dot{\boldsymbol{g}}|_{1,t}\,,$$

the estimate (18) easily follows. Q. E. D.

2.2. The operator B. We next define the operator B by;

$$D(B)=X_1 \quad\text{and}\quad Bv=P(\boldsymbol{w}\cdot\nabla)\boldsymbol{v}+P(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}$$

where $\boldsymbol{w}$ is a function on $\mathcal{E}$ satisfying the Assumption 2. The works of Finn [**3**, **4**] underlies the Assumption 2. We now show the following

LEMMA 7. *We have*

(21) $$\|B\boldsymbol{v}\|\leqq M_1\|\boldsymbol{v}\|_A\,,$$

(22) $$(B\boldsymbol{v}, \boldsymbol{v})=((\boldsymbol{v}\cdot\nabla)\boldsymbol{w}, \boldsymbol{w})\geqq -2C\|\boldsymbol{v}\|_A^2$$

for $\boldsymbol{v}\in X_1$, *where*

$$M_1=\|\boldsymbol{w}\|_{L^\infty}+\frac{2}{\sqrt{3}}\|\nabla\boldsymbol{w}\|_{L^3}$$

and

$$C=\sup_{x\in\mathcal{E}}(|x|\,|\boldsymbol{w}(x)-\boldsymbol{w}^\infty|)\,.$$

PROOF. We have

$$\|(\boldsymbol{w}\cdot\nabla)\boldsymbol{v}\|\leqq\|\boldsymbol{w}\|_{L^\infty}\|\nabla\boldsymbol{v}\|=\|\boldsymbol{w}\|_{L^\infty}\|\boldsymbol{v}\|_A\,.$$

By the Sobolev inequality: $\|\boldsymbol{v}\|_{L^6}\leqq\frac{2}{\sqrt{3}}\|\nabla\boldsymbol{v}\|=\frac{2}{\sqrt{3}}\|\boldsymbol{v}\|_A$, we have

$$\|(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\|\leqq\|\nabla\boldsymbol{w}\|_{L^3}\|\boldsymbol{v}\|_{L^6}\leqq\frac{2}{\sqrt{3}}\|\nabla\boldsymbol{w}\|_{L^3}\|\boldsymbol{v}\|_A\,;$$

note that $\nabla\boldsymbol{w}\in L^3(\mathcal{E})$ by the Assumption 2. Hence we have (21). Integrating by parts, we have, by $\operatorname{div}\boldsymbol{w}=0$, $((\boldsymbol{w}\cdot\nabla)\boldsymbol{v}, \boldsymbol{v})=-(\boldsymbol{v}, (\boldsymbol{w}\cdot\nabla)\boldsymbol{v})$ and so $((\boldsymbol{w}\cdot\nabla)\boldsymbol{v}, \boldsymbol{v})=0$. Similarly, $((\boldsymbol{v}\cdot\nabla)\boldsymbol{w}, \boldsymbol{v})=-(\boldsymbol{w}-\boldsymbol{w}^\infty, (\boldsymbol{v}\cdot\nabla)\boldsymbol{v})$. Hence

$$\begin{aligned}(B\boldsymbol{v}, \boldsymbol{v})&=-(\boldsymbol{w}-\boldsymbol{w}^\infty, (\boldsymbol{v}\cdot\nabla)\boldsymbol{v})\\ &\geqq-(\sup_{x\in\mathcal{E}}|x|\,|\boldsymbol{w}(x)-\boldsymbol{w}^\infty|)\left\|\frac{\boldsymbol{v}}{|x|}\right\|\|\nabla\boldsymbol{v}\|\\ &\geqq -2C\|\nabla\boldsymbol{v}\|^2\,,\end{aligned}$$

showing (22). Here we used the Poincaré inequality;

(23) $$\int_{\mathcal{E}}\frac{|\boldsymbol{v}(x)|^2}{|x-y|^2}dx\leqq 4\|\nabla\boldsymbol{v}\|^2 \qquad (y\in\mathcal{E})\,.$$

This completes the proof of the lemma.

2.3. The nonlinear operator F. We formally define F by $F[\boldsymbol{u}]=A^{-1/4}P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$. More rigorously, we define the nonlinear operator F_ε from $X_{1,\infty}\times X_{1,\infty}$ to X_0 by

$$F_\varepsilon[\boldsymbol{u},\boldsymbol{v}]=(A+\varepsilon)^{-1/4}P[(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}]\,,\qquad \varepsilon>0\,,\quad (\boldsymbol{u},\boldsymbol{v}\in X_{1,\infty})$$

where $X_{1,\infty}=X_1\cap L^\infty(\mathcal{E})$. Then we show:

LEMMA 8 (Sobolevski).

(i) *For each fixed $\boldsymbol{u},\boldsymbol{v}$ in $X_{1,\infty}$, $F_\varepsilon[\boldsymbol{u},\boldsymbol{v}]$ converges to some element in the weak topology of X_0 as $\varepsilon\to 0$; we denote the limit by $\tilde{F}[\boldsymbol{u},\boldsymbol{v}]$.*

(ii) *$\tilde{F}[\boldsymbol{u},\boldsymbol{v}]$ thus defined can be extended to the nonlinear operator $F[\boldsymbol{u},\boldsymbol{v}]$ from $X_1\times X_1$ to X_0 with the estimates*

$$\|F[\boldsymbol{u},\boldsymbol{v}]\|\leqq M_2\|\boldsymbol{u}\|_A\|\boldsymbol{v}\|_A\qquad (\textit{for } \boldsymbol{u},\boldsymbol{v}\in X_1)\,,\tag{24}$$

$$\|F[\boldsymbol{u}_1,\boldsymbol{v}_1]-F[\boldsymbol{u}_2,\boldsymbol{v}_2]\|\leqq M_2\{\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_A\|\boldsymbol{v}_1\|_A+\|\boldsymbol{u}_2\|_A\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_A\}\tag{25}$$

(for $\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{v}_1,\boldsymbol{v}_2\in X_1$), where $M_2=\sqrt{2}\pi^{-1}$. We set

$$F[\boldsymbol{u}]=F[\boldsymbol{u},\boldsymbol{u}]\qquad (\boldsymbol{u}\in X_1)\,.$$

PROOF. The following proof is a modification of the proof given by Fujita-Kato [5] which treated the case that the domain is bounded. **I.** Let L be the Friedrichs extension of $-\Delta$ defined for (3-dimensional) C^∞-vector function with compact support in $\mathcal{E}$. Let $\varepsilon>0$. We have, similarly to A,

$$D((L+\varepsilon)^{1/2})=\mathring{W}_2^1(\mathcal{E})\,,\tag{26}$$

$$\|\nabla\boldsymbol{v}\|^2+\varepsilon\|\boldsymbol{v}\|^2=\|(L+\varepsilon)^{1/2}\boldsymbol{v}\|^2\qquad (\boldsymbol{v}\in\mathring{W}_2^1(\mathcal{E}))\,.\tag{27}$$

By means of the maximum principle, applied to each component, for the Helmholtz equation $-\Delta+\varepsilon+\lambda$ ($\lambda>0$), we can easily show that $(L+\varepsilon+\lambda)^{-1}$ is an integral operator with some kernel $G(x,y;\lambda+\varepsilon)$, satisfying

$$0\leqq G(x,y;\lambda+\varepsilon)\leqq\frac{1}{4\pi}\frac{e^{-\sqrt{\lambda+\varepsilon}|x-y|}}{|x-y|}\leqq\frac{1}{4\pi}\frac{e^{-\sqrt{\lambda}|x-y|}}{|x-y|}\qquad (x,y\in\mathcal{E})\,.\tag{28}$$

By Lemma 3 with $\gamma=\dfrac{1}{2}$ and with A replaced by L,

$$\begin{aligned}(\varepsilon+L)^{-1/2}\boldsymbol{v}(x)&=\frac{1}{\pi}\int_0^\infty\lambda^{-1/2}\Big(\int_{\mathcal{E}}G(x,y;\lambda+\varepsilon)\boldsymbol{v}(y)dy\Big)d\lambda\\&=\int_{\mathcal{E}}G^{(1/2)}(x,y;\varepsilon)\boldsymbol{v}(y)dy\end{aligned}$$

where

$$G^{(1/2)}(x,y;\varepsilon)=\frac{1}{\pi}\int_0^\infty\lambda^{-1/2}G(x,y;\lambda+\varepsilon)d\lambda\,.\tag{29}$$

Substitution of (28) into the right hand side of (29) yields the estimate

$$0 \leqq G^{(1/2)}(x, y; \varepsilon) \leqq \frac{1}{2\pi^2} |x-y|^{-2}. \tag{30}$$

II. Let j be the injection operator from X_0 to $L^2(\mathcal{E})$, i.e., $j(\boldsymbol{u})=\boldsymbol{u}$ for $\boldsymbol{u}\in X_0$. Setting $A_\varepsilon=A+\varepsilon$, $L_\varepsilon=L+\varepsilon$, we have, by (9) and (27),

$$\begin{cases} jD(A_\varepsilon^{1/2})\subset D(L_\varepsilon^{1/2}); \\ \|A_\varepsilon^{1/2}\boldsymbol{v}\|^2=\|L_\varepsilon^{1/2}j\boldsymbol{v}\|^2(=\|\nabla\boldsymbol{v}\|^2+\varepsilon\|\boldsymbol{v}\|^2) \end{cases}$$

for $\boldsymbol{v}\in X_1$. Hence, in virtue of the Heinz inequality (T. Kato [7]; see also [16]), this gives

$$\|L_\varepsilon^{1/4}j\boldsymbol{v}\|\leqq\|A_\varepsilon^{1/4}\boldsymbol{v}\| \qquad (\boldsymbol{v}\in D(A^{1/4}))$$

which implies that $L_\varepsilon^{1/4}jA_\varepsilon^{-1/4}$ is a bounded operator from X_0 to $L^2(\mathcal{E})$ with a bound $\|L_\varepsilon^{1/4}jA_\varepsilon^{-1/4}\|\leqq 1$. Therefore, the operator $S_\varepsilon=A_\varepsilon^{-1/4}PL_\varepsilon^{1/4}$, defined on $D(L_\varepsilon^{1/4})$, admits of the bounded extension $\bar{S}_\varepsilon$ with a bound

$$\|\bar{S}_\varepsilon\|\leqq 1 \tag{31}$$

which is a bounded operator from $L^2(\mathcal{E})$ to X_0: note $S_\varepsilon^*=L_\varepsilon^{1/4}jA_\varepsilon^{-1/4}$. **III.** Let $\boldsymbol{u}, \boldsymbol{v}\in X_{1,\infty}$, we then set $\boldsymbol{g}=(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}$. By the Schwarz inequality,

$$\begin{aligned} \|(L+\varepsilon)^{-1/4}\boldsymbol{g}\|^2 &= ((L+\varepsilon)^{-1/2}\boldsymbol{g}, \boldsymbol{g}) = \iint_{\mathcal{E}\times\mathcal{E}} G^{(1/2)}(x, y; \varepsilon)\boldsymbol{g}(y)\boldsymbol{g}(x)dx\,dy \\ &\leqq \frac{1}{2\pi^2}\iint_{\mathcal{E}\times\mathcal{E}} \frac{|\boldsymbol{g}(y)||\boldsymbol{g}(x)|}{|x-y|^2}dx\,dy \qquad [\text{by (30)}] \\ &\leqq \frac{1}{2\pi^2}\Big(\iint_{\mathcal{E}\times\mathcal{E}} h(x, y)dx\,dy\Big)^{1/2}\Big(\iint_{\mathcal{E}\times\mathcal{E}} h(y, x)dx\,dy\Big)^{1/2} \end{aligned}$$

where $h(x, y)=|\boldsymbol{u}(x)|^2|\nabla\boldsymbol{v}(y)|^2/|x-y|^2$. Using (23), we get

$$\iint_{\mathcal{E}\times\mathcal{E}} h(x, y)dx\,dy=\iint_{\mathcal{E}\times\mathcal{E}} h(y, x)dx\,dy\leqq 4\|\nabla\boldsymbol{u}\|^2\|\nabla\boldsymbol{v}\|^2,$$

whence

$$\|L_\varepsilon^{-1/4}\boldsymbol{g}\|\leqq\frac{\sqrt{2}}{\pi}\|\nabla\boldsymbol{u}\|\|\nabla\boldsymbol{v}\|. \tag{32}$$

Hence, it follows from (31) and (32) that

$$\|F_\varepsilon[\boldsymbol{u}, \boldsymbol{v}]\|=\|\bar{S}_\varepsilon L_\varepsilon^{-1/4}\boldsymbol{g}\|\leqq\|\bar{S}_\varepsilon\|\|L_\varepsilon^{-1/4}\boldsymbol{g}\|\leqq\frac{\sqrt{2}}{\pi}\|\boldsymbol{u}\|_A\|\boldsymbol{v}\|_A. \tag{33}$$

IV. For any $\boldsymbol{\varphi}\in D(A^{1/4})$, we have

$$\begin{aligned} &(F_\varepsilon[\boldsymbol{u}, \boldsymbol{v}], A^{1/4}\boldsymbol{\varphi}) \\ &\quad=(P(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, A^{1/4}A_\varepsilon^{-1/4}\boldsymbol{\varphi})\longrightarrow(P(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{\varphi})=((\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{\varphi}) \\ &\qquad\qquad(\boldsymbol{u}, \boldsymbol{v}\in X_{1,\infty}) \end{aligned} \tag{34}$$

for $\varepsilon \to 0$. Since the range of $A^{1/4}$ is dense in X_0, implied by the non-existence of the zero eigen-value of A (by (9)), and since $F_\varepsilon[\boldsymbol{u}, \boldsymbol{v}]$ is, by (33), uniformly bounded for $0 < \varepsilon \leqq 1$, the weak convergence of $F_\varepsilon[\boldsymbol{u}, \boldsymbol{v}]$ as $\varepsilon \to 0$ follows from (34). If we denote its limit by $\tilde{F}[\boldsymbol{u}, \boldsymbol{v}]$, we have, by the resonance theorem (see [**15**]) and (33),

$$\|\tilde{F}[\boldsymbol{u}, \boldsymbol{v}]\| \leqq \frac{\sqrt{2}}{\pi}\|\boldsymbol{u}\|_A \|\boldsymbol{v}\|_A . \tag{35}$$

For each fixed $\boldsymbol{u}$ (or $\boldsymbol{v}$), $F_\varepsilon[\boldsymbol{u}, \boldsymbol{v}]$ is linear in $\boldsymbol{v}$ (or $\boldsymbol{u}$), and so is its limit $\tilde{F}[\boldsymbol{u}, \boldsymbol{v}]$. Hence, by (35),

$$\begin{aligned}\|\tilde{F}[\boldsymbol{u}_1, \boldsymbol{v}_1]-\tilde{F}[\boldsymbol{u}_2, \boldsymbol{v}_2]\| &\leqq \|\tilde{F}[\boldsymbol{u}_1-\boldsymbol{u}_2, \boldsymbol{v}_1]\|+\|\tilde{F}[\boldsymbol{u}_2, \boldsymbol{v}_1-\boldsymbol{v}_2]\| \\ &\leqq \frac{\sqrt{2}}{\pi}\{\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_A\|\boldsymbol{v}_1\|_A+\|\boldsymbol{u}_2\|_A\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_A\} .\end{aligned}$$

Since $C_{0,\sigma}^\infty(\mathcal{E})$ is dense in X_1, $X_{1,\infty}$ is dense in X_1. Hence, by the above inequality, we can define the nonlinear operator $F[\boldsymbol{u}, \boldsymbol{v}]$ by

$$F[\boldsymbol{u}, \boldsymbol{v}] = s\text{-}\lim_{n\to\infty} \tilde{F}[\boldsymbol{u}_n, \boldsymbol{v}_n] \quad \text{(strong limit in } X_0)$$

for $\boldsymbol{u}, \boldsymbol{v} \in X_1$, where $\boldsymbol{u}_n$ and $\boldsymbol{v}_n$ are sequences in $X_{1,\infty}$ with $\boldsymbol{u}_n \to \boldsymbol{u}$, $\boldsymbol{v}_n \to \boldsymbol{v}$ (in the norm of X_1). From the above inequality, it follows that the $F[\boldsymbol{u}, \boldsymbol{v}]$ thus defined satisfies the desired estimates (24), (25). This completes the proof of the lemma.

LEMMA 9. *For $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in X_1$, we have*

$$(F[\boldsymbol{v}_1, \boldsymbol{v}_2], A^{1/4}\boldsymbol{v}_3) = ((\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_2, \boldsymbol{v}_3) . \tag{36}$$

In particular,

$$(F[\boldsymbol{v}_1, \boldsymbol{v}_2], A^{1/4}\boldsymbol{v}_2) = 0 . \tag{37}$$

PROOF. Since $X_{1,\infty}$ is dense in X_1, and since $F[\boldsymbol{u}, \boldsymbol{v}]$ (and $A^{1/4}\boldsymbol{u}$) are continuous operators from $X_1 \times X_1$ (and X_1) to X_0 respectively, it suffices to prove (36) and (37) for $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in X_{1,\infty}$. If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in X_{1,\infty}$, then, by (34),

$$(F[\boldsymbol{v}_1, \boldsymbol{v}_2], A^{1/4}\boldsymbol{v}_3) = \lim_{\varepsilon\to\infty} (F_\varepsilon[\boldsymbol{v}_1, \boldsymbol{v}_2], A^{1/4}\boldsymbol{v}_3) = ((\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_2, \boldsymbol{v}_3) ,$$

whence (36) holds. By integrating by parts, and by noting that $\operatorname{div} \boldsymbol{v}_1 = 0$, we have $((\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_2, \boldsymbol{v}_2) = -(\boldsymbol{v}_2, (\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_2)$, which implies $((\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_2, \boldsymbol{v}_2) = 0$. Hence (37) holds for $\boldsymbol{v}_1, \boldsymbol{v}_2 \in X_{1,\infty}$. This completes the proof of the lemma.

LEMMA 10. *If $\boldsymbol{u}, \boldsymbol{v} \in X_{1,\infty}$, then we have $F[\boldsymbol{u}, \boldsymbol{v}] \in D(A^{1/4})$ and $A^{1/4}F[\boldsymbol{u}, \boldsymbol{v}] = P((\boldsymbol{u} \cdot \nabla)\boldsymbol{v})$.*

PROOF. By (34),

$$(F[\boldsymbol{u}, \boldsymbol{v}], A^{1/4}\boldsymbol{\varphi}) = \lim_{\varepsilon\to 0} (F_\varepsilon[\boldsymbol{u}, \boldsymbol{v}], A^{1/4}\boldsymbol{\varphi}) = (P(\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi}) \qquad (\boldsymbol{\varphi} \in D(A^{1/4})) .$$

Since $|(P(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}, \boldsymbol{\varphi})| \leqq \|\boldsymbol{u}\|_{L^\infty}\|\nabla\boldsymbol{v}\|\|\boldsymbol{\varphi}\|$, and since $A^{1/4}$ is self-adjoint, we see from the above relation that $F[\boldsymbol{u}, \boldsymbol{v}] \in D(A^{1/4})$ and that $A^{1/4}F[\boldsymbol{u}, \boldsymbol{v}] = P(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}$.

LEMMA 11. *If $\boldsymbol{u} \in D(A)$, then $F[\boldsymbol{v}] \in D(A^{1/4})$. Moreover, the estimate*

$$\begin{aligned} (38) \qquad & \|A^{1/4}F[\boldsymbol{u}] - A^{1/4}F[\boldsymbol{v}]\| \\ & \leqq M_0\|A\boldsymbol{u}-A\boldsymbol{v}\|^{1/2}\|\boldsymbol{u}-\boldsymbol{v}\|_A^{1/2}\|\boldsymbol{u}\|_A + M_0'\|\boldsymbol{u}-\boldsymbol{v}\|^{1/2}\|\boldsymbol{u}-\boldsymbol{v}\|_A^{1/2}\|\boldsymbol{u}\|_A \\ & \quad + M_0\|A\boldsymbol{v}\|^{1/2}\|\boldsymbol{v}\|_A^{1/2}\|\boldsymbol{u}-\boldsymbol{v}\|_A + M_0'\|\boldsymbol{v}\|^{1/2}\|\boldsymbol{v}\|_A^{1/2}\|\boldsymbol{u}-\boldsymbol{v}\|_A \end{aligned}$$

holds for all $\boldsymbol{u}, \boldsymbol{v}$ in $D(A)$, where M is a constant appeared in (11).

PROOF. By Proposition 1, $\boldsymbol{u} \in X_{1,\infty}$. Hence, by Lemma 10, $F[\boldsymbol{u}] \in D(A^{1/4})$. Also, by Lemma 10

$$\begin{aligned} \|A^{1/4}F[\boldsymbol{u}] - A^{1/4}F[\boldsymbol{v}]\| &= \|P((\boldsymbol{u}-\boldsymbol{v})\cdot\nabla)\boldsymbol{u} + P(\boldsymbol{v}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{v})\| \\ &\leqq \|((\boldsymbol{u}-\boldsymbol{v})\cdot\nabla)\boldsymbol{u}\| + \|(\boldsymbol{v}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{v})\| \\ &\leqq \|\boldsymbol{u}-\boldsymbol{v}\|_{L^\infty}\|\nabla\boldsymbol{u}\| + \|\boldsymbol{v}\|_{L^\infty}\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|\,. \end{aligned}$$

Applying the estimate (11) to $\|\boldsymbol{u}-\boldsymbol{v}\|_{L^\infty}$ and $\|\boldsymbol{v}\|_{L^\infty}$, we get the desired estimate (38). Q. E. D.

LEMMA 12. *Let t_1, t_2 be positive numbers with $t_1 < t_2$. Then we have:*

(i) *if $\boldsymbol{u}, \boldsymbol{v} \in C([t_1, t_2]; X_1)$, then $F[\boldsymbol{u}(t), \boldsymbol{v}(t)] \in C([t_1, t_2]; X_0)$;*

(ii) *if $\boldsymbol{u}(t)$ is X_1-continuously differentiable for $t_1 < t < t_2$, then $F[\boldsymbol{u}(t)]$ is X_0-continuously differentiable for $t_1 < t < t_2$, and satisfies*

$$(39) \qquad (d/dt)F[\boldsymbol{u}(t)] = F[\dot{\boldsymbol{u}}(t), \boldsymbol{u}(t)] + F[\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)]\,;$$

(iii) *if $\boldsymbol{u}(t) \in D(A)$ and $\boldsymbol{u}(t)$, $A\boldsymbol{u}(t)$ are X_0-continuous in t $(t_1 < t < t_2)$, then $F[\boldsymbol{u}(t)] \in D(A^{1/4})$ and $A^{1/4}F[\boldsymbol{u}(t)]$ is X_0-continuous for $t_1 < t < t_2$.*

PROOF. (i) follows from (25) with $\boldsymbol{u}_1 = \boldsymbol{u}(t)$, $\boldsymbol{u}_2 = \boldsymbol{u}(s)$, $\boldsymbol{v}_1 = \boldsymbol{v}(t)$, $\boldsymbol{v}_2 = \boldsymbol{v}(s)$. We next show (ii). Let t_1', t_2' be such that $t_1 < t_1' < t_2' < t_2$. We define the operator Δ_h by

$$(40) \qquad \Delta_h\boldsymbol{g}(t) = h^{-1}[\boldsymbol{g}(t+h) - \boldsymbol{g}(t)]\,.$$

Then we have

$$\begin{aligned} (41) \qquad & \Delta_h F[\boldsymbol{u}(t)] - F[\dot{\boldsymbol{u}}(t), \boldsymbol{u}(t)] - F[\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)] \\ & = F[\Delta_h\boldsymbol{u}(t) - \dot{\boldsymbol{u}}(t), \boldsymbol{u}(t+h)] + F[\dot{\boldsymbol{u}}(t), \boldsymbol{u}(t+h) - \boldsymbol{u}(t)] \\ & \quad + F[\boldsymbol{u}(t), \Delta_h\boldsymbol{u}(t) - \dot{\boldsymbol{u}}(t)]\,. \end{aligned}$$

Since $\Delta_h\boldsymbol{u}(t) - \dot{\boldsymbol{u}}(t)$ and $\boldsymbol{u}(t+h) - \boldsymbol{u}(t)$ converge to zero as $h \to 0$ uniformly on $[t_1', t_2']$ in the norm of X_1, applying Lemma 8 ((24)), we see that each term of the right hand side of (41) tends to zero as $h \to 0$ (in X_0) uniformly on $[t_1', t_2']$. Hence the left hand side of (41) converges to zero uniformly on $[t_1', t_2']$ in the norm of X_0. Hence it is easy to see (ii). The continuity in t of $A^{1/2}\boldsymbol{u}(t)$

follows, by (13), from that of $\boldsymbol{u}(t)$ and $A\boldsymbol{u}(t)$. Hence, (iii) follows from (38) with $\boldsymbol{u}=\boldsymbol{u}(t)$, $\boldsymbol{v}=\boldsymbol{u}(s)$. Q.E.D.

2.4. Estimate of $\boldsymbol{f}$. We prove:

LEMMA 13. *Suppose that* $\boldsymbol{f}$ *satisfies the Assumption* 3. *Then* $\boldsymbol{f}(t)$ *is a uniformly bounded and continuous* X_0*-valued function of* $t \geqq 0$ *with*

$$\|\boldsymbol{f}(t)\| \leqq Mt^{-1/2} \qquad (t>0) \tag{42}$$

where M *is a constant independent of* t. *In particular,* $\boldsymbol{f} \in L^p((0, \infty); X_0)$ *for* $1 \leqq p \leqq \infty$.

PROOF. Since $\boldsymbol{f}(t)-\boldsymbol{f}(s)=\int_s^t \dot{\boldsymbol{f}}(\sigma)d\sigma$ for all s, t with $0 \leqq s \leqq t < \infty$, we have:

$$\|\boldsymbol{f}(t)\| \leqq \|\boldsymbol{f}(s)\| + \int_s^t \|\dot{\boldsymbol{f}}(\sigma)\| d\sigma\,, \tag{43}$$

showing that $\|\boldsymbol{f}(t)\|$ is uniformly bounded for $t \geqq 0$, since $t^{1/2}\dot{\boldsymbol{f}} \in L^1((0, \infty); X_0)$. $\boldsymbol{f} \in L^p((0, \infty); X_0)$ follows from the uniform boundedness of $\|\boldsymbol{f}(t)\|$ and from $\boldsymbol{f} \in L^1((0, \infty); X_0)$. By (43) with $s=0$ and with $\boldsymbol{f}$ replaced by $t^{1/2}\boldsymbol{f}(t)$, we have

$$t^{1/2}\|\boldsymbol{f}(t)\| \leqq \frac{1}{2}\int_0^t \sigma^{-1/2}\|\boldsymbol{f}(\sigma)\| d\sigma + \int_0^t \sigma^{1/2}\|\dot{\boldsymbol{f}}(\sigma)\| d\sigma\,,$$

from which (42) follows (see (8)). Q.E.D.

2.5. Uniqueness Theorem. Let us consider the initial value problem (4), (4_a), (4_b) in some finite interval $(0, T)$. Then for this problem we can define also a weak solution of (4), (4_a), (4_b) in the finite interval $(0, T)$ in the same way as in the infinite interval $(0, \infty)$, (replace $t>0$ by $0<t<T$).

We now state the uniqueness theorem, due to Sather-Serrin [**12**].

PROPOSITION 2. *Let* $\boldsymbol{u}, \boldsymbol{v}$ *be weak solutions of the initial value problem* (4), (4_a), (4_b) *in the interval* $(0, T)$. *Let the Assumptions* 1–3 *be satisfied. Suppose that*

$$\int_0^T \|\boldsymbol{v}(\cdot, t)\|_{L^s}^{s'} dt < \infty \tag{44}$$

for some pair (s, s') *of exponent with* $3s^{-1}+2(s')^{-1}=1$, *and with* $s>3$. *Suppose also that*

$$\|\boldsymbol{v}(t)\|^2 + 2\int_0^t \|\nabla \boldsymbol{v}(s)\|^2 ds + 2\int_0^t ((\boldsymbol{v}\cdot\nabla)\boldsymbol{w}, \boldsymbol{v})ds = \|\boldsymbol{v}(0)\|^2 + 2\int_0^t (\boldsymbol{f}(s), \boldsymbol{v}(s))ds$$

for $0<t<T$. *Then we have*

$$\|\boldsymbol{u}(t)-\boldsymbol{v}(t)\| \leqq \|\boldsymbol{u}(0)-\boldsymbol{v}(0)\| \exp\left\{M\int_0^t \|\boldsymbol{v}\|_{L^s}^{s'} d\sigma\right\}.$$

In particular, if $\boldsymbol{u}(0)=\boldsymbol{v}(0)$, *then* $\boldsymbol{u}=\boldsymbol{v}$ *in* $0<t<T$.

Sather-Serrin proved the above proposition for the case that $\boldsymbol{f}=0$ and $\boldsymbol{w}=0$.

The extension of their proof to the general case is, however, straight forward; see also Prodi [**10**].

§ 3. Regularity of weak solutions.

Let $\boldsymbol{u}$ be a weak solution of (4), (4_a), (4_b). Then, for the $\boldsymbol{u}$ we shall denote, in this and the subsequent sections 3-5, by $\boldsymbol{S}$ the set of all $s \geqq 0$ such that $\boldsymbol{u}(s) \in X_1$, and such that the inequality (6) holds for all $t > s$. From the definition of weak solutions, it is clear that the $\boldsymbol{S}^c$ is of the Lebesgue measure zero ($\boldsymbol{S}^c = [0, \infty) - \boldsymbol{S}$). The purpose of this section is to show:

LEMMA 14. *Let $s_0 \in \boldsymbol{S}$. Let N be any number with*

$$\|\boldsymbol{u}(s_0)\| + \|\boldsymbol{u}(s_0)\|_A + \int_0^\infty \|\boldsymbol{f}(s)\| ds + \Big(\int_0^\infty \|\boldsymbol{f}(s)\|^2 ds\Big)^{1/2} \leqq N. \tag{45}$$

We set

$$t_0 = \min\Big\{\frac{1}{2}, \Big[2M_1 B\Big(\frac{1}{8}, \frac{3}{8}\Big) + 8M_2 NB\Big(\frac{1}{8}, \frac{1}{8}\Big)\Big]^{-4}\Big\}. \tag{46}$$

Then:

(i) $\boldsymbol{u} \in C([s_0, s_0+t_0]; X_1)$,

(ii) *$\boldsymbol{u}(t)$ is X_1-continuously differentiable for t $(s_0 < t < s_0+t_0)$,*

(iii) *$\boldsymbol{u}(t) \in D(A)$ and $A\boldsymbol{u}(t)$ is X_0-continuous for t $(s_0 < t < s_0+t_0)$,*

(iv) *$\boldsymbol{u}$ satisfies the equation*

$$\frac{d\boldsymbol{u}}{dt} = -A\boldsymbol{u} - B\boldsymbol{u} - A^{1/4}F[\boldsymbol{u}(t)] + \boldsymbol{f}(t), \qquad (s_0 < t < s_0+t_0), \tag{47}$$

and the relation

$$\|\boldsymbol{u}(t)\|^2 + 2\int_s^t \|\boldsymbol{u}(\sigma)\|_A^2 d\sigma + 2\int_s^t (B\boldsymbol{u}(\sigma), \boldsymbol{u}(\sigma)) d\sigma = \|\boldsymbol{u}(s)\|^2 + 2\int_s^t (\boldsymbol{f}(\sigma), \boldsymbol{u}(\sigma)) d\sigma \tag{48}$$

for any s, t with $s_0 \leqq s \leqq t < s_0+t_0$.

To prove this lemma, we shall first construct an approximating solution of (4), (4_a), (4_b) with the initial value $\boldsymbol{u}(s_0)$. We define a sequence $\{\boldsymbol{v}_n\}$, inductively, by

$$\boldsymbol{v}_1(t) = 0;$$

$$\boldsymbol{v}_{n+1}(t) = I_1(t) + I_2(t) + I_3^n(t) + I_4^n(t)$$

where

$$\begin{aligned} &I_1(t) = e^{-tA}\boldsymbol{u}(s_0); \qquad I_2(t) = \int_0^t e^{-(t-s)A}\boldsymbol{f}(s+s_0) ds; \\ &I_3^n(t) = -\int_0^t e^{-(t-s)A} B\boldsymbol{v}_n(s) ds; \\ &I_4^n(t) = -\int_0^t A^{1/4} e^{-(t-s)A} F[\boldsymbol{v}_n(s)] ds. \end{aligned} \tag{49}$$

Then, to show the convergence of the $\{\boldsymbol{v}_n\}$, we shall prepare two lemmas.

LEMMA 15. *For* $n=1, 2, \cdots$, *the* $\boldsymbol{v}_n(t)$ *defined above has the properties;*

(a) $\boldsymbol{v}_n \in C([0, t_0]; X_1)$ *and satisfies the estimate*

$$\|A^{\alpha}\boldsymbol{v}_n(t)\| \leqq 2N \qquad (0 \leqq \alpha \leqq 1/2,\ 0 \leqq t \leqq t_0). \tag{50}$$

(b) $\boldsymbol{v}_n(t)$ *is* X_1*-continuously differentiable for* t $(0<t<t_0)$ *and satisfies the estimate*

$$t^{\alpha+1/2}\|A^{\alpha}\dot{\boldsymbol{v}}_n(t)\| \leqq 2N_1 \qquad (0 \leqq \alpha \leqq 1/2,\ 0<t<t_0). \tag{51}$$

(c) $\boldsymbol{v}_n(t) \in D(A)$ *and* $A\boldsymbol{v}_n(t)$ *is* X_0*-continuous for* t $(0<t<t_0)$. *Moreover* $A\boldsymbol{v}_n(t)$ *satisfies the estimate*

$$t^{1/2}\|A\boldsymbol{v}_n(t)\| \leqq 2N_2 \qquad (0<t<t_0) \tag{52}$$

where

$$N_1 = N+2M_1N+4M_2N^2+\sup_{t>0}\|\boldsymbol{f}(t)\|+\sup_{t>0}\Big(\int_t^{t+1}\|\dot{\boldsymbol{f}}(t)\|^2dt\Big)^{1/2},$$

$$N_2 = 4N_1+4M_1N+4M_0^2N^3+4M_0'N^2+2\sup_{s>0}\|\boldsymbol{f}(s)\|.$$

PROOF. It is clear that (a), (b) and (c) hold with $n=1$. We first note that $\boldsymbol{v}_n(t)$ is X_1-continuous (X_1-continuously differentiable) if and only if $A^{\alpha}\boldsymbol{v}_n(t)$ $(0 \leqq \alpha \leqq 1/2)$ is X_0-continuous (X_0-continuously differentiable) respectively. We suppose that (a) holds with $n=k$. By Lemma 12 and (21) $B\boldsymbol{v}_k(t)$ and $F[\boldsymbol{v}_k(t)]$ are both X_0-continuous for $0 \leqq t \leqq t_0$. By Assumption 3, $\boldsymbol{f}(t)$ is also X_0-continuous for t. Hence, by applying Lemma 5 with $\alpha=0$, $\alpha=\frac{1}{2}$ or $\alpha=\frac{3}{4}$ we see that $I_2(t)$, $I_3^k(t)$ and $I_4^k(t)$ are all X_1-continuous for t $(0 \leqq t \leqq t_0)$. Clearly, $I_1(t)$ is also X_1-continuous in t. Hence, $\boldsymbol{v}_{k+1} \in C([0, t_0]; X_1)$. We shall show (50) is true for $n=k+1$. Now, by Lemmas 1 and 2, we have: $\|A^{\alpha}I_1\| \leqq \|A^{\alpha}\boldsymbol{u}(s_0)\| \leqq \|\boldsymbol{u}(s_0)\|^{1-2\alpha}\|\boldsymbol{u}(s_0)\|_A^{2\alpha}$. Changing the variable $s \to s'=s+s_0$, and applying Lemma 4 with $\lambda=0$ and with $\boldsymbol{g}(s)=\boldsymbol{f}(s+s_0)$, we have

$$\|A^{\alpha}I_2\| \leqq \Big(\int_0^t\|\boldsymbol{f}(s+s_0)\|^2ds\Big)^{\alpha}\Big(\int_0^t\|\boldsymbol{f}(s+s_0)\|ds\Big)^{1-2\alpha}.$$

Hence, by (45), $\|A^{\alpha}I_1\|+\|A^{\alpha}I_2\| \leqq N$. By (24) and (50) with $n=k$,

$$\|F[\boldsymbol{v}_k(s)]\| \leqq M_2\|\boldsymbol{v}_k(s)\|_A^2 \leqq 4M_2N^2, \tag{53}$$

from which it follows, by Lemma 1, that

$$\|A^{\alpha+1/4}e^{-(t-s)A}F[\boldsymbol{v}_k(s)]\| \leqq (t-s)^{-\alpha-1/4}\|F[\boldsymbol{v}_k(s)]\| \leqq 4M_2N^2(t-s)^{-\alpha-1/4}.$$

Hence,

$$\begin{aligned}\|A^{\alpha}I_4^k\| &\leqq \int_0^t\|A^{\alpha+1/4}e^{-(t-s)A}F[\boldsymbol{v}_k(s)]\|ds \\ &\leqq 4M_2N^2\Big(\frac{3}{4}-\alpha\Big)^{-1}t^{-\alpha+3/4} \leqq 16M_2N^2t_0^{1/4}\end{aligned}$$

for $0 \leqq t \leqq t_0$ (<1) and $0 \leqq \alpha \leqq 1/2$. Similarly, we have $\|A^\alpha I_3^k\| \leqq 4M_1 N t_0^{1/4}$. Hence, noting $B\left(\frac{1}{8}, \frac{1}{8}\right) > B\left(\frac{1}{8}, \frac{3}{8}\right) > 2$, we have, by (46),

$$\begin{aligned}\|A^\alpha \boldsymbol{v}_{k+1}(t)\| &\leqq \|A^\alpha I_1(t)\| + \|A^\alpha I_2(t)\| + \|A^\alpha I_3^k(t)\| + \|A^\alpha I_4^k(t)\| \\ &\leqq N + 4N(M_1 + 4M_2 N) t_0^{1/4} \leqq 2N,\end{aligned}$$

showing that (a) holds with $n=k+1$. This proves (a). We next show (b). We suppose that (b) holds with $n=k$. As can be easily seen, $I_1(t)$ is X_1-continuously differentiable for $t>0$, and satisfies $\dot{I}_1(t) = Ae^{-tA}\boldsymbol{u}(s_0)$. Hence, by Lemma 1 and (45),

$$t^{\alpha+1/2}\|A^\alpha \dot{I}_1(t)\| = t^{\alpha+1/2}\|A^{\alpha+1/2}e^{-tA}A^{1/2}\boldsymbol{u}(s_0)\| \leqq \|\boldsymbol{u}(s_0)\|_A \leqq N.$$

Next, since $\boldsymbol{f}(t)$ is, by the Assumption 3, X_0-continuously differentiable for $0 \leqq t < \infty$, it follows from Lemma 6 that $A^\alpha I_2(t)$ is X_0-continuously differentiable for $0 < t < \infty$. Changing the variable $s \rightarrow s' = t-s$, and differentiating in t, we obtain

$$A^\alpha \dot{I}_2(t) = A^\alpha e^{-tA}\boldsymbol{f}(s_0) - \int_0^t A^\alpha e^{-sA}\boldsymbol{f}_s(t-s+s_0)ds (\equiv \boldsymbol{f}^{(1)}(t) + \boldsymbol{f}^{(2)}(t)).$$

By Lemma 1, $\|\boldsymbol{f}^{(1)}(t)\| \leqq t^{-\alpha}\|\boldsymbol{f}(s_0)\|$. After changing the variable $s \rightarrow s' = t-s$ again in the integrand of $\boldsymbol{f}^{(2)}(t)$, we apply Lemma 4 to $\boldsymbol{f}^{(2)}(t)$. The result is:

$$\|\boldsymbol{f}^{(2)}(t)\| \leqq \left(\int_0^t \|\dot{\boldsymbol{f}}(s+s_0)\|^2 ds\right)^\alpha \left(\int_0^t \|\dot{\boldsymbol{f}}(s+s_0)\| ds\right)^{1-2\alpha}, \qquad (0 \leqq t \leqq 1).$$

Hence, using the Schwarz inequality, we have

$$\|\boldsymbol{f}^{(2)}(t)\| \leqq \sup_{t>0}\left(\int_t^{t+1} \|\boldsymbol{f}(s+s_0)\|^2 ds\right)^{1/2}.$$

Hence, we have

$$t^{\alpha+1/2}\|A^\alpha \dot{I}_2(t)\| \leqq t^\alpha \|\boldsymbol{f}^{(1)}(t)\| + \|\boldsymbol{f}^{(2)}(t)\| \leqq N_3 \qquad (0<t<t_0<1)$$

where

$$N_3 = \sup_{0<s} \|\dot{\boldsymbol{f}}(s)\| + \sup_{t>0}\left(\int_t^{t+1} \|\dot{\boldsymbol{f}}(s)\|^2 ds\right)^{1/2}.$$

By Lemma 12, and by the assumption that (b) holds with $n=k$, it is easy to see that $B\boldsymbol{v}_k(t)$, $F[\boldsymbol{v}_k(t)]$ are in $C_1^1((0, t_0); X_0)$ (this space is defined in section 1). Applying Lemma 6 with $\beta = 1/8$, with α replaced by $\alpha + 1/4$, and with $\boldsymbol{g}(t) = F[\boldsymbol{v}_k(t)]$, we see that $A^\alpha I_4^k(t)$ is X_0-continuously differentiable for t and satisfies

$$t^{\alpha+1/4}\|\dot{I}_4^k(t)\| \leqq \sup_{0<s<t}\|F[\boldsymbol{v}_k(t)]\| + B\left(\frac{1}{8}, \frac{5}{8} - \alpha\right) \sup_{0<s<t} s\|(d/ds)F[\boldsymbol{v}_k(s)]\|.$$

Since by (24), (50) and (51),

$$s\|(d/ds)F[\boldsymbol{v}_k(s)]\| \leqq 2M_2 s\|\dot{\boldsymbol{v}}_k(s)\|_A \|\boldsymbol{v}_k(s)\|_A \leqq 8M_2 N N_1,$$

and since, by (24) and (50),

$$\|F[\boldsymbol{v}_k(s)]\| \leqq M_2 \|\boldsymbol{v}_k(s)\|_A^2 \leqq 4M_2 N^2$$

we have

$$t^{\alpha+1/4}\|\dot{I}_4^k(t)\| \leqq N_4$$

where $N_4 = 4M_2N^2 + 8M_2NN_1B\left(\frac{1}{8}, \frac{5}{8}-\alpha\right)$. Similarly, it can be shown that $A^\alpha I_3^k(t)$ is X_0-continuously differentiable for t and satisfies

$$t^\alpha \|A^\alpha \dot{I}_3^k(t)\| \leqq N_5, \qquad (0<t<t_0<1),$$

where $N_5 = 2M_1N + 2M_1N_1B\left(\frac{1}{8}, \frac{7}{8}-\alpha\right)$. Hence, we see that $\boldsymbol{v}_{k+1}(t)$ is, by (49), X_1-continuously differentiable for t and satisfies, by (46),

$$\begin{aligned} t^{\alpha+1/2}\|A^\alpha \dot{\boldsymbol{v}}_{k+1}\| &\leqq t^{\alpha+1/2}(\|A^\alpha \dot{I}_1\| + \|A^\alpha \dot{I}_2\| + \|A^\alpha \dot{I}_3^k\| + \|A^\alpha \dot{I}_4^k\|) \\ &\leqq 2N_1 \qquad (0<t<t_0<1), \end{aligned}$$

which shows (b). We finally show (c). Suppose that (c) holds with $n=k$. If $\boldsymbol{\varphi} \in D(A)$, then $(\boldsymbol{v}_{k+1}(t), \boldsymbol{\varphi})$ is, by (b), continuously differentiable for t. A direct calculation gives:

$$\begin{aligned} \text{(54)} \qquad \frac{d}{dt}(\boldsymbol{v}_{k+1}(t), \boldsymbol{\varphi}) &= -(\boldsymbol{v}_{k+1}(t), A\boldsymbol{\varphi}) - (B\boldsymbol{v}_k(t), \boldsymbol{\varphi}) \\ &\quad - (A^{1/4}F[\boldsymbol{v}_k(t)], \boldsymbol{\varphi}) + (\boldsymbol{f}(t+s_0), \boldsymbol{\varphi}). \end{aligned}$$

We estimate all terms other than $(\boldsymbol{v}_{k+1}(t), A\boldsymbol{\varphi})$. Applying Lemma 11 with $\boldsymbol{u} = \boldsymbol{v}_k(t)$, $\boldsymbol{v}=0$, and using (50), we have, by the assumption on $\boldsymbol{v}_k$,

$$\begin{aligned} t^{1/2}\|A^{1/4}F[\boldsymbol{v}_k(t)]\| &\leqq M_0 t^{1/2}\|A\boldsymbol{v}_k(t)\|^{1/2}\|\boldsymbol{v}_k(t)\|_A^{3/2} \\ &\quad + M_0' t^{1/2}\|\boldsymbol{v}_k(t)\|^{1/2}\|\boldsymbol{v}_k(t)\|_A^{3/2} \\ &\leqq 2M_0N_2^{1/2}N^{3/2} + 2M_0'N^2 \qquad (0<t<t_0<1). \end{aligned}$$

By (50), (21) and $0<t\leqq t_0<1$, we have $t^{1/2}\|B\boldsymbol{v}_k(t)\| \leqq \|B\boldsymbol{v}_k(t)\| \leqq 2M_1N$. By (51), $t^{1/2}\|\dot{\boldsymbol{v}}_{k+1}(t)\| \leqq 2N_1$. It is clear that $t^{1/2}\|\boldsymbol{f}(t)\| \leqq \sup_{s>0}\|\boldsymbol{f}(s)\|$. Hence, by (54),

$$\begin{aligned} t^{1/2}|(\boldsymbol{v}_{k+1}(t), A\boldsymbol{\varphi})| &\leqq t^{1/2}\{\|\dot{\boldsymbol{v}}_{k+1}(t)\| + \|A^{1/4}F[\boldsymbol{v}_k(t)]\| \\ &\quad + \|B\boldsymbol{v}_k(t)\| + \|\boldsymbol{f}(t)\|\}\|\boldsymbol{\varphi}\| \\ &\leqq N_6\|\boldsymbol{\varphi}\|, \qquad (0<t<t_0<1), \end{aligned}$$

where $N_6 = 2(N_1 + M_1N + M_0N^{3/2}N_2^{1/2} + M_0'N^2) + \sup_{s>0}\|\boldsymbol{f}(s)\|$. By the self-adjointness of A, this implies that $\boldsymbol{v}_{k+1}(t) \in D(A)$, and that $t^{1/2}\|A\boldsymbol{v}_{k+1}(t)\| \leqq N_5$. By elementary calculation, $N_6 \leqq 2N_2$. Hence we have (52) with $n=k+1$. Moreover, by (54),

$$\dot{\boldsymbol{v}}_{k+1}(t) = -A\boldsymbol{v}_{k+1}(t) - B\boldsymbol{v}_k(t) - A^{1/4}F[\boldsymbol{v}_k(t)] + \boldsymbol{f}(t+s_0)\,. \tag{55}$$

It remains only to prove the X_0-continuity in t of $A\boldsymbol{v}_{k+1}(t)$. $\dot{\boldsymbol{v}}_{k+1}(t)$, $B\boldsymbol{v}_k(t)$ and $\boldsymbol{f}(t)$ are all X_0-continuous, because of (a), (b) and the Assumption 3. By (b) and (c) with $n=k$, it follows from Lemma 12 that $A^{1/4}F[\boldsymbol{v}_k(t)]$ is X_0-continuous. Hence, by (55) we can see that the right side of (55), and so $A\boldsymbol{v}_{k+1}(t)$ is X_0-continuous. Thus the lemma is proved.

We next show the $\{\boldsymbol{v}_n\}$ is convergent. To this end, we set

$$a_n = \sup_{t,\alpha} \|A^\alpha \boldsymbol{v}_{n+1}(t) - A^\alpha \boldsymbol{v}_n(t)\|\,;$$

$$b_n = \sup_{t,\alpha} \{t^{\alpha+1/2}\|A^\alpha \dot{\boldsymbol{v}}_{n+1}(t) - A^\alpha \dot{\boldsymbol{v}}_n(t)\|\}\,;$$

$$c_n = \sup_{t,\alpha} \{t^{1/2}\|A\boldsymbol{v}_{n+1}(t) - A\boldsymbol{v}_n(t)\|\}\,;$$

where the supremum is taken over $0 \leqq \alpha \leqq 1/2$, $0 < t < t_0$. Then:

LEMMA 16. *We have:*

$$\sum_{n=1}^{\infty} \{a_n + b_n + c_n\} < \infty\,. \tag{56}$$

PROOF. By (49),

$$\boldsymbol{v}_{n+1}(t) - \boldsymbol{v}_n(t) = \int_0^t e^{-(t-s)A}\boldsymbol{g}_{n-1}(s)ds + \int_0^t A^{1/4}e^{-(t-s)A}\boldsymbol{h}_{n-1}(s)ds \tag{57}$$

$$(\equiv \boldsymbol{v}^{(1)}(t) + \boldsymbol{v}^{(2)}(t))$$

where

$$\boldsymbol{g}_n(t) = B(\boldsymbol{v}_n(t) - \boldsymbol{v}_{n+1}(t))\,; \qquad \boldsymbol{h}_n(t) = F[\boldsymbol{v}_n(t)] - F[\boldsymbol{v}_{n+1}(t)]\,.$$

By (25) and (50),

$$\begin{aligned} \|\boldsymbol{h}_{n-1}(t)\| &\leqq M_2\|\boldsymbol{v}_n(t) - \boldsymbol{v}_{n-1}(t)\|_A \{\|\boldsymbol{v}_{n-1}(t)\|_A + \|\boldsymbol{v}_n(t)\|_A\} \\ &\leqq 2M_2 N a_{n-1}\,. \end{aligned} \tag{58}$$

Hence, by Lemma 1,

$$\begin{aligned} \|A^{\alpha+1/4}e^{-(t-s)A}\boldsymbol{h}_{n-1}(s)\| &\leqq (t-s)^{-\alpha-1/4}\|\boldsymbol{h}_{n-1}(s)\| \\ &\leqq 4M_2N(t-s)^{-\alpha-1/4}a_{n-1} \end{aligned}$$

and so

$$\begin{aligned} \|A^\alpha \boldsymbol{v}^{(2)}(t)\| &\leqq \int_0^t \|A^{\alpha+1/4}e^{-(t-s)A}\boldsymbol{h}_{n-1}(s)\|ds \\ &\leqq 4M_2N\Big(\frac{3}{4}-\alpha\Big)^{-1} t^{-\alpha+3/4}a_{n-1} \leqq 16M_2Nt_0^{1/4}a_{n-1} \end{aligned}$$

for $0 \leqq t \leqq t_0$ (<1) and $0 \leqq \alpha \leqq 1/2$. Similarly, we have $\|A^\alpha \boldsymbol{v}^{(1)}(t)\| \leqq 2M_1 t_0^{1/2} a_{n-1}$. Hence,

$$\|A^\alpha[\boldsymbol{v}_{n+1}(t) - \boldsymbol{v}_n(t)]\| \leqq \|A^\alpha \boldsymbol{v}^{(1)}(t)\| + \|A^\alpha \boldsymbol{v}^{(2)}(t)\| \leqq \delta_1 a_{n-1} \ (0<t<t_0)$$

where $\delta_1 = 2M_1 t_0^{1/2} + 16M_2Nt_0^{1/4}$. Hence we have; $a_n \leqq \delta_1 a_{n-1}$, and so $a_n \leqq \delta_1^{n-2} a_2$.

Since, by (46) and $t_0^{1/2}<t_0^{1/4}$, $0<\delta_1<1$, we have $\sum_{n=1}^{\infty} a_n<\infty$. We next turn to the proof of $\sum_n b_n<\infty$. By (39),

$$\dot{\boldsymbol{h}}_{n-1}(s)=F[\dot{\boldsymbol{v}}_{n-1}(s)-\dot{\boldsymbol{v}}_n(s),\ \boldsymbol{v}_n(s)]+F[\dot{\boldsymbol{v}}_{n-1}(s),\ \boldsymbol{v}_{n-1}(s)-\boldsymbol{v}_n(s)]$$
$$+F[\boldsymbol{v}_{n-1}(s)-\boldsymbol{v}_n(s),\ \dot{\boldsymbol{v}}_n(s)]+F[\boldsymbol{v}_{n-1}(s),\ \dot{\boldsymbol{v}}_{n-1}(s)-\dot{\boldsymbol{v}}_n(s)]\,.$$

Applying Lemma 8 to each term on the right, and using the estimates (50), (51), we find

$$s\|\dot{\boldsymbol{h}}_{n-1}(s)\|\leqq 4M_2N_1a_{n-1}+4M_2Nb_{n-1} \qquad (0<s<t_0<1)\,.$$

Hence, using this estimate and (58), and applying Lemma 6 with $\boldsymbol{g}(s)=\boldsymbol{h}_{n-1}(s)$, with $\beta=1/8$ and with α replaced by $\alpha+1/4$, we have

$$t^{\alpha+1/4}\|A^\alpha\dot{\boldsymbol{v}}^{(2)}(t)\|\leqq|\boldsymbol{h}_{n-1}|_t+B(1/8,\,(5/8)-\alpha)|\dot{\boldsymbol{h}}_{n-1}|_{1,t}$$
$$\leqq 4M_2Na_{n-1}+4B(1/8,\,(3/8)-\alpha)M_2(N_1a_{n-1}+Nb_{n-1})\,.$$

Similarly it can be shown that

$$t^\alpha\|A^\alpha\dot{\boldsymbol{v}}^{(1)}(t)\|\leqq 2M_1a_{n-1}+M_1B(1/8,\,(7/8)-\alpha)b_{n-1}\,.$$

Hence,

$$t^{\alpha+1/2}\|A^\alpha\dot{\boldsymbol{v}}_{n+1}(t)-A^\alpha\dot{\boldsymbol{v}}_n(t)\|\leqq t^{\alpha+1/2}\|A^\alpha\dot{\boldsymbol{v}}^{(1)}(t)\|+t^{\alpha+1/2}\|A^\alpha\dot{\boldsymbol{v}}^{(2)}(t)\|$$
$$\leqq Ma_{n-1}+\delta_2b_{n-1}\,,$$
$$(0\leqq\alpha\leqq 1/2,\ 0<t<t_0)$$

and so

$$b_n\leqq Ma_{n-1}+\delta_2b_{n-1} \tag{59}$$

where M is some positive constant and $\delta_2=4t_0^{1/4}M_2NB(1/8,\,1/8)+t_0^{1/2}M_1B(1/8,\,3/8)$. Since $\delta_2<1$ by definition of t_0 and $t_0^{1/2}<t_0^{1/4}$, summing (59) from $n=2$ to k gives $(1-\delta_2)\sum_{n=2}^{k} b_n\leqq M\sum_{n=1}^{k-1} a_n+b_1$, which implies $\sum_{n=1}^{\infty} b_n<\infty$.

We finally show $\sum_n c_n<\infty$. Applying Lemma 11 with $\boldsymbol{u}=\boldsymbol{v}_n$ and $\boldsymbol{v}=\boldsymbol{v}_{n-1}$, and using the estimates (50), (51) and (52), we have, for $0<t<t_0$ (<1),

$$\begin{aligned} t^{1/2}\|A^{1/4}\boldsymbol{h}_{n-1}(t)\| &\leqq 2t^{1/4}M_0Na_{n-1}^{1/2}c_{n-1}^{1/2}+2t^{1/2}[2M_0'N+M_0N^{1/2}N_2^{1/2}]a_{n-1} \\ &\leqq Ma_{n-1}+\frac{1}{2}c_{n-1}\,, \qquad (M\text{: positive constant}) \end{aligned} \tag{60}$$

In the last inequality, we used the Schwarz inequality. Clearly $t^{1/2}\|\boldsymbol{g}_n\|\leqq Ma_n$. Hence, by (55) and (60), we have ($\boldsymbol{g}_n=B\boldsymbol{v}_n-B\boldsymbol{v}_{n+1}$)

$$t^{1/2}\|A\boldsymbol{v}_{n+1}(t)-A\boldsymbol{v}_n(t)\|\leqq t^{1/2}\|\boldsymbol{g}_{n-1}(t)\|+t^{1/2}\|\dot{\boldsymbol{v}}_{n+1}(t)-\dot{\boldsymbol{v}}_n(t)\|+t^{1/2}\|A^{1/4}\boldsymbol{h}_{n-1}(t)\|$$
$$\leqq Ma_{n-1}+b_n+\frac{1}{2}c_{n-1}$$

and hence

$$c_n \leqq Ma_{n-1} + b_n + \frac{1}{2} c_{n-1},$$

from which it easily follows that $\sum_{n=1}^{\infty} c_n < \infty$. Q. E. D.

We now prove Lemma 14. By (56), $\{\boldsymbol{v}_n\}$, $\{\dot{\boldsymbol{v}}_n\}$, $\{A\boldsymbol{v}_n\}$ are Cauchy sequences in $C([0, t_0]; X_1)$, $C([\varepsilon, t_0-\varepsilon]; X_1)$, $C([\varepsilon, t_0-\varepsilon]; X_0)$ (with the maximum norm) respectively where ε is any small positive number. Hence, if we denote the limit (in X_1) of $\boldsymbol{v}_n$ by $\boldsymbol{v}$, then $\boldsymbol{v}(t)$ is X_1-continuously differentiable for $0<t<t_0$ and $\dot{\boldsymbol{v}}(t) = \lim \dot{\boldsymbol{v}}_n(t)$ (the limit in the sense of X_1). Also, because of the closedness of the operator A, $\boldsymbol{v}(t) \in D(A)$ $(0<t<t_0)$, $A\boldsymbol{v}(t)$ is X_0-continuous for $0<t<t_0$, and $A\boldsymbol{v}(t) = \lim A\boldsymbol{v}_n(t)$ (the limit in the sense of X_0). Letting $n \to \infty$ in (55), and using (38), we find

$$\left\{ \begin{array}{l} \dot{\boldsymbol{v}}(t) = -A\boldsymbol{v}(t) - B\boldsymbol{v}(t) - P((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) + \boldsymbol{f}(t+s_0) \\ \boldsymbol{v}(0) = \boldsymbol{u}(s_0) \end{array} \right. \tag{61}$$

$(0<t<t_0)$, from which it follows that $\boldsymbol{v}$ is a weak solution of (4), (4_a), (4_b) in $(0, t_0)$ with the initial value $\boldsymbol{u}(s_0)$, and with $\boldsymbol{f}(t)$ replaced by $\boldsymbol{f}(t+s_0)$. Since $\|\boldsymbol{v}(t)\|_A$ is continuous for $0 \leqq t \leqq t_0$, and since $\|\boldsymbol{v}(t)\|_{L^6} \leqq \frac{2}{\sqrt{3}} \|\boldsymbol{v}(t)\|_A$, we see that $\boldsymbol{v}$ satisfies the estimate (44) with $s=6$, $s'=4$. Differentiating $\|\boldsymbol{v}(t)\|^2$ with respect to t, and using (37), (47), we get

$$(d/dt)\|\boldsymbol{v}(t)\|^2 = -2\|A^{1/2}\boldsymbol{v}(t)\|^2 + 2(B\boldsymbol{v}(t), \boldsymbol{v}(t)) + 2(\boldsymbol{f}(t+s_0), \boldsymbol{v}(t)).$$

Integrating both sides with respect to t, we see that (48) holds with $\boldsymbol{u}$ replaced by $\boldsymbol{v}$ for $0<s<t<t_0$. Letting $s \to 0$, $t \to t_0$, we see that (48) holds with $\boldsymbol{u}$ replaced by $\boldsymbol{v}$ for $0 \leqq s < t \leqq t_0$. On the other hand, since $s_0 \in \boldsymbol{S}$, $\boldsymbol{u}(t+s_0)$ is also a weak solution in $(0, t_0)$ with the initial value $\boldsymbol{u}(s_0)$, and with $\boldsymbol{f}(t)$ replaced by $\boldsymbol{f}(t+s_0)$. Hence, applying Proposition 2, we have $\boldsymbol{u}(t+s_0) = \boldsymbol{v}(t)$. This implies that $\boldsymbol{u}$ has the properties of $\boldsymbol{v}$ stated above. Thus Lemma 14 is completely proved.

Finally, we show

LEMMA 17. *Let t_1, t_2 be such that $0<t_1<t_2$. Let $\boldsymbol{u}$ be a weak solution of* (4), (4_a), (4_b). *Suppose that $\boldsymbol{u}(t)$ is X_1-continuous for t $(t_1<t<t_2)$. Then we have* $(t_1, t_2) \subset \boldsymbol{S}$.

PROOF. Let t_* be any number in (t_1, t_2). Let $t>t_*$. Then, since $\boldsymbol{S}^c$ is of the Lebesgue measure zero, there is a sequence $t_n \in \boldsymbol{S}$ with $t_1 < t_n < \min(t_2, t)$, and with $t_n \to t_*$. By $t_n \in \boldsymbol{S}$, (6) holds with $s=t_n$ and $t>t_n$. Since $\|\boldsymbol{u}(t)\|_A$ is continuous for t $(t_1<t<t_2)$, letting $n \to \infty$ in the inequality (6) with $s=t_n$, we see that (6) holds with $s=t_*$ and $t>t_*$. This, together with $\boldsymbol{u}(t_*) \in X_1$, gives the proof of the lemma.

§ 4. Global estimates of weak solutions.

4.1. Estimate of $\|\boldsymbol{u}(t)\|$. We begin with the so-called energy inequality.

LEMMA 18. *Let* $\boldsymbol{u}$ *be a weak solution of* (4), (4_a), (4_b). *Let* s *be such that the inequality* (6) *holds for all* $t>s$. *Then the energy inequality:*

$$\|\boldsymbol{u}(t)\|^2+(2-4C)\int_s^t\|\boldsymbol{u}(\sigma)\|_A^2 d\sigma \leqq \left(\|\boldsymbol{u}(s)\|+\int_s^t\|\boldsymbol{f}(\sigma)\|d\sigma\right)^2 \tag{62}$$

holds for all $t>s$. *In particular, we have*

$$\|\boldsymbol{u}(t)\|^2+(2-4C)\int_0^t\|\boldsymbol{u}(\sigma)\|_A^2 d\sigma \leqq K_0^2, \qquad (t>0), \tag{63}$$

where $K_0=\|\boldsymbol{u}_0\|+\int_0^\infty \|\boldsymbol{f}(\sigma)\|d\sigma$.

PROOF. By (6), (22) and the Schwarz inequality,

$$\|\boldsymbol{u}(t)\|^2+(2-4C)\int_s^t\|\boldsymbol{u}(\sigma)\|_A^2 d\sigma \leqq \|\boldsymbol{u}(s)\|^2+2\int_s^t\|\boldsymbol{f}(\sigma)\|\,\|\boldsymbol{u}(\sigma)\|d\sigma\,. \tag{64}$$

Hence the function $x(t)$ defined by

$$\begin{cases} x(t)=\|\boldsymbol{u}(t)\|^2+(2-4C)\displaystyle\int_s^t\|\boldsymbol{u}(\sigma)\|_A^2 d\sigma\,, & (t>s) \\ x(s)=\|\boldsymbol{u}(s)\|^2\,. \end{cases}$$

satisfies $x(t)\leqq x(s)+2\int_s^t\|\boldsymbol{f}(\sigma)\|x(\sigma)^{1/2}d\sigma$. Then, $x(t)$ is, by the comparison theorem (Remark stated below), dominated by the solution $\bar{x}(t)$ of the equation: $\bar{x}(t)=x(s)+2\int_s^t \|\boldsymbol{f}(\sigma)\|\bar{x}(\sigma)^{1/2}d\sigma$, which is explicitly given by $\bar{x}(t)=\left(x(s)^{1/2}+\int_s^t\|\boldsymbol{f}(\sigma)\|d\sigma\right)^2$. Hence, $x(t)\leqq\bar{x}(t)$ implies the desired estimate (62). Q.E.D.

REMARK. Here we show; $x(t)\leqq\bar{x}(t)$. We set

$$y(t)=a^2+2\int_s^t b(\sigma)x(\sigma)^{1/2}d\sigma\,;\ \bar{x}_\varepsilon(t)=\left((a^2+\varepsilon)^{1/2}+\int_s^t b(\sigma)d\sigma\right)^2, \quad \varepsilon>0\,,$$

where $a=\|\boldsymbol{u}(s)\|$, $b(t)=\|\boldsymbol{f}(t)\|$. Since $x(t)$ is a bounded measurable function of $t\geqq s$, $y(t)$ is continuous for $t\geqq s$. On the other hand, since $\bar{x}_\varepsilon(t)$ satisfies $\dot{\bar{x}}_\varepsilon(t)=2b(t)\bar{x}_\varepsilon(t)^{1/2}$, $\bar{x}_\varepsilon(t)$ satisfies: $\bar{x}_\varepsilon(t)=a^2+\varepsilon+2\int_s^t b(\sigma)\bar{x}_\varepsilon(\sigma)^{1/2}d\sigma$. We now assume that $\bar{x}_\varepsilon(t)=y(t)$ for some t $(\geqq s)$. Then there would be a t_* such that $\bar{x}_\varepsilon(t)>y(t)$ $(s<t<t_*)$ and $\bar{x}_\varepsilon(t_*)=y(t_*)$. Hence, by $x(t)\leqq y(t)\leqq\bar{x}_\varepsilon(t)$ $(s<t<t_*)$,

$$\begin{aligned} y(t_*)&=a^2+2\int_s^{t_*}b(\sigma)x(\sigma)^{1/2}d\sigma\leqq a^2+2\int_s^{t_*}b(\sigma)\bar{x}_\varepsilon(\sigma)^{1/2}d\sigma \\ &=\bar{x}_\varepsilon(t_*)-\varepsilon<\bar{x}_\varepsilon(t_*) \end{aligned}$$

which is a contradiction. Hence $y(t)<\bar{x}_\varepsilon(t)$ for all $t>s$. Letting $\varepsilon\to 0$, we get

$x(t) \leqq y(t) \leqq \bar{x}(t)$.

4.2. Estimate of $\|\nabla u(t)\|$. By Lemma 13 and (63), we can take s_1 so large that

$$\left(\int_{s_1}^{\infty}\|f(s)\|^2 ds\right)^{1/2}+M_1\left(\int_{s_1}^{\infty}\|u(s)\|_A^2 ds\right)^{1/2}<K_1 \tag{65}$$

where $K_1=(2-4C)^{1/4}\left[64\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)^{3/4}\Gamma\left(\frac{1}{4}\right)^2 K_0 M_2^2\right]^{-1}$. After choosing such a s_1, we set $s_2=s_1+K_0^2[(1-2C)K_1^2]^{-1}$. Then for some T_0 in $S\cap(s_1, s_2)$ (S is defined in the preceding section), we have

$$\|u(T_0)\|_A^2<K_1^2 . \tag{66}$$

Indeed, if $\|u(t)\|_A^2 \geqq K_1^2$ for all t in $S\cap(s_1, s_2)$, then we integrate over $S\cap(s_1, s_2)$ with respect to s:

$$\int_{S\cap(s_1,s_2)}\|u(s)\|_A^2 ds \geqq K_1^2\int_{S\cap(s_1,s_2)} ds .$$

Since S^c (and so $S^c\cap(s_1, s_2)$) is of the Lebesgue measure zero, we obtain

$$\int_{s_1}^{s_2}\|u(s)\|_A^2 ds \geqq K_1^2\int_{s_1}^{s_2} ds=K_1^2(s_2-s_1)=(1-2C)^{-1}K_0^2$$

which is a contradiction to (63).

We now show:

LEMMA 19. *Let u be a weak solution of* (4), (4_a), (4_b). *Let T_0 be as above. Then $u\in C([T_0, \infty) ; X_1)$ and satisfies the estimate*

$$\|u(t)\|_A<8K_1 \qquad (\textit{for all} \quad t>T_0) . \tag{67}$$

PROOF. Let T_* be the least upper bound for t_* ($>T_0$) such that $u(t)$ is X_1-continuous for $T_0 \leqq t<t_*$ and satisfies the estimate (67) for $T_0<t<t_*$; note $\|u(T_0)\|_A<K_1$. By Lemma 14, we have $T_*>T_0$. Suppose that T_* is finite. By Lemma 17, we have $[T_0, T_*)\subset S$. By Lemma 14, for each s_0 in (T_0, T_*), (and so $s_0\in S$) there is a t_0, depending on $\|u(s_0)\|$ and $\|u(s_0)\|_A$, such that $u(t)$ is X_1-continuous for $s_0 \leqq t \leqq s_0+t_0$. Hence, since $\|u(s_0)\|$ is, by (63), bounded by K_0, and since $\|u(s_0)\|_A$ is bounded by $8K_1$ (note $T_0<s_0<T_*$), there is a $\delta>0$, depending on K_0 and K_1, such that $u(t)$ is X_1-continuous for $T_0 \leqq t \leqq T_*+\delta$. If it is shown that $\|u(T_*)\|_A<8K_1$, we have $\|u(t)\|_A<8K_1$ for $T_*<t<T_*+\delta'$, where δ' is some positive constant. This is a contradiction to the definition of T_*. We now show $\|u(T_*)\|_A<8K_1$. For simplicity, we set

$$v(t)=u(t+T_0) \quad \text{and} \quad U_\lambda(t)=\exp[-t(A+\lambda)] , \qquad \lambda>0 .$$

Then by (47),

$$\boldsymbol{v}(t)=U_\lambda(t)\boldsymbol{u}(T_0)+\int_0^t \frac{\partial}{\partial s}[U_\lambda(t-s)\boldsymbol{v}(s)]\,ds$$

(68)
$$=U_\lambda(t)\boldsymbol{u}(T_0)+\int_0^t U_\lambda(t-s)\boldsymbol{f}(s+T_0)ds-\int_0^t U_\lambda(t-s)B\boldsymbol{v}(s)\,ds$$

$$+\lambda\int_0^t U_\lambda(t-s)\boldsymbol{v}(s)\,ds-\int_0^t A^{1/4}U_\lambda(t-s)F[\boldsymbol{v}(s)]\,ds$$

$$(\equiv \boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3+\boldsymbol{v}_4+\boldsymbol{v}_5)$$

for $0<t<T_*-T_0$. Using

(69)
$$\|A^\alpha U_\lambda(t-s)\|\leqq(t-s)^{-\alpha}e^{-\lambda(t-s)}\qquad(0\leqq\alpha\leqq 2)$$

we estimate each term on the right. Clearly, $\|\boldsymbol{v}_1\|_A\leqq\|\boldsymbol{u}(T_0)\|_A$. By (15), we have

$$\|\boldsymbol{v}_2\|_A\leqq\Big(\int_0^t\|\boldsymbol{f}(s+T_0)\|^2ds\Big)^{1/2}\leqq\Big(\int_{T_0}^\infty\|\boldsymbol{f}(s)\|^2ds\Big)^{1/2};$$

$$\|\boldsymbol{v}_3\|_A\leqq\Big(\int_0^t\|B\boldsymbol{v}(s)\|^2ds\Big)^{1/2}\leqq M_1\Big(\int_{T_0}^\infty\|\boldsymbol{u}(s)\|_A^2ds\Big)^{1/2};$$

$$\|\boldsymbol{v}_4\|_A\leqq\lambda\int_0^t\|A^{1/4}U_\lambda(t-s)A^{1/4}\boldsymbol{v}(s)\|ds$$

$$\leqq\lambda\Big(\int_0^t\|A^{1/4}U_\lambda(t-s)\|^{4/3}ds\Big)^{3/4}\Big(\int_0^t\|A^{1/4}\boldsymbol{v}(s)\|^4ds\Big)^{1/4}.$$

Since, by (69),

$$\int_0^t\|A^{1/4}U_\lambda(t-s)\|^{4/3}ds\leqq\Big(\frac{3}{4\lambda}\Big)^{2/3}\Gamma\Big(\frac{2}{3}\Big),$$

and since, by (13) and (63),

$$\int_0^t\|A^{1/4}\boldsymbol{v}(s)\|^4ds\leqq\int_0^t\|A^{1/2}\boldsymbol{v}(s)\|^2\|\boldsymbol{v}(s)\|^2ds$$

$$\leqq\int_0^t\|\boldsymbol{v}(s)\|_A^2ds\sup_{0<s<t}\|\boldsymbol{v}(s)\|^2\leqq(2-4C)^{-1}K_0^4$$

we have $\|\boldsymbol{v}_4\|_A\leqq\lambda^{1/2}\Big(\frac{9}{32-64C}\Big)^{1/4}\Gamma\Big(\frac{2}{3}\Big)^{3/4}K_0$. Similarly, by (69) and (24),

$$\|\boldsymbol{v}_5\|_A\leqq\int_0^t\|A^{3/4}U_\lambda(t-s)\|\,\|F[\boldsymbol{v}(s)]\|ds\leqq\lambda^{-1/4}\Gamma\Big(\frac{1}{4}\Big)M_2\sup_{0<s<t}\|\boldsymbol{v}(s)\|_A^2.$$

Collecting all these terms, and setting $x(t)=\sup_{0<s<t}\|\boldsymbol{v}(s)\|_A$, we obtain

(70)
$$x(t)\leqq K_2+\lambda^{1/2}K_3+\lambda^{-1/4}K_4x(t)^2$$

where

$$K_2=\|\boldsymbol{u}(T_0)\|_A+\Big(\int_{T_0}^\infty\|\boldsymbol{f}(s)\|^2ds\Big)^{1/2}+M_1\Big(\int_{T_0}^\infty\|\boldsymbol{u}(s)\|_A^2ds\Big)^{1/2};$$

$$K_3=\left(\frac{9}{32-64C}\right)^{1/4}\Gamma\left(\frac{2}{3}\right)^{3/4}K_0\,; \qquad K_4=\Gamma\left(\frac{1}{4}\right)M_2\,.$$

In particular, if we take $\lambda=K_1^2K_3^{-2}$ in (70), then

$$\begin{aligned} x(t) &\leqq K_2+K_1+K_1^{-1/2}K_3^{1/2}K_4x(t)^2 \\ &\leqq K_2+K_1+64K_1K_1^{1/2}K_3^{1/2}K_4\,. \end{aligned} \tag{71}$$

Here we used the assumption that (67) holds for $T_0<t<T_*$, i.e., $x(t)<8K_1$ $(0<t<T_*-T_0)$. Since $K_2<2K_1$ by (65) and (66), and since $128K_1K_3K_4^2=1$ by direct calculation, we have, by (71),

$$\sup_{0<s<t}\|\boldsymbol{u}(s+T_0)\|_A=x(t)<7K_1 \qquad (0<t<T_*-T_0)\,.$$

Letting $t\to T_*-T_0$, we get $\|\boldsymbol{u}(T_*)\|_A\leqq 7K_1<8K_1$ which is a contradiction to the definition of T_*. Hence T_* must be the infinity. This proves the lemma.

Combining Lemma 19 just proved, with Proposition 1, Lemma 14 and Lemma 17, we can prove, in the standard argument, the following

LEMMA 20. *Let* $\boldsymbol{u}$ *be a weak solution of* (4), (4_a), (4_b). *Then there is a* $T_0>0$ *such that* $\boldsymbol{u}(t)\in W_2^2(\mathcal{E})\cap X_1$, *is* X_1*-continuously differentiable for* t $(t>T_0)$, *and satisfies* (47).

REMARK. The T_0 is estimated by $s_1+K_0^2[(1-2C)K_1^2]^{-1}$, where s_1 and K_1 are defined in (65).

§5. Decay of weak solutions.

We now proceed to the final step: the decay problem of weak solutions. We first show $\|\boldsymbol{u}(t)\|_A\to 0$ $(t\to\infty)$. To this end, we prepare the following lemma.

LEMMA 21. *Let* $\lambda>0$. *Let* α *be a number with* $0\leqq\alpha<1$. *Let* $\boldsymbol{g}\in L^p((0,\infty)\,;X_0)$ *for some* p *with* $(1-\alpha)^{-1}<p<\infty$. *Then the function* $\boldsymbol{h}_\alpha(t)$ *defined by*

$$\boldsymbol{h}_\alpha(t)=\int_0^t A^\alpha e^{-(t-s)[A+\lambda]}\boldsymbol{g}(s)\,ds \tag{72}$$

converges to zero as $t\to\infty$ *in the norm of* X_0.

PROOF. We set

$$\boldsymbol{h}^{(1)}(t)=\int_0^N A^\alpha e^{-(t-s)[A+\lambda]}\boldsymbol{g}(s)\,ds$$

$$\boldsymbol{h}^{(2)}(t)=\int_N^t A^\alpha e^{-(t-s)[A+\lambda]}\boldsymbol{g}(s)\,ds\,.$$

Then, by (69),

$$\|\boldsymbol{h}^{(1)}(t)\|\leqq\int_0^N\|A^\alpha e^{-(t-s)[A+\lambda]}\boldsymbol{g}(s)\|\,ds\leqq\int_0^N(t-s)^{-\alpha}e^{-\lambda(t-s)}\|\boldsymbol{g}(s)\|\,ds$$

$$\leqq \int_0^N (t-s)^{-\alpha q} e^{-q\lambda(t-s)} ds)^{1/q} \Big(\int_0^N \|\boldsymbol{g}(s)\|^p ds\Big)^{1/p}, \quad (p^{-1}+q^{-1}=1),$$

which, for each fixed N, tends to zero as $t\to\infty$. Changing the variable $s\to s'=s-N$ in the integrand of $\boldsymbol{h}^{(2)}(t)$, noting $g\in L^p$, and using the Schwarz inequality, we have, by (69),

$$\|\boldsymbol{h}^{(2)}(t)\| \leqq \Big(\int_0^{t-N} s^{-\alpha q} e^{-\lambda q s} ds\Big)^{1/q} \Big(\int_0^{t-N} \|\boldsymbol{g}(s+N)\|^p ds\Big)^{1/p}$$

$$\leqq M\Big(\int_N^\infty \|\boldsymbol{g}(s)\|^p ds\Big)^{1/p}$$

whence $\|\boldsymbol{h}^{(2)}(t)\|\to 0$ uniformly in t as $N\to\infty$. Hence, $\|\boldsymbol{h}(t)\|\to 0$ for $t\to\infty$, since $\boldsymbol{h}(t)=\boldsymbol{h}^{(1)}(t)+\boldsymbol{h}^{(2)}(t)$. Q. E. D.

LEMMA 22. *Let* $\boldsymbol{u}$ *be a weak solution of* (4), (4_a), (4_b). *Then* $\|\boldsymbol{u}(t)\|_A$ *tends to zero as* $t\to\infty$.

PROOF. Let T_0, $\boldsymbol{v}_j$ be as in (68). Clearly $\|\boldsymbol{v}_1\|_A \leqq \|A^{1/2}e^{-tA}\boldsymbol{u}(T_0)\| \leqq t^{-1/2}\|\boldsymbol{u}(T_0)\|$, which tends to zero ($t\to\infty$). From Lemma 13, (24), (63) and (67) it follows that $\boldsymbol{f}(s+T_0)$, $B\boldsymbol{u}(s+T_0)$, $A^{1/2}\boldsymbol{u}(s+T_0)$, $F[\boldsymbol{u}(s+T_0)]\in L^p((0,\infty);X)$ (for all p with $4<p\leqq\infty$). Applying Lemma 21 with $\alpha=0$ and with $\boldsymbol{g}(s)=A^{1/2}\boldsymbol{u}(s+T_0)$, we get $\|\boldsymbol{v}_4\|_A\to 0$. Similarly, applying Lemma 21 with $\alpha=1/2$ and $\alpha=3/4$, we see that $\|\boldsymbol{v}_2\|_A$, $\|\boldsymbol{v}_3\|_A$ and $\|\boldsymbol{v}_5\|_A$ tend to zero as $t\to\infty$. This proves the Lemma. Using this lemma just proved, we shall give the fundamental estimate for $\boldsymbol{u}$ similar to (62), which state:

LEMMA 23. *Let* $\boldsymbol{u}$ *be a weak solution of* (4), (4_a), (4_b). *Then there is a* $T_1(>T_0)$ *such that the estimate*

$$\|\dot{\boldsymbol{u}}(t)\|^2+(1-2C)\int_s^t \|\dot{\boldsymbol{u}}(\sigma)\|_A^2 d\sigma \leqq \Big\{\|\dot{\boldsymbol{u}}(s)\|+\int_s^t \|\dot{\boldsymbol{f}}(\sigma)\| d\sigma\Big\}^2 \tag{73}$$

holds for all s, t *with* $T_1<s<t$.

PROOF. We set $\boldsymbol{u}_h(t)=\Delta_h\boldsymbol{u}(t)$ and $\boldsymbol{f}_h(t)=\Delta_h\boldsymbol{f}(t)$ (for the definition of Δ_h, see (40)). We then have

$$\dot{\boldsymbol{u}}_h(t) = -A\boldsymbol{u}_h(t)-B\boldsymbol{u}_h(t)-P(\boldsymbol{u}(t+h)\cdot\nabla)\boldsymbol{u}_h(t)$$

$$-P(\boldsymbol{u}_h(t)\cdot\nabla)\boldsymbol{u}(t)+\boldsymbol{f}_h(t).$$

Hence,

$$\frac{d}{dt}\|\boldsymbol{u}_h(t)\|^2 = -2\|\nabla\boldsymbol{u}_h(t)\|^2-2(B\boldsymbol{u}_h(t),\boldsymbol{u}_h(t))-2((\boldsymbol{u}(t+h)\cdot\nabla)\boldsymbol{u}_h(t),\boldsymbol{u}_h(t))$$

$$-2((\boldsymbol{u}_h(t)\cdot\nabla)\boldsymbol{u}(t),\boldsymbol{u}_h(t))+2(\boldsymbol{f}_h(t),\boldsymbol{u}_h(t)).$$

$$(\equiv \boldsymbol{w}_1+\boldsymbol{w}_2+\boldsymbol{w}_3+\boldsymbol{w}_4+\boldsymbol{w}_5).$$

We shall estimate each term on the right. By (22), $\boldsymbol{w}_2\leqq 4C\|\nabla\boldsymbol{u}_h(t)\|^2$. By (37), $\boldsymbol{w}_3=0$. By partial integration,

$$\begin{aligned} \boldsymbol{w}_4 &= 2(\boldsymbol{u}(t), (\boldsymbol{u}_h(t)\cdot\nabla)\boldsymbol{u}_h(t)) \\ &\leqq 2\|\boldsymbol{u}(t)\|_{L^3}\|\boldsymbol{u}_h(t)\|_{L^6}\|\nabla\boldsymbol{u}_h(t)\| \\ &\leqq 2\Big(\frac{4}{3}\Big)^{3/4}\|\boldsymbol{u}(t)\|^{1/2}\|\nabla\boldsymbol{u}(t)\|^{1/2}\|\nabla\boldsymbol{u}_h(t)\|^2 . \end{aligned}$$

Here we used the inequalities: $\|\boldsymbol{u}\|_{L^3}\leqq\|\boldsymbol{u}\|^{1/2}\|\boldsymbol{u}\|^{1/2}_{L^6}$ and $\|\boldsymbol{u}\|_{L^6}\leqq\frac{2}{\sqrt{3}}\|\nabla\boldsymbol{u}\|$. Since $\|\boldsymbol{u}(t)\|$ is, by (63), uniformly bounded for $t>T_0$, and since $\|\nabla\boldsymbol{u}(t)\|\to 0$ for $t\to\infty$ by Lemma 22, we can take T_1 so large that

$$2\Big(\frac{4}{3}\Big)^{3/4}\|\boldsymbol{u}(t)\|^{1/2}\|\nabla\boldsymbol{u}(t)\|^{1/2}<(1-2C)$$

for all $t>T_1$. Hence, $\boldsymbol{w}_4\leqq(1-2C)\|\nabla\boldsymbol{u}_h(t)\|^2$ $(t>T_1)$. By the Schwarz inequality $\boldsymbol{w}_5\leqq 2\|\boldsymbol{f}_h\|\|\boldsymbol{u}_h\|$.

Consequently,

$$\frac{d}{dt}\|\boldsymbol{u}_h(t)\|^2\leqq-(1-2C)\|\nabla\boldsymbol{u}_h(t)\|^2+2\|\boldsymbol{f}_h(t)\|\|\boldsymbol{u}_h(t)\| \qquad (t>T_1).$$

Hence, if s is any fixed number $>T_1$, then the function $x(t)$ defined by

$$x(t)=\|\boldsymbol{u}_h(t)\|^2+(1-2C)\int_s^t\|\nabla\boldsymbol{u}_h(\sigma)\|^2d\sigma \qquad (t>s)$$

satisfies $\dot{x}(t)\leqq 2\|\boldsymbol{f}_h(t)\|x(t)^{1/2}$. Then, the solution $x(t)$ is dominated by the solution $\bar{x}(t)$ of the differential equation: $(d/dt)\bar{x}(t)=2\|\boldsymbol{f}_h(t)\|(\bar{x}(t))^{1/2}$ with the initial value $\bar{x}(s)=x(s)$. Since $\bar{x}(t)$ is explicitly given by: $\bar{x}(t)=\Big\{\|\boldsymbol{u}_h(s)\|+\int_s^t\|\boldsymbol{f}_h(\sigma)\|d\sigma\Big\}^2$, we have

$$\|\boldsymbol{u}_h(t)\|^2+(1-2C)\int_s^t\|\nabla\boldsymbol{u}_h(\sigma)\|^2d\sigma\leqq\Big\{\|\boldsymbol{u}_h(s)\|+\int_s^t\|\boldsymbol{f}_h(\sigma)\|d\sigma\Big\}^2. \tag{74}$$

By Lemma 20, it is easy to see that $\|\boldsymbol{u}_h(t)\|\to\|\dot{\boldsymbol{u}}(t)\|$, and $\|\nabla\boldsymbol{u}_h(t)\|\to\|\nabla\dot{\boldsymbol{u}}(t)\|$ uniformly on any compact interval in (T_0,∞). From the Assumption 2, it follows that $\int_s^t\|\boldsymbol{f}_h(\sigma)\|d\sigma\to\int_s^t\|\dot{\boldsymbol{f}}(\sigma)\|d\sigma$ for any s, t with $T_0<s<t<\infty$. Hence, letting $h\to 0$ in (74), we get the desired estimate. Q. E. D.

As the first consequence of (73), we have:

LEMMA 24. *Let* $\boldsymbol{u}$ *be a weak solution of* (4), (4_a), (4_b). *Then* $\|\dot{\boldsymbol{u}}(t)\|$ *is square-integrable over* (T_0+1,∞) *with respect to* t.

PROOF. By (47),

$$\begin{aligned} (\dot{\boldsymbol{u}}(t),\dot{\boldsymbol{u}}(t)) &= -(\dot{\boldsymbol{u}}(t), A\boldsymbol{u}(t))-(\dot{\boldsymbol{u}}(t), B\boldsymbol{u}(t)) \\ &\quad -(\dot{\boldsymbol{u}}(t), P[(\boldsymbol{u}(t)\cdot\nabla)\boldsymbol{u}(t)])+(\dot{\boldsymbol{u}}(t),\boldsymbol{f}(t)) \\ &\quad (\equiv\varphi_1+\varphi_2+\varphi_3+\varphi_4). \end{aligned}$$

We shall estimate each term on the right. We have:

$$\varphi_1(t) = -(A^{1/2}\dot{\boldsymbol{u}}(t), A^{1/2}\boldsymbol{u}(t)) \leqq \|\dot{\boldsymbol{u}}(t)\|_A^2 + \|\boldsymbol{u}(t)\|_A^2 ;$$

$$\varphi_2(t) \leqq M_1^2 \|\boldsymbol{u}(t)\|_A^2 + \frac{1}{4} \|\dot{\boldsymbol{u}}(t)\|^2 \qquad [\text{by (21)}] .$$

By (24) and (13),

$$\begin{aligned}\varphi_3(t) &= (A^{1/4}\dot{\boldsymbol{u}}(t), F[\boldsymbol{u}(t), \boldsymbol{u}(t)]) \leqq \|A^{1/4}\dot{\boldsymbol{u}}(t)\| \, \|F[\boldsymbol{u}(t)]\| \\ &\leqq M_2 \|\dot{\boldsymbol{u}}(t)\|^{1/2} \|\dot{\boldsymbol{u}}(t)\|_A^{1/2} \|\boldsymbol{u}(t)\|_A^2 \\ &\leqq \frac{1}{4} \|\dot{\boldsymbol{u}}(t)\|^2 + \|\dot{\boldsymbol{u}}(t)\|_A^2 + M_2^2 \|\boldsymbol{u}(t)\|_A^4 .\end{aligned}$$

By the Schwarz inequality,

$$\varphi_4(t) \leqq \frac{1}{4} \|\dot{\boldsymbol{u}}\|^2 + \|\boldsymbol{f}(t)\|^2 .$$

Hence,

$$\|\dot{\boldsymbol{u}}(t)\|^2 \leqq M \|\boldsymbol{u}(t)\|_A^2 + M \|\dot{\boldsymbol{u}}(t)\|_A^2 + M \|\boldsymbol{u}(t)\|_A^4 + M \|\boldsymbol{f}(t)\|^2$$

where M is constant independent of t. The second term on the right is integrable over (T_1, ∞) by Lemma 23. The remaining terms on the right are, by (63), (67) and the Assumption 3, integrable over (T_1+1, ∞). Hence, $\|\dot{\boldsymbol{u}}(t)\|^2$ is integrable over (T_1+1, ∞). By the continuity in $t > T_0$ of $\|\dot{\boldsymbol{u}}(t)\|$, the lemma is proved.

Using (73), we shall now prove:

LEMMA 25. *Let* $\boldsymbol{u}$ *be a weak solution of* (4), (4_a), (4_b). *Then the estimates*

$$\|\boldsymbol{u}(t)\|_A \leqq M t^{-1/4} ; \tag{75}$$

$$\|\dot{\boldsymbol{u}}(t)\| \leqq M t^{-1/2} ; \tag{76}$$

$$\|A\boldsymbol{u}(t)\| \leqq M t^{-1/4} ; \tag{77}$$

$$\sup_{x \in \mathcal{E}} |\boldsymbol{u}(x, t)| \leqq M t^{-1/8} \tag{78}$$

hold for all $t > T_0+1$, *where* M *is a constant independent of* t.

PROOF. To prove the lemma, it is sufficient to show that the estimates hold for large t, since $\|\boldsymbol{u}\|_A$, $\|\dot{\boldsymbol{u}}\|$, $\|A\boldsymbol{u}\|$, and $\sup|\boldsymbol{u}|$ in (75–78) are all continuous for $t > T_0$. Integrating both sides of (73) over (T_1, t) with respect to s, we have

$$\begin{aligned}(t-T_1)\|\dot{\boldsymbol{u}}(t)\|^2 &\leqq 2\int_{T_1}^t \|\dot{\boldsymbol{u}}(s)\|^2 ds + 2\int_{T_1}^t ds \Big(\int_s^t \|\dot{\boldsymbol{f}}(\sigma)\| d\sigma\Big)^2 \\ &\leqq 2\int_{T_1}^\infty \|\dot{\boldsymbol{u}}(s)\|^2 ds + 4\Big(\int_{T_1}^\infty \sigma^{1/2} \|\dot{\boldsymbol{f}}(\sigma)\| d\sigma\Big)^2 .\end{aligned}$$

The right side is, by Lemma 24 and (5), finite. This shows (76). Differentiating $\|\boldsymbol{u}(t)\|^2$ with respect to t, and using (22) and (37), we have:

$$(\dot{\boldsymbol{u}}, \boldsymbol{u}) \leqq -(A\boldsymbol{u}, \boldsymbol{u})+2C\|\boldsymbol{u}\|_A^2+(\boldsymbol{f}, \boldsymbol{u})$$
$$= -(1-2C)\|\boldsymbol{u}\|_A^2+(\boldsymbol{f}, \boldsymbol{u}) .$$

Hence, by the Schwarz inequality,

$$\|\boldsymbol{u}(t)\|_A^2 \leqq (1-2C)^{-1}(\|\dot{\boldsymbol{u}}(t)\|+\|\boldsymbol{f}(t)\|)\|\boldsymbol{u}(t)\| .$$

By (63), (76) and (42), the right hand side of the above inequality is $O(t^{-1/2})$ (as $t\to\infty$), showing (75). By (47),

$$\|A\boldsymbol{u}\| \leqq \|\dot{\boldsymbol{u}}\|+\|B\boldsymbol{u}\|+\|A^{1/4}F[\boldsymbol{u}]\|+\|\boldsymbol{f}\| . \tag{79}$$

Since, by (38),

$$\|A^{1/4}F[\boldsymbol{u}]\| \leqq M_0\|A\boldsymbol{u}\|^{1/2}\|\boldsymbol{u}\|_A^{3/2}+M_0'\|\boldsymbol{u}\|^{1/2}\|\boldsymbol{u}\|_A^{3/2}$$
$$\leqq \frac{1}{2}\|A\boldsymbol{u}\|+\frac{1}{2}M_0^2\|\boldsymbol{u}\|_A^3+M_0'\|\boldsymbol{u}\|^{1/2}\|\boldsymbol{u}\|_A^{3/2} ,$$

we have

$$\|A\boldsymbol{u}\| \leqq 2\|\dot{\boldsymbol{u}}\|+2\|B\boldsymbol{u}\|+M_0^2\|\boldsymbol{u}\|_A^3+2M_0'\|\boldsymbol{u}\|^{1/2}\|\boldsymbol{u}\|_A^{3/2}+2\|\boldsymbol{f}\| .$$

By (75), (76), (63) and (42), the right side is $O(t^{-1/4})$ (as $t\to\infty$), showing (77). Finally, (78) follows from (77), (75) and (11), i.e.

$$|\boldsymbol{u}(x, t)| \leqq \text{const.}\, t^{-1/4}+\text{const.}\, M_0' t^{-1/8} . \tag{80}$$

This completes the proof of the lemma.

Now, Lemma 20, together with Lemma 25 just proved, gives the theorem. Thus the theorem is proved.

Appendix.

PROOF OF PROPOSITION 1.

We begin with the proof of (i), which is an easy consequence of the theorems on the interior regularity of solutions of the Stokes problem for the exterior domain, and on the regularity up to the boundary of solutions for the interior problem. Let $\mathcal{E}_0$ be a subdomain of $\mathcal{E}$ with $\mathcal{E}_0\subset\mathcal{E}$, and such that $\mathcal{E}-\mathcal{E}_0$ is a bounded domain bounded by disjoint compact Lyapounov surfaces of type $\Lambda_{2,h}$ $(0<h<1)$. We denote the boundary of $\mathcal{E}_0$ by Σ_0. Let $\boldsymbol{u}$ be any function in $D(A)$. Let $\boldsymbol{g}=A\boldsymbol{u}$. Then we have $(\nabla\boldsymbol{u}, \nabla\boldsymbol{\varphi})=(\boldsymbol{g}, \boldsymbol{\varphi})$ (for all $\boldsymbol{\varphi}$ in X_1). By the interior regularity theorem (see Ladyzhenskaya [**9**; p. 40]), $\boldsymbol{u}\in W_2^2(\mathcal{E}')$ for any subdomain $\mathcal{E}'$ of $\mathcal{E}$ with $\overline{\mathcal{E}'}\subset\mathcal{E}$. Since $\boldsymbol{u}\in W_2^2(\mathcal{E}')$ and $\operatorname{div}\boldsymbol{u}=0$, we have $\boldsymbol{u}\in W_2^{3/2}(\Sigma_0)$ and satisfies the relation $\oint_{\Sigma_0}\boldsymbol{u}\boldsymbol{n}d\sigma=0$ (for the definition of $W_2^{3/2}(\Sigma_0)$ see, e.g., Ladyzhenskaya [**9**; p. 65]). Then Ladyzhenskaya [**9**; p. 24] constructed a solenoidal vector field $\boldsymbol{a}(x)\in W_2^2(\mathcal{E}-\mathcal{E}_0)$ which takes $\boldsymbol{a}|_{\Sigma_0}=\boldsymbol{u}|_{\Sigma_0}$, and $\boldsymbol{a}|_{\Sigma}=0$. We now consider the Stokes problem in $\mathcal{E}-\mathcal{E}_0$:

$$\begin{cases} -\Delta \boldsymbol{u}_c - \nabla \boldsymbol{p}_c = \boldsymbol{g} - \Delta \boldsymbol{a}\,; \qquad \operatorname{div} \boldsymbol{u}_c = 0 \ (x \in \mathcal{E} - \mathcal{E}_0) \\ \boldsymbol{u}_c|_{\Sigma_0} = \boldsymbol{u}_c|_{\Sigma} = 0\,. \end{cases}$$

It is well-known that this problem has a unique solution $\boldsymbol{u}_c$ in $W_2^2(\mathcal{E}-\mathcal{E}_0)$, since $\boldsymbol{g}-\Delta\boldsymbol{a} \in L^2(\mathcal{E}-\mathcal{E}_0)$ and since $\mathcal{E}-\mathcal{E}_0$ is a bounded domain with smooth boundary. On the other hand, $\boldsymbol{v}=\boldsymbol{u}-\boldsymbol{a}$ is in $\mathring{W}_2^1(\mathcal{E}-\mathcal{E}_0)$, $\operatorname{div} \boldsymbol{v}=0$ and satisfies the equation $(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi})=(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi})-(\nabla \boldsymbol{a}, \nabla \boldsymbol{\varphi})=(\boldsymbol{g}-\Delta \boldsymbol{a}, \boldsymbol{\varphi})$ (for all $\boldsymbol{\varphi} \in \mathring{W}_2^1(\mathcal{E}-\mathcal{E}_0)$ with $\operatorname{div} \boldsymbol{\varphi}=0$). Clearly, $\boldsymbol{u}_c$ satisfies $(\nabla \boldsymbol{u}_c, \nabla \boldsymbol{\varphi})=(\boldsymbol{g}-\Delta \boldsymbol{a}, \boldsymbol{\varphi})$. Hence, $(\nabla(\boldsymbol{u}_c-\boldsymbol{v}), \nabla \boldsymbol{\varphi})=0$ (for all $\boldsymbol{\varphi} \in \mathring{W}_2^1(\mathcal{E}-\mathcal{E}_0)$ with $\operatorname{div} \boldsymbol{\varphi}=0$), from which it follows that $\boldsymbol{u}_c=\boldsymbol{v}$. This implies that $\boldsymbol{v} \in W_2^2(\mathcal{E}-\mathcal{E}_0)$, and hence (because of $\boldsymbol{a} \in W_2^2(\mathcal{E}-\mathcal{E}_0)$) that $\boldsymbol{u} \in W_2^2(\mathcal{E}-\mathcal{E}_0)$. Since $\boldsymbol{u} \in W_2^2(\mathcal{E}') \cap W_2^2(\mathcal{E}-\mathcal{E}_0)$ for any subdomain $\mathcal{E}_0$ and $\mathcal{E}'$ with the properties mentioned above, we have $\boldsymbol{u} \in W_2^2(\mathcal{E})$. Since $D(A) \subset X_1$, we have $D(A) \subset W_2^2(\mathcal{E}) \cap X_1$. By definition, $D(A) \supset W_2^2(\mathcal{E}) \cap X_1$. Hence we have (i). Since A is a Friedrichs extension of $\mathcal{A}\boldsymbol{u}=-P\Delta \boldsymbol{u}$ with $D(\mathcal{A})=W_2^2(\mathcal{E}) \cap X_1$, and since $D(A)=D(\mathcal{A})$, we have $A=\mathcal{A}$. We next show (iii). $W_2^2(\mathcal{E}) \cap X_1$ is the closed subspace of $W_2^2(\mathcal{E})$, which we denote by $X_{1,2}$. Then $X_{1,2}$ is a Hilbert space with the scalar product in $W_2^2(\mathcal{E})$. Since $\|(1+A)\boldsymbol{u}\| \leqq \|\boldsymbol{u}\|+\|P\Delta \boldsymbol{u}\| \leqq \|\boldsymbol{u}\|+\|\Delta \boldsymbol{u}\| \leqq \|\boldsymbol{u}\|_{X_{1,2}}$, it follows from (i) that $1+A$ is a bounded operator from $X_{1,2}$ to X_0. Since $1+A$ is one to one and onto, it follows from the closed graph theorem (see, e.g., Yosida [**15**; p. 79]) that $(1+A)^{-1}$ is a bounded operator from X_0 to $X_{1,2}$. Hence, if $\boldsymbol{u} \in D(A)$, then, using the relation $\boldsymbol{u}=(1+A)^{-1}(\boldsymbol{u}+A\boldsymbol{u})$, we have

$$\|\boldsymbol{u}\|_{X_{1,2}}=\|(1+A)^{-1}(\boldsymbol{u}+A\boldsymbol{u})\|_{X_{1,2}} \leqq M\|\boldsymbol{u}+A\boldsymbol{u}\|_{X_0} \leqq M\|\boldsymbol{u}\|_{X_0}+M\|A\boldsymbol{u}\|_{X_0}$$

where M is a constant independent of $\boldsymbol{u}$. This shows (10). We finally show (11). We follow the notations in the proof of Lemma 8. Let $\boldsymbol{u} \in D(A)$. Let $\varepsilon>0$. We set $\boldsymbol{g}=(\varepsilon+L)^{1/2}\boldsymbol{u}$; note $D(A) \subset D(L)$. Since $(\varepsilon+L)^{-1/2}$ is the integral operator with the kernel $G^{(1/2)}(x, y\,; \varepsilon)$, we have

$$\boldsymbol{u}(x)=\int_{\mathcal{E}} G^{(1/2)}(x, y\,; \varepsilon)\boldsymbol{g}(y)dy\,.$$

Hence, by (30),

$$|\boldsymbol{u}(x)| \leqq \int_{\mathcal{E}} |G^{(1/2)}(x, y\,; \varepsilon)|\,|\boldsymbol{g}(y)|\,dy \leqq \frac{1}{2\pi^2}\int_{\mathcal{E}} \frac{|\boldsymbol{g}(y)|}{|x-y|^2}\,dy\,.$$

We devide the integration domain $\mathcal{E}$ into two parts;

$$[\mathrm{I}]:\ |x-y|>r\,, \qquad [\mathrm{II}]:\ |x-y| \leqq r \qquad (r>0)\,.$$

Then, by the Schwarz inequality,

$$\int_{[\mathrm{I}]} \frac{|\boldsymbol{g}(y)|}{|x-y|^2}\,dy \leqq \Big(\int_{[\mathrm{I}]} \frac{dy}{|x-y|^4}\Big)^{1/2}\Big(\int_{\mathcal{E}} |\boldsymbol{g}(y)|^2 dy\Big)^{1/2}$$
$$=|\omega_2|^{1/2} r^{-1/2}\|\boldsymbol{g}\|$$

($|\omega_2|$; the surface area of the 3-dimensional sphere). Also,

$$\int_{[II]} \frac{|\boldsymbol{g}(y)|}{|x-y|^2} dy \leqq \Big(\int_{[II]} \frac{dy}{|x-y|^{12/5}}\Big)^{5/6} \Big(\int_{\mathcal{E}} |\boldsymbol{g}(y)|^6 dy\Big)^{1/6}$$

$$= |\omega_2|^{5/6}(5/3)^{5/6} r^{1/2} \|\boldsymbol{g}\|_{L6} .$$

Hence,

$$|\boldsymbol{u}(x)| \leqq |\omega_2|^{1/2} r^{-1/2} \|\boldsymbol{g}\| + |\omega_2|^{5/6}(5/3)^{5/6} r^{1/2} \|\boldsymbol{g}\|_{L6} .$$

If we take $r = |\omega_2|^{-1/3}(5/3)^{-5/6}\|\boldsymbol{g}\|/\|\boldsymbol{g}\|_{L6}$, then we have

(81) $$|\boldsymbol{u}(x)| \leqq (\sqrt{2}\pi)^{-2} |\omega_2|^{2/3} (5/3)^{5/12} \|\boldsymbol{g}\|^{1/2} \|\boldsymbol{g}\|_{L6}^{1/2} .$$

On the other hand, we have

$$\|\boldsymbol{g}\|_{L6} \leqq \frac{2}{\sqrt{3}} \|\nabla \boldsymbol{g}\| = \frac{2}{\sqrt{3}} \|L^{1/2}\boldsymbol{g}\| = \frac{2}{\sqrt{3}} \|L^{1/2}(L+\varepsilon)^{1/2}\boldsymbol{u}\|$$

and

$$\|\boldsymbol{g}\| = \|(L+\varepsilon)^{1/2}\boldsymbol{u}\| .$$

Substituting these inequalities into (81), and then letting $\varepsilon \to 0$, we have

$$|\boldsymbol{u}(x)| \leqq (\sqrt{2}\pi)^{-2} |\omega_2|^{2/3} (5/3)^{5/12} (4/3)^{1/4} \|L\boldsymbol{u}\|^{1/2} \|L^{1/2}\boldsymbol{u}\|^{1/2} .$$

Noting the well-known fact that $D(L) = W_2^2(\mathcal{E}) \cap \mathring{W}_2^1(\mathcal{E})$, $L\boldsymbol{u} = -\Delta \boldsymbol{u}$, and $\|L^{1/2}\boldsymbol{u}\| = \|\nabla \boldsymbol{u}\|$, we have (11), by (9) and (10).

REMARK. For $\boldsymbol{f} \in X_0$, the function $\boldsymbol{u} = (\lambda + A)^{-1}\boldsymbol{f}$, $(\lambda > 0)$, is in $W_2^2(\mathcal{E})$, and is the solution of the Stokes problem:

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \\ \operatorname{div} \boldsymbol{u} = 0 \qquad (x \in \mathcal{E}) \\ \quad \boldsymbol{u} = 0 \qquad (\text{on } \Sigma) . \end{cases}$$

Using the Green function, we can write $\boldsymbol{u}$ in the form

$$\boldsymbol{u}(x) = \int_{\mathcal{E}} K(x, y ; \lambda) \boldsymbol{f}(y) dy .$$

If it is shown that the kernel $K(x, y ; \lambda)$ satisfies the estimate

(82) $$|K(x, y ; \lambda)| \leqq \text{const.} \frac{e^{-\sqrt{\lambda}|x-y|}}{|x-y|} ,$$

which implies $|K^{(1/2)}(x, y ; \lambda)| \leqq \text{const.} |x-y|^{-2}$, then, in the same way as L, we can show that

(83) $$|\boldsymbol{u}(x)| \leqq \text{const.} \|A\boldsymbol{u}\|^{1/2} \|A^{1/2}\boldsymbol{u}\|^{1/2} ,$$

which shows (11) holds with $M_0'=0$. This sharp estimate enables one to improve the decay rate of $\sup|\boldsymbol{u}|$, which can be seen from (80). On the other hand, the estimate (82) can be proved, similarly to Odqvist [**1**], Weyl [**17**]. (Here we do not enter into the detail.)

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