J. Math. Soc. Japan
Vol. 26, No. 1, 1974

Markov subshifts and realization of β-expansions

By Shunji ITO and Yōichirō TAKAHASHI

(Received June 20, 1972)
(Revised March 12, 1973)

§ 0. Introduction.

The β-transformations, which originate in a number-theoretical concept, β-expansions, have been supplying numerous results to ergodic theory. They correspond naturally to the symbolic dynamics called β-subshifts through the coefficients in the β-expansions, and have natural invariant measures, which are unique as measures of maximal entropy. The ergodic properties of β-transformations have been studied by A. Renyi [**12**], W. Parry [**10**] et al. All these results depend intrinsically upon the symbolical structure of β-subshifts, though it may be obscure. In this paper we will study the symbolical structure in detail, from which we deduce the dynamical and ergodic properties.

The notion of Markov subshifts is not only an indispensable tool from our stand point, but also furnishes a class of simple but interesting topological dynamics; the section 1 is devoted to the study of them; a characterization by open-ness of mappings, irreducibility and aperiodicity, their topological entropy. Several properties of topological entropy will be treated in the section 2. It will be interesting to note that the topological entropy is in a close relation to the numbers of periodic points. It gives an information on the character of topological entropy, which is, however, not a complete invariant even for Markov subshifts. It is also known that there corresponds an invariant measure to each Markov subshift by the maximality of entropy and that general subshifts and the invariant measures for them can be investigated through approximation by Markov subshifts.

In the section 3, we will discuss on the natural realization by subshifts of β-transformations. Here we must emphasize on the role played by the sequence ω_β called expansion of one and by the lexicographical order structure. Using them, we obtain a classification of words and an asymptotic estimate of mumbers of words.

In the consequent section 4 one can find a necessary and sufficient condition for β-transformations to be Markov. As a consequence we know that β-subshifts are not Markov except for those β's which satisfy a certain kind of algebraic equations. At the same time the family of β-subshift forms a one-parameter continuous family of increasing closed invariant set, and each of them can be closely approximated by Markovian β-subshifts.

In the last section 5 the Ornstein's weak Bernoulli condition will be verified for β-automorphisms with the help of our information on their symbolic structure. This result is also obtained by Smorodinsky [**15**]. We finally note that one of the authors constructed isomorphism of β-automorphisms to mixing Markov automorphisms [**16**].

ACKNOWLEDGEMENT. The authors express deep thanks to Professor Yuji Ito of the Brown University for his encouragement and the discussions with him during his visit to Japan in 1969-1970, when the original form of the present paper can be established. We also thank sincerely to Professor H. Totoki and Mr. I. Shiokawa for their advices and customary discussions. Without them we could not have completed the present paper.

§ 1. Markov subshift.

Let A be a finite set with discrete topology, $A^{\boldsymbol{Z}}$ and $A^{\boldsymbol{N}}$ the infinite product spaces where $\boldsymbol{N}=\{0, 1, 2, \cdots\}$, $\boldsymbol{Z}=\{0, \pm 1, \pm 2, \cdots\}$. The n-th coordinate of an element $\omega\in A^{\boldsymbol{Z}}$ or $A^{\boldsymbol{N}}$ is denoted by $\omega(n)$. The shift transformation σ is considered both on $A^{\boldsymbol{Z}}$ and on $A^{\boldsymbol{N}}$, and is defined by the relation:

$$(\sigma\omega)(n)=\omega(n+1)\,. \tag{1}$$

DEFINITION 1.1. A subshift is a pair (X, σ) where X is a σ-invariant closed subset of the product space $A^{\boldsymbol{Z}}$ or $A^{\boldsymbol{N}}$ and the letter σ stands for the restriction $\sigma|X$ to the subset X of the shift transformation.

Thus subshifts are topological dynamics with canonical generator $\{[a] \mid a\in A\}$. Here for a word u over the alphabet set A, in other words for $u\in \bigcup_{n\geqq 1} A^n$, $[u]$ denotes the corresponding cylinder set:

$$[u]=\{\omega \mid \omega(k)=a_k,\ 0\leqq k\leqq n\}$$

if $u=(a_0, \cdots, a_n)$ $(a_k\in A)$.

For a subshift (X, σ), the topological entropy $e(X, \sigma)$ can be computed by the formula:

$$e(X, \sigma)=\lim_{n\to\infty}\frac{1}{n}\log \operatorname{card}(W_n(X)) \tag{1}$$

where $\operatorname{card}(W)$ is the cardinality of a set W and

$$W_n(X)=\{(\omega(0), \cdots, \omega(n-1)) \mid \omega\in X\}\,. \tag{2}$$

DEFINITION 1.2. An invariant measure μ for a subshift (X, σ) will be called maximal if the metrical entropy $h(\mu)=h(X, \mu, \sigma)$ coincides with the topological entropy $e(X, \sigma)$.

REMARK 1.3. The existence of maximal measure for any subshift is well-known. A proof can be given as follows. From the inequalities

$$tH_{\mu}(\alpha)+(1-t)H_{\nu}(\alpha) \geqq H_{t\mu+(1-t)\nu}(\alpha) \geqq tH_{\mu}(\alpha)+(1-t)H_{\nu}(\alpha)-\log 2$$

for any finite Borel partition α and probability measures μ, ν, we conclude that the metrical entropy $h(\mu)$ is upper semi-continuous affine function on the set of invariant probability measures because the symbol set A is finite. Since this set is compact in the vague topology, the function $h(\mu)$ admits its maximum there.

DEFINITION 1.4. A subshift (X, σ) will be called Markov subshift of order p if there is a subset W of A^{p+1} $(p>0)$ such that

$$X=\mathcal{M}(W)=\{\omega \mid (\omega(n), \cdots, \omega(n+p)) \in W \text{ for any } n\}. \tag{4}$$

The set W will be called the structure set of Markov subshift (X, σ). The structure matrix $M=(M_{uv})_{u,v \in A^p}$ is defined as follows;

$$M_{uv}=\begin{cases} 1 & \text{if } u=(a_0, \cdots, a_{p-1}),\ v=(a_1, \cdots, a_p) \text{ for some } (a_0, \cdots, a_p) \in W \\ 0 & \text{otherwise}. \end{cases} \tag{5}$$

REMARK 1.5. i) This notion is equivalent to subshifts of finite type in Smale [**14**] if $X \subset A^Z$ and to intrinsic Markov chain in W. Parry [**11**] if $X \subset A^N$. The reason why we call such subshifts Markovian is that any maximal invariant measure μ is Markovian in the sense that $x_n(\omega)=\omega(n)$ form a Markov chain with state space A.

ii) The topological entropy of a Markov subshift (X, σ) of order p is computable from its structure matrix M via the following obvious identity:

$$\text{card}\,(W_n(X))=(M^{n-p-1}\mathbf{1}, \mathbf{1}) \qquad (n>p+1)$$

where $\mathbf{1}={}^t(1, 1, \cdots, 1)$.

Consequently, as is shown in [**1**],

$$e(X, \sigma)=\log \rho(M) \tag{6}$$

where $\rho(M)$ is the maximal modulus of eigenvalues of matrix M, which is the spectral radius of the operator M on the vector space $\boldsymbol{C}^{A^p}$.

iii) Let (X, σ) be a subshift. We can define Markov subshifts (X^p, σ) $(p \geqq 0)$ setting

$$X^p=\mathcal{M}(W_{p+1}(X)). \tag{7}$$

Then it is obvious that $X^p \supset X^{p+1}$, $X=\bigcap_{p \geqq 0} X^p$, and

$$e(X, \sigma) = \lim_{p\to\infty} e(X^p, \sigma) = \inf_{p\geq 0} e(X^p, \sigma). \tag{8}$$

The following theorem, which is an extension of Parry's result ([11]), characterizes Markov subshifts. We use the following notations: σ_+ denotes the shift transformation on one-sided sequence space A^N in order to distinguish it from the shift transformation σ on bilateral sequence space A^Z and the natural projection of A^Z onto A^N is denoted by π.

THEOREM 1. *Let X be a σ-invariant closed subset of the product space A^Z and $X_+ = \pi(X)$ the projection of X to A^N. Then the following four conditions are mutually equivalent:*

(a) *(X, σ) is a Markov subshift.*

(b) *(X_+, σ_+) is a Markov subshift.*

(c) *The restriction $\pi|X$ to the set X of projection π is an open map.*

(d) *The restriction $\sigma_+|X_+$ to the set X_+ of shift transformation σ_+ is an open map.*

PROOF. The equivalence of (a) and (b) follows from the closedness of the set X; in fact

$$X = \{\omega \in A^Z \mid (\omega(n+k))_{k\geq 0} \in X_+ \text{ for any } n \in \boldsymbol{Z}\}.$$

(c) implies (d): it is obvious from the homeomorphy of σ and the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\sigma|X"] \arrow[d, "\pi|X"'] & X \arrow[d, "\pi|X"] \\ X_+ \arrow[r, "\sigma_+|X_+"'] & X_+ \end{tikzcd} \qquad \pi\cdot\sigma = \sigma_+\cdot\pi$$

We now need the following:

LEMMA 1.6. *A subset of the countable product space A^Z is open and closed if and only if it is a finite union of cylinder sets.*

In fact, the "if" part is obvious. Suppose that an open and closed subset X is not a finite union of cylinder sets. Then there must exist a sequence of cylinder sets

$$C_n = \{\omega \in A^Z \mid \omega(k) = a_k^n,\ |k| \leq n\} \qquad (a_k^n \in A,\ n \geq 0)$$

which intersect both of X and its complement X^c. Since the set A is finite, we can choose a divergent subsequence (n') for which $a_k^{n'}$ does not depend on n', say $a_k^{n'} = a_k$. Thus we obtain a sequence $\omega = (a_k)_{k\in Z} \in A^Z$ which is contained both in the closed sets X and X^c, which is absurd. We have completed the proof.

Now we continue the proof of Theorem 1.

(a) implies (c): It suffices to prove that $\pi(U)$ is an open set for any cylinder set U of form

$$U=\{\omega\in X \mid \omega(-n)=a_0, \cdots, \omega(m-n)=a_m\}(n\in \boldsymbol{Z}, m\in \boldsymbol{N}, a_0, \cdots, a_m\in A).$$

Take an arbitrary point $\omega_+^0\in\pi(U)$ and let $\omega^0\in U$ be such that $\pi(\omega^0)=\omega_+^0$. Then the set $V=\{\omega_+\in X_+ \mid \omega_+(k)=\omega_+^0(k)\ 0\leqq k\leqq l\}$ is an open neighbourhood of ω_+^0 in X_+ where $l=\max\{p, m-n+p\}$ and p is the order of Markov subshift (X, σ). We will show that $\pi(U)\supset V$. For any element $\omega_+\in V$, the sequence $\omega\in A^{\boldsymbol{Z}}$ defined by $\omega(n)=\omega^0(n)$ $(n<0)$, $=\omega_+(n)$ $(n\geqq 0)$ does belong to the set X by its Markovianness. Furthermore $\omega\in U$ since $\omega^0\in U$ and U is of the form mentioned above. Hence $\omega_+=\pi(\omega)\in\pi(U)$.

(c) implies (b): This implication is essentially due to W. Parry [**11**], but we must show it since we do not give a proof of the equivalence of Parry's intrinsic Markov chains and our Markov subshifts. From (c) and the continuity of σ_+ it follows that the set $\sigma_+([a]\cap X_+)$ is open and closed for any $a\in A$. Therefore we can choose a family $\{P_a \mid a\in A\}$ of subsets of the word set A^p with the following properties:

1) $\sigma_+([a]\cap\pi(X))=\bigcup_{u\in P_a}[u]\cap\pi(X)$ $(a\in A)$.

2) $[u]\cap\pi(X)\neq\emptyset$ for any $u\in\bigcup_{a\in A}P_a$.

We set

$$W=\{(a_0, a_1, \cdots, a_p)\in A^{p+1} \mid (a_1, \cdots, a_p)\in P_{a_0}\}$$

and show that $X_+=\mathcal{M}(W)$. Noting that, for an arbitrary subset Y of $A^{\boldsymbol{N}}$ and for $n\geqq 1$, $a_0, \cdots, a_n\in A$,

$$\sigma_+([a_0, \cdots, a_n]\cap Y)=[a_1, \cdots, a_n]\cap\sigma_+([a_0]\cap Y), \tag{9}$$

we have

$$\sigma_+([a_0, \cdots, a_n]\cap X_+)=[a_1, \cdots, a_n]\cap X_+$$

if $n\geqq p$. From this it follows immediately by induction that $C\cap X_+\neq\emptyset$ for any cylinder set C in $\mathcal{M}(W)$. Consequently, for any $\omega\in\mathcal{M}(W)$, we can choose ω^n, $n\geqq 0$ such that $\omega^n\in X_+$ and that $\omega^n(k)=\omega(k)$ for $0\leqq k\leqq n$. Hence

$$\mathcal{M}(W)\subset \mathrm{cl.}(X_+)=X_+.$$

The inverse inclusion follows from the inclusion relation

$$\bigcup_{a\in A}\bigcup_{u\in P_a}[a\cdot u]=\bigcup_{w\in W}[w]\supset X_+.$$

Thus we have shown the Markovianness of (X_+, σ_+).

REMARK 1.7. As corollaries to Lemma 1.6 which was used in the proof above, we obtain the followings.

a) (Hedlund's theorem) For any continuous homomorphism ϕ of a shift $(A^{\boldsymbol{Z}}, \sigma)$ into a shift $(B^{\boldsymbol{Z}}, \sigma)$, there exists a partition P_b, $b\in B$, to A^p for some integer $p \geqq 0$, such that

$$\phi(\omega)(n) = F[\omega(n+k), \cdots, \omega(n+k+p-1)] \tag{10}$$

where $F(a_0, \cdots, a_{p-1}) = b$ if $(a_0, \cdots, a_{p-1}) \in P_b$.

b) Any continuous homomorphism ϕ of a subshift (X, σ) of (A^Z, σ) into (B^Z, σ) can be extended to a continuous homomorphism $\bar{\phi}$ of shift (A^Z, σ) into (B^Z, σ). In fact $\phi^{-1}[b]$, $b \in B$, are open subset of the set X and form a partition of X; in particular they are open and closed in X. It follows from Lemma 1.6 that there are mutually disjoint subsets Q_b, $b \in B$, of A^p for some $p \geqq 0$ such that $\phi^{-1}[b] = \bigcup_{u \in Q_b} [u] \cap X$. Let P_b, $b \in B$, be a partition of A^p for which $P_b \supset Q_b$. Then the map $\bar{\phi}$ defined through the formula (10) is a continuous homomorphism of (A^Z, σ) into (B^Z, σ) which agree with ϕ on the set X.

c) A Markov subshift (X, σ) of a shift (A^Z, σ) can be expressed as

$$X = \phi(B^Z) \cap A^Z \tag{11}$$

for some continuous homomorphism of a shift (B^Z, σ) into (A^Z, σ), where B is an alphabet set containing A. In fact, let W be the structure set of the given Markov subshift, $B = A \cup \{*\}$ ($*$ being an additional point) and

$$F \mid W = \text{identity}, \qquad F \mid W^c = * .$$

Then the map ϕ given by (10) possesses the property (11).

§ 2. Properties of topological entropy.

DEFINITION 2.1. (i) A subshift (X, σ) is called transitive if, for any cylinder sets C and D, there exists a positive integer n such that

$$(C \cap X) \cap \sigma^n (D \cap X) \neq \emptyset . \tag{1}$$

(ii) A subshift (X, σ) is called uniformly transitive if, for any finite partition α by cylinder sets, there exists a positive integer n such that (1) holds for any two members C, D of α.

We recall that a matrix $M = (M_{ij})_{1 \leqq i,j \leqq n}$ is called permutation-irreducible if there is no permutation τ of the set $\{1, \cdots, n\}$ for which the matrix $M' = (M_{\tau(i)\tau(j)})_{1 \leqq i,j \leqq n}$ is of form

$$\begin{pmatrix} N_1 & N_3 \\ 0 & N_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} N_1 & 0 \\ N_3 & N_2 \end{pmatrix}$$

where N_1 and N_2 are non-zero square matrices (See Gantmacher [5]), and that a nonnegative permutation-irreducible matrix $M \geqq 0$ has a positive eigenvalue $\rho(M)$ which is simple and of maximal modulus among its eigenvalues (theorem of Perron-Frobenius).

We also note that any Markov subshift (X, σ) of order $p \geqq 2$, over an alphabet set A, is naturally isomorphic to a Markov subshift of order 1 over alphabet set A^p.

DEFINITION 2.2. (i) A Markov subshift (X, σ) is called irreducible if its structure matrix is permutation-irreducible. (ii) (X, σ) is called aperiodic if some power of its structure matrix is strictly positive.

REMARK 2.3. It is proved in [1], [11] that the maximal invariant measure is unique for a irreducible Markov subshift and that such a Markov measure is ergodic and is determined by its structure matrix M as follows: Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be right and left eigen-vectors corresponding to the maximal eigenvalue $\rho(M)$, i.e. $M\boldsymbol{x} = \rho(M)\boldsymbol{x}$ and $\boldsymbol{y}M = \rho(M)\boldsymbol{y}$. Since the components of $\boldsymbol{x}$ or $\boldsymbol{y}$ must have common sign, we may assume that they are positive. Then the maximal Markov measure is defined by the transition matrix $P = (P_{uv})$, $P_{uv} = \dfrac{M_{uv}x_v}{\rho(M)x_u}$ whose unique invariant probability vector is $\boldsymbol{\pi} = (\pi_u)$, $\pi_u = \dfrac{x_u y_u}{\sum_{v \in A} x_v y_v}$.

The following lemma is due to N. Iwahori [7].

LEMMA 2.4. *Let (X, σ) be a simple Markov subshift over alphabet set A. Then the set A is decomposed as follows:*

(i) $A = A_0 \cup A_1 \cup \cdots \cup A_m$ *(disjoint union)* $(m \geqq 1)$.

(ii) *If we define* $X_k = \{\omega \in X \mid \omega(n) \in A_k$ *for all* $n\}$, *then* (X_k, σ) *is an irreducible Markov subshift for* $1 \leqq k \leqq m$.

(iii) $\{\omega \in X \mid \omega(n) \in A_0$ *for all* $n\} = \emptyset$.

PROOF. We define a binary relation R on A: aRb if $a = b$ or if there is a word $(a_0, \cdots, a_n) \in W_{n+1}(X)$ such that $a_0 = a$ and $a_n = b$, and another relation $a\bar{R}b$ by aRb and bRa. Then the relation $\bar{R}$ is an equivalence relation. Let $A(X)$ be the space of equivalence classes with respect to $\bar{R}$, $A_1, \cdots, A_m$ be such equivalence classes that the sets X_k defined in (ii) are not empty, and A_0 the subset of those alphabets which belong to none of $A_1, \cdots, A_m$. Then (i) and (iii) are trivial and (ii) follows from the fact that the irreducibility of a Markov subshift (Y, σ) is equivalent to the condition: if a, b are any two alphabets appearing in Y, then there exists a word $(a_0, \cdots, a_n)$ of Y for which $a = a_0$ and $b = a_n$.

COROLLARY 2.5. *Let $\chi(t)$ and $\chi_k(t)$ be characteristic polynomials of the structure matrices of Markov subshifts (X, σ) and (X_k, σ) in Lemma 2.4 $(k = 1, \cdots, m)$.*

Then,

$$\chi(t) = l \prod_{k=1}^{m} \chi_k(t) \tag{2}$$

where $l = \text{card}(A_0)$.

REMARK 2.6. i) The structure matrix of (X_k, σ) is minor matrix $(M_{ab})_{a,b\in A_k}$.

ii) Consequently,

$$e(X, \sigma) = \max_{1\leq k\leq m} e(X_k, \sigma).$$

iii) In particular, the maximal invariant measures for a subshift (X, σ) are supported by union of X_k's for which $e(X_k, \sigma) = e(X, \sigma)$ and they span a convex set of dimension n where n is the number of such X_k's.

PROPOSITION 2.7. (i) *For a subshift* (X, σ),

$$\limsup_{n\to\infty} \frac{1}{n} \log P_n(X, \sigma) \leqq e(X, \sigma) \tag{3}$$

where $P_n(X, \sigma) = \text{card}\,\{\omega \in X \mid \sigma^n\omega = \omega\}$.

(ii) *The equality* (3) *holds if* (X, σ) *is Markovian and the limit in the left-hand side exists if it is aperiodic.*

REMARK 2.8. (a) If (Y, σ) is an expansive dynamical system with finite topological entropy, then the periodic points increases at most exponentially. In fact there exists a (topological) generator $\{U_a \mid a \in A\}$ for an expansive system ([8]) where A is a finite index set and we can define an isomorphism R of a subshift (X, σ) onto (Y, ϕ) as follows:

$$\{R(\omega)\} = \cap\, \phi^n(\bar{U}_{\omega(n)}) \qquad \text{(one-point set)}$$

and

$$X = \{\omega \in A^Z \mid \{R(\omega)\} \neq \phi\}.$$

(b) In particular, the Anosov system for which the existence of Markov partition is proved ([3], also [13]) has periodic points which increases at most exponentially and the bound of increasing order is given by its topological entropy. This result is found in Bowen [4].

(c) For β-subshifts, which will be studied in sections 3, 4 and 5 the equality (3) does hold even when it is not Markovian.

PROOF. We first show (ii). Let M be the structure matrix of a Markov subshift (X, σ). Then

$$P_n(X, \sigma) = \text{Tr}\, M^n = \sum_{a\in A} (M^n)_{aa}$$

and

$$\text{card}\, W_n(X) = \sum_{a,b\in A} (M^n)_{ab}.$$

These two quantities can be expressed as linear combination of powers of eigenvalues of the matrix M. It is now obvious that $\lim_{n\to\infty} \frac{1}{n} \log P_n(X, \sigma)$ and $\lim_{n\to\infty} \frac{1}{n} \log \text{card}\, W_n(X)$ exist and coincide with $\log \rho(M)$ when M is aperiodic. If (X, σ) is an irreducible Markov subshift, then the eigenvalue $\rho(M)$ is simple so that a suitable choice of subsequence of $n_k \to \infty$ shows the statement (ii). Now (ii) follows from the Lemma 2.4 and its Corollary 2.5.

To prove (i), we approximate (X, σ) by the Markov subshifts (X^p, σ) where $X^p = \mathcal{M}(W_{p+1}(X))$ (see § 1, (7)). Then $X^p \supset X$ implies that

$$P_n(X^p, \sigma) \geqq P_n(X, \sigma)$$

and that

$$\limsup_{n\to\infty} P_n(X, \sigma) \leqq \limsup_{n\to\infty} P_n(X^p, \sigma) \leqq e(X^p, \sigma).$$

Since $\lim_{p\to\infty} e(X^p, \sigma) = e(X, \sigma)$, we have (i).

REMARK 2.9. i) There exist two Markov subshifts (X, σ) and (Y, σ) with common topological entropy which are not topologically isomorphic. Let $X = \mathcal{M}(M)$ and $Y = \mathcal{M}(N)$ where

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then $\rho(M) = \rho(N) = 1 + \sqrt{2}$, but $P_1(X, \sigma) = 2$, $P_1(Y, \sigma) = 1$.

ii) Even if the zeta functions ([2]) of two Markov subshifts (X, σ) and (Y, σ) coincide, i.e., $P_n(X, \sigma) = P_n(Y, \sigma)$ for all $n \geqq 1$, (X, σ) and (Y, σ) may not be isomorphic. An example can be given by the matrices:

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomials are common and given by $t^3 - 2t^2 + 1 = 0$. $\mathrm{Tr}\, M = \mathrm{Tr}\, N = 2$, $\mathrm{Tr}\, M^2 = \mathrm{Tr}\, N^2 = 4$. $\mathcal{M}(N)$ has an isolated fixed point while $\mathcal{M}(M)$ has none.

§ 3. Realization of β-transformations.

Let β be a real number such that $s-1 < \beta \leqq s$ for some integer $s \geqq 2$. The β-expansion of a real number t is the expression of the form:

$$t = a_{-1} + \sum_{n\geqq 0} a_n \beta^{-n-1} \tag{1}$$

where a_{-1} is an integer and, for $n \geqq 0$, $a_n \in A = \{0, 1, \cdots, s-1\}$. The expression can be uniquely determined via β-transformation T_β which has been studied by A. Renyi [12] and W. Parry [10], and is defined on the unit interval $[0, 1)$ by the relation:

(2) $$T_\beta t \equiv \beta t \pmod 1 .$$

Let π_β be the map of the unit interval $[0, 1)$ into the infinite product space $\Omega = A^N = \{0, \cdots, s-1\}^N$ defined as follows:

(3) $$\pi_\beta(t)(n) = k, \quad \text{if} \quad k\beta^{-1} \leqq T_\beta^n t < (k+1)\beta^{-1}$$

where $T_\beta^0 t = t$, $T_\beta^{n+1} t = T_\beta(T_\beta^n t)$ $(n \geqq 0)$.

Then it is proved in [**10**] that the expression (1) holds for $a_n = \pi_\beta(t)(n)$, $n \geqq 0$, $a_{-1} = 0$ in the case of $t \in [0, 1)$; in other words

(1′) $$t = \rho_\beta(\pi_\beta(t))$$

where

(4) $$\rho_\beta(\omega) = \sum_{n \geqq 0} \omega(n) \beta^{-n-1}$$

for $\omega \in A^N$. (See 5) of the Proposition 3.2 below.)

Let Y_β be the image $\pi_\beta([0, 1))$ and X_β its closure in the product space Ω with the product topology.

DEFINITION 3.1. The subshift (X_β, σ) will be called β-subshift.

The space Ω is endowed with the lexicographical order $\omega > \omega'$: if and only if there exists an integer n such that $\omega(k) = \omega'(k)$ for $k < n$ and $\omega(n) > \omega'(n)$. The shift transformation on the space $\Omega = A^N$ will be denoted by σ. We set

(5) $$T_\beta^n 1 = \lim_{t \uparrow 1} T_\beta^n t$$

and

(6) $$\pi_\beta(1) = \max X_\beta = \omega_\beta .$$

PROPOSITION 3.2.

1) *$\sigma \circ \pi_\beta = \pi_\beta \circ T_\beta$ on $[0, 1)$.*

2) *$\pi_\beta : [0, 1] \to X_\beta$ is an injection and is strictly order-preserving,* i.e. *$t < s$ implies that $\pi_\beta(t) < \pi_\beta(s)$.*

3) *$\rho_\beta \circ \pi_\beta$ is identity on $[0, 1]$.*

4) *$\rho_\beta \circ \sigma = T_\beta \circ \rho_\beta$ on Y_β.*

5) *$\rho_\beta : X_\beta \to [0, 1]$ is a continuous surjection and is order-preserving,* i.e. *$\omega < \omega'$ implies that $\rho_\beta(\omega) \leqq \rho_\beta(\omega')$.*

6) *The inverse image $\rho_\beta^{-1}(t)$ of $t \in [0, 1]$ consists either of a one point $\pi_\beta(t)$ or of two points $\pi_\beta(t)$ and $\sup_{s<t} \pi_\beta(s)$. The latter case occurs only when $T_\beta^n t = 0$ for some $n > 0$.*

7) *In particular, $\pi_\beta(\omega)$ is one-to-one except for a countable number of points $\omega \in X_\beta$.*

PROOF. We first note that

(7) $$\beta t = T_\beta t + \pi_\beta(t)(0)$$

and that

(8) $$T_\beta^n t = \beta^n\Big(t - \sum_{k=0}^{n-1} \pi_\beta(t)(k)\beta^{-k-1}\Big)$$

for any $t \in [0, 1)$ and $n \geqq 1$. (7) is the definition itself of T_β and π_β, and (8) is an immediate consequence of (7).

The assertion 1) and 4) are trivial. 3) follows from (8), since $\lim_{n\to\infty} \sum_{k=0}^{n-1} \pi_\beta(t)(k)\beta^{-k-1} = \rho_\beta(\pi_\beta(t))$, and 3) implies the injectivity of π_β and the surjectivity of ρ_β.

Now we prove that the map π_β preserves the order. Then we have the assertion 2). Suppose that there exist $t, s \in [0, 1]$ such that $t < s$ and that $\pi_\beta(t) > \pi_\beta(s)$, i. e., that there exists an integer $n \geqq 0$ with the properties:

$$\pi_\beta(t)(k) = \pi_\beta(s)(k) \qquad \text{for} \quad k < n$$

and

$$\pi_\beta(t)(n) > \pi_\beta(s)(n) .$$

Then we would have, by the identity (8),

$$\begin{aligned} t &= \sum_{k=0}^{n} \pi_\beta(t)(k)\beta^{-k-1} + \beta^{-n-1}T^{n+1}t \\ &\geqq \sum_{k=0}^{n} \pi_\beta(s)(k)\beta^{-k-1} + \beta^{-n-1}(1+T^{n+1}t) \\ &= s + \beta^{-n-1}(1+T^{n+1}t - T^{n+1}s) . \end{aligned}$$

This is absurd since $t < s$.

Recalling that the completion of the set Y_β with respect to the lexicographical order coincides with its closure X_β in the product topology, we have 5).

Finally we prove the assertion 6). It is now obvious that

$$\rho_\beta^{-1}(t) = \{\omega \in X_\beta \mid \sup_{s<t} \pi_\beta(s) \leqq \omega \leqq \inf_{s>t} \pi_\beta(s)\} .$$

Since $\lim_{s\downarrow t} T_\beta s = T_\beta t$, we have

$$\lim_{s\downarrow t} \pi_\beta(t)(0) = \pi_\beta(t)(0)$$

and so

$$\inf_{s>t} \pi_\beta(s) = \pi_\beta(t) .$$

Consequently,

$$\rho_\beta^{-1}(t) = \{\sup_{s<t} \pi_\beta(s), \pi_\beta(t)\} .$$

Assume now that $\omega = \sup_{s<t} \pi_\beta(s)$ and $\omega' = \pi_\beta(t)$ are distinct. Then there is an

integer n for which

$$\omega(k)=\omega'(k) \qquad (k<n)$$

and

$$\omega(n)\neq\omega'(n).$$

Since $\omega<\omega'$ by 2), and since $\rho_\beta(\omega'')<1$ for any $\omega''\in X_\beta$, we have

$$\omega(n)=\omega'(n)-1.$$

Then, from the identities

$$t=\rho_\beta(\omega)=\sum_{k=0}^{n}(k)\beta^{-k-1}+\beta^{-n-1}\rho_\beta(\sigma^n\omega)$$

and

$$t=\rho_\beta(\omega')=\sum_{k=0}^{n}\omega'(k)\beta^{-k-1}+\beta^{-n-1}\rho_\beta(\sigma^n\omega')$$

it follows that

$$\beta^{-n-1}=\beta^{-n-1}(\rho_\beta(\sigma^n\omega)-\rho_\beta(\sigma^n\omega')).$$

But this equality holds if and only if $\rho_\beta(\sigma^n\omega)=1$ and $\rho_\beta(\sigma^n\omega')=0$. Hence $\omega'(k)=0$ for any $k>n$ and

$$\sigma^n\omega=\omega_\beta=\max X_\beta$$

since $\rho_\beta^{-1}\{1\}$ consists of one point ω_β. The proof of Proposition is completed.

REMARK 3.3. The Proposition implies that the orders induced from the usual order on the unit interval $[0,1]$ by the maps ρ_β and ρ_s $(s-1<\beta\leqq s)$ coincide on the set Y_β, since the order induced by ρ_s is lexicographical on A^N.

This remark enables us to classify the set $W_n(X_\beta)$ of those words of length n which appears in β-subshift (X_β,σ) $(n\geqq 1)$. These word sets are also endowed with lexicographical order. Let

$$\text{(9)}\qquad \begin{aligned} &W_n^0=\{(a_1,\cdots,a_n)\in W_n(X_\beta)\mid(a_1,\cdots,a_{n-1},a_n+1)\in W_n(X_\beta)\},\\ &W_n^0(u)=\{u\cdot v\in W_{n+k}(X_\beta)\mid u\cdot v\in W_{n+k}^0\},\qquad W_n(u)=\{u\cdot v\mid u\cdot v\in W_{n+k}\}\end{aligned}$$

where $n\geqq 1$, $u\in W_k(X_\beta)$, $k\geqq 0$, and the symbol "$\cdot$" denotes the concatenation, i. e.

$$\text{(10)}\qquad u\cdot v=(a_1,\cdots,a_n,b_1,\cdots,b_m)$$

if $u=(a_1,\cdots,a_n)$ and $b=(b_1,\cdots,b_m)$ (the number m may be infinite). The empty word ε is a symbol such that $\varepsilon\cdot u=u\cdot\varepsilon=u$ for any word u. Finally we set $W_0(X_\beta)=W_0^0(u)=\{\varepsilon\}$.

PROPOSITION 3.4. *For any $k\geqq 0$ and a word $u\in W_k(X_\beta)$*

$$W_n(u)=\bigcup_{j=1}^{n}W_j^0(u)\cdot\omega_\beta[0,n-j)\cup\{\max W_n(u)\}$$

where

(11) $$\omega_\beta[0, j) = \begin{cases} (\omega_\beta(0), \cdots, \omega_\beta(j-1)) & (j > 1), \\ \varepsilon \text{ (empty word)} & (j = 0). \end{cases}$$

REMARK 3.5. If $u=\varepsilon$, then, $W_n^0(\varepsilon)=W_n^0$ and $\max W_n(\varepsilon)=\omega_\beta[0, n)$.

PROOF. We only prove the proposition for the case $u=\varepsilon$, since the general assertion is then deduced immediately. We first note that, if a word $(a_1, \cdots, a_n)$ belongs to the set $W_n(X_\beta)$ and if $a_n > 1$, then any words $(a_1, \cdots, a_{n-1}, b)$ with $0 \leqq b \leqq a_n$ also belong to $W_n(X_\beta)$. In fact it is the initial word of length n of $\pi_\beta(t)$ where

$$t = a_1\beta^{-1} + \cdots + a_{n-1}\beta^{-(n-1)} + b\beta^{-n}.$$

Consequently, if $u=(a_1, \cdots, a_n) \in W_n(X_\beta) \setminus W_n^0(X_\beta)$ and if $u \neq \max W_n(X_\beta)$, then the word $\min \{v \in W_n(X_\beta) \mid v > u\}$ must be of the form

$$(b_1, \cdots, b_k, 0, 0, \cdots, 0)$$

for some $1 \leqq k \leqq n$ and some $b_1, \cdots, b_k \in A$, $b_k \neq 0$. Since there is no word in $W_n(X_\beta)$ which lies between u and this word, we have $a_1=b_1, \cdots, a_{k-1}=b_{k-1}$, $a_k=b_k-1$. Comparing the words $(a_k, \cdots, a_n)$ and $(a_k+1, 0, \cdots, 0)$, we can conclude that

$$(a_{k+1}, \cdots, a_n) = \omega_\beta[0, n-k).$$

On the other hand, the word $(a_1, \cdots, a_k+1)$ belongs to the set $W_k(X_\beta)$ as well as the word $(a_1, \cdots, a_k)$. This means that $(a_1, \cdots, a_k) \in W_k^0(X_\beta)$. Hence

$$u = (a_1, \cdots, a_k)\cdot(a_{k+1}, \cdots, a_n) \in W_k^0(X_\beta)\cdot\omega_\beta[0, n-k).$$

Finally, if $u=\max W_n(X_\beta)$, then it is obvious that $u=\omega_\beta[0, n)$. The inverse inclusion is trivial. The proof is completed.

We note that the sets $[w]=\{\omega \in X_\beta \mid (\omega(0), \cdots, \omega(n))=w\}$, $w \in W_{n+1}(X_\beta)$ form a partition of the set X_β and that $\rho_\beta([w])=\{\rho_\beta(\omega) \mid \omega \in [w]\}$, $w \in W_{n+1}(X_\beta)$ form a covering of the unit interval by intervals, any two of which have at most one common point. Let $R_\beta(w)$ be the length of interval $\rho_\beta([w])$. For any $u \in W_k(X_\beta)$,

(12) $$\sum_{u\cdot v \in W_n(u)} R_\beta(u\cdot v) = R_\beta(u)$$

and

(13) $$R_\beta(u\cdot v) = \begin{cases} \rho_\beta((u\cdot v)')-\rho_\beta(u\cdot v) & \text{if} \quad u\cdot v \neq \max W_{n+k}(X_\beta) \\ 1-\rho_\beta(u\cdot v) & \text{if} \quad u\cdot v = \max W_{n+1}(X_\beta) \end{cases}$$

where we denote, for a word $w \in W_l(X_\beta)$,

$$w' = \min \{v \in W_l(X_\beta) \mid v > w\}$$

if it exists.

COROLLARY 3.6. *Let* $u\in W_k(X_\beta)$ *and* $M_\beta=\sum_{n\geqq 0}(n+1)\omega_\beta(n)\beta^{-n-1}$.

a) $\lim_{n\to\infty}\beta^{-n}\operatorname{card}(W_n^0(u))=\dfrac{\beta^k R_\beta(u)}{M_\beta}$

b) $\lim_{n\to\infty}\beta^{-n}\operatorname{card}(W_n(u))=\dfrac{\beta^k R_\beta(u)}{M_\beta(1-\beta^{-1})}$

where the convergence is uniform in $k\geqq 1$ *and* $u\in W_k(X_\beta)$.

c) *In particular, the topological entropy of the subshift* (X_β, σ) *is* $\log\beta$.

PROOF. If $u\cdot v\in W_{n-j}^0(u)\omega_\beta[0, j)$ for some $0\leqq j\leqq n-1$, then $(u\cdot v)'=u\cdot v'$ and we obtain from Proposition 3.4

$$R_\beta(u\cdot v)=\beta^{-(k+n-j)}\Big(1-\sum_{m=0}^{j-1}\omega_\beta(m)\beta^{-m-1}\Big) \tag{14}$$
$$=\beta^{-k-n}T_\beta^j 1\,.$$

On the other hand, if $u\in W_{k-l}^0\omega_\beta[0, l)$ and $u\cdot v=\max W_n(u)$, then

$$R_\beta(u\cdot v)=\beta^{-k-n}T_\beta^{n+l}1\,.$$

Now it follows from the expressions (12) and (14) that

$$R_\beta(u)=\sum_{j=0}^{n-1}\beta^{-k-n}T_\beta^j 1\cdot N_{n-j}^0(u)+\beta^{-k-n}T_\beta^{n+l}1 \qquad (n\geqq 1)\,, \tag{15}$$

where $N_m^0(u)=\operatorname{card}(W_m^0(u))$ $(m\geqq 1)$. Let us consider a formal power series:

$$\sum_{n\geqq 1}t^n\sum_{j=0}^{n-1}\beta^{-k-n}T_\beta^j 1 N_{n-j}^0(u)\,.$$

Then it is easy to see that the series converges for $|t|<1$ and we can deduce from (15) that

$$\sum_{n\geqq 1}\beta^{-n}N_n^0(u)t^n=\frac{\beta^k R_\beta(u)\,t}{1-\phi_\beta(t)}-g_l(t) \tag{16}$$

where

$$g_l(t)=\frac{1-t}{1-\phi_\beta(t)}\sum_{n\geqq 1}t^n\beta^{-n}T_\beta^{n+l}1\,,$$

$$\phi_\beta(t)=\sum_{n\geqq 0}\omega_\beta(n)\beta^{-n-1}t^{n+1}\,. \tag{17}$$

But the series in (17) converges in a neighbourhood of the unit disk $\{t\in \boldsymbol{C}: |t|\leqq 1\}$ and the function $1-\phi_\beta(t)$ has only one simple root at $t=1$ in a disk $\{t\in\boldsymbol{C}: |t|<1+\varepsilon\}$ for some $\varepsilon>0$. Consequently the function

$$f_u(t)\equiv\sum_{n\geqq 1}\Big(\beta^{-n}N_n^0(u)-\frac{\beta^k R_\beta(u)}{M_\beta}\Big)t^n\,,$$

which is equal to

$$\frac{\beta^k R_\beta(u)\,t}{1-\phi_\beta(t)}-\frac{t\beta^k R_\beta(u)}{(1-t)\phi_\beta'(1)}-g_l(t)\,,$$

is analytic in $\{t\in\boldsymbol{C}\mid t\neq 1,\ |t|<1+\varepsilon\}$ and the singular point $t=1$ is removal.

In particular the function $f_u(t)$ has bounded derivative on the circle $|t|=1$. Furthermore

$$\sup_k \sup_{u\in W_k(X_\beta)} \sup_{|t|=1} |f'_u(t)|<\infty ,$$

since the function g_l and its derivative g'_l is uniformly bounded in $l\geqq 1$ and $\beta^k R(u)\leqq 1$ for any $u\in W_k(X_\beta)$, $k\geqq 1$. Using the estimate

$$\begin{aligned} n\times|\beta^{-n}N_n^0(u)-\beta^k R_\beta(u)/M_\beta| &= \left|(2\pi r^n)^{-1}\int_0^{2\pi} f'_u(re^{i\theta})e^{-in\theta}d\theta\right| \\ &\leqq r^{-n}\sup_{|t|\leqq 1}|f'_u(t)| \end{aligned}$$

for $0<r<1$ and $n\geqq 0$, we obtain

$$\sup_{\substack{n\geqq 1,\, u\in W_k(X_\beta) \\ k\geqq 1}} n\,|\beta^{-n}N_n^0(u)-\beta^k R_\beta(u)/M_\beta|<\infty .$$

Hence a). The assertion b) follows from the obvious identity

$$\operatorname{card} W_n(u)=\sum_{k=1}^{n}\operatorname{card} W_k^0(u)+1 .$$

c) is now obvious.

Now we investigate the Markov subshift $(\mathcal{M}(W_{p+1}(X_\beta)),\sigma)$ whose structure set is given by $W_{p+1}(X_\beta)$. Let $M=M_{\beta,p}$ be its structure matrix.

PROPOSITION 3.7. *Let q be the minimal positive integer n for which $(\omega_\beta(n),\cdots,\omega_\beta(p-1))=(\omega_\beta(0),\cdots,\omega_\beta(p-n-1))$ if such an n exists and $q=p$ otherwise. Assume that there exist a vector $\boldsymbol{x}=(x_u)_{u\in A^p}\neq\boldsymbol{0}$ and a complex number $\lambda\neq 0$ such that $M_{\beta,p}\boldsymbol{x}=\lambda\boldsymbol{x}$. Then*

(i) $1=\sum_{k=0}^{q=1}\omega_\beta(k)\lambda^{-k-1}+\lambda^{-q}$ *if* $\omega_\beta(p)=\omega_\beta(p-q)$,

$1=\sum_{k=0}^{p}\omega_\beta(k)\lambda^{-k-1}+\lambda^{-p-1}$ *if* $\omega_\beta(p)<\omega_\beta(p-q)$.

(ii) *For* $u\in W_{p-k}^0\cdot\omega_\beta[0,k)$,

$$x_u=c\cdot\lambda^k\Big(1-\sum_{j=0}^{k-1}\omega_\beta(j)\lambda^{-j-1}\Big)$$

where c is some complex number.

PROOF. We first show that

$$x_u=\xi_k \qquad \text{if} \quad u\in W_{p-k}^0\cdot\omega_\beta[0,k)$$

where $\xi_k=x_{(0,\cdots,0)\omega_\beta[0,k)}$. We appeal to the induction on the value $\delta(u,v)$ where

$$\delta(u,v)=\begin{cases} 0 & \text{if} \quad u=v \\ \max\{j\mid u(j)\neq v(j)\}+1 & \text{if} \quad u\neq v , \end{cases}$$

and prove that, if $u,v\in W_{p-k}^0\cdot\omega_\beta[0,k)$, then $x_u=x_v$. If $\delta(u,v)=0$, then it is trivial. We assume that this is true for $\delta(u,v)\leqq j$. Let $u,v\in W_{p-k}^0\cdot\omega_\beta[0,k)$

and $\delta(u, v)=j+1$. If we define words u' and v' setting

$$u'(i)=u(i+1)\,, \qquad v'(i)=v(i+1) \qquad (0\leqq i\leqq p-1)\,,$$

then, if one of $u'\cdot a$ and $v'\cdot a$ belongs to $W_p(X_\beta)$, so do both words and $\delta(u'\cdot a, v'\cdot a)=j$. From the definition of structure matrix and the method of classification of words it follows that

$$\lambda x_u = M_{uw}x_w = \sum_{a=0}^{\omega_\beta(k)} x_{u'a}$$

and

$$\lambda x_v = M_{vw}x_w = \sum_{a=0}^{\omega_\beta(k)} x_{v'a}\,.$$

Consequently, from the induction assumption, we conclude that $x_u = x_v$ since $\lambda \neq 0$.

We next show that the common value ξ_k is given by (ii). If $0\leqq k<p$, then

$$\begin{aligned}\lambda\xi_k &= \lambda x_{(0,\cdots,0)\cdot\omega_\beta[0,k)} \\ &= \sum_{a=0}^{\omega_\beta(k)} x_{(0,\cdots,0)\cdot\omega_\beta[0,k)\cdot a} \\ &= \xi_{k+1}+\omega_\beta(k)\xi_0\,.\end{aligned}$$

From these recurrence formulas the statement (ii) follows immediately. In particular, if a component x_n is zero, then the vector $\boldsymbol{x}$ is zero. For $k=p$, we obtain from the definition of q that

$$\begin{aligned}\lambda\xi_p = \lambda x_{\omega_\beta[0,p)} &= \sum_{a=0}^{\omega_\beta(p)} x_{\omega_\beta[1,p)a} \\ &= \begin{cases} (\omega_\beta(p)+1)\xi_0 & \text{if} \quad \omega_\beta(p)<\omega_\beta(p-q) \\ \xi_{p-q+1}+\omega_\beta(p)\xi_0 & \text{if} \quad \omega_\beta(p)=\omega_\beta(p-q)\,. \end{cases}\end{aligned}$$

Consequently (i) follows from this together with the reccurrence formulas (ii).

§ 4. Symbolical properties of β-subshifts.

In this section we show two important properties of β-subshifts; one is the property of the family $\{X_\beta\colon \beta>1\}$, the other is the characterization of those β's for which the subshifts (X_β, σ) are Markovian.

PROPOSITION 4.1.

(i) *If* $1<\beta\leqq\alpha$, *then* $Y_\beta\subset Y_\alpha$.

(ii) *For any* $\beta>1$,

$$\text{a)}\quad X_\beta=\overline{\bigcup_{\alpha>\beta} X_\alpha}\,, \qquad \text{b)}\quad X_\beta=\bigcap_{\alpha>\beta} X_\alpha\,.$$

REMARK 4.2. The family $\{X_\beta \mid \beta>1\}$ gives an example of increasing family of shift invariant closed sets with topological entropy $\log \beta$ (see corollary 3.6).

In order to prove the Proposition we need the following lemma, which asserts that the set $Y_\beta=\pi_\beta([0, 1))$ is a G_δ-set.

LEMMA 4.3.

$$Y_\beta=\pi_\beta([0, 1))=\{\omega\in A^N \mid \rho_\beta(\sigma^n\omega)<1 \ ({}^\forall n\geqq 0)\} .$$

PROOF. Let Z_β be the set in the right-hand side. Then it is obvious that $Y_\beta\subset Z_\beta$. Conversely, if $\omega\in Z_\beta$, then

$$\rho(\sigma\omega)=\beta\rho(\omega)-\omega(0)$$

since $\rho(\omega)<1$ and $\rho(\sigma\omega)<1$. This implies that $\pi_\beta(\rho_\beta(\omega))(0)=\omega(0)$. Consequently $\pi_\beta(\rho_\beta(\omega))=\omega$ because of the shift-invariance of the set Z_β.

LEMMA 4.4.

$$X_\beta=\{\omega\in A^N \mid \sigma^n\omega\leqq\omega_\beta \textit{ for any } n\geqq 0\} .$$

PROOF. Assume that $\sigma^n\omega\leqq\omega_\beta$ for any $n\geqq 0$. Let ω_k be a sequence such that

$$\omega_k(n)=\begin{cases}\omega(n) & \text{for } n<k \\ 0 & \text{for } n\geqq k .\end{cases}$$

Then $\sigma^k\omega_k\in Y_\beta$. We prove that $\sigma^{k-j}\omega_k\in Y_\beta$ by induction on $j=0, 1, 2, \cdots, k$. For this it suffices to show that if $\omega'\in Y_\beta$ and $a\cdot\omega'<\omega_\beta$, then $a\cdot\omega'\in Y_\beta$. But in case of $a=\omega_\beta(0)$,

$$\rho_\beta(a\cdot\omega')=\beta^{-1}(a+\rho(\omega'))<\beta^{-1}(a+\rho(\sigma\omega_\beta))=1 ,$$

and in case of $a<\omega_\beta(0)$,

$$\rho_\beta(a\cdot\omega')=\beta^{-1}(a+\rho(\omega'))<\beta^{-1}(a+1)\leqq\beta^{-1}\omega_\beta(0)<1 .$$

Thus we get $\omega_k\in Y_\beta$ for any $k\geqq 1$. Consequently $\omega=\lim\omega_k\in X_\beta$.

THEOREM 2. *Let $\beta>1$. Then the following three conditions are equivalent* $(p\geqq 1)$*:*

1) *The subshift (X_β, σ) is Markov and its order is strictly equal to p.*

2) *There exist integers a_i, $i=0, \cdots, p$, $0\leqq a_i<s$, such that*

$$\text{a)} \qquad 1-\beta^{-p-1}=\sum_{j=0}^{p} a_j\beta^{-j-1} ,$$

$$\text{b)} \qquad 1-\beta^{-p-1}>\sum_{j=0}^{p} a_{j+k}\beta^{-j-1} \qquad (k=1, \cdots, p)$$

where we set $a_{n+p+1}=a_n$ for $n\geqq 0$.

3) *The sequence ω_β is periodic with period $p+1$,* i.e.

$$\text{a}') \qquad \sigma^{p+1}\omega_\beta = \omega_\beta$$

and

$$\text{b}') \qquad \sigma^q\omega_\beta < \omega_\beta \quad \textit{for any} \quad q=1, \cdots, p.$$

PROOF. We first show that 3) follows from 1). Let (X, σ) be a Markov subshift whose order is strictly equal to p. Then by Proposition 3.7 the following dichotomy occurs:

$$\alpha) \qquad 1 = \sum_{k=0}^{p} \omega_\beta(k)\beta^{-k-1} + \beta^{-p-1}.$$

β) there exists an integer $q < p$ such that

$$1 = \sum_{k=0}^{q-1} \omega_\beta(k)\beta^{-k-1} + \beta^{-q}.$$

Then it follows easily that

$$\sum_{k=0}^{\infty} \omega_\beta(k+r)\beta^{-k-1} = 1$$

where $r = p+1$ in case α) and $r = q$ in case β). Therefore

$$\sigma^r\omega_\beta = \omega_\beta.$$

Assume that β) holds. Then, $r = q$ and

$$\sigma^n\omega \leqq \omega_\beta = \sigma^q\omega_\beta$$

for any $\omega \in \mathcal{M}(W_q(X_\beta))$ and any $n \geqq 0$. Hence

$$X_\beta = \mathcal{M}(W_q(X_\beta))$$

This is a contradiction. Consequently the case β) never occurs. In other words a′) and b′) holds.

Next we show that the statement 3) implies 2). In fact the equality a) follows from a′) since

$$1 = \rho_\beta(\omega_\beta) = \sum_{j=0}^{p} \omega_\beta(j)\beta^{-j-1} + \beta^{-p-1}\rho_\beta(\sigma^{p+1}\omega_\beta).$$

To show b) it suffices to rewrite the inequality

$$\rho_\beta(\sigma^q\omega_\beta) < \rho_\beta(\omega_\beta) \qquad \text{for any} \quad q=1, \cdots, p$$

which hold in virtue of the uniqueness of the sequence ω_β.

Finally we show that 1) follows from 2). Let

$$\omega_n(k) = \begin{cases} a_k & \text{for} \quad k < n(p+1) \\ 0 & \text{for} \quad k \geqq n(p+1). \end{cases}$$

Then it is easy to see by a) and b) that $\rho_\beta(\sigma^k\omega_n) < 1$ for any $k \geqq 0$. Therefore $\omega_n \in Y_\beta$ by Lemma 4.3. And then $\lim_{n\to\infty} \omega_n \in X_\beta$ and $\rho_\beta(\lim_{n\to\infty} \omega_n) = 1$. This shows that the sequence $(a_n)_{n\geqq 0}$ is the β-expansion of one, ω_β.

Let $W=W_{p+1}(X_\beta)$. If $\omega\in\mathcal{M}(W)$, then $\sigma^n\omega\leqq\omega_\beta$ for any $n\geqq 0$ since $(a_0,\cdots,a_p)=\max W$. Consequently $\omega\in X_\beta$ by Lemma 4.4. Hence $X_\beta=\mathcal{M}(W)$.

Now we claim that the subshift (X_β,σ) is not Markovian of order $q<p$. Suppose that it is Markovian of order q. Then the statements a′) and b′) and therefore a) and b) hold with p replaced by q. But this is absurd.

§ 5. Metrical properties of β-automorphisms.

Renyi [12] showed that there exists a maximal invariant probability measure for β-transformation on the unit interval [0, 1) which is absolutely continuous with respect to Lebesgue measure. That measure induces a shift-invariant probability measure $d\mu_\beta(\omega)=f_\beta(\omega)d\rho_\beta(\omega)$ on X_β, where

(1)
$$\rho_\beta(\omega)=\sum_{n\geqq 0}\omega(n)\beta^{-n-1},$$

$d\rho_\beta$ is Stieltjes integral on the ordered space X_β,

(2)
$$f_\beta(\omega)=M_\beta^{-1}\sum_{n\geqq 0}\beta^{-n}I(\omega\leqq\sigma^n\omega_\beta),$$

(2)
$$M_\beta=\sum_{n\geqq 0}(n+1)\omega_\beta(n)\beta^{-n-1}$$

and $I(\omega\leqq\eta)$ denotes the indicator function of the set $\{\omega\mid\omega\leqq\eta\}$. The proof of invariance of μ_β is immediate in our symbolical form.

DEFINITION 5.1. The endomorphism $(X_\beta,\mu_\beta,\sigma)$ will be called β-endomorphism and its natural extension $(\bar{X}_\beta,\bar{\mu}_\beta,\bar{\sigma})$ β-automorphism.

THEOREM 3. *The β-automorphism $(\bar{X}_\beta,\bar{\mu}_\beta,\bar{\sigma})$ is Bernoullian.*

The proof will be given by a series of lemmas, where following convention is used:

$$\phi(u)=\phi(u\cdot(000\ldots))\text{ for any function }\phi\text{ on }X_\beta.$$

LEMMA 5.1. *Let*

(4)
$$S_\beta\phi(\omega)=\beta^{-1}\sum\phi(a\omega)$$

where the sum is taken over $\{a\in A\mid a\cdot\omega\in X\}$. Then S_β is a nonnegative operator on the spaces of Borel functions on X_β and satisfies the following properties:

a)
$$\int_{X_\beta}S_\beta\phi(\omega)\cdot\psi(\omega)d\rho_\beta(\omega)=\int_{X_\beta}\phi(\omega)\psi(\sigma\omega)d\rho_\beta(\omega)$$

whenever $\phi\in L^1(X_\beta,d\rho_\beta)$ and $\psi\in L^\infty(X_\beta,d\rho_\beta)$.

b) *In particular, S_β is a nonnegative contraction operator on $L^1(X_\beta,d\rho_\beta)$ such that*

$$\int_{X_\beta} S_\beta \phi(\omega) d\rho_\beta(\omega) = \int_{X_\beta} \phi(\omega) d\rho_\beta(\omega).$$

c) *S_β is a bounded operator on $L^\infty(X_\beta, d\rho_\beta)$ and*

$$\lim_{n\to\infty} \sup_{\|\phi\|_\infty \leq 1} |S_\beta^n \phi(\omega)| = \frac{1}{M_\beta(1-\beta^{-1})}.$$

PROOF.

a)
$$\int_{X_\beta} S_\beta \phi(\omega)\psi(\omega) d\rho_\beta(\omega) = \beta^{-1} \sum_a \int_{X_\beta} I_{X_\beta}(a\cdot\omega)\phi(a\cdot\omega)\psi(\omega) d\rho_\beta(\omega)$$
$$= \sum_a \int_{[a]} \phi(\omega)\psi(\sigma\omega) d\rho_\beta(\omega)$$
$$= \int_{X_\beta} \phi(\omega)\psi(\sigma\omega) d\rho_\beta(\omega).$$

b) is obvious from a).

c) follows from the Corollary 3.5.

Now we prove the main lemma.

LEMMA 5.2. *Let ϕ be a continuous function on X_β. Then*

$$\lim_{k\to\infty} \|S_\beta^k \phi - c(\phi) f_\beta\|_\infty = 0 \qquad \text{where} \quad c(\phi) = \int \phi \, d\rho_\beta,$$

and the convergence is uniform on the set $\Phi = \bigcup_{n\geq 1} S_\beta^n \Phi_n$ where

(5) $$\Phi_n = \{\phi \mid \|\phi\|_\infty \leq 1, \ \phi(\omega) \text{ depends only on } \omega(k), \ 0 \leq k < n\}.$$

PROOF. Let ϕ be a function on X_β such that $\phi(\omega) = \phi(\omega')$ if $\omega(k) = \omega'(k)$ for $0 \leq k < n$. We will prove the lemma for such ϕ's.

The classification of the word set $W_k = W_k(X_\beta)$ guarantees that

$$S_k \phi(\omega) = \beta^{-k} \sum_{w\in W_k} \phi(w\cdot\omega)$$
$$= \sum_{j=0}^{k} \beta^{-k} \sum_{v\in W_{k-j}^0} \phi(v\cdot\omega_\beta^*[0, j)\cdot\omega)\cdot I(\sigma^j\omega_\beta > \omega)$$

where $W_j^0 = W_j^0(X_\beta)$ and the sums is understood to be taken over w's such that $w\cdot\omega \in X_\beta$ here and hereafter. Let us define

(6) $$S_\beta^k(m)\phi(\omega) = \sum_{j=0}^{m\wedge k} \beta^{-k} \sum_{v\in W_{k-j}^0} \phi(v\cdot\omega_\beta[0, j)\cdot\omega).$$

Then

(7) $$\|S_\beta^k \phi - S_\beta^k(m)\phi\|_\infty \leq \sum_{j=m+1}^{k} \beta^{-k} \|\phi\|_\infty \operatorname{card}(W_{k-j}^0)$$
$$\leq \|\phi\|_\infty \beta^{-k} \operatorname{card}(W_{k-m-1})$$
$$\leq C_0 \|\phi\|_\infty \beta^{-m-1}.$$

We now assume that $k-m>n$. Then, since the function ϕ depends only on the first n coordinates, we have

$$S_\beta^k(m)\phi(\omega)=\sum_{j=0}^{m}\beta^{-k}\sum_{v\in W_{k-j}^0}\phi(v)I(\sigma^j\omega_\beta\geqq\omega)$$

$$=\sum_{j=0}^{m}\beta^{-k}\sum_{u\in W_n}\phi(u)N_{k-j-n}^0(u)I(\sigma^j\omega_\beta\geqq\omega)$$

where $N_k^0(u)=\mathrm{card}\,(W_k^0(u))$. Therefore

$$S_\beta^k(m)\phi(\omega)=\sum_{j=0}^{m}\beta^{-j}I(\sigma^j\omega_\beta\geqq\omega)\cdot\beta^{-n}\sum_{u\in W_n}\phi(u)\beta^{-(k-j-n)}N_{k-j-n}^0(u)\,.$$

Consequently,

$$\begin{aligned}&S_\beta^{k+n}(m)\phi(\omega)-C(\phi)f_\beta(\omega)\\&\quad=S_\beta^{k+n}(m)\phi(\omega)-M_\beta f_\beta(\omega)\beta^{-n}\sum_{u\in W_n}\phi(u)\beta^nR_\beta(u)M_\beta^{-1}\\&\quad=-\sum_{j\geqq m+1}\beta^{-j}I(\sigma^j\omega_\beta\geqq\omega)\beta^{-n}\sum_{u\in W_n}\phi(u)\beta^nR_\beta(u)M_\beta^{-1}\\&\qquad+\sum_{j=0}^{m}\beta^{-j}I(\sigma^j\omega_\beta\geqq\omega)\beta^{-n}\sum_{u\in W_n}\phi(u)\{\beta^{-(k-j)}N_{k-j}^0(u)-\beta^nR_\beta(u)M_\beta^{-1}\}\,.\end{aligned}$$

The first term in the last is majorated by

$$M_\beta^{-1}\|\phi\|_\infty\sum_{j\geqq m}\beta^{-j-1}=M_\beta^{-1}\|\phi\|_\infty\frac{\beta^{-m-1}}{1-\beta^{-1}}$$

which tends to zero as $m\to\infty$, and the second term by

$$\begin{aligned}&\|\phi\|_\infty\sum_{j=0}^{m}\beta^{-j}I(\sigma^j\omega_\beta\geqq\omega)\beta^{-n}\sum_{u\in W_n}|\beta^{-(k-j)}N_{k-j}^0(u)-\beta^nR_\beta(u)M_\beta^{-1}|\\&\leqq\|\phi\|_\infty\sum_{j=0}\beta^{-j}I(\sigma^j\omega_\beta\geqq\omega)\beta^{-n}\,\mathrm{card}\,(W_n)\cdot\sup_{u\in W_n}\sup_{k\geqq i\geqq k-m}|\beta^{-i}N_i^0(u)-\beta^nR_\beta(u)M_\beta^{-1}|\\&\leqq\mathrm{const.}\,\|\phi\|_\infty\sup_{i\geqq k-m}\sup_{n\geqq1}\sup_{u\in W_n}|\beta^{-i}N_i^0(u)-\beta^nR_\beta(u)M_\beta^{-1}|\,,\end{aligned}$$

which tends to zero as $k\to\infty$ uniformly in n and $u\in W_n$ so long as m is fixed.

Thus we have shown that $\lim S^k\phi$ exists in the topology of uniform convergence and the convergence is uniform in the set Φ.

Now the rest to be proved is easily verified if we approximate general ϕ's by functions which depend only on a finite number of coordinates.

PROOF OF THE THEOREM. Let $(\bar{X}_\beta, \bar{\mu}_\beta, \sigma)$ be β-automorphism, i.e., the natural extension of β-endomorphism $(X_\beta, \mu_\beta, \sigma)$. We recall that a shift transformation (A^Z, σ, μ) is Bernoulli if it satisfies the Ornstein's weak Bernoulli condition:

$$\lim_{k\to\infty}\sup_{n\geqq1}\sum_{u\in A^n}\sum_{v\in A^n}|\mu([u]\cap\sigma^{-n-k}[v])-\mu([u])\mu([v])|=0\,.$$

This condition is equivalent to the following condition:

$$(*) \qquad \lim_{k\to\infty} \sup_{n} \sup_{\phi\in\Phi_n} \sum_{v\in A^n} \left| \int_{\sigma^{-k-n}[v]} \phi d\mu_\beta - \int \phi d\mu \cdot \mu[v] \right| = 0 .$$

Now we prove that the measure $\bar{\mu}_\beta$ satisfies this condition. It suffices to show (*) for μ_β. It follows from Lemma 5.2 a)

$$\int_{\sigma^{-k-n}[v]} \phi d\mu_\beta = \int \phi(\omega) f_\beta(\omega) 1_{[v]}(\sigma^{k+n}\omega) d\rho_\beta(\omega)$$

$$= \int_{[v]} S_\beta^{k+n}(\phi f_\beta) d\rho_\beta .$$

Therefore

$$\Delta_n^k(\phi) \equiv \sum_{v\in A^n} \left| \int_{\sigma^{-k-n}[v]} \phi\, d\mu_\beta - \int \phi\, d\mu_\beta \cdot \mu_\beta[v] \right|$$

$$= \sum_{v\in A^n} \left| \int_{[v]} \left\{ S_\beta^{k+n}(\phi f_\beta) - \left(\int \phi d\mu_\beta \right) f_\beta \right\} d\rho_\beta \right|$$

$$\leqq \int \left| S_\beta^{k+n}(\phi\cdot f_\beta) - \left(\int \phi d\mu_\beta \right) f_\beta \right| d\rho_\beta .$$

Since the function f_β is integrable with respect to the measure ρ_β, for any $\varepsilon > 0$ we can find a continuous function g which depends only on the coordinates $\omega(k)$, $0 \leqq k < m$ for some m such that

$$\int |f_\beta - g| d\rho_\beta < \varepsilon .$$

Then by Lemma 5.2 b) we have

$$\int | S_\beta^{k+n}(\phi\cdot f_\beta) - S_\beta^{k+n}(\phi\cdot g) | d\rho_\beta \leqq \int S_\beta^{k+n} |\phi f_\beta - \phi\cdot g| d\rho_\beta$$

$$= \int |\phi\cdot f_\beta - \phi\cdot g| d\rho_\beta < \varepsilon .$$

Thus

$$\Delta_n^k(\phi) \leqq 2\varepsilon + \int \left| S_\beta^{k+n}(\phi\cdot g) - \left(\int \phi\cdot g d\rho_\beta \right) f_\beta \right| d\rho_\beta$$

$$\leqq 2\varepsilon + \left\| S_\beta^k [S_\beta^n(\phi\cdot g)] - \left(\int \phi\cdot g d\rho_\beta \right) f_\beta \right\| .$$

The last term converges to zero as $k \to 0$ uniformly in $n \geqq 1$ and $\phi \in \Phi_n$ by Lemma 5.3 since the function $\phi\cdot g$ belongs to Φ_n for $n \geqq m$ up to constant multiplication. Hence the condition (*) is verified and the proof is completed.

References

[1] R. L. Adler and B. Weiss, Similarity of automorphisms of the torus, Memoires of the Amer. Math. Soc., No. 98.

[2] M. Artin and B. Mazur, On periodic points, Ann. of Math., **81** (1965), 82-99.

[3] R. Bowen, Markov partitions for Axiom A diffeomorphisms, Amer. J. Math., **92** (1970), 725–747.
[4] R. Bowen, Periodic points and measures for Axiom A diffeomorphisms, Trans. Amer. Math. Soc., **154** (1971), 377–397.
[5] F.R. Gantmacher, Theory of matrices, Part II, New York, 1960, (English translation).
[6] G.A. Hedlund, Transformations commuting with the shift, in "Topological dynamics", (Auslander and Gottchalk ed), 1968, 259–289.
[7] N. Iwahori, On stochastic matrices of a given Frobenius type, J. Fac. Sci. Univ. Tokyo Sect. I, **13** (1966), 139–161.
[8] H.B. Keynes and J.B. Robertson, Generators for topological entropy and expansiveness, Math. Systems Theory, **3** (1969), 51–59.
[9] K.R. Meyer, Periodic points of diffeomorphisms, Bull. Amer. Math. Soc., **73** (1967), 615–617.
[10] W. Parry, On the β-expansions of real numbers, Acta Math. Acad. Sci. Hungar., **11** (1960), 401–416.
[11] W. Parry, Intrinsic Markov chains, Trans. Amer. Math. Soc., **112** (1964), 55–66.
[12] A. Renyi, Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hungar., **8** (1957), 447–493.
[13] Ya. G. Sinai, Markov partitions and C-diffeomorphisms, Funkcional. Anal. i Priložen, **2**, No. 1 (1968), 64–89.
[14] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc., **73** (1967), 747–817.
[15] Smorodinsky, β-automorphisms are Bernoulli shifts, (preprint).
[16] Y. Takahashi, Isomorphisms of β-automorphisms to Markov automorphisms, Osaka Math. J.

Shunji Ito
Department of the Foundations
of Mathematical Sciences
Tokyo University of Education
Otsuka, Bunkyo-ku
Tokyo, Japan

Yōichirō Takahashi
Department of Mathematics
Hiroshima University
Present address:
Department of Mathematics
College of General Education
University of Tokyo
Komaba, Meguro-ku
Tokyo, Japan