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On the nonlinear semi-groups associated with $u_t = \Delta\beta(u)$ and $\varphi(u_t) = \Delta u$

By Yoshio KONISHI

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Introduction.

The purpose of the present paper is to study the *differentiability of the nonlinear contraction semi-groups* generated in the sense of Crandall and Liggett [4], which are associated with the following nonlinear problems of diffusion:

$$(1)\quad \begin{cases} u_t = \Delta\beta(u) & \text{in} \quad \Omega\times(0, \infty), \\ u = 0 & \text{on} \quad \partial\Omega\times(0, \infty), \\ u(0) = a & \text{in} \quad \Omega; \end{cases}$$

$$(2)\quad \begin{cases} \varphi(u_t) = \Delta u & \text{in} \quad \Omega\times(0, \infty), \\ u = 0 & \text{on} \quad \partial\Omega\times(0, \infty), \\ u(0) = a & \text{in} \quad \Omega, \end{cases}$$

where Δ is the Laplace operator on a bounded domain $\Omega \subset R^d$ with smooth boundary $\partial\Omega$, a's are given initial data and β and φ are strictly monotone increasing continuous functions on R^1 such that

$$\beta(0) = \varphi(0) = 0$$

and that the range of φ is R^1. (Concerning the problem (2), see also Strauss [14, 15].) It has been already known that we can study the problems (1) and (2) from the point of view of the theory established by Crandall and Liggett in [4]. Crandall [2] and Konishi [6], for example, associated with the problem (1) a nonlinear dissipative[1)] (accretive) operator $Au = \Delta\beta(u)$ $(Au = -\Delta\beta(u))$

1) A (possibly) nonlinear operator $\mathcal{A}$ in a real Banach space $\mathcal{X}$ is said to be *dissipative* if

$$\|u-v-\lambda(\mathcal{A}u-\mathcal{A}v)\|_{\mathcal{X}} \geqq \|u-v\|_{\mathcal{X}}$$

for $\lambda>0$, $u, v\in D(\mathcal{A})$, or equivalently, if

$$\tau(u-v,\ -\mathcal{A}u+\mathcal{A}v)\geqq 0$$

whenever $u, v\in D(\mathcal{A})$; where

$$\tau(f, g) = \lim_{\varepsilon\downarrow 0}(\|f+\varepsilon g\|_{\mathcal{X}} - \|f\|_{\mathcal{X}})/\varepsilon, \qquad f, g\in\mathcal{X}.$$

By definition, $\mathcal{A}$ is *accretive* if $-\mathcal{A}$ is dissipative.

with the domain $D(A)$ and the range $R(A)$ contained in the separable Banach space $X=L^1(\Omega)$, and constructed the corresponding nonlinear semi-group. In order to study its differentiability, we shall introduce a natural extension $\tilde{A}$ of A in the dual space X'^* of a suitable Banach space X' which is strongly separable and weakly* dense in the dual space X^* of X.[2] The same idea will be applied to the problem (2), which has been grasped by Konishi [**7**] within the scope of the semi-group theory. Special nonlinearities in our problems permit us to use arguments which seem somewhat peculiar, especially in the study of (2). Nevertheless, we hope that our result can be a contribution to the construction of an abstract general theory on the differentiability of nonlinear semi-groups in non-reflexive Banach spaces. See also the recent work of Crandall [**3**].

§ 1. On $u_t=\Delta\beta(u)$.

We denote by $C_0(\Omega)$ the Banach space of all real-valued continuous functions f on $\bar{\Omega}$ satisfying $f(x)=0$ for $x\in\partial\Omega$, normed with the maximum of the absolute value. Then the dual space $C_0(\Omega)^*$ of $C_0(\Omega)$ is the Banach space of all bounded Baire measures on Ω, with the norm of total variation. The space $L^1(\Omega)$ can be regarded as a subspace of $C_0(\Omega)^*$. Let us consider the problem (1) in $C_0(\Omega)^*$. We define an operator Δ_0 in $C_0(\Omega)$:

$$D(\Delta_0)=\{f\in C_0(\Omega)\cap W^{2,d+1}(\Omega)\,;\ \Delta f\in C_0(\Omega)\}\,,$$

$$\Delta_0 f=\Delta f \quad \text{for} \quad f\in D(\Delta_0)\,.$$

Thus Δ_0 is the infinitesimal generator of a contraction semi-group of class (C_0) in $C_0(\Omega)$ (see Masuda [**11**]). We denote its dual operator by Δ_0^*, which is dissipative in $C_0(\Omega)^*$ by the well known theory on dual semi-groups (see, for example, Yosida [**16**]). Next we define a nonlinear strongly closed operator β_1 in $L^1(\Omega)$:

$$(1.1)\qquad \begin{aligned} &D(\beta_1)=\{f\in L^1(\Omega)\,;\ \beta(f(\cdot))\in L^1(\Omega)\}\,,\\ &(\beta_1 f)(x)=\beta(f(x))\,,\quad x\in\Omega\,,\qquad \text{for}\quad f\in D(\beta_1)\,. \end{aligned}$$

Then we obtain

LEMMA 1. *The product $\Delta_0^*\beta_1$ of the operators Δ_0^* and β_1 is a dissipative operator in $C_0(\Omega)^*$ and satisfies the relation:*

$$(1.2)\qquad R(I-\lambda\Delta_0^*\beta_1)\supset\overline{D(\Delta_0^*\beta_1)}=L^1(\Omega)\qquad \textit{for}\quad \lambda>0\,^{3)}\,.$$

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2) This idea is due to Kōmura [5] (see Problem I) and is used also in addendum II of Konishi [**9**]. See also Konishi [**8, 10**].

3) $\overline{D(\Delta_0^*\beta_1)}$ denotes the closure of $D(\Delta_0^*\beta_1)$ relative to the strong topology.

PROOF. In the case where $\mathcal{X}=C_0(\Omega)^*$, we have

$$\tau(f, g)=g_f^c(\Omega_f^+)-g_f^c(\Omega_f^-)+\|g_f^s\|_{C_0(\Omega)^*}, \qquad f, g\in C_0(\Omega)^*,$$

where g_f^c and g_f^s are, respectively, the absolutely continuous part and the singular part of g with respect to $|f|$ and Ω_f^+ and Ω_f^- denote, respectively, the positivity set and the negativity set in the Hahn decomposition of Ω relative to f (see Sato [13], 6.7). Hence, for $u, v\in D(\Delta_0^*\beta_1)$, we have

$$\tau(u-v, -\Delta_0^*\beta_1 u+\Delta_0^*\beta_1 v)=\tau(\beta_1 u-\beta_1 v, -\Delta_0^*\beta_1 u+\Delta_0^*\beta_1 v)\geqq 0,$$

i.e., $\Delta_0^*\beta_1$ is again dissipative in $C_0(\Omega)^*$. Next we show (1.2). We know that $R(I-\lambda\Delta_0^*\beta_1)$ $(\lambda>0)$ is strongly dense in $L^1(\Omega)$ (see Theorem 4.12 of Crandall [2]). Moreover, since $(\Delta_0^*)^{-1}$ $(=(\Delta_0^{-1})^*)$ is a strongly continuous operator of $C_0(\Omega)^*$ into $L^1(\Omega)$, $\Delta_0^*\beta_1$ is strongly closed. Thus $R(I-\lambda\Delta_0^*\beta_1)$ $(\lambda>0)$ is strongly closed in $C_0(\Omega)^*$. Consequently we have

$$R(I-\lambda\Delta_0^*\beta_1)\supset L^1(\Omega) \qquad \text{for each} \quad \lambda>0.$$

Moreover, since $\beta_1^{-1}(\mathcal{D}(\Omega))$ is strongly dense in $L^1(\Omega)$, we have

$$\overline{D(\Delta_0^*\beta_1)}=L^1(\Omega). \qquad \text{Q.E.D.}$$

By virtue of Lemma 1, the operator $\Delta_0^*\beta_1$ generates a nonlinear contraction semi-group $\{\exp(t\Delta_0^*\beta_1)\}_{t\geqq 0}$ on $L^1(\Omega)\subset C_0(\Omega)^*$ in the sense of Theorem I of Crandall and Liggett [4]. We shall study the differentiability of this semi-group.

THEOREM 1. *We assume that*

$$a\in D(\Delta_0^*\beta_1).$$

Then

(1.3) $$\exp(t\Delta_0^*\beta_1)\cdot a\in D(\Delta_0^*\beta_1) \qquad \textit{for each} \quad t\geqq 0,$$

the function

$$t\in[0, \infty)\longmapsto \exp(t\Delta_0^*\beta_1)\cdot a\in L^1(\Omega)\subset C_0(\Omega)^*$$

is weakly continuously differentiable and*

(1)' $$\begin{cases} \text{w*-}\dfrac{d}{dt}\exp(t\Delta_0^*\beta_1)\cdot a=\Delta_0^*\beta_1\exp(t\Delta_0^*\beta_1)\cdot a, & t\geqq 0, \\ \exp(0\Delta_0^*\beta_1)\cdot a=a. \end{cases}$$

PROOF OF THEOREM 1. We know that

(1.4) $$\text{s-}\lim_{\lambda\downarrow 0}(I-\lambda\Delta_0^*\beta_1)^{-[t/\lambda]}a=\exp(t\Delta_0^*\beta_1)\cdot a \quad \text{in} \quad L^1(\Omega), \qquad t\geqq 0,$$

and that

(1.5) $$\|\Delta_0^*\beta_1(I-\lambda\Delta_0^*\beta_1)^{-[t/\lambda]}a\|_{C_0(\Omega)^*}\leqq\|\Delta_0^*\beta_1 a\|_{C_0(\Omega)^*}, \qquad t\geqq 0, \quad \lambda>0.$$

Hence, by the strong compactness of $(\Delta_0^*)^{-1}$, the set

$$\{\beta_1(I-\lambda\Delta_0^*\beta_1)^{-[t/\lambda]}a\,;\ \lambda>0\}$$

is strongly relatively compact in $L^1(\Omega)$ for each $t\geqq 0$. Accordingly

$$\exp(t\Delta_0^*\beta_1)\cdot a\in D(\beta_1)$$

and

$$\text{s-}\lim_{\lambda\downarrow 0}\beta_1(I-\lambda\Delta_0^*\beta_1)^{-[t/\lambda]}a=\beta_1\exp(t\Delta_0^*\beta_1)\cdot a\quad\text{in}\quad L^1(\Omega) \tag{1.6}$$

for each $t\geqq 0$. In view of (1.5) and (1.6) and by the weak* closedness of Δ_0^*. we have (1.3) and

$$\text{w*-}\lim_{\lambda\downarrow 0}\Delta_0^*\beta_1(I-\lambda\Delta_0^*\beta_1)^{-[t/\lambda]}a=\Delta_0^*\beta_1\exp(t\Delta_0^*\beta_1)\cdot a\quad\text{in}\quad C_0(\Omega)^* \tag{1.7}$$

for each $t\geqq 0$. Moreover (1.5) and (1.7) imply the estimate:

$$\|\Delta_0^*\beta_1\exp(t\Delta_0^*\beta_1)\cdot a\|_{C_0(\Omega)^*}\leqq\|\Delta_0^*\beta_1 a\|_{C_0(\Omega)^*}\,,\qquad t\geqq 0\,,$$

from which follows the weak* continuity of the function

$$t\in[0,\infty)\longmapsto\Delta_0^*\beta_1\exp(t\Delta_0^*\beta_1)\cdot a\in C_0(\Omega)^*\,.$$

Now letting λ tend to zero in the following equality due to Ôharu (see, for example, [**12**]):

$$\begin{aligned}(I-\lambda\Delta_0^*\beta_1)^{-[t/\lambda]}a-a=&\int_0^t\Delta_0^*\beta_1(I-\lambda\Delta_0^*\beta_1)^{-[s/\lambda]}a\,ds\\&+\lambda\{\Delta_0^*\beta_1(I-\lambda\Delta_0^*\beta_1)^{-[t/\lambda]}a-\Delta_0^*\beta_1 a\}\\&-\int_{[t/\lambda]\lambda}^t\Delta_0^*\beta_1(I-\lambda\Delta_0^*\beta_1)^{-[s/\lambda]}a\,ds\,,\qquad\lambda>0\,,\ \ t\geqq 0\,,\end{aligned} \tag{1.8}$$

we have, by (1.4), (1.5) and (1.7),

$$\exp(t\Delta_0^*\beta_1)\cdot a-a=\text{w*-}\int_0^t\Delta_0^*\beta_1\exp(s\Delta_0^*\beta_1)\cdot a\,ds\quad\text{in}\quad C_0(\Omega)^*$$

for each $t\geqq 0$, from which follows (1)$'$. Q. E. D.

§2. On $\varphi(u_t)=\Delta u$.

We define an operator Δ_1 in $L^1(\Omega)$:

$$D(\Delta_1)=\{f\in W_0^{1,1}(\Omega)\,;\ \Delta f\in L^1(\Omega)\}\,,$$

$$\Delta_1 f=\Delta f\quad\text{for}\quad f\in D(\Delta_1)\,.$$

Thus Δ_1 is the infinitesimal generator of a contraction semi-group of class (C_0) in $L^1(\Omega)$ (see Brezis and Strauss [**1**]). We denote its dual operator by Δ_1^*, which is dissipative in $L^\infty(\Omega)=L^1(\Omega)^*$. We define a nonlinear homeomorphism φ_∞ of $L^\infty(\Omega)$ onto itself:

$$D(\varphi_\infty)=L^\infty(\Omega),$$
$$(\varphi_\infty f)(x)=\varphi(f(x)), \quad x\in\Omega, \qquad \text{for} \quad f\in L^\infty(\Omega).$$

Then we have

LEMMA 2. *$\varphi_\infty^{-1}\Delta_1^*$ is dissipative in $L^\infty(\Omega)$ and*

$$R(I-\lambda\varphi_\infty^{-1}\Delta_1^*)\supset\overline{D(\varphi_\infty^{-1}\Delta_1^*)}=C_0(\Omega), \qquad \lambda>0. \tag{2.1}$$

PROOF. (2.1) is a direct consequence of Proposition 2 of Konishi [7] and the fact that $D(\Delta_0)\subset D(\Delta_1^*)\subset C_0(\Omega)$. The dissipativity of $\varphi_\infty^{-1}\Delta_1^*$ follows from the concrete form of τ for $\mathcal{X}=L^\infty(\Omega)$ (cf. Lemma 3 of Konishi [7]):

$$\tau(f, g)=\lim_{\varepsilon\downarrow 0}\ \underset{x\in\Omega(f,\varepsilon)}{\text{ess sup}}\ (\text{sgn}\, f(x))g(x), \qquad f, g\in L^\infty(\Omega), \quad f\neq 0,$$

here

$$\Omega(f, \varepsilon)=\{x\in\Omega;\ |f(x)|>\|f\|_{L^\infty(\Omega)}-\varepsilon\}$$

(see Sato [13], 6.4). Q. E. D.

We denote by $\{\exp(t\varphi_\infty^{-1}\Delta_1^*)\}_{t\geqq 0}$ the semi-group on $C_0(\Omega)$ generated by $\varphi_\infty^{-1}\Delta_1^*$ in the sense of Crandall and Liggett [4]. Concerning its differentiability, we have:

THEOREM 2. *Suppose that*

$$a\in D(\Delta_1^*).$$

Then

$$\exp(t\varphi_\infty^{-1}\Delta_1^*)\cdot a\in D(\Delta_1^*) \qquad \textit{for each} \quad t\geqq 0, \tag{2.2}$$

the function

$$t\in[0, \infty)\longmapsto\exp(t\varphi_\infty^{-1}\Delta_1^*)\cdot a\in C_0(\Omega)\subset L^\infty(\Omega)=L^1(\Omega)^*$$

is weakly continuously differentiable, and*

$$\left\{\begin{array}{l}\varphi_\infty\left(\text{w}^*\text{-}\dfrac{d}{dt}\exp(t\varphi_\infty^{-1}\Delta_1^*)\cdot a\right)=\Delta_1^*\exp(t\varphi_\infty^{-1}\Delta_1^*)\cdot a, \qquad t\geqq 0, \\ \exp(0\varphi_\infty^{-1}\Delta_1^*)\cdot a=a.\end{array}\right. \tag{2)'}$$

PROOF. We have the following:

$$\text{s-}\lim_{\lambda\downarrow 0}(I-\lambda\varphi_\infty^{-1}\Delta_1^*)^{-[t/\lambda]}a=\exp(t\varphi_\infty^{-1}\Delta_1^*)\cdot a \quad \text{in} \quad C_0(\Omega), \qquad t\geqq 0, \tag{2.3}$$

$$\begin{aligned}&\|\Delta_1^*(I-\lambda\varphi_\infty^{-1}\Delta_1^*)^{-[t/\lambda]}a\|_{L^\infty(\Omega)}\\ &\qquad\leqq\max(\varphi(\|\varphi_\infty^{-1}\Delta_1^*a\|_{L^\infty(\Omega)}), -\varphi(-\|\varphi_\infty^{-1}\Delta_1^*a\|_{L^\infty(\Omega)})), \qquad t\geqq 0, \ \lambda>0.\end{aligned} \tag{2.4}$$

Thus we have (2.2) and

$$\text{w}^*\text{-}\lim_{\lambda\downarrow 0}\Delta_1^*(I-\lambda\varphi_\infty^{-1}\Delta_1^*)^{-[t/\lambda]}a=\Delta_1^*\exp(t\varphi_\infty^{-1}\Delta_1^*)\cdot a \quad \text{in} \quad L^\infty(\Omega). \tag{2.5}$$

On the other hand,

$$\Delta_1^*(I-\lambda\varphi_\infty^{-1}\Delta_1^*)^{-[t/\lambda]}a=(I-\lambda\Delta_0^*(\varphi^{-1})_1)^{-[t/\lambda]}\Delta_1^*a\,,\qquad t\geqq 0\,,\quad \lambda>0\,\text{[4]}\,, \tag{2.6}$$

here $(\varphi^{-1})_1$ is an operator in $L^1(\Omega)$ defined by (1.1) with $\beta=\varphi^{-1}$. Hence, by the result of § 1, (2.5) shows

$$\text{s-}\lim_{\lambda\downarrow 0}\Delta_1^*(I-\lambda\varphi_\infty^{-1}\Delta_1^*)^{-[t/\lambda]}a=\Delta_1^*\exp(t\varphi_\infty^{-1}\Delta_1^*)\cdot a \quad\text{in}\quad L^1(\Omega)\,,\qquad t\geqq 0\,.$$

Thus, in view of (2.4) and (2.6), we have

$$\text{w}^*\text{-}\lim_{\lambda\downarrow 0}\varphi_\infty^{-1}\Delta_1^*(I-\lambda\varphi_\infty^{-1}\Delta_1^*)^{-[t/\lambda]}a=\varphi_\infty^{-1}\Delta_1^*\exp(t\varphi_\infty^{-1}\Delta_1^*)\cdot a \quad\text{in}\quad L^\infty(\Omega)\,,\qquad t\geqq 0\,, \tag{2.7}$$

and that

$$\varphi_\infty^{-1}\Delta_1^*\exp(t\varphi_\infty^{-1}\Delta_1^*)\cdot a=(\varphi^{-1})_1\exp(t\Delta_0^*(\varphi^{-1})_1)\cdot\Delta_1^*a\,,\qquad t\geqq 0\,. \tag{2.8}$$

(2.8) implies the weak* continuity of the function $t\in[0,\infty)\mapsto\varphi_\infty^{-1}\Delta_1^*\exp(t\varphi_\infty^{-1}\Delta_1^*)\cdot a\in L^\infty(\Omega)$. Letting λ tend to 0 in the equality:

$$\begin{aligned}(I-\lambda\varphi_\infty^{-1}\Delta_1^*)^{-[t/\lambda]}a-a&=\int_0^t\varphi_\infty^{-1}\Delta_1^*(I-\lambda\varphi_\infty^{-1}\Delta_1^*)^{-[s/\lambda]}a\,ds\\&\quad+\lambda\{\varphi_\infty^{-1}\Delta_1^*(I-\lambda\varphi_\infty^{-1}\Delta_1^*)^{-[t/\lambda]}a-\varphi_\infty^{-1}\Delta_1^*a\}\\&\quad-\int_{[t/\lambda]\lambda}^t\varphi_\infty^{-1}\Delta_1^*(I-\lambda\varphi_\infty^{-1}\Delta_1^*)^{-[s/\lambda]}a\,ds\,,\qquad t\geqq 0\,,\quad\lambda>0\,,\end{aligned}$$

we have, by (2.3), (2.4) and (2.7),

$$\exp(t\varphi_\infty^{-1}\Delta_1^*)\cdot a-a=\text{w}^*\text{-}\int_0^t\varphi_\infty^{-1}\Delta_1^*\exp(s\varphi_\infty^{-1}\Delta_1^*)\cdot a\,ds \quad\text{in}\quad L^\infty(\Omega)\,,\qquad t\geqq 0\,.$$

Consequently we have (2)′. Q. E. D.

CONCLUDING REMARK. $D(\Delta_0^*\beta_1)$ and $D(\varphi_\infty^{-1}\Delta_1^*)$ themselves coincide with what Crandall [3] calls the "generalized domains" $\hat{D}(\Delta_0^*\beta_1)$ and $\hat{D}(\varphi_\infty^{-1}\Delta_1^*)$ respectively. Hence we can conclude that $\Delta_0^*\beta_1$ *and* $\varphi_\infty^{-1}\Delta_1^*$ *are "weak* infinitesimal generators" of* $\{\exp(t\Delta_0^*\beta_1)\}_{t\geqq 0}$ *and* $\{\exp(t\varphi_\infty^{-1}\Delta_1^*)\}_{t\geqq 0}$ *respectively:*

$$D(\Delta_0^*\beta_1)=\{f\in L^1(\Omega)\,;\ \text{w}^*\text{-}\lim_{h\downarrow 0}(\exp(t\Delta_0^*\beta_1)\cdot f-f)/h \text{ exists in } C_0(\Omega)^*\}\,,$$

$$\text{w}^*\text{-}\lim_{h\downarrow 0}(\exp(t\Delta_0^*\beta_1)\cdot f-f)/h=\Delta_0^*\beta_1 f\,,\qquad f\in D(\Delta_0^*\beta_1)\,;$$

$$D(\varphi_\infty^{-1}\Delta_1^*)=\{f\in C_0(\Omega)\,;\ \text{w}^*\text{-}\lim_{h\downarrow 0}(\exp(t\varphi_\infty^{-1}\Delta_1^*)\cdot f-f)/h \text{ exists in } L^\infty(\Omega)\}\,,$$

$$\text{w}^*\text{-}\lim_{h\downarrow 0}(\exp(t\varphi_\infty^{-1}\Delta_1^*)\cdot f-f)/h=\varphi_\infty^{-1}\Delta_1^*f\,,\qquad f\in D(\varphi_\infty^{-1}\Delta_1^*)\,.$$

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4) This equality has its origin in a kind remark by Prof. S. Ôharu.

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Yoshio KONISHI
Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Bunkyo-ku, Tokyo
Japan