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Approximation of solutions of differential equations in Hilbert space

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§ 1. Introduction.

The present note is concerned with the limiting behavior as $\varepsilon\to 0_+$ of solutions u_ε of the equation

$$(1-\varepsilon A)u_\varepsilon'(t)-Bu_\varepsilon(t)=f_\varepsilon(t)\,. \tag{1.1}$$

A and B are maximal dissipative linear operators in a complex Hilbert space H, A is self-adjoint and $D(A)\subset D(B)$. We wish to show that if $f_\varepsilon\to f$ and $u_\varepsilon(0)\to x$ in a suitable fashion, then u_ε converges to the solution u of

$$u'(t)-Bu(t)=f(t)\,,\qquad u(0)=x\,, \tag{1.2}$$

that $u_\varepsilon'\to u'$ and that the rate of convergence is $O(\sqrt{\varepsilon})$. The conditions imposed on A and B imply that B is relatively bounded with respect to A so that (1.1) is a singular perturbation of (1.2). Our results apply in particular when (1.2) is a partial differential equation of parabolic or of Schroedinger type.

Equation (1.1) arises in a variety of physical problems including fluid flow through a fissured rock [**1**], shear in second order fluids [**3**, **10**], soil mechanics [**9**], thermodynamics [**2**] and many others [**4**], and (1.2) is often used as an approximating model when the physical constant ε is small. Convergence of solutions of (1.1) to solutions of (1.2) was considered by T. W. Ting [**11**] in the following special situation: $H=L^2(\Omega)$ where Ω is a bounded open set in $\boldsymbol{R}^n$ with smooth boundary and A and B are, respectively, the realizations in $L^2(\Omega)$ of the partial differential operators

$$\mathcal{A}=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial}{\partial x_j}\Big)-a(x)\,,\qquad a(x)\geqq 0\,,$$

$$\mathcal{B}=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(b_{ij}\frac{\partial}{\partial x_j}\Big)-b(x)\,,\qquad b(x)\geqq 0\,,$$

under Dirichlet boundary conditions. The matrices (a_{ij}) and (b_{ij}) were assumed to be real, symmetric and uniformly positive definite in $\bar{\Omega}$ and the coefficients in $\mathcal{A}$ and in $\mathcal{B}$ were assumed to be smooth in $\bar{\Omega}$. If $f_\varepsilon(t)=f(t)\equiv 0$ and $u_\varepsilon(0)\equiv x\in H^4(\Omega)\cap H_0^2(\Omega)$ $(\varepsilon>0)$, Ting proved that $u_\varepsilon(t)$ converges in $L^2(\Omega)$ as $\varepsilon\to 0_+$, uniformly on $[0, \infty)$, and that the rate of convergence is $O(\sqrt{\varepsilon})$. Although it was claimed that the limit of u_ε is the solution of (1.2), Ting's proof does not show this and it turns out that the proof of this fact is decidedly nontrivial and in particular makes use of the convergence of u_ε', which is also not proved in [11]. We remark that our proof of the convergence of u_ε in the general case is substantially simpler and more direct than the proof given in [11] for the special case just discussed.

The main results of this paper are contained in Section 3. Our assumptions and the necessary preliminary results are given in Section 2.

§ 2. Preliminaries.

In this section we state our assumptions and a number of results which will be needed in Section 3. Many of the results of this section are known so we omit or only briefly sketch their proofs.

Let V and H be complex Hilbert spaces with $V\subset H$ algebraically and topologically such that V is dense in H. We denote the inner product and norm in H by $(\cdot,\cdot)$ and $|\cdot|$ respectively, and the norm in V by $\|\cdot\|$. Let $a(u, v)$ and $b(u, v)$ be continuous sesquilinear forms on V which satisfy the following conditions.

$$\begin{aligned} &a(u, v)=\overline{a(v, u)}, \qquad a(v, v)\leqq 0, \\ &\lambda_0|v|^2-a(v, v)\geqq c_0\|v\|^2, \qquad v\in V. \end{aligned} \tag{2.1}$$

$$\operatorname{Re} b(v, v)\leqq 0, \qquad |b(v, v)|\geqq c_1\|v\|^2, \qquad v\in V, \tag{2.2}$$

where λ_0, c_0 and c_1 are positive constants independent of v.

Let $D(A)$ be the set of $u\in V$ such that the mapping $v\to a(u, v)$ is continuous on V in the topology induced by H. $D(A)$ is the domain of a linear, self-adjoint, dissipative operator A in H such that

$$a(u, v)=(Au, v)$$

for all $u\in D(A)$ and $v\in V$ (see e.g., Lions [6]). If $\varepsilon>0$, $(1-\varepsilon A)$ is an isomorphism of $D(A)$ onto H and

$$\|(1-\varepsilon A)^{-1}f\|\leqq(\text{const.})\varepsilon^{-1/2}|f|,$$

$$|(1-\varepsilon A)^{-1}f|\leqq|f|, \qquad f\in H.$$

In a similar way, $b(u, v)$ defines a linear, maximal dissipative operator B in H such that

$$b(u, v) = (Bu, v)$$

for all $u \in D(B) \subset V$ and $v \in V$, and B is an isomorphism of $D(B)$ onto H. We remark that the domains of A and B are dense both in H and in V.

Since B is maximal dissipative, it is the generator of a (C_0)-semigroup $\{e^{tB} : t \geqq 0\}$ of contractions on H. If in (2.2) the condition $\operatorname{Re} b(v, v) \equiv 0$ is satisfied, then B is skew-adjoint and this semigroup may be extended to a (C_0)-group $\{e^{tB} : -\infty < t < +\infty\}$ of unitary operators on H.

THEOREM 2.1. *Suppose B satisfies* (2.2). *Let $f \in C([0, \infty); H)$ such that f is locally strongly absolutely continuous and differentiable a.e. with derivative $f' \in L^{\infty}_{\text{loc}}([0, \infty); H)$. Let $x \in D(B)$. There is one and only one function $u \in C'([0, \infty); H)$ such that $u(0) = x$, $u(t) \in D(B)$ and*

$$u'(t) - Bu(t) = f(t) \tag{2.3}$$

for all $t > 0$.

This result is proved in [7] when $f \in C'([0, \infty); H)$. A slight modification of that proof yields Theorem 2.1. $u(t)$ is given by

$$u(t) = e^{tB}x + \int_0^t e^{(t-s)B} f(s) ds . \tag{2.4}$$

COROLLARY 2.1. *Suppose B satisfies* (2.2) *and* $\operatorname{Re} b(v, v) \equiv 0$. *Let $f \in C((-\infty, \infty); H)$ such that f is locally strongly absolutely continuous and differentiable a.e. with derivative $f' \in L^{\infty}_{\text{loc}}((-\infty, \infty); H)$. Let $x \in D(B)$. Then* (2.4) *is a function in $C'((-\infty, \infty); H)$, $u(t) \in D(B)$ and satisfies* (2.3) *everywhere in $(-\infty, \infty)$.*

We now turn to the equation

$$(1 - \varepsilon A) u'_{\varepsilon}(t) - Bu_{\varepsilon}(t) = f_{\varepsilon}(t) . \tag{2.5}$$

DEFINITION. A *solution* of (2.5) is a function $u_{\varepsilon} \in C((-\infty, \infty); V)$ which is locally strongly absolutely continuous and differentiable a.e. such that $u_{\varepsilon}(t) \in D(B)$, $u'_{\varepsilon}(t) \in D(A)$ and (2.5) is satisfied a.e. in $(-\infty, \infty)$.

The proof of the following result is essentially contained in [8] (c.f. [5]).

THEOREM 2.2. *Suppose A and B satisfy* (2.1) *and* (2.2) *and* $D(A) \subset D(B)$. *Let $\varepsilon > 0$, $f_{\varepsilon} \in L^1_{\text{loc}}((-\infty, \infty); H)$ and $x_{\varepsilon} \in D(B)$. Then* (2.5) *has a unique solution $u_{\varepsilon}(t)$ such that $u_{\varepsilon}(0) = x_{\varepsilon}$. Moreover, $u_{\varepsilon} \in C((-\infty, \infty); D_B)$ and $u'_{\varepsilon} \in L^1_{\text{loc}}((-\infty, \infty); D_A)$ where D_A (respectively, D_B) is the Hilbert space $D(A)$ (respectively, $D(B)$) equipped with the norm $(|Au|^2 + |u|^2)^{1/2}$ (respectively, $|Bu|$). If $f_{\varepsilon} \in C((-\infty, \infty); H)$ then $u'_{\varepsilon} \in C((-\infty, \infty); D_A)$ and* (2.5) *is satisfied everywhere in $(-\infty, \infty)$.*

In fact the unique solution of (2.5) is given by

$$u_{\varepsilon}(t) = e^{tB_{\varepsilon}} x_{\varepsilon} + \int_0^t e^{(t-s)B_{\varepsilon}} (1 - \varepsilon A)^{-1} f_{\varepsilon}(s) ds \tag{2.6}$$

where B_{ε} is the bounded linear operator on V given by

$$B_\varepsilon = \text{cl}\,(1-\varepsilon A)^{-1}B$$

where $\text{cl}\,(1-\varepsilon A)^{-1}B$ means the closure of $(1-\varepsilon A)^{-1}B$ as an operator in V and $\{e^{tB_\varepsilon}: -\infty < t < +\infty\}$ is the corresponding uniformly continuous group generated by B_ε.

LEMMA 2.1. *Let A and B satisfy* (2.1) *and* (2.2). *Then* $D_B \subset V$ *algebraically and topologically and if* $D(A) \subset D(B)$, *then* $D_A \subset D_B$ *topologically.*

PROOF. From (2.2), for $u \in D(B)$ we have

$$\|u\|^2 \leqq c_1^{-1}|b(u, u)| = c_1^{-1}|(Bu, u)| \leqq (\text{const.})|Bu|\,\|u\|$$

which proves the first part of the lemma. If also $D(A) \subset D(B)$ then as a consequence of the closed graph theorem we have (see e.g. Yosida [12])

$$|Bu| \leqq (\text{const.})(|Au| + |u|)$$

which proves the second part.

From this lemma follows that $(1-\varepsilon A)^{-1}B$ is a bounded linear operator on D_B whenever $D(A) \subset D(B)$, since

$$|B(1-\varepsilon A)^{-1}Bu| \leqq (\text{const.})(|A(1-\varepsilon A)^{-1}Bu| + |(1-\varepsilon A)^{-1}Bu|) \leqq K_\varepsilon |Bu|$$

where K_ε is a constant depending only on ε. It also follows from the lemma and the definition of B_ε that e^{tB_ε} along with B_ε maps $D(B)$ into itself and

$$e^{tB_\varepsilon}x = \exp(t(1-\varepsilon A)^{-1}B)x\,, \qquad x \in D(B)\,,$$

again provided that $D(A) \subset D(B)$, where $\{\exp(t(1-\varepsilon A)^{-1}B): -\infty < t < +\infty\}$ is the uniformly continuous group of operators on D_B generated by $(1-\varepsilon A)^{-1}B$.

Next we study the growth properties of e^{tB_ε}. For this we introduce for each $\varepsilon > 0$ and u, v in V

$$(u, v)_\varepsilon = (u, v) - \varepsilon a(u, v)\,. \tag{2.7}$$

(2.7) defines an inner product on V and the corresponding norm defined by

$$|u|_\varepsilon = \sqrt{(u, u)_\varepsilon}$$

is equivalent to $\|\cdot\|$ because of (2.1).

LEMMA 2.2. *Let A and B satisfy* (2.1) *and* (2.2). *Then for each* $\varepsilon > 0$ *and* $x \in V$,

$$|e^{tB_\varepsilon}x|_\varepsilon \leqq |x|_\varepsilon\,, \qquad t \geqq 0\,. \tag{2.8}$$

If also $\text{Re}\, b(v, v) \equiv 0$, *then*

$$|e^{tB_\varepsilon}x|_\varepsilon = |x|_\varepsilon\,, \qquad -\infty < t < +\infty\,.$$

PROOF. For $v \in D(B)$ we have

$$\operatorname{Re}(B_\varepsilon v, v)_\varepsilon = \operatorname{Re}((1-\varepsilon A)^{-1}Bv, v)_\varepsilon = \operatorname{Re}(Bv, v)$$
$$= \operatorname{Re} b(v, v) \leqq 0 .$$

Since $D(B)$ is dense in V we see that B_ε is dissipative on V with respect to the inner product $(\cdot,\cdot)_\varepsilon$. This implies (2.8). If $\operatorname{Re} b(v, v) \equiv 0$, then B_ε is skew-adjoint with respect to $(\cdot,\cdot)_\varepsilon$ so that the operators e^{tB_ε} are all unitary with respect to $|\cdot|_\varepsilon$.

The next two lemmas will be needed in the proof of Theorem 3.1.

LEMMA 2.3. *Let A satisfy* (2.1). *Then for $\varepsilon > 0$,*

$$((1-\varepsilon A)^{-1}Au, Au) \leqq -\varepsilon^{-1}(Au, u), \qquad u \in D(A). \tag{2.9}$$

PROOF. For $w \in D(A)$ we have

$$(Aw, w) - \varepsilon |Aw|^2 = (Aw, (1-\varepsilon A)w) \leqq 0 .$$

Setting $w = (1-\varepsilon A)^{-1}v$, $v \in D(A)$, we obtain

$$(A(1-\varepsilon A)^{-1}v, v) = ((1-\varepsilon A)^{-1}(\varepsilon Av + (1-\varepsilon A)v), Av) \leqq 0$$

which is the same as (2.9).

LEMMA 2.4. *Let A and B satisfy* (2.1) *and* (2.2). *Then $D(AB)$ is dense in D_B and, therefore, in V.*

PROOF. Suppose $(Bu, Bv) = 0$ for all $u \in D(AB)$ and some $v \in D(B)$. Set $w = (1-\varepsilon A)^{-1}Bv$, $\varepsilon > 0$. Then

$$0 = (Bu, (1-\varepsilon A)w) = ((1-\varepsilon A)Bu, w)$$

for all $u \in D(AB) = D((1-\varepsilon A)B)$. For $\varepsilon > 0$, $(1-\varepsilon A)B$ is an isomorphism of $D(AB)$ onto H and therefore $w = 0$ which implies $v = 0$.

§ 3. Main results.

THEOREM 3.1. *Suppose A and B satisfy* (2.1) *and* (2.2) *and that $D(A) \subset D(B)$. Then for each $x \in V$*

$$\lim_{\varepsilon \to 0+} |e^{tB_\varepsilon}x_\varepsilon - e^{tB}x| = 0$$

for any filter $\{x_\varepsilon\} \subset V$ such that $|x_\varepsilon - x|_\varepsilon \to 0$. Moreover, for each $x \in D(B)$ and any filter $\{x_\varepsilon\} \subset D(B)$ such that $|Bx_\varepsilon - Bx| \to 0$ we have

$$\lim_{\varepsilon \to 0+} |B_\varepsilon e^{tB_\varepsilon}x_\varepsilon - Be^{tB}x| = 0 .$$

In each case the convergence is uniform on bounded subsets of $[0, \infty)$ or, when $\operatorname{Re} b(v, v) \equiv 0$, *on bounded subsets of $(-\infty, \infty)$. In addition we have the estimate*

$$|e^{tB_\varepsilon}x - e^{tB}x| \leqq (\text{const.})\,|t|\sqrt{\varepsilon}\,(|Bx| + |ABx|)$$

for each $x \in D(AB)$ and for such x

$$\lim_{\varepsilon \to 0_+} |e^{tB_\varepsilon}x - e^{tB}x|_\varepsilon = 0 .$$

Theorem 3.1 is the key to the comparison of solutions of

(3.1) $$(1-\varepsilon A)u_\varepsilon'(t) - Bu_\varepsilon(t) = f_\varepsilon(t) , \qquad u_\varepsilon(0) = x_\varepsilon ,$$

with solutions of

(3.2) $$u'(t) - Bu(t) = f(t) , \qquad u(0) = x .$$

THEOREM 3.2. *Suppose A and B satisfy* (2.1) *and* (2.2), *that $D(A) \subset D(B)$ and that $\{f_\varepsilon\} \subset L^1_{\mathrm{loc}}((-\infty, \infty); H)$, $|f_\varepsilon(t)| \leqq F(t)$ where $F \in L^1_{\mathrm{loc}}(-\infty, \infty)$ and $|f_\varepsilon(t) - f(t)| \to 0$ a.e. in $(-\infty, \infty)$ as $\varepsilon \to 0_+$. Then*

$$\left| \int_0^t e^{(t-s)B_\varepsilon}(1-\varepsilon A)^{-1} f_\varepsilon(s)ds - \int_0^t e^{(t-s)B} f(s)ds \right| \to 0$$

as $\varepsilon \to 0_+$ for each $t \geqq 0$ or, if $\mathrm{Re}\, b(v, v) \equiv 0$, *for each $t \in (-\infty, \infty)$.*

COROLLARY 3.1. *In addition to the hypothesis of Theorem* 3.2, *let f satisfy the conditions of Corollary* 2.1, *$\{x_\varepsilon\} \subset D(B)$, $x \in D(B)$ and $|x_\varepsilon - x|_\varepsilon \to 0$. If u_ε is the solution of* (3.1) *and u the solution of* (3.2), *then $u_\varepsilon(t) \to u(t)$ in H as $\varepsilon \to 0_+$ for each $t \geqq 0$ or, if* $\mathrm{Re}\, b(v, v) \equiv 0$, *for each t in $(-\infty, \infty)$.*

THEOREM 3.3. *Suppose A and B satisfy* (2.1) *and* (2.2), *that $D(A) \subset D(B)$ and that*

(i) *$\{x_\varepsilon\} \subset D(B)$, $x \in D(B)$ and $|Bx_\varepsilon - Bx| \to 0$ as $\varepsilon \to 0_+$.*

(ii) *f and f_ε are locally strongly absolutely continuous and differentiable a.e. in $(-\infty, \infty)$,*

$$f' \in L^\infty_{\mathrm{loc}}((-\infty, \infty); H) , \qquad |f_\varepsilon(t)| + |f_\varepsilon'(t)| \leqq F(t)$$

where $F \in L^1_{\mathrm{loc}}(-\infty, \infty)$, and as $\varepsilon \to 0_+$

$$|f_\varepsilon(t) - f(t)| \to 0, \qquad |f_\varepsilon'(t) - f'(t)| \to 0$$

a.e. in $(-\infty, \infty)$.

Let u_ε be the solution of (3.1) *and u the solution of* (3.2). *Then as $\varepsilon \to 0_+$*

$$|u_\varepsilon(t) - u(t)| \to 0 , \qquad |u_\varepsilon'(t) - u'(t)| \to 0$$

for each $t \geqq 0$ or, if $\mathrm{Re}\, b(v, v) \equiv 0$, *for each t in $(-\infty, \infty)$.*

PROOF OF THEOREM 3.1. Since

$$|e^{tB_\varepsilon}x_\varepsilon - e^{tB}x| \leqq |e^{tB_\varepsilon}(x_\varepsilon - x)| + |e^{tB_\varepsilon}x - e^{tB}x|$$
$$\leqq |x_\varepsilon - x|_\varepsilon + |e^{tB_\varepsilon}x - e^{tB}x| ,$$

to prove the first part it suffices to show that $|e^{tB_\varepsilon}x - e^{tB}x|$ tends to zero as $\varepsilon \to 0_+$, uniformly on bounded t sets.

Let $\varepsilon > \eta > 0$ and $x \in V$. We write

$$e^{tB_\eta}x - e^{tB_\varepsilon}x = \int_0^t \frac{d}{ds}(e^{(t-s)B_\varepsilon}e^{sB_\eta}x)ds$$

$$= \int_0^t e^{(t-s)B_\varepsilon}(B_\eta - B_\varepsilon)e^{sB_\eta}x\, ds\,.$$

By Lemma 2.2 we have

$$|e^{tB_\varepsilon}x - e^{tB_\eta}x|_\varepsilon \leqq \int_0^t |(B_\varepsilon - B_\eta)e^{sB_\eta}x|_\varepsilon ds\,.$$

Suppose $x \in D(B)$. Then $v_\eta(s) \equiv e^{sB_\eta}x \in D(B)$ and

$$\begin{aligned}(B_\varepsilon - B_\eta)v_\eta(s) &= ((1-\varepsilon A)^{-1} - (1-\eta A)^{-1})Bv_\eta(s)\\ &= (\varepsilon-\eta)(1-\varepsilon A)^{-1}A(1-\eta A)^{-1}Bv_\eta(s)\\ &= (\varepsilon-\eta)(1-\varepsilon A)^{-1}AB_\eta v_\eta(s)\,.\end{aligned}$$

Therefore

$$\begin{aligned}|(B_\varepsilon - B_\eta)v_\eta(s)|_\varepsilon^2 &= ((1-\varepsilon A)(B_\varepsilon - B_\eta)v_\eta(s),\ (B_\varepsilon - B_\eta)v_\eta(s))\\ &= (\varepsilon-\eta)^2(AB_\eta v_\eta(s),\ (1-\varepsilon A)^{-1}AB_\eta v_\eta(s))\,.\end{aligned}$$

Applying Lemma 2.3 we have

$$|(B_\varepsilon - B_\eta)v_\eta(s)|_\varepsilon^2 \leqq -\varepsilon\Big(1-\frac{\eta}{\varepsilon}\Big)^2(AB_\eta v_\eta(s),\ B_\eta v_\eta(s))\,.$$

The negative of the inner product on the right equals

$$\begin{aligned}|a(B_\eta v_\eta(s),\ B_\eta v_\eta(s))| &\leqq (\text{const.})\|B_\eta v_\eta(s)\|^2\\ &\leqq (\text{const.})|b(B_\eta v_\eta(s),\ B_\eta v_\eta(s))|\\ &= (\text{const.})|(BB_\eta v_\eta(s),\ B_\eta v_\eta(s))|\\ &= (\text{const.})|((1-\eta A)B_\eta^2 v_\eta(s),\ B_\eta v_\eta(s))|\\ &= (\text{const.})|(B_\eta^2 v_\eta(s),\ B_\eta v_\eta(s))_\eta|\\ &= (\text{const.})|(e^{sB_\eta}B_\eta^2 x,\ e^{sB_\eta}B_\eta x)_\eta|\end{aligned}$$

where (const.) does not depend on ε, η, s or x. Applying the Schwarz inequality and Lemma 2.2 we obtain

$$|(B_\varepsilon - B_\eta)v_\eta(s)|_\varepsilon^2 \leqq (\text{const.})\varepsilon\Big(1-\frac{\eta}{\varepsilon}\Big)^2 |B_\eta^2 x|_\eta |B_\eta x|_\eta$$

provided $x \in D(B)$. For such x

$$|B_\eta x|_\eta^2 = (Bx,\ (1-\eta A)^{-1}Bx) \leqq |Bx|^2\,.$$

Now suppose $x \in D(AB)$. By Lemma 2.1

$$\begin{aligned}|B_\eta^2 x|_\eta^2 &= (B(1-\eta A)^{-1}Bx,\ (1-\eta A)^{-1}B(1-\eta A)^{-1}Bx)\\ &\leqq |B(1-\eta A)^{-1}Bx|^2\end{aligned}$$

$$\leqq (\text{const.})(|A(1-\eta A)^{-1}Bx|^2+|(1-\eta A)^{-1}Bx|^2)$$
$$\leqq (\text{const.})(|ABx|^2+|Bx|^2).$$

We have therefore proved that for $x \in D(AB)$ and $t \geqq 0$,

$$|e^{tB_\varepsilon}x - e^{tB_\eta}x|_\varepsilon \leqq (\text{const.})|t|\sqrt{\varepsilon}\left(1-\frac{\eta}{\varepsilon}\right)(|Bx|+|ABx|) \tag{3.3}$$

provided $\varepsilon > \eta > 0$. An analogous argument gives the same inequality for all t whenever $\operatorname{Re} b(v, v) \equiv 0$. Since $|\cdot| \leqq |\cdot|_\varepsilon$ for each $\varepsilon > 0$, we see that $e^{tB_\varepsilon}x$ converges in H as $\varepsilon \to 0_+$ for each $x \in D(AB)$, uniformly on bounded subsets of $[0, \infty)$ or, if $\operatorname{Re} b(v, v) \equiv 0$, uniformly on bounded subsets of $(-\infty, \infty)$. Therefore, for each $x \in D(AB)$ we may define

$$S(t)x = \lim_{\varepsilon \to 0_+} e^{tB_\varepsilon}x \tag{3.4}$$

the limit being taken in H. From (3.3) we have

$$|e^{tB_\varepsilon}x - S(t)x| \leqq (\text{const.})|t|\sqrt{\varepsilon}(|Bx|+|ABx|).$$

We wish to show that $S(t)x \in V$ and that

$$\lim_{\varepsilon \to 0_+} |e^{tB_\varepsilon}x - S(t)x|_\varepsilon = 0, \qquad x \in D(AB). \tag{3.5}$$

Let $\varepsilon > 0$ be fixed. From (3.3) we obtain

$$|e^{tB_\eta}x|_\varepsilon \leqq K(t, \varepsilon, x)$$

for all η in $(0, \varepsilon]$. Thus if V_ε is the Hilbert space $(V, |\cdot|_\varepsilon)$, then the set $\{e^{tB_\eta}x : 0 < \eta \leqq \varepsilon\}$ is bounded in V_ε and therefore $\{\exp(tB_{\eta_k})x\}$ converges weakly in V_ε for some sequence $\eta_k \downarrow 0$. Since $V_\varepsilon = V$ algebraically and topologically, this sequence converges weakly in V and, therefore, in H since V is continuously embedded in H. The limit of this sequence must be $S(t)x$ because of (3.4). It follows that $e^{tB_\eta}x$ converges weakly in V to $S(t)x$ as $\eta \to 0_+$, for each $x \in D(AB)$. Therefore $S(t)x \in V$ and $\{\|e^{tB_\eta}x\| : 0 < \eta \leqq \varepsilon\}$ is bounded. Since

$$|e^{tB_\eta}x - S(t)x|_\eta^2 = |e^{tB_\eta}x - S(t)x|^2$$
$$-\eta a(e^{tB_\eta}x - S(t)x, e^{tB_\eta}x - S(t)x)$$

and the right side goes to zero as $\eta \to 0_+$ by what we have proved, the desired conclusion follows.

Now let $x \in V$ and $\{x_k\} \subset D(AB)$ such that $\|x - x_k\| < \frac{1}{k}$. Then $|x - x_k|_\varepsilon < \frac{1}{k}$ for all sufficiently small $\varepsilon > 0$. If $\varepsilon > \eta > 0$ we have

$$|e^{tB_\varepsilon}x - e^{tB_\eta}x| \leqq |e^{tB_\varepsilon}(x-x_k)|_\varepsilon + |(e^{tB_\varepsilon} - e^{tB_\eta})x_k| + |e^{tB_\eta}(x-x_k)|_\eta$$
$$\leqq |x-x_k|_\varepsilon + |x-x_k|_\eta + |(e^{tB_\varepsilon} - e^{tB_\eta})x_k|.$$

Thus

$$\limsup_{\varepsilon\to 0+} |e^{tB_\varepsilon}x - e^{tB_\eta}x| \leqq \frac{2}{k}, \qquad k=1, 2, \cdots,$$

uniformly on bounded t sets, and so $S(t)x$ may be defined by (3.4) for each $x\in V$. The mapping $t\to S(t)x$ of $[0, \infty)$ (or of $(-\infty, \infty)$) into H is continuous since for each of the approximations e^{tB_ε} we have

$$|e^{tB_\varepsilon}x - e^{sB_\varepsilon}x| \leqq |x - e^{(s-t)B_\varepsilon}x|_\varepsilon \to 0$$

as $t-s\to 0$. Also, for $x\in V$ and each t,

$$|S(t)x| = \lim_{\varepsilon\to 0+} |e^{tB_\varepsilon}x| \leqq \lim_{\varepsilon\to 0+} |x|_\varepsilon = |x| .$$

Since V is dense in H, $S(t)$ may be extended by continuity to a bounded linear operator on H. We next show that $S(t+s)=S(t)S(s)$. Let $x\in D(AB)$. Then $S(\tau)x\in V$ and we have

$$|S(t+s)x - S(t)S(s)x| \leqq |S(t+s)x - e^{(t+s)B_\varepsilon}x| + |e^{tB_\varepsilon}(e^{sB_\varepsilon}x - S(s)x)| + |(e^{tB_\varepsilon} - S(t))S(s)x| .$$

The first and third terms on the right go to zero as $\varepsilon\to 0_+$ and the second is bounded by $|e^{sB_\varepsilon}x - S(s)x|_\varepsilon$ so that it too goes to zero by (3.5). Since $D(AB)$ is dense in H and $|S(\tau)x|\leqq |x|$, the desired conclusion follows easily.

We have therefore proved that $\{S(t): 0\leqq t<\infty\}$ is a (C_0)-semigroup of contractions and, if $\operatorname{Re} b(v, v)\equiv 0$, $\{S(t): -\infty<t<\infty\}$ is a (C_0)-group of unitary operators (since for a group, $|S(t)x|\leqq |x|$ for all t implies $|S(t)x|\geqq |x|$ for all t). We next prove that

$$\lim_{\varepsilon\to 0+} |B_\varepsilon e^{tB_\varepsilon}x_\varepsilon - S(t)Bx| = 0$$

for each $x\in D(B)$, uniformly on bounded t sets, provided $\{x_\varepsilon\}\subset D(B)$ and $|Bx_\varepsilon - Bx|\to 0$. Since

$$|e^{tB_\varepsilon}B_\varepsilon x_\varepsilon - S(t)Bx| \leqq |B_\varepsilon(x_\varepsilon - x)|_\varepsilon + |e^{tB_\varepsilon}B_\varepsilon x - S(t)Bx| \leqq |Bx_\varepsilon - Bx| + |e^{tB_\varepsilon}B_\varepsilon x - S(t)Bx|,$$

we may assume that $x_\varepsilon\equiv x$. Let $\{x_k\}\subset D(AB)$ such that $|Bx_k - Bx| < \frac{1}{k}$. We have

$$\begin{aligned} |e^{tB_\varepsilon}B_\varepsilon x - S(t)Bx| &\leqq |e^{tB_\varepsilon}(B_\varepsilon x - B_\varepsilon x_k)|_\varepsilon \\ &\quad + |e^{tB_\varepsilon}(B_\varepsilon x_k - Bx_k)|_\varepsilon + |(e^{tB_\varepsilon} - S(t))Bx_k| \\ &\quad + |S(t)(Bx_k - Bx)| \\ &\leqq |B_\varepsilon(x - x_k)|_\varepsilon + |(B_\varepsilon - B)x_k|_\varepsilon + |Bx_k - Bx| \\ &\quad + |(e^{tB_\varepsilon} - S(t))Bx_k| . \end{aligned}$$

The first term on the right does not exceed $|Bx-Bx_k|$ and the second and fourth terms approach zero with ε. Therefore

$$\limsup_{\varepsilon\to 0+} |e^{tB_\varepsilon}B_\varepsilon x-S(t)Bx|\leqq\frac{2}{k}, \qquad k=1, 2, \cdots$$

for each $x\in D(B)$, uniformly on bounded t sets.

The only thing left to prove is $S(t)=e^{tB}$. Let $x\in D(B)$. Then

$$S(t)x-x=\lim_{\varepsilon\to 0+}\int_0^t e^{sB_\varepsilon}B_\varepsilon x\,ds=\int_0^t \lim_{\varepsilon\to 0+} e^{sB_\varepsilon}B_\varepsilon x\,ds$$
$$=\int_0^t S(s)Bx\,ds$$

and so

$$\lim_{t\to 0+}\frac{1}{t}(S(t)x-x)=Bx.$$

Thus the infinitesimal generator $\hat{B}$ of $S(t)$ is an extension of B. But both B and $\hat{B}$ are maximal dissipative operators and must therefore coincide. It follows that $S(t)=e^{tB}$.

PROOF OF THEOREM 3.2. The conclusion of the theorem is a consequence of the dominated convergence theorem. In fact we have

$$|e^{(t-s)B_\varepsilon}(1-\varepsilon A)^{-1}f_\varepsilon(s)|\leqq|(1-\varepsilon A)^{-1}f_\varepsilon(s)|_\varepsilon$$
$$\leqq|f_\varepsilon(s)|\leqq F(s)$$

and

$$|e^{(t-s)B_\varepsilon}(1-\varepsilon A)^{-1}f_\varepsilon(s)-e^{(t-s)B}f(s)|\leqq|(1-\varepsilon A)^{-1}(f_\varepsilon(s)-f(s))|_\varepsilon$$
$$+|(B_\varepsilon e^{(t-s)B_\varepsilon}-Be^{(t-s)B})B^{-1}f(s)|.$$

The right side clearly converges to zero a.e. in $(-\infty, \infty)$ as $\varepsilon\to 0_+$.

PROOF OF COROLLARY 3.1. This is a consequence of (2.4), (2.6) and Theorems 3.1 and 3.2.

PROOF OF THEOREM 3.3. The only thing left to prove is that $u'_\varepsilon(t)\to u'(t)$. Under the stated hypotheses we may write ([7], Section 6)

$$u'_\varepsilon(t)=B_\varepsilon e^{tB_\varepsilon}x_\varepsilon+e^{tB_\varepsilon}(1-\varepsilon A)^{-1}f_\varepsilon(0)+\int_0^t e^{(t-s)B_\varepsilon}(1-\varepsilon A)^{-1}f'_\varepsilon(s)ds, \tag{3.6}$$

$$u'(t)=Be^{tB}x+e^{tB}f(0)+\int_0^t e^{(t-s)B}f'(s)ds. \tag{3.7}$$

The first term in (3.6) converges to the first term in (3.7) by Theorem 3.1. Using the same argument as above one shows that the integral in (3.6) converges to the integral in (3.7) and that

$$\lim_{\varepsilon\to 0+}|e^{tB_\varepsilon}(1-\varepsilon A)^{-1}f_\varepsilon(0)-e^{tB}f(0)|=0.$$

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Addendum

Professor T. Kato has kindly communicated to the present author the following, very direct, proof of the convergence of u_ε to u. Kato defines

$$P_\varepsilon = B(1-\varepsilon A)^{-1} \in \mathcal{L}(H),$$

$$Q_\varepsilon = (1-\varepsilon A)^{-1} B \in \mathcal{L}(D_B)$$

and writes the solution of (2.5) as

$$u_\varepsilon(t) = e^{tQ_\varepsilon} x_\varepsilon + (1-\varepsilon A)^{-1} \int_0^t e^{(t-s)P_\varepsilon} f_\varepsilon(s) ds .$$

As is known, assumptions (2.1) and (2.2) imply the existence of a dissipative operator $R_\varepsilon \in \mathcal{L}(H)$ such that

$$B = (1-\varepsilon A)^{1/2} R_\varepsilon (1-\varepsilon A)^{1/2},$$

the domain relation being exact. In terms of R_ε the solution of (2.5) is

$$u_\varepsilon(t)=(1-\varepsilon A)^{-1/2}e^{tR_\varepsilon}(1-\varepsilon A)^{1/2}x_\varepsilon+(1-\varepsilon A)^{-1/2}\int_0^t e^{(t-s)R_\varepsilon}(1-\varepsilon A)^{-1/2}f_\varepsilon(s)ds$$

provided $x_\varepsilon\in V=D((1-\varepsilon A)^{1/2})$. It is to be noted that B^{-1} and R_ε^{-1} both belong to $\mathcal{L}(H)$. This follows from (2.2) since, for example,

$$\begin{aligned}|b(v, v)|&=(R_\varepsilon(1-\varepsilon A)^{1/2}v, (1-\varepsilon A)^{1/2}v)\\ &\geqq C_1\|v\|^2\geqq C_\varepsilon|(1-\varepsilon A)^{1/2}v|^2,\ v\in V,\end{aligned}$$

from some $C_\varepsilon>0$. Hence $|R_\varepsilon^{-1}|_H\leqq C_\varepsilon^{-1}$ and similarly one proves $|B^{-1}|_H<\infty$.

Kato next proves that $e^{tR_\varepsilon}\to e^{tB}$ as $\varepsilon\to 0$, strongly and uniformly in any finite subinterval of $[0, \infty)$. As is well known, this is equivalent to $R_\varepsilon^{-1}\to B^{-1}$ strongly as $\varepsilon\to 0$. To prove strong convergence of R_ε^{-1} to B^{-1}, fix an $\varepsilon_0>0$ and note that

$$\begin{aligned}R_\varepsilon^{-1}&=(1-\varepsilon_0 A)^{-1/2}(1-\varepsilon A)^{1/2}R_{\varepsilon 0}^{-1}(1-\varepsilon A)^{1/2}(1-\varepsilon_0 A)^{-1/2}\\ &\to(1-\varepsilon_0 A)^{-1/2}R_{\varepsilon 0}^{-1}(1-\varepsilon_0 A)^{-1/2}=B^{-1}\end{aligned}$$

as $\varepsilon\to 0$. It follows from what has been proved that $u_\varepsilon(t)\to u(t)$ strongly in H provided

$$(*)\qquad (1-\varepsilon A)^{1/2}x_\varepsilon\to x \quad \text{in } H$$

and $f_\varepsilon\to f$ in $L^1(H)$. Certain other results of the present paper can be proved in a similar way. Note that in (*), x need not be in V. If x is in V, then (*) is equivalent to the condition $|x_\varepsilon-x|_\varepsilon\to 0$ which appears earlier.